Logic and Metalogic:
Logic, Metalogic, Fuzzy and Quantum Logics
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Introduction

Note. This book is based on the Wikipedia article, "Logic." The supporting articles are those referenced as major expansions of selected sections.
Logic

Logic is the art and science of reasoning which seeks to identify and understand the principles of valid demonstration and inference. Logic is a branch of philosophy and was part of the classical trivium. As a discipline, logic dates back to Aristotle and remains integral to fields such as mathematics, computer science, and linguistics. The word derives from the Greek λογική (logike), fem. of λογικός (logikos), "possessed of reason, intellectual, dialectical, argumentative", and from λόγος logos, "word, thought, idea, argument, account, reason, or principle".[1][2]

Logic concerns the structure of statements and arguments, in formal systems of inference and natural language. Topics include validity, fallacies and paradoxes, reasoning using probability and arguments involving causality. Logic is also commonly used today in argumentation theory.[3]

Nature of logic

The concept of logical form is central to logic; it being held that the validity of an argument is determined by its logical form, not by its content. Traditional Aristotelian syllogistic logic and modern symbolic logic are examples of formal logics.

• **Informal logic** is the study of natural language arguments. The study of fallacies is an especially important branch of informal logic. The dialogues of Plato[4] are a good example of informal logic.

• **Formal logic** is the study of inference with purely formal content, where that content is made explicit. (An inference possesses a purely formal content if it can be expressed as a particular application of a wholly abstract rule, that is, a rule that is not about any particular thing or property. The works of Aristotle contain the earliest known formal study of logic, which were incorporated in the late nineteenth century into modern formal logic.[5] In many definitions of logic, logical inference and inference with purely formal content are the same. This does not render the notion of informal logic vacuous, because no formal logic captures all of the nuance of natural language.)

• **Symbolic logic** is the study of symbolic abstractions that capture the formal features of logical inference.[6][7] Symbolic logic is often divided into two branches, propositional logic and → predicate logic.

• **Mathematical logic** is an extension of symbolic logic into other areas, in particular to the study of model theory, proof theory, set theory, and recursion theory.
Logic

Consistency, soundness, and completeness
Among the important properties that logical systems can have:

- **Consistency**, which means that no theorem of the system contradicts another.
- **Soundness**, which means that the system's rules of proof will never allow a false inference from a true premise. If a system is sound and its axioms are true then its theorems are also guaranteed to be true.
- **Completeness**, which means that there are no true sentences in the system that cannot, at least in principle, be proved in the system.

Some logical systems do not have all three properties. As an example, Kurt Gödel's incompleteness theorems show that no standard formal system of arithmetic can be consistent and complete. At the same time his theorems for first-order predicate logics not extended by specific axioms to be arithmetic formal systems with equality show those to be complete and consistent.

Rival conceptions of logic
Logic arose (see below) from a concern with correctness of argumentation. Modern logicians usually wish to ensure that logic studies just those arguments that arise from appropriately general forms of inference; so for example the Stanford Encyclopedia of Philosophy says of logic that it "does not, however, cover good reasoning as a whole. That is the job of the theory of rationality. Rather it deals with inferences whose validity can be traced back to the formal features of the representations that are involved in that inference, be they linguistic, mental, or other representations".

By contrast, Immanuel Kant argued that logic should be conceived as the science of judgment, an idea taken up in Gottlob Frege's logical and philosophical work, where thought (German: Gedanke) is substituted for judgement (German: Urteil). On this conception, the valid inferences of logic follow from the structural features of judgements or thoughts.

Deductive and inductive reasoning
Deductive reasoning concerns what follows necessarily from given premises. However, inductive reasoning—the process of deriving a reliable generalization from observations—has sometimes been included in the study of logic. Correspondingly, we must distinguish between deductive validity and inductive validity (called "cogency"). An inference is deductively valid if and only if there is no possible situation in which all the premises are true and the conclusion false.

The notion of deductive validity can be rigorously stated for systems of formal logic in terms of the well-understood notions of semantics. Inductive validity on the other hand requires us to define a reliable generalization of some set of observations. The task of providing this definition may be approached in various ways, some less formal than others; some of these definitions may use mathematical models of probability. For the most part this discussion of logic deals only with deductive logic.
History of logic

The earliest sustained work on the subject of logic is that of Aristotle,\(^{[10]}\) In contrast with other traditions, \(\rightarrow\) Aristotelian logic became widely accepted in science and mathematics, ultimately giving rise to the formally sophisticated systems of modern logic.

Several ancient civilizations have employed intricate systems of reasoning and asked questions about logic or propounded logical paradoxes. In India, the Nasadiya Sukta of the Rigveda (RV 10.129) contains ontological speculation in terms of various logical divisions that were later recast formally as the four circles of catuskoti: "A", "not A", "A and not A", and "not A and not not A".\(^{[11]}\) The Chinese philosopher Gongsun Long (ca. 325–250 BC) proposed the paradox "One and one cannot become two, since neither becomes two."\(^{[12]}\)

Also, the Chinese 'School of Names' is recorded as having examined logical puzzles such as "A White Horse is not a Horse" as early as the fifth century BCE.\(^{[13]}\) In China, the tradition of scholarly investigation into logic, however, was repressed by the Qin dynasty following the legalist philosophy of Han Feizi.

Logic in Islamic philosophy also contributed to the development of modern logic, which included the development of "Avicennian logic" as an alternative to Aristotelian logic. Avicenna's system of logic was responsible for the introduction of hypothetical syllogism,\(^{[14]}\) temporal \(\rightarrow\) modal logic,\(^{[15]}\)\(^{[16]}\) and inductive logic.\(^{[17]}\)\(^{[18]}\) The rise of the Asharite school, however, limited original work on logic in Islamic philosophy, though it did continue into the 15th century and had a significant influence on European logic during the Renaissance.

In India, innovations in the scholastic school, called Nyaya, continued from ancient times into the early 18th century, though it did not survive long into the colonial period. In the 20th century, Western philosophers like Stanislaw Schayer and Klaus Glashoff have tried to explore certain aspects of the Indian tradition of logic.

During the later medieval period, major efforts were made to show that Aristotle's ideas were compatible with Christian faith. During the later period of the Middle Ages, logic became a main focus of philosophers, who would engage in critical logical analyses of philosophical arguments.

The syllogistic logic developed by Aristotle predominated until the mid-nineteenth century when interest in the foundations of mathematics stimulated the development of symbolic logic (now called \(\rightarrow\) mathematical logic). In 1854, George Boole published An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities, introducing symbolic logic and the principles of what is now known as Boolean logic. In 1879 Frege published Begriffsschrift which inaugurated modern logic with the invention of quantifier notation. In 1903 Alfred North Whitehead and Bertrand Russell published Principia Mathematica\(^{[6]}\) on the foundations of mathematics, attempting to derive mathematical truths from axioms and inference rules in symbolic logic. In 1931 Gödel raised serious problems with the foundationalist program and logic ceased to focus on such issues.

The development of logic since Frege, Russell and Wittgenstein had a profound influence on the practice of philosophy and the perceived nature of philosophical problems (see Analytic philosophy), and Philosophy of mathematics. Logic, especially sentential logic, is implemented in computer logic circuits and is fundamental to computer science. Logic is commonly taught by university philosophy departments often as a compulsory discipline.
Topics in logic

Syllogistic logic

The *Organon* was Aristotle's body of work on logic, with the *Prior Analytics* constituting the first explicit work in formal logic, introducing the syllogistic. The parts of syllogistic, also known by the name term logic, were the analysis of the judgements into propositions consisting of two terms that are related by one of a fixed number of relations, and the expression of inferences by means of syllogisms that consisted of two propositions sharing a common term as premise, and a conclusion which was a proposition involving the two unrelated terms from the premises.

Aristotle's work was regarded in classical times and from medieval times in Europe and the Middle East as the very picture of a fully worked out system. It was not alone: the Stoics proposed a system of propositional logic that was studied by medieval logicians; nor was the perfection of Aristotle's system undisputed; for example the problem of multiple generality was recognised in medieval times. Nonetheless, problems with syllogistic logic were not seen as being in need of revolutionary solutions.

Today, some academics claim that Aristotle's system is generally seen as having little more than historical value (though there is some current interest in extending term logics), regarded as made obsolete by the advent of propositional logic and the predicate calculus. Others use Aristotle in argumentation theory to help develop and critically question argumentation schemes that are used in artificial intelligence and legal arguments.

Sentential (propositional) logic

A propositional calculus or logic (also a sentential calculus) is a formal system in which formulae representing propositions can be formed by combining atomic propositions using logical connectives, and a system of formal proof rules allows certain formulæ to be established as "theorems".

Predicate logic

Predicate logic is the generic term for symbolic formal systems like first-order logic, Second-order logic, many-sorted logic or infinitary logic.

Whereas Aristotelian syllogistic logic specified the forms that the relevant part of the involved judgements took, predicate logic allows sentences to be analysed into subject and argument in several different ways, thus allowing predicate logic to solve the problem of multiple generality that had perplexed medieval logicians. Predicate logic provides an account of quantifiers general enough to express a wider set of arguments occurring in natural language.

The development of predicate logic is usually attributed to Gottlob Frege, who is also credited as one of the founders of analytical philosophy, but the formulation of predicate logic most often used today is the first-order logic presented in *Principles of Mathematical Logic* by David Hilbert and Wilhelm Ackermann in 1928. The analytical generality of the predicate logic allowed the formalisation of mathematics, and drove the investigation of set theory, allowed the development of Alfred Tarski's approach to model theory; it is no exaggeration to say that it is the foundation of modern mathematical logic.
Frege's original system of predicate logic was not first-, but second-order. Second-order logic is most prominently defended (against the criticism of Willard Van Orman Quine and others) by George Boolos and Stewart Shapiro.

**Modal logic**

In languages, modality deals with the phenomenon that sub-parts of a sentence may have their semantics modified by special verbs or modal particles. For example, "We go to the games" can be modified to give "We should go to the games", and "We can go to the games" and perhaps "We will go to the games". More abstractly, we might say that modality affects the circumstances in which we take an assertion to be satisfied.

The logical study of modality dates back to Aristotle, who was concerned with the alethic modalities of necessity and possibility, which he observed to be dual in the sense of De Morgan duality. While the study of necessity and possibility remained important to philosophers, little logical innovation happened until the landmark investigations of Clarence Irving Lewis in 1918, who formulated a family of rival axiomatizations of the alethic modalities. His work unleashed a torrent of new work on the topic, expanding the kinds of modality treated to include deontic logic and epistemic logic. The seminal work of Arthur Prior applied the same formal language to treat temporal logic and paved the way for the marriage of the two subjects. Saul Kripke discovered (contemporaneously with rivals) his theory of frame semantics which revolutionised the formal technology available to modal logicians and gave a new graph-theoretic way of looking at modality that has driven many applications in computational linguistics and computer science, such as dynamic logic.

**Informal reasoning**

The motivation for the study of logic in ancient times was clear: it is so that one may learn to distinguish good from bad arguments, and so become more effective in argument and oratory, and perhaps also to become a better person. Half of the works of Aristotle's Organon treat inference as it occurs in an informal setting, side by side with the development of the syllogistic, and in the Aristotelian school, these informal works on logic were seen as complementary to Aristotle's treatment of rhetoric.

This ancient motivation is still alive, although it no longer takes centre stage in the picture of logic; typically dialectical logic will form the heart of a course in critical thinking, a compulsory course at many universities.

Argumentation theory is the study and research of informal logic, fallacies, and critical questions as they relate to every day and practical situations. Specific types of dialogue can be analyzed and questioned to reveal premises, conclusions, and fallacies. Argumentation theory is now applied in artificial intelligence and law.
Mathematical logic

Mathematical logic really refers to two distinct areas of research: the first is the application of the techniques of formal logic to mathematics and mathematical reasoning, and the second, in the other direction, the application of mathematical techniques to the representation and analysis of formal logic.[19]

The earliest use of mathematics and geometry in relation to logic and philosophy goes back to the ancient Greeks such as Euclid, Plato, and Aristotle. [20] Many other ancient and medieval philosophers applied mathematical ideas and methods to their philosophical claims. [21]

The boldest attempt to apply logic to mathematics was undoubtedly the logicism pioneered by philosopher-logicians such as Gottlob Frege and Bertrand Russell: the idea was that mathematical theories were logical tautologies, and the programme was to show this by means to a reduction of mathematics to logic.[6] The various attempts to carry this out met with a series of failures, from the crippling of Frege’s project in his Grundgesetze by Russell’s paradox, to the defeat of Hilbert’s program by Gödel's incompleteness theorems.

Both the statement of Hilbert's program and its refutation by Gödel depended upon their work establishing the second area of mathematical logic, the application of mathematics to logic in the form of proof theory.[22] Despite the negative nature of the incompleteness theorems, Gödel’s completeness theorem, a result in model theory and another application of mathematics to logic, can be understood as showing how close logicism came to being true: every rigorously defined mathematical theory can be exactly captured by a first-order logical theory; Frege's proof calculus is enough to describe the whole of mathematics, though not equivalent to it. Thus we see how complementary the two areas of mathematical logic have been.

If proof theory and model theory have been the foundation of mathematical logic, they have been but two of the four pillars of the subject. Set theory originated in the study of the infinite by Georg Cantor, and it has been the source of many of the most challenging and important issues in mathematical logic, from Cantor's theorem, through the status of the Axiom of Choice and the question of the independence of the continuum hypothesis, to the modern debate on large cardinal axioms.

Recursion theory captures the idea of computation in logical and arithmetic terms; its most classical achievements are the undecidability of the Entscheidungsproblem by Alan Turing, and his presentation of the Church-Turing thesis.[23] Today recursion theory is mostly concerned with the more refined problem of complexity classes — when is a problem efficiently solvable? — and the classification of degrees of unsolvability.[24]

Philosophical logic

Philosophical logic deals with formal descriptions of natural language. Most philosophers assume that the bulk of “normal” proper reasoning can be captured by logic, if one can find the right method for translating ordinary language into that logic. Philosophical logic is essentially a continuation of the traditional discipline that was called "Logic" before the invention of mathematical logic. Philosophical logic has a much greater concern with the connection between natural language and logic. As a result, philosophical logicians have contributed a great deal to the development of non-standard logics (e.g., free logics, tense logics) as well as various extensions of classical logic (e.g., → modal logics), and non-standard semantics for such logics (e.g., Kripke’s technique of supervaluations in the
Logic and the philosophy of language are closely related. Philosophy of language has to do with the study of how our language engages and interacts with our thinking. Logic has an immediate impact on other areas of study. Studying logic and the relationship between logic and ordinary speech can help a person better structure their own arguments and critique the arguments of others. Many popular arguments are filled with errors because so many people are untrained in logic and unaware of how to correctly formulate an argument.

**Logic and computation**

Logic cut to the heart of computer science as it emerged as a discipline: Alan Turing’s work on the Entscheidungsproblem followed from Kurt Gödel’s work on the incompleteness theorems, and the notion of general purpose computers that came from this work was of fundamental importance to the designers of the computer machinery in the 1940s.

In the 1950s and 1960s, researchers predicted that when human knowledge could be expressed using logic with mathematical notation, it would be possible to create a machine that reasons, or artificial intelligence. This turned out to be more difficult than expected because of the complexity of human reasoning. In logic programming, a program consists of a set of axioms and rules. Logic programming systems such as Prolog compute the consequences of the axioms and rules in order to answer a query.

Today, logic is extensively applied in the fields of artificial intelligence, and computer science, and these fields provide a rich source of problems in formal and informal logic. Argumentation theory is one good example of how logic is being applied to artificial intelligence. The ACM Computing Classification System in particular regards:

- Section F.3 on Logics and meanings of programs and F.4 on Mathematical logic and formal languages as part of the theory of computer science: this work covers formal semantics of programming languages, as well as work of formal methods such as Hoare logic
- Boolean logic as fundamental to computer hardware: particularly, the system's section B.2 on Arithmetic and logic structures, relating to operands AND, NOT, and OR;
- Many fundamental logical formalisms are essential to section I.2 on artificial intelligence, for example → modal logic and default logic in Knowledge representation formalisms and methods, Horn clauses in logic programming, and description logic.

Furthermore, computers can be used as tools for logicians. For example, in symbolic logic and mathematical logic, proofs by humans can be computer-assisted. Using automated theorem proving the machines can find and check proofs, as well as work with proofs too lengthy to be written out by hand.

**Criticisms of logic**

Hegel was deeply critical of any simplified notion of the Law of Non-Contradiction. It was based on Leibniz’s idea that this law of logic also requires a sufficient ground in order to specify from what point of view (or time) one says that something cannot contradict itself, a building for example both moves and does not move, the ground for the first is our solar system for the second the earth. In Hegelian dialectic the law of non-contradiction, of identity, itself relies upon difference and so is not independently assertable.
Hegel developed his own dialectic logic that extended Kant's transcendental logic but also brought it back to ground by assuring us that "neither in heaven nor in earth, neither in the world of mind nor of nature, is there anywhere such an abstract 'either--or' as the understanding maintains. Whatever exists is concrete, with difference and opposition in itself."[25]

Nietzsche: "Logic, too, also rests on assumptions that do not correspond to anything in the real world."[26]

**Controversies in logic**

Just as we have seen there is disagreement over what logic is about, so there is disagreement about what logical truths there are.

**Bivalence and the law of the excluded middle**

The logics discussed above are all "→ bivalent" or "two-valued"; that is, they are most naturally understood as dividing propositions into true and false propositions. Non-classical logics are those systems which reject bivalence.

In 1910 Nicolai A. Vasiliev rejected the law of excluded middle and the law of contradiction and proposed the law of excluded fourth and logic tolerant to contradiction. In the early 20th century Jan Łukasiewicz investigated the extension of the traditional true/false values to include a third value, "possible", so inventing ternary logic, the first → multi-valued logic.

Logics such as → fuzzy logic have since been devised with an infinite number of "degrees of truth", represented by a real number between 0 and 1.[27]

Intuitionistic logic was proposed by L.E.J. Brouwer as the correct logic for reasoning about mathematics, based upon his rejection of the law of the excluded middle as part of his intuitionism. Brouwer rejected formalisation in mathematics, but his student Arend Heyting studied intuitionistic logic formally, as did Gerhard Gentzen. Intuitionistic logic has come to be of great interest to computer scientists, as it is a constructive logic, and is hence a logic of what computers can do.

→ Modal logic is not truth conditional, and so it has often been proposed as a non-classical logic. However, modal logic is normally formalised with the principle of the excluded middle, and its relational semantics is bivalent, so this inclusion is disputable.

**Implication: strict or material?**

It is obvious that the notion of implication formalised in classical logic does not comfortably translate into natural language by means of "if... then...", due to a number of problems called the paradoxes of material implication.

The first class of paradoxes involves counterfactuals, such as "If the moon is made of green cheese, then 2+2=5", which are puzzling because natural language does not support the principle of explosion. Eliminating this class of paradoxes was the reason for C. I. Lewis's formulation of strict implication, which eventually led to more radically revisionist logics such as relevance logic.

The second class of paradoxes involves redundant premises, falsely suggesting that we know the succedent because of the antecedent: thus "if that man gets elected, granny will die" is materially true if granny happens to be in the last stages of a terminal illness, regardless of the man's election prospects. Such sentences violate the Gricean maxim of
relevance, and can be modelled by logics that reject the principle of monotonicity of entailment, such as relevance logic.

**Tolerating the impossible**

Closely related to questions arising from the paradoxes of implication comes the suggestion that logic ought to tolerate inconsistency. Relevance logic and → paraconsistent logic are the most important approaches here, though the concerns are different: a key consequence of classical logic and some of its rivals, such as intuitionistic logic, is that they respect the principle of explosion, which means that the logic collapses if it is capable of deriving a contradiction. Graham Priest, the main proponent of dialetheism, has argued for paraconsistency on the grounds that there are in fact, true contradictions.\[^{28}\]

**Is logic empirical?**

What is the epistemological status of the laws of logic? What sort of argument is appropriate for criticising purported principles of logic? In an influential paper entitled "Is logic empirical?"\[^{29}\] Hilary Putnam, building on a suggestion of W.V. Quine, argued that in general the facts of propositional logic have a similar epistemological status as facts about the physical universe, for example as the laws of mechanics or of general relativity, and in particular that what physicists have learned about quantum mechanics provides a compelling case for abandoning certain familiar principles of classical logic: if we want to be realists about the physical phenomena described by quantum theory, then we should abandon the principle of distributivity, substituting for classical logic the → quantum logic proposed by Garrett Birkhoff and John von Neumann.\[^{30}\]

Another paper by the same name by Sir Michael Dummett argues that Putnam's desire for realism mandates the law of distributivity.\[^{31}\] Distributivity of logic is essential for the realist's understanding of how propositions are true of the world in just the same way as he has argued the principle of bivalence is. In this way, the question, "Is logic empirical?" can be seen to lead naturally into the fundamental controversy in metaphysics on realism versus anti-realism.

**See also**

- Logics
- Aristotle
- Plato
- Artificial intelligence
- Deductive reasoning
- Digital electronics (also known as digital logic)
- Indian Logic
- Inductive reasoning
- Logic puzzle
- Logical consequence
- Mathematical logic
- Metalogic
- Philosophy
  - List of basic philosophy topics
  - List of philosophy topics
- Probabilistic logic
- Propositional logic
- Reason
- Straight and Crooked Thinking (book)
- Table of logic symbols
- Term logic
- Truth
- False
• Mathematics
  • List of basic mathematics topics
  • List of mathematics articles

Notes

[8] Mendelson, "Quantification Theory: Completeness Theorems"
[22] Mendelson, "Formal Number Theory: Gödel's Incompleteness Theorem"
[23] Brookshear, "Computability: Foundations of Recursive Function Theory"
[24] Brookshear, "Complexity"

References

- Raymond m. Smullyan, First-order logic

Further reading

- The London Philosophy Study Guide (http://www.ucl.ac.uk/philosophy/LPSG/) offers many suggestions on what to read, depending on the student's familiarity with the subject:
  - Logic & Metaphysics (http://www.ucl.ac.uk/philosophy/LPSG/L&M.htm)
  - Set Theory and Further Logic (http://www.ucl.ac.uk/philosophy/LPSG/SetTheory.htm)
  - Mathematical Logic (http://www.ucl.ac.uk/philosophy/LPSG/MathLogic.htm)
- Carroll, Lewis
  - "Symbolic Logic" (http://durendal.org:8080/icsl/), 1896.

**External links**

• An Introduction to Philosophical Logic (http://www.galilean-library.org/manuscript.php?postid=43782), by Paul Newall, aimed at beginners.
• forall x: an introduction to formal logic (http://www.fecundity.com/logic/), by P.D. Magnus, covers sentential and quantified logic.
• Logic Self-Taught: A Workbook (http://www.filozofia.uw.edu.pl/kpaprzycka/Publ/xLogicSelfTaught.html) (originally prepared for on-line logic instruction).
• The limits of reason (http://www_integralbuddha_net/topic/general-science/the-limits-of-reason.php), by Gregory Chaitin
• Test your logic skills (http://www.think-logically.co.uk/lt.htm).
• Translation Tips (http://www.earlham.edu/~peters/courses/log/transtip.htm), by Peter Suber, for translating from English into logical notation.
• Ontology and History of Logic in Western Thought. An Introduction (http://www.formalontology.it/history-of-logic.htm) Annotated bibliography
History of logic

The history of logic is the study of the development of the science of valid inference (logic). While many cultures have employed intricate systems of reasoning, and logical methods are evident in all human thought, an explicit analysis of the principles of reasoning was developed only in three traditions: those of China, India, and Greece. Of these, only the treatment of logic descending from the Greek tradition, particularly Aristotelian logic, found wide application and acceptance in science and mathematics. The Greek tradition was further developed by Islamic logicians and then medieval European logicians. Not until the 19th century does the next great advance in logic arise, with the development of symbolic logic by George Boole and its subsequent development into formal calculable logical systems by Gottlob Frege and set theorists such as Georg Cantor and Giuseppe Peano, ushering in the Information Age.

Logic was known as 'dialectic' or 'analytic' in Ancient Greece. The word 'logic' (from the Greek logos, meaning discourse or sentence) does not appear in the modern sense until the commentaries of Alexander of Aphrodisias, writing in the third century A.D.

Logic in ancient Greece

Before Plato

People have employed valid reasoning in all periods of human history. However, logic studies the principles of valid reasoning, inference and demonstration, and there is almost no historic evidence of such study before the time of Plato and Aristotle. It is probable that the idea of demonstrating a conclusion first developed in connection with geometry, which originally meant the same as 'land measurement'. The ancient Egyptians discovered some truths of geometry, such as the formula for a truncated pyramid, empirically, but the great achievement of the ancient Greeks was to replace empirical methods by demonstrative science. The systematic study of this seems to have begun with the school of Pythagoras in the late sixth century B.C. The three basic principles of geometry are that certain propositions must be accepted as true...
History of logic

without demonstration, that all other propositions of the system are derived from these, and that the derivation must be formal, i.e. independent of the special subject matter in question. Fragments of early proofs are preserved in the works of Plato and Aristotle,[2] and it is probable that the idea of a deductive system was known in the Pythagorean school, and in the Platonic Academy.

Separately from geometry, the idea of a standard argument pattern is found in the *Reductio ad absurdum* used by Zeno of Elea, a pre-Socratic philosopher of the fifth century B.C. This is the technique of drawing an obviously false, absurd or impossible conclusion from an assumption, thus demonstrating that the assumption is false. Plato's Parmenides portrays Zeno as claiming to have written a book defending the monism of Parmenides by demonstrating the absurd consequence of assuming that there is plurality. Other philosophers who practised such *dialectic* reasoning were the so-called Minor Socratics, including Euclid of Megara, who were probably followers of Parmenides and Zeno. The members of this school were called 'dialecticians' (from a Greek word meaning 'to discuss').

Further evidence that pre-Aristotelian thinkers were concerned with the principles of reasoning is found in the fragment called *Dissoi Logoi*, probably written at the beginning of the fourth century B.C.[3] This is part of a protracted debate about truth and falsity.

**Plato's logic**

None of the surviving works of the great fourth century philosopher Plato (428 – 347) include any formal logic, but he is certainly the first major thinker in the field of → philosophical logic. Plato raises three important logical questions:

- What is it that can properly be called true or false?
- What is the nature of the connection between the assumptions of a valid argument and its conclusion?
- What is the nature of definition?

The first question arises in the dialogue Theaetetus in the attempt to define knowledge. Plato identifies thought or opinion with talk or discourse (*logos*): 'forming an opinion is talking, and opinion is speech that is held not with someone else or aloud but in silence with oneself' (*Theaetetus* 189E-190A).

The second question arises with Plato's theory of Forms. Forms are not things in the ordinary sense, nor strictly ideas in the mind, but they correspond to what philosophers later called universals, namely an abstract entity common to each set of things that have the same name. Both in The Republic and The Sophist it is strongly suggested by Plato that correct thinking is following out the connection between forms. The necessary connection between the premises and the conclusion of an argument is a relation between thoughts...
determined by the 'forms' which underlie the thoughts. The third question involves the nature of definition. Many of Plato's dialogues concern the search for a definition of some important concept (justice, truth, the Good), and it is likely that Plato was impressed by the importance of definition in mathematics. What underlies every definition is a Platonic Form, the common nature present in different particular things. Thus a definition reflects the ultimate object of our understanding, and is the foundation of all valid inference. Plato's conception of definition had a great influence on Aristotle, in particular Aristotle's notion of the essence of a thing, the 'what it is to be' a particular thing of a certain kind.

Aristotle's logic

The logic of Aristotle, and particularly his theory of the syllogism, has had an enormous influence in Western thought. His logical works, called the Organon, are the earliest formal study of logic that have come down to modern times. Though it is difficult to determine the dates, the probable order of writing of Aristotle's logical works is:

- The Categories, a study of the ten kinds of primitive term.
- Topics, with an appendix called On Sophistical Refutations, a discussion of dialectics.
- On Interpretation - an analysis of simple categorical propositions, into simple terms, nouns and verbs, negation, and signs of quantity.
- Prior Analytics a formal analysis of valid argument or 'syllogism'.
- Posterior Analytics a study of scientific demonstration, containing Aristotle's mature views on logic.

These works are of outstanding importance in the history of logic. Aristotle is the first logician to attempt a systematic analysis of logical syntax, into noun or term, and verb. In the Categories, he attempts to classify all the possible things that a term can refer to. This idea underpins his philosophical work, the Metaphysics, which later had a great influence on Western thought. Aristotle was the first formal logician (i.e. he gives the principles of reasoning using variables to show the underlying logical form of arguments). He is looking for relations of dependence which characterise necessary inference, and distinguishes the validity of these relations, from the truth of the premisses (the soundness of the argument). The Prior Analytics contains his exposition of the 'syllogistic', where three important principles are applied for the first time in history, the use of variables, a purely formal treatment, and the use of an axiomatic system.
Stoic logic

The other great school of Greek logic is that of the Stoics. Stoic logic traces its roots back to the late fifth century philosopher, Euclid of Megara, a pupil of Socrates and slightly older contemporary of Plato. He was probably a disciple of Parmenides. His pupils and successors were called 'Megarians', or 'Eristics', and later the 'Dialecticians'. Among his pupils were Eubulides (according to tradition), and Stilpo. Unlike with Aristotle, we have no complete works by writers of this school, and have to rely on accounts (sometimes hostile) of Sextus Empiricus, writing in the third century A.D. The three most important contributions of the Stoic school were (i) their account of modality, (ii) their theory of the Material conditional, and (iii) their account of meaning and truth.

(1) Modality. According to Aristotle, the Megarians of his day claimed there was no distinction between potentiality and actuality. Diodorus Cronus (2nd half 4th century BC) defined the possible as that which either is or will be, the impossible as what will not be true, and the contingent as that which either is already, or will be false. Diodorus is also famous for his so-called Master argument, that the three propositions 'everything that is past is true and necessary', 'The impossible does not follow from the impossible', and 'What neither is nor will be is possible' are inconsistent. Diodorus used the plausibility of the first two to prove that nothing is possible if it neither is nor will be true. Chrysippus (c.280–c.207 BC), by contrast, denied the second premiss and said that the impossible could follow from the possible.

(2) Conditional statements. The first logicians to debate conditional statements were Diodorus and his pupil Philo of Megara (fl. 300 BC). Sextus Empiricus refers three times to a debate between Diodorus and Philo. Philo argued that a true conditional is one that does not begin with a truth and end with a falsehood, such as 'if it is day, then I am talking'. But Diodorus argued that a true conditional is what could not possibly begin with a truth and end with falsehood - thus the conditional quoted above could be false if it were day and I became silent. Philo's criterion of truth is what would now be called a truth-functional definition of 'if ... then'. In a second reference, Sextus says 'According to him there are three ways in which a conditional may be true, and one in which it may be false'.

(3) Meaning and truth. The most important and striking difference between Megarian-Stoic logic and Aristotelian logic is that it concerns propositions, not terms, and is thus closer to modern propositional logic. The Stoics distinguished between utterance (phone), which may be noise, speech (lexis), which is articulate but which may be meaningless, and discourse (logos), which is meaningful utterance. The most original part of their theory is the idea that what is expressed by a sentence, called a lekton, is something real. This corresponds to what is now called a proposition. Sextus says that according to the Stoics, three things are linked together, that which is signified, that which signifies, and the object.
For example, what signifies is the word 'Dion', what is signified is what Greeks understand but barbarians do not, and the object is Dion himself.[11]

## Logic in ancient Asia

### Logic in India

Formal logic also developed in India, without the influence, so far as is known, of Greek logic.[12] Two of the six Indian schools of thought deal with logic: Nyaya and Vaisheshika. The Nyaya Sutras of Aksapada Gautama constitute the core texts of the Nyaya school, one of the six orthodox schools of Hindu philosophy. This realist school developed a rigid five-member schema of inference involving an initial premise, a reason, an example, an application and a conclusion. The idealist Buddhist philosophy became the chief opponent to the Naiyayikas. Nagarjuna, the founder of the Madhyamika "Middle Way" developed an analysis known as the "catuskoti" or tetralemma. This four-cornered argumentation systematically examined and rejected the affirmation of a proposition, its denial, the joint affirmation and denial, and finally, the rejection of its affirmation and denial. But it was with Dignaga and his successor Dharmakirti that Buddhist logic reached its height. Their analysis centered on the definition of necessary logical entailment, "vyapti", also known as invariable concomitance or pervasion. To this end a doctrine known as "apoha" or differentiation was developed. This involved what might be called inclusion and exclusion of defining properties. The difficulties involved in this enterprise, in part, stimulated the neo-scholastic school of Navya-Nyāya, which developed a formal analysis of inference in the sixteenth century.

### Logic in China

In China, a contemporary of Confucius, Mozi, "Master Mo", is credited with founding the Mohist school, whose canons dealt with issues relating to valid inference and the conditions of correct conclusions. In particular, one of the schools that grew out of Mohism, the Logicians, are credited by some scholars for their early investigation of formal logic. Unfortunately, due to the harsh rule of Legalism in the subsequent Qin Dynasty, this line of investigation disappeared in China until the introduction of Indian philosophy by Buddhists.
Logic in Islamic philosophy

For a time after the Prophet Muhammad's death, Islamic law placed importance on formulating standards of argument, which gave rise to a novel approach to logic in Kalam, but this approach was later displaced to some extent by ideas from Greek philosophy and Hellenistic philosophy with the rise of the Mu'tazili theologians, who highly valued Aristotle's *Organon*. The works of Hellenistic-influenced Islamic philosophers were crucial in the reception of Aristotelian logic in medieval Europe, along with the commentaries on the *Organon* by Averroes. The works of al-Farabi, Avicenna, al-Ghazali and other Muslim logicians who often criticized and corrected Aristotelian logic and introduced their own forms of logic, also played a central role in the subsequent development of medieval European logic.

Islamic logic not only included the study of formal patterns of inference and their validity but also elements of the philosophy of language and elements of epistemology and metaphysics. Due to disputes with Arabic grammarians, Islamic philosophers were interested in working out the relationship between logic and language, and they devoted much discussion to the question of the subject matter and aims of logic in relation to reasoning and speech. In the area of formal logical analysis, they elaborated upon the theory of terms, propositions and syllogisms. They considered the syllogism to be the form to which all rational argumentation could be reduced, and they regarded syllogistic theory as the focal point of logic. Even poetics was considered as a syllogistic art in some fashion by many major Islamic logicians.

Al-Farabi's logic

Though Al-Farabi (Alfarabi) (873–950) was mainly an Aristotelian logician, he introduced a number of non-Aristotelian elements of logic. He discussed the topics of future contingents, the number and relation of the categories, the relation between logic and grammar, and non-Aristotelian forms of inference. He is credited for categorizing logic into two separate groups, the first being "idea" and the second being "proof".\[13\] Al-Farabi also introduced the theories of conditional syllogism and analogical inference, which were not part of the Aristotelian tradition.\[14\] Another addition al-Farabi made to the Aristotelian tradition was his introduction of the concept of poetic syllogism in a commentary on Aristotle's *Poetics*.\[15\]
Avicennian logic

<table>
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Ibn Sina (Avicenna) (980–1037) developed his own system of logic known as "Avicennian logic" as an alternative to Aristotelian logic. After the Latin translations of the 12th century, Avicennian logic also influenced early medieval European logicians such as Albertus Magnus,[16] though Aristotelian logic later became more popular in Europe due to the strong influence of Averroism.

Avicenna developed an early theory on hypothetical syllogism, which formed the basis of his early risk factor analysis.[17] He also developed an early theory on → propositional calculus, which was an area of logic not covered in the Aristotelian tradition.[18] The first criticisms on Aristotelian logic were also written by Avicenna, who developed an original theory on temporal → modal syllogism.[13] He also contributed inventively to the development of inductive logic, being the first to describe the methods of agreement, difference and concomitant variation which are critical to inductive logic and the scientific method.[17]

Fakhr al-Din al-Razi (b. 1149) criticised Aristotle's "first figure" and formulated an early system of inductive logic, foreshadowing the system of inductive logic developed by John Stuart Mill (1806-1873).[19] Systematic refutations of Greek logic were written by the Illuminationist school, founded by Shahab al-Din Suhravardi (1155-1191), who developed the idea of "decisive necessity", which refers to the reduction of all modalities (necessity, possibility, contingency and impossibility) to the single mode of necessity.[20] Ibn al-Nafis (1213-1288) wrote a book on Avicennian logic, which was a commentary of Avicenna's Al-Isharat (The Signs) and Al-Hidayah (The Guidance).[21] Another systematic refutation of Greek logic was written by Ibn Taymiyyah (1263-1328), who wrote the ar-Radd 'ala al-Mantiqiyyin (Refutation of Greek Logicians), in which he gave a proof for induction being the only true form of argument, which had an important influence on the development of the scientific method of observation and experimentation.[19] The Sharh al-takmil fi'l-mantiq written by Muhammad ibn Fayd Allah ibn Muhammad Amin al-Shawrani in the 15th century was the last major Arabic work on logic.[22]
History of logic

Logic in medieval Europe

"Medieval logic" (also known as "Scholastic logic") generally means the form of Aristotelian logic developed in medieval Europe throughout the period c 1200–1600. For centuries after Stoic logic had been formulated, it was the dominant system of logic in the classical world. When the study of logic resumed after the Dark Ages, the main source was the work of the Christian philosopher Boethius. Alcuin, who taught at York in the eighth century AD, mentions that the library there contained Aristotle, Marius Victorinus, and Boethius (although we do not know how much of Aristotle was included there). Until the twelfth century the only works of Aristotle available in the West were the Categories, On Interpretation and Boethius' translation of the Isagoge of Porphyry (a commentary on the Categories). These works were known as the 'Old Logic' (Logica Vetus or Ars Vetus). A significant and original work on the old logic was the Logica Ingredientibus of Peter Abelard.

The 'rediscovery' of the works of antiquity began in the Latin West in the late twelfth century, when Arabic texts on Aristotelian logic and works by Islamic logicians were translated into Latin. While the logic of Arabic writers such as Avicenna had an influence on early medieval European logicians such as Albertus Magnus,[23] the Aristotelian tradition became more dominant due to the strong influence of Averroism.

After the initial translation phase, the tradition of Medieval logic was developed through textbooks such as that by Peter of Spain (fl. thirteenth century), whose exact identity is unknown, who was the author of a standard textbook on logic, the Tractatus, which was well known in Europe for many centuries. The tradition reached its high point in the fourteenth century, with the works of William of Ockham (c. 1287–1347) and Jean Buridan.

One feature of the development of Aristotelian logic through what is known as supposition theory, a study of the semantics of the terms of the proposition.

The last great works in this tradition are the Logic of John Poinsot (1589–1644, known as John of St Thomas), and the Metaphysical Disputations of Francisco Suarez (1548–1617).
Traditional logic

"Traditional Logic" generally means the textbook tradition that begins with Antoine Arnauld and Pierre Nicole's *Logic, or the Art of Thinking*, better known as the *Port-Royal Logic*. Published in 1662, it was the most influential work on logic in England until Mill's *System of Logic* in 1825. The book presents a loosely Cartesian doctrine (that the proposition is a combining of ideas rather than terms, for example) within a framework that is broadly derived from Aristotelian and medieval term logic. Between 1664 and 1700 there were eight editions, and the book had considerable influence after that. It was frequently reprinted in English up to the end of the nineteenth century. It is still reprinted today, but mostly for historical purposes.

The account of propositions that Locke gives in the *Essay* is essentially that of Port-Royal: "Verbal propositions, which are words, [are] the signs of our ideas, put together or separated in affirmative or negative sentences. So that proposition consists in the putting together or separating these signs, according as the things which they stand for agree or disagree." (Locke, *An Essay Concerning Human Understanding*, IV. 5. 6)

Works in this tradition include Isaac Watts' *Logick: Or, the Right Use of Reason* (1725), Richard Whately's *Logic* (1826), and John Stuart Mill's *A System of Logic* (1843), which was one of the last great works in the tradition.

Another influential work was the *Novum Organum* by Francis Bacon, published in 1620. The title translates as "new instrument". This is a reference to Aristotle's work *Organon*. In this work, Bacon repudiated the syllogistic method of Aristotle in favour of an alternative procedure 'which by slow and faithful toil gathers information from things and brings it into understanding' (Farrington, 1964, 89). This method is known as Induction. The inductive method starts from empirical observation and proceeds to lower axioms or propositions. From the lower axioms more general ones can be derived (by induction). In finding the cause of a *phenomenal nature* such as heat, one must list all of the situations where heat is found. Then another list should be drawn up, listing situations that are similar to those of the first list except for the lack of heat. A third table lists situations where heat can vary. The *form nature*, or cause, of heat must be that which is common to all instances in the first table, is lacking from all instances of the second table and varies by degree in instances of the third table.
Non-Traditional logic

Some major philosophers made moves to step outside traditional logic, see Kant's Transcendental Logic and Hegel's Science of Logic.

Outside of modern philosophy, for example in computer science, there have been a number of attempts at non-classical logic, non-classical logics are those that lack one or more of the following properties:

1. Law of the excluded middle and Double negative elimination;
2. Law of noncontradiction;
3. Monotonicity of entailment and Idempotency of entailment;
4. Commutativity of conjunction;
5. De Morgan duality: every logical operator is dual to another.

The advent of modern logic

Descartes proposed using algebra, especially techniques for solving for unknown quantities in equations, as a vehicle for scientific exploration. The idea of a calculus of reasoning was also developed by Gottfried Wilhelm Leibniz. Leibniz was the first to formulate the notion of a broadly applicable system of mathematical logic. However, the relevant documents were not published until 1901 or remain unpublished to the present day, and the current understanding of the power of Leibniz's discoveries did not emerge until the 1980s. See Lenzen's chapter in Gabbay and Woods (2004).

In 1854, George Boole published the Laws of Thought, widely introducing a symbolic system of logic and calculation. Gottlob Frege in his 1879 Begriffsschrift proposed a calculus which would encompass propositional logic and, by the use of quantifiers to represent "all" and "some" as in the propositions of Aristotelian logic. He showed how the use of variables and quantifiers reveals the logical structure of sentences which may have been obscured by their grammatical structure. For instance, "All humans are mortal" becomes "For every x, if x is human then x is mortal." and "Some Greeks are just" as "There is an x such that x is Greek and x is just". This made Aristotelian logic and syllogism redundant after some 2000 years. The critical importance of Frege's work was recognised by Russell and Wittgenstein who wrote respectively, Principia Mathematica and Tractatus Logico-Philosophicus.

According to Putnam, in "On the Algebra of Logic: A Contribution to the Philosophy of Notation" [24] (1885), read by Peano, Ernst Schröder, and others, Charles Sanders Peirce introduced the term "second-order logic" and provided us with much of our modern logical notation, including prefixed symbols for universal and existential quantification. Logicians in the late 19th and early 20th centuries were thus more familiar with the Peirce-Schröder system of logic, although Frege is generally recognized today as being the "Father of modern logic".

In 1889 Giuseppe Peano published the first version of the logical axiomatization of arithmetic. Five of the nine axioms he came up with are now known as the Peano axioms. One of these axioms was a formalized statement of the principle of mathematical induction.
See also

- Term Logic
- Ernst Schröder
- Charles Sanders Peirce

Notes

[1] Kneale & Kneale, p.2
[2] Heath
[4] Kneale & Kneale p. 21
[7] Boethius, Commentary on the Perihermenias, Meiser p. 234
[12] Bochenski p.446
[15] Ludescher, Tanyss (February 1996), "The Islamic roots of the poetic syllogism (http://findarticles.com/p/articles/mi_qa3709/is_199602/ai_n8749610)", College Literature, , retrieved on 2008-02-29
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• Epictetus, *Dissertationes* ed. Schenkl.

External links

• Ontology and History of Logic in Western Thought. An Introduction (http://www.formalontology.it/history-of-logic.htm) Annotated bibliography
• Petrus Hispanus (http://plato.stanford.edu/entries/peter-spain) (Stanford Encyclopedia of Philosophy)
• Paul Spade’s "Thoughts Words and Things" (http://pvspade.com/Logic/docs/thoughts1_1a.pdf)
• John of St Thomas (http://www.newadvent.org/cathen/08479b.htm)
• Joyce’s Principles of Logic (Traditional Logic Primer) (http://uk.geocities.com/frege@btinternet.com/joyce/principlesoflogic.htm)
• Article on Logic in Britannica 1911 - a good summary of developments in logic before Frege-Russell (http://uk.geocities.com/frege@btinternet.com/cantor/Logic1911.htm)
Propositional calculus

In logic and mathematics, a propositional calculus or logic (also a sentential calculus) is a formal system in which formulae representing propositions can be formed by combining atomic propositions using logical connectives, and a system of formal proof rules allows certain formulae to be established as theorems.

Terminology

In general terms, a calculus is a formal system that consists of a set of syntactic expressions (well-formed formulæ or wffs), a distinguished subset of these expressions (axioms), plus a set of formal rules that define a specific binary relation, intended to be interpreted as logical equivalence, on the space of expressions.

When the formal system is intended to be a logical system, the expressions are meant to be interpreted as statements, and the rules, known as inference rules, are typically intended to be truth-preserving. In this setting, the rules (which may include axioms) can then be used to derive ("infer") formulæ representing true statements from given formulæ representing true statements.

The set of axioms may be empty, a nonempty finite set, a countably infinite set, or be given by axiom schemata. A formal grammar recursively defines the expressions and well-formed formulæ (wffs) of the language. In addition a semantics may be given which defines truth and valuations (or interpretations).

The language of a propositional calculus consists of (1) a set of primitive symbols, variously referred to as atomic formulae, placeholders, proposition letters, or variables, and (2) a set of operator symbols, variously interpreted as logical operators or logical connectives. A well-formed formula (wff) is any atomic formula or any formula that can be built up from atomic formulæ by means of operator symbols according to the rules of the grammar.
Outline
The following outlines a standard propositional calculus. Many different formulations exist which are all more or less equivalent but differ in the details of (1) their language, that is, the particular collection of primitive symbols and operator symbols, (2) the set of axioms, or distinguished formulæ, and (3) the set of inference rules.

Generic description of a propositional calculus
A propositional calculus is a formal system $\mathcal{L} = (\mathcal{A}, \mathcal{O}, \mathcal{Z}, \mathcal{I})$, where:

- The alpha set $\mathcal{A}$ is a finite set of elements called proposition symbols or propositional variables. Syntactically speaking, these are the most basic elements of the formal language $\mathcal{L}$, otherwise referred to as atomic formulæ or terminal elements. In the examples to follow, the elements of $\mathcal{A}$, are typically the letters $p$, $q$, $r$, and so on.

- The omega set $\mathcal{O}$ is a finite set of elements called operator symbols or logical connectives. The set $\mathcal{O}$ is partitioned into disjoint subsets as follows:

$$\mathcal{O} = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \ldots \cup \mathcal{O}_j \cup \ldots \cup \mathcal{O}_n.$$  

In this partition, $\mathcal{O}_j$ is the set of operator symbols of arity $j$.

In the more familiar propositional calculi, $\mathcal{O}$ is typically partitioned as follows:

$$\mathcal{O}_1 = \{ \neg \},$$

$$\mathcal{O}_2 \subseteq \{ \land, \lor, \rightarrow, \leftrightarrow \}.$$  

A frequently adopted convention treats the constant logical values as operators of arity zero, thus:

$$\mathcal{O}_0 = \{ 0, 1 \}.$$  

Some writers use the tilde ($\sim$) instead of ($\neg$); and some use the ampersand ($\&$) or $\cdot$ instead of $\land$. Notation varies even more for the set of logical values, with symbols like $\{ \text{false}, \text{true} \}$, $\{ \text{F}, \text{T} \}$, or $\{ \bot, \top \}$ all being seen in various contexts instead of $\{ 0, 1 \}$.

- The zeta set $\mathcal{Z}$ is a finite set of transformation rules that are called inference rules when they acquire logical applications.

- The iota set $\mathcal{I}$ is a finite set of initial points that are called axioms when they receive logical interpretations.

The language of $\mathcal{L}$, also known as its set of formulæ, well-formed formulas or wffs, is inductively or recursively defined by the following rules:

1. Base: Any element of the alpha set $\mathcal{A}$ is a formula of $\mathcal{L}$.
2. If $p_1, p_2, \ldots, p_j$ are formulæ and $f$ is in $\mathcal{O}_j$, then $(f(p_1, p_2, \ldots, p_j))$ is a formula.
3. Closed: Nothing else is a formula of $\mathcal{L}$.

Repeated applications of these rules permits the construction of complex formulæ. For example:

1. By rule 1, $p$ is a formula.
2. By rule 2, $\neg p$ is a formula.
3. By rule 1, $q$ is a formula.
4. By rule 2, $(\neg p \lor q)$ is a formula.
Example 1. Simple axiom system

Let $\mathcal{L}_1 = \mathcal{L} (A, \Omega, Z, I)$, where $A$, $\Omega$, $Z$, $I$ are defined as follows:

- The alpha set $A$, is a finite set of symbols that is large enough to supply the needs of a given discussion, for example:
  $$A = \{ p, q, r, s, t, u \}.$$ 

- Of the three connectives for conjunction, disjunction, and implication ($\land$, $\lor$, and $\rightarrow$), one can be taken as primitive and the other two can be defined in terms of it and negation ($\neg$). Indeed, all of the logical connectives can be defined in terms of a sole sufficient operator. The biconditional ($\leftrightarrow$) can of course be defined in terms of conjunction and implication, with $a \leftrightarrow b$ defined as $(a \rightarrow b) \land (b \rightarrow a)$.

Adopting negation and implication as the two primitive operations of a propositional calculus is tantamount to having the omega set $\Omega = \Omega_1 \cup \Omega_2$ partition as follows:

$$\Omega_1 = \{ \neg \}$$
$$\Omega_2 = \{ \rightarrow \}$$

- An axiom system discovered by Jan Łukasiewicz formulates a propositional calculus in this language as follows. The axioms are all substitution instances of:

  - $(p \rightarrow (q \rightarrow p))$
  - $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$
  - $((\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p))$

- The rule of inference is modus ponens (i.e. from $P$ and $(p \rightarrow q)$, infer $q$). Then $a \lor b$ is defined as $\neg a \rightarrow b$, and $a \land b$ is defined as $\neg (a \rightarrow \neg b)$.

Example 2. Natural deduction system

Let $\mathcal{L}_2 = \mathcal{L} (A, \Omega, Z, I)$, where $A$, $\Omega$, $Z$, $I$ are defined as follows:

- The alpha set $A$, is a finite set of symbols that is large enough to supply the needs of a given discussion, for example:
  $$A = \{ p, q, r, s, t, u \}$$

- The omega set $\Omega = \Omega_1 \cup \Omega_2$ partitions as follows:
  $$\Omega_1 = \{ \neg \}$$
  $$\Omega_2 = \{ \land, \lor, \rightarrow, \leftrightarrow \}$$

In the following example of a propositional calculus, the transformation rules are intended to be interpreted as the inference rules of a so-called natural deduction system. The particular system presented here has no initial points, which means that its interpretation for logical applications derives its theorems from an empty axiom set.

- The set of initial points is empty, that is, $I = \emptyset$

- The set of transformation rules, $Z$, is described as follows:

  Our propositional calculus has ten inference rules. These rules allow us to derive other true formulae given a set of formulae that are assumed to be true. The first nine simply state that we can infer certain wffs from other wffs. The last rule however uses hypothetical reasoning in the sense that in the premise of the rule we temporarily assume an (unproven) hypothesis to be part of the set of inferred formulae to see if we can infer a certain other formula. Since the first nine rules don’t do this they are usually described as
non-hypothetical rules, and the last one as a hypothetical rule.

**Reductio ad absurdum (negation introduction)**

From \((p \rightarrow q)\), if accepting \(q\) leads to a proof that \(\neg(p \rightarrow q)\), infer \(\neg q\).

**Double negative elimination**

From \(\neg\neg p\), infer \(p\).

**Conjunction introduction**

From \(p\) and \(q\), infer \((p \land q)\).

From \(p\) and \(q\), infer \((q \land \neg p)\).

**Conjunction elimination**

From \((p \land q)\), infer \(p\)

From \((p \land q)\), infer \(q\).

**Disjunction introduction**

From \(p\), infer \((p \lor q)\)

From \(p\), infer \((q \lor \neg p)\).

**Disjunction elimination**

From \((p \lor q)\), \((p \rightarrow r)\), \((q \rightarrow r)\), infer \(r\).

**Biconditional introduction**

From \((p \rightarrow q)\), \((q \rightarrow p)\), infer \((p \leftrightarrow q)\).

**Biconditional elimination**

From \((p \leftrightarrow q)\), infer \((p \rightarrow q)\);

From \((p \leftrightarrow q)\), infer \((q \rightarrow p)\).

**Modus ponens (conditional elimination)**

From \(p\), \((p \rightarrow q)\), infer \(q\).

**Conditional proof (conditional introduction)**

If accepting \(p\) allows a proof of \(q\), infer \((p \rightarrow q)\).

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<td><strong>Name</strong></td>
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<td>Modus Tollens</td>
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<td>Hypothetical Syllogism</td>
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Propositional calculus
### Propositional Calculus

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<th>Proposition</th>
<th>Expression</th>
<th>Description</th>
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<tr>
<td><strong>Conjunction</strong></td>
<td>( p \land q )</td>
<td>( p ) and ( q ) are true separately; therefore they are true conjointly</td>
</tr>
<tr>
<td><strong>Addition</strong></td>
<td>( p \lor q )</td>
<td>( p ) is true; therefore the disjunction (( p ) or ( q )) is true</td>
</tr>
<tr>
<td><strong>Composition</strong></td>
<td>( (p \rightarrow q) \land (p \rightarrow r) \lor (p \rightarrow (q \land r)) )</td>
<td>If ( p ) then ( q ); and if ( p ) then ( r ); therefore if ( p ) is true then ( q ) and ( r ) are true</td>
</tr>
<tr>
<td><strong>De Morgan’s Theorem (1)</strong></td>
<td>( \neg(p \land q) \lor \neg q )</td>
<td>The negation of (( p ) and ( q )) is equiv. to (not ( p ) or not ( q ))</td>
</tr>
<tr>
<td><strong>De Morgan’s Theorem (2)</strong></td>
<td>( \neg(p \lor q) \lor \neg p )</td>
<td>The negation of (( p ) or ( q )) is equiv. to (not ( p ) and not ( q ))</td>
</tr>
<tr>
<td><strong>Commutation (1)</strong></td>
<td>( p \land q \lor q \land p )</td>
<td>(( p ) or ( q )) is equiv. to (( q ) or ( p ))</td>
</tr>
<tr>
<td><strong>Commutation (2)</strong></td>
<td>( p \lor q \land q \lor p )</td>
<td>(( p ) and ( q )) is equiv. to (( q ) and ( p ))</td>
</tr>
<tr>
<td><strong>Commutation (3)</strong></td>
<td>( p \leftrightarrow q \lor q \leftrightarrow p )</td>
<td>(( p ) is equiv. to ( q )) is equiv. to (( q ) is equiv. to ( p ))</td>
</tr>
<tr>
<td><strong>Association (1)</strong></td>
<td>( p \lor (q \lor r) \lor (q \lor p) )</td>
<td>(( p ) or ( q ) or ( r )) is equiv. to (( p ) or ( q )) or ( r )</td>
</tr>
<tr>
<td><strong>Association (2)</strong></td>
<td>( p \land (q \land r) \lor (q \land p) )</td>
<td>(( p ) and ( q ) and ( r )) is equiv. to (( p ) and ( q )) and ( r )</td>
</tr>
<tr>
<td><strong>Distribution (1)</strong></td>
<td>( p \land (q \lor r) \lor (p \land q) \lor (p \land r) )</td>
<td>(( p ) and ( q ) or ( r )) is equiv. to (( p ) and ( q )) or ( r )</td>
</tr>
<tr>
<td><strong>Distribution (2)</strong></td>
<td>( p \lor (q \land r) \lor (p \lor q) \lor (p \lor r) )</td>
<td>(( p ) or ( q ) and ( r )) is equiv. to (( p ) or ( q )) and ( r )</td>
</tr>
<tr>
<td><strong>Double Negation</strong></td>
<td>( p \lor \neg p )</td>
<td>( p ) is equivalent to the negation of not ( p )</td>
</tr>
<tr>
<td><strong>Transposition</strong></td>
<td>( p \rightarrow q \lor \neg q \rightarrow \neg p )</td>
<td>If ( p ) then ( q ) is equiv. to if not ( q ) then not ( p )</td>
</tr>
<tr>
<td><strong>Material Implication</strong></td>
<td>( p \lor q \land q \lor p )</td>
<td>(( p ) or ( q )) is true</td>
</tr>
<tr>
<td><strong>Material Equivalence (1)</strong></td>
<td>( (p \leftrightarrow q) \lor (p \leftrightarrow q) \lor (p \leftrightarrow q) )</td>
<td>(( p ) is equiv. to ( q )) means (if ( p ) is true then ( q ) is true) and (if ( q ) is true then ( p ) is true)</td>
</tr>
<tr>
<td><strong>Material Equivalence (2)</strong></td>
<td>( (p \leftrightarrow q) \lor (p \leftrightarrow q) \lor (p \leftrightarrow q) )</td>
<td>(( p ) is equiv. to ( q )) means either (( p ) and ( q ) are true) or (both ( p ) and ( q ) are false)</td>
</tr>
<tr>
<td><strong>Material Equivalence (3)</strong></td>
<td>( (p \leftrightarrow q) \lor (p \leftrightarrow q) \lor (p \leftrightarrow q) )</td>
<td>(( p ) is equiv. to ( q )) means, both (( p ) or not ( q ) is true) and (not ( p ) or ( q ) is true)</td>
</tr>
<tr>
<td><strong>Exportation</strong></td>
<td>( (p \rightarrow q) \lor (q \rightarrow r) \lor (p \rightarrow (q \rightarrow r)) )</td>
<td>from (if ( p ) and ( q ) are true then ( r ) is true) we can prove (if ( q ) is true then ( r ) is true, if ( p ) is true)</td>
</tr>
<tr>
<td><strong>Importation</strong></td>
<td>( (p \rightarrow (q \rightarrow r)) \lor (p \rightarrow q) \lor (p \rightarrow r) )</td>
<td></td>
</tr>
<tr>
<td><strong>Tautology (1)</strong></td>
<td>( p \lor (p \lor p) )</td>
<td>( p ) is true is equiv. to ( p ) is true or ( p ) is true</td>
</tr>
<tr>
<td><strong>Tautology (2)</strong></td>
<td>( p \lor (p \lor p) )</td>
<td>( p ) is true is equiv. to ( p ) is true and ( p ) is true</td>
</tr>
<tr>
<td><strong>Tertium non datur (Law of Excluded Middle)</strong></td>
<td>( p \lor \neg p )</td>
<td>( p ) or not ( p ) is true</td>
</tr>
<tr>
<td><strong>Law of Non-Contradiction</strong></td>
<td>( \neg (p \land \neg p) )</td>
<td>( p ) and not ( p ) is false, is a true statement</td>
</tr>
</tbody>
</table>
**Proofs in propositional calculus**

One of the main uses of a propositional calculus, when interpreted for logical applications, is to determine relations of logical equivalence between propositional formulae. These relationships are determined by means of the available transformation rules, sequences of which are called derivations or proofs.

In the discussion to follow, a proof is presented as a sequence of numbered lines, with each line consisting of a single formula followed by a reason or justification for introducing that formula. Each premise of the argument, that is, an assumption introduced as an hypothesis of the argument, is listed at the beginning of the sequence and is marked as a "premise" in lieu of other justification. The conclusion is listed on the last line. A proof is complete if every line follows from the previous ones by the correct application of a transformation rule. (For a contrasting approach, see proof-trees).

**Example of a proof**

- To be shown that $A \rightarrow A$.
- One possible proof of this (which, though valid, happens to contain more steps than are necessary) may be arranged as follows:

<table>
<thead>
<tr>
<th>Number</th>
<th>Formula</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A$</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>$A \lor A$</td>
<td>From (1) by disjunction introduction</td>
</tr>
<tr>
<td>3</td>
<td>$(A \lor A) \land A$</td>
<td>From (1) and (2) by conjunction introduction</td>
</tr>
<tr>
<td>4</td>
<td>$A$</td>
<td>From (3) by conjunction elimination</td>
</tr>
<tr>
<td>5</td>
<td>$A \vdash A$</td>
<td>Summary of (1) through (4)</td>
</tr>
<tr>
<td>6</td>
<td>$\vdash A \rightarrow A$</td>
<td>From (5) by conditional proof</td>
</tr>
</tbody>
</table>

Interpret $A \vdash A$ as "Assuming $A$, infer $A". Read $\vdash A \rightarrow A$ as "Assuming nothing, infer that $A$ implies $A"", or "It is a tautology that $A$ implies $A"", or "It is always true that $A$ implies $A"".

**Soundness and completeness of the rules**

The crucial properties of this set of rules are that they are sound and complete. Informally this means that the rules are correct and that no other rules are required. These claims can be made more formal as follows.

We define a truth assignment as a function that maps propositional variables to true or false. Informally such a truth assignment can be understood as the description of a possible state of affairs (or possible world) where certain statements are true and others are not. The semantics of formulae can then be formalized by defining for which "state of affairs" they are considered to be true, which is what is done by the following definition.

We define when such a truth assignment $A$ satisfies a certain wff with the following rules:

- $A$ satisfies the propositional variable $P$ if and only if $A(P) = true$
- $A$ satisfies $\neg \phi$ if and only if $A$ does not satisfy $\phi$
- $A$ satisfies $(\phi \land \psi)$ if and only if $A$ satisfies both $\phi$ and $\psi$
Propositional calculus

- \( A \) satisfies \((\phi \lor \psi)\) if and only if \( A \) satisfies at least one of either \( \phi \) or \( \psi \).
- \( A \) satisfies \((\phi \rightarrow \psi)\) if and only if it is not the case that \( A \) satisfies \( \phi \) but not \( \psi \).
- \( A \) satisfies \((\phi \leftrightarrow \psi)\) if and only if \( A \) satisfies both \( \phi \) and \( \psi \) or satisfies neither one of them.

With this definition we can now formalize what it means for a formula \( \phi \) to be implied by a certain set \( S \) of formulae. Informally this is true if in all worlds that are possible given the set of formulae \( S \) the formula \( \phi \) also holds. This leads to the following formal definition: We say that a set \( S \) of wffs semantically entails (or implies) a certain wff \( \phi \) if all truth assignments that satisfy all the formulae in \( S \) also satisfy \( \phi \).

Finally we define syntactical entailment such that \( \phi \) is syntactically entailed by \( S \) if and only if we can derive it with the inference rules that were presented above in a finite number of steps. This allows us to formulate exactly what it means for the set of inference rules to be sound and complete:

**Soundness**

If the set of wffs \( S \) syntactically entails wff \( \phi \) then \( S \) semantically entails \( \phi \)

**Completeness**

If the set of wffs \( S \) semantically entails wff \( \phi \) then \( S \) syntactically entails \( \phi \)

For the above set of rules this is indeed the case.

**Sketch of a soundness proof**

(For most logical systems, this is the comparatively "simple" direction of proof)

Notational conventions: Let "\( G \)" be a variable ranging over sets of sentences. Let "\( A \)", "\( B \)",

and "\( C \)" range over sentences. For "\( G \) syntactically entails \( A \)" we write "\( G \) proves \( A \)". For "\( G \) semantically entails \( A \)" we write "\( G \) implies \( A \)".

We want to show: \((A)(G)(if \ G \ proves \ A, \ then \ G \ implies \ A)\).

We note that "\( G \) proves \( A \)" has an inductive definition, and that gives us the immediate resources for demonstrating claims of the form "If \( G \) proves \( A \), then ...". So our proof proceeds by induction.

- I. Basis. Show: If \( A \) is a member of \( G \), then \( G \) implies \( A \).
- II. Basis. Show: If \( A \) is an axiom, then \( G \) implies \( A \).
- III. Inductive step (induction on \( n \), the length of the proof):
  
  (a) Assume for arbitrary \( G \) and \( A \) that if \( G \) proves \( A \) in \( n \) or fewer steps, then \( G \) implies \( A \).
  
  (b) For each possible application of a rule of inference at step \( n+1 \), leading to a new theorem \( B \), show that \( G \) implies \( B \).

Notice that Basis Step II can be omitted for natural deduction systems because they have no axioms. When used, Step II involves showing that each of the axioms is a (semantic) logical truth.

The Basis step(s) demonstrate(s) that the simplest provable sentences from \( G \) are also implied by \( G \), for any \( G \). (The is simple, since the semantic fact that a set implies any of its members, is also trivial.) The Inductive step will systematically cover all the further sentences that might be provable—by considering each case where we might reach a logical conclusion using an inference rule—and shows that if a new sentence is provable, it is also logically implied. (For example, we might have a rule telling us that from "\( A \)" we can
derive "A or B". In III.(a) We assume that if A is provable it is implied. We also know that if A is provable then "A or B" is provable. We have to show that then "A or B" too is implied. We do so by appeal to the semantic definition and the assumption we just made. A is provable from $G$, we assume. So it is also implied by $G$. So any semantic valuation making all of $G$ true makes $A$ true. But any valuation making $A$ true makes "A or B" true, by the defined semantics for "or". So any valuation which makes all of $G$ true makes "A or B" true. So "A or B" is implied.) Generally, the Inductive step will consist of a lengthy but simple case-by-case analysis of all the rules of inference, showing that each "preserves" semantic implication.

By the definition of provability, there are no sentences provable other than by being a member of $G$, an axiom, or following by a rule; so if all of those are semantically implied, the deduction calculus is sound.

**Sketch of completeness proof**

(This is usually the much harder direction of proof.)

We adopt the same notational conventions as above.

We want to show: If $G$ implies $A$, then $G$ proves $A$. We proceed by contraposition: We show instead that If $G$ does not prove $A$ then $G$ does not imply $A$.

- I. $G$ does not prove $A$. (Assumption)
- II. If $G$ does not prove $A$, then we can construct an (infinite) "Maximal Set", $G^*$, which is a superset of $G$ and which also does not prove $A$.
  - (a)Place an "ordering" on all the sentences in the language. (e.g., alphabetical ordering), and number them $E_1$, $E_2$, ...
  - (b)Define a series $G_n$ of sets ($G_0$, $G_1$, ...) inductively, as follows. (i) $G_0 = G$. (ii) If $\{G_k', E_{(k+1)}\}$ proves $A$, then $G_{(k+1)} = G_k$. (iii) If $\{G_k', E_{(k+1)}\}$ does not prove $A$, then $G_{(k+1)} = \{G_k', E_{(k+1)}\}$
  - (c)Define $G^*$ as the union of all the $G_n$. (That is, $G^*$ is the set of all the sentences that are in any $G_n$).
  - (d) It can be easily shown that (i) $G^*$ contains (is a superset of) $G$ (by (b.1)); (ii) $G^*$ does not prove $A$ (because if it proves $A$ then some sentence was added to some $G_n$ which caused it to prove $A$; but this was ruled out by definition); and (iii) $G^*$ is a "Maximal Set" (with respect to $A$): If any more sentences whatever were added to $G^*$, it would prove $A$. (Because if it were possible to add any more sentences, they should have been added when they were encountered during the construction of the $G_n$, again by definition)
- III. If $G^*$ is a Maximal Set (wrt $A$), then it is "truth-like". This means that it contains the sentence "C" only if it does not contain the sentence not-C; If it contains "C" and contains "If C then B" then it also contains "B"; and so forth.
- IV. If $G^*$ is truth-like there is a "$G^*$-Canonical" valuation of the language: one that makes every sentence in $G^*$ true and everything outside $G^*$ false while still obeying the laws of semantic composition in the language.
- V. A $G^*$-canonical valuation will make our original set $G$ all true, and make $A$ false.
- VI. If there is a valuation on which $G$ are true and $A$ is false, then $G$ does not (semantically) imply $A$.

QED
Another outline for a completeness proof

If a formula is a tautology, then there is a truth table for it which shows that each valuation yields the value true for the formula. Consider such a valuation. By mathematical induction on the length of the subformulæ, show that the truth or falsity of the subformula follows from the truth or falsity (as appropriate for the valuation) of each propositional variable in the subformula. Then combine the lines of the truth table together two at a time by using "(P is true implies S) implies ((P is false implies S) implies S)". Keep repeating this until all dependences on propositional variables have been eliminated. The result is that we have proved the given tautology. Since every tautology is provable, the logic is complete.

Interpretation of a truth-functional propositional calculus

An interpretation of a truth-functional propositional calculus \( \mathcal{P} \) is an assignment to each propositional symbol of \( \mathcal{P} \) of one or the other (but not both) of the truth values truth (T) and falsity (F), and an assignment to the connective symbols of \( \mathcal{P} \) of their usual truth-functional meanings. An interpretation of a truth-functional propositional calculus may also be expressed in terms of truth tables.\(^{[1]}\)

For \( n \) distinct propositional symbols there are \( 2^n \) distinct possible interpretations. For any particular symbol \( a \), for example, there are \( 2^1 = 2 \) possible interpretations: 1) \( a \) is assigned \( T \), or 2) \( a \) is assigned \( F \). For the pair \( a, b \) there are \( 2^2 = 4 \) possible interpretations: 1) both are assigned \( T \), 2) both are assigned \( F \), 3) \( a \) is assigned \( T \) and \( b \) is assigned \( F \), or 4) \( a \) is assigned \( F \) and \( b \) is assigned \( T \).\(^{[1]}\)

Since \( \mathcal{P} \) has \( \aleph_0 \), that is, denumerably many propositional symbols, there are \( 2^{\aleph_0} = \mathfrak{c} \), and therefore uncountably many distinct possible interpretations of \( \mathcal{P} \).\(^{[1]}\)

Interpretation of a sentence of truth-functional propositional logic

If \( \Phi \) and \( \Psi \) are formulas of \( \mathcal{P} \) and \( \mathcal{T} \) is an interpretation of \( \mathcal{P} \) then:

- A sentence of propositional logic is true under an interpretation \( \mathcal{T} \) iff \( \mathcal{T} \) assigns the truth value T to that sentence. If a sentence is true under an interpretation, then that interpretation is called a model of that sentence.
- \( \Phi \) is false under an interpretation \( \mathcal{T} \) iff \( \Phi \) is not true under \( \mathcal{T} \).\(^{[1]}\)
- A sentence of propositional logic is logically valid iff it is true under every interpretation \( \mathcal{T} \).
- \( \Phi \) means that \( \Phi \) is logically valid
- A sentence \( \Psi \) of propositional logic is a semantic consequence of a sentence \( \Phi \) iff there is no interpretation under which \( \Phi \) is true and \( \Psi \) is false.
- A sentence of propositional logic is consistent iff it is true under at least one interpretation. It is inconsistent if it is not consistent.

Some consequences of these definitions:

- For any given interpretation a given formula is either true or false.\(^{[1]}\)
- No formula is both true and false under the same interpretation.\(^{[1]}\)
- \( \Phi \) is false for a given interpretation iff \( \neg \Phi \) is true for that interpretation; and \( \Phi \) is true under an interpretation iff \( \neg \Phi \) is false under that interpretation.\(^{[1]}\)
- If \( \Phi \) and \( (\Phi \rightarrow \Psi) \) are both true under a given interpretation, then \( \Psi \) is true under that interpretation.\(^{[1]}\)
- If \( \models_{\mathcal{P}} \Phi \) and \( \models_{\mathcal{P}} (\Phi \rightarrow \Psi) \), then \( \models_{\mathcal{P}} \Psi \).\(^{[1]}\)
- \( \neg \Phi \) is true under \( \mathcal{T} \) iff \( \Phi \) is not true under \( \mathcal{T} \).
• \((\Phi \rightarrow \Psi)\) is true under \(\mathcal{T}\) iff either \(\Phi\) is not true under \(\mathcal{T}\) or \(\Psi\) is true under \(\mathcal{T}\).\(^\text{[1]}\)
• A sentence \(\Psi\) of propositional logic is a semantic consequence of a sentence \(\Phi\) iff \((\Phi \rightarrow \Psi)\) is logically valid, that is, \(\Phi \models \Psi\) iff \(\vdash (\Phi \rightarrow \Psi)\).\(^\text{[1]}\)

**Alternative calculus**

It is possible to define another version of propositional calculus, which defines most of the syntax of the logical operators by means of axioms, and which uses only one inference rule.

**Axioms**

Let \(\phi\), \(\chi\) and \(\psi\) stand for well-formed formulae. (The wffs themselves would not contain any Greek letters, but only capital Roman letters, connective operators, and parentheses.) Then the axioms are as follows:

<table>
<thead>
<tr>
<th>Name</th>
<th>Axiom Schema</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>THEN-1</td>
<td>(\phi \rightarrow (\chi \rightarrow \phi))</td>
<td>Add hypothesis (\chi), implication introduction</td>
</tr>
<tr>
<td>THEN-2</td>
<td>((\phi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \rightarrow \psi)))</td>
<td>Distribute hypothesis (\phi) over implication</td>
</tr>
<tr>
<td>AND-1</td>
<td>(\phi \land \chi \rightarrow \phi)</td>
<td>Eliminate conjunction</td>
</tr>
<tr>
<td>AND-2</td>
<td>(\phi \land \chi \rightarrow \chi)</td>
<td></td>
</tr>
<tr>
<td>AND-3</td>
<td>(\phi \rightarrow ((\chi \land \phi) \rightarrow \chi))</td>
<td>Introduce conjunction</td>
</tr>
<tr>
<td>OR-1</td>
<td>(\phi \rightarrow \phi \lor \chi)</td>
<td>Introduce disjunction</td>
</tr>
<tr>
<td>OR-2</td>
<td>(\chi \rightarrow \phi \lor \chi)</td>
<td></td>
</tr>
<tr>
<td>OR-3</td>
<td>((\phi \rightarrow \psi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\phi \lor \chi)))</td>
<td>Eliminate disjunction</td>
</tr>
<tr>
<td>NOT-1</td>
<td>(\phi \rightarrow (\phi \rightarrow \chi))</td>
<td>Introduce negation</td>
</tr>
<tr>
<td>NOT-2</td>
<td>(\phi \rightarrow (\neg \phi \rightarrow \chi))</td>
<td>Exclude middle, classical logic</td>
</tr>
<tr>
<td>NOT-3</td>
<td>(\phi \lor \neg \phi)</td>
<td>Eliminate equivalence</td>
</tr>
<tr>
<td>IFF-1</td>
<td>((\phi \leftrightarrow \chi) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \leftrightarrow \chi)))</td>
<td>Introduce equivalence</td>
</tr>
<tr>
<td>IFF-2</td>
<td>((\phi \leftrightarrow \chi) \rightarrow ((\chi \rightarrow \phi) \rightarrow (\phi \leftrightarrow \chi)))</td>
<td></td>
</tr>
</tbody>
</table>

Axiom THEN-2 may be considered to be a "distributive property of implication with respect to implication."

Axioms AND-1 and AND-2 correspond to "conjunction elimination". The relation between AND-1 and AND-2 reflects the commutativity of the conjunction operator.

Axiom AND-3 corresponds to "conjunction introduction."

Axioms OR-1 and OR-2 correspond to "disjunction introduction." The relation between OR-1 and OR-2 reflects the commutativity of the disjunction operator.

Axiom NOT-1 corresponds to "reductio ad absurdum."

Axiom NOT-2 says that "anything can be deduced from a contradiction."

Axiom NOT-3 is called "tertium non datur" (Latin: "a third is not given") and reflects the semantic valuation of propositional formulae: a formula can have a truth-value of either true or false. There is no third truth-value, at least not in classical logic. Intuitionistic logicians do not accept the axiom NOT-3.
Inference rule
The inference rule is modus ponens:

\[ \phi, \phi \rightarrow \chi \vdash \chi. \]

Meta-inference rule
Let a demonstration be represented by a sequence, with hypotheses to the left of the turnstile and the conclusion to the right of the turnstile. Then the deduction theorem can be stated as follows:

*If the sequence*

\[ \phi_1, \phi_2, ..., \phi_n, \chi \vdash \psi \]

*has been demonstrated, then it is also possible to demonstrate the sequence*

\[ \phi_1, \phi_2, ..., \phi_n \vdash \chi \rightarrow \psi. \]

This deduction theorem (DT) is not itself formulated with propositional calculus: it is not a theorem of propositional calculus, but a theorem about propositional calculus. In this sense, it is a meta-theorem, comparable to theorems about the soundness or completeness of propositional calculus.

On the other hand, DT is so useful for simplifying the syntactical proof process that it can be considered and used as another inference rule, accompanying modus ponens. In this sense, DT corresponds to the natural conditional proof inference rule which is part of the first version of propositional calculus introduced in this article.

The converse of DT is also valid:

*If the sequence*

\[ \phi_1, \phi_2, ..., \phi_n \vdash \chi \rightarrow \psi \]

*has been demonstrated, then it is also possible to demonstrate the sequence*

\[ \phi_2, \phi_3, ..., \phi_n, \chi \vdash \psi \]

in fact, the validity of the converse of DT is almost trivial compared to that of DT:

*If*

\[ \phi_1, ..., \phi_n \vdash \chi \rightarrow \psi \]

*then*

1. \[ \phi_2, ..., \phi_n, \chi \vdash \chi \rightarrow \psi \]
2. \[ \phi_1, ..., \phi_n, \chi \vdash \chi \]

*and from (1) and (2) can be deduced*

3. \[ \phi_2, ..., \phi_n, \chi \vdash \psi \]

*by means of modus ponens, Q.E.D.*

The converse of DT has powerful implications: it can be used to convert an axiom into an inference rule. For example, the axiom AND-1,

\[ \vdash \phi \land \chi \rightarrow \phi \]

can be transformed by means of the converse of the deduction theorem into the inference rule

\[ \phi \land \chi \vdash \phi \]
which is conjunction elimination, one of the ten inference rules used in the first version (in this article) of the propositional calculus.

**Example of a proof**

The following is an example of a (syntactical) demonstration, involving only axioms THEN-1 and THEN-2:

**Prove:** $A \rightarrow A$ (Reflexivity of implication).

**Proof:**

1. $(A \rightarrow ((B \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (B \rightarrow A)) \rightarrow (A \rightarrow A))$

   Axiom THEN-2 with $\varphi = A$, $\chi = B \rightarrow A$, $\psi = A$

2. $A \rightarrow ((B \rightarrow A) \rightarrow A)$

   Axiom THEN-1 with $\varphi = A$, $\chi = B \rightarrow A$

3. $(A \rightarrow (B \rightarrow A)) \rightarrow (A \rightarrow A)$

   From (1) and (2) by modus ponens.

4. $A \rightarrow (B \rightarrow A)$

   Axiom THEN-1 with $\varphi = A$, $\chi = B$

5. $A \rightarrow A$

   From (3) and (4) by modus ponens.

**Equivalence to equational logics**

The preceding alternative calculus is an example of a Hilbert-style deduction system. In the case of propositional systems the axioms are terms built with logical connectives and the only inference rule is modus ponens. Equational logic as standardly used informally in high school algebra is a different kind of calculus from Hilbert systems. Its theorems are equations and its inference rules express the properties of equality, namely that it is a congruence on terms that admits substitution.

Classical propositional calculus as described above is equivalent to Boolean algebra, while intuitionistic propositional calculus is equivalent to Heyting algebra. The equivalence is shown by translation in each direction of the theorems of the respective systems. Theorems $\Phi$ of classical or intuitionistic propositional calculus are translated as equations $\Phi = 1$ of Boolean or Heyting algebra respectively. Conversely theorems $x = y$ of Boolean or Heyting algebra are translated as theorems $(x \rightarrow y) \land (y \rightarrow x)$ of classical or propositional calculus respectively, for which $x \equiv y$ is a standard abbreviation. In the case of Boolean algebra $x \neq y$ can also be translated as $(x \land y) \lor (\neg x \land \neg y)$, but this translation is incorrect intuitionistically.

In both Boolean and Heyting algebra, inequality $x \leq y$ can be used in place of equality. The equality $x = y$ is expressible as a pair of inequalities $x \leq y$ and $y \leq x$. Conversely the inequality $x \leq y$ is expressible as the equality $x \land y = x$, or as $x \lor y = y$. The significance of inequality for Hilbert-style systems is that it corresponds to the latter's deduction or entailment symbol $\vdash$. An entailment

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

is translated in the inequality version of the algebraic framework as

$$\phi_1 \land \phi_2 \land \ldots \land \phi_n \leq \psi$$
Conversely the algebraic inequality \( x \leq y \) is translated as the entailment
\[ x \models y \]
The difference between implication \( x \rightarrow y \) and inequality or entailment \( x \leq y \) or \( x \models y \) is that the former is internal to the logic while the latter is external. Internal implication between two terms is another term of the same kind. Entailment as external implication between two terms expresses a metatruth outside the language of the logic, and is considered part of the metalanguage. Even when the logic under study is intuitionistic, entailment is ordinarily understood classically as two-valued: either the left side entails, or is less-or-equal to, the right side, or it is not.

Similar but more complex translations to and from algebraic logics are possible for natural deduction systems as described above and for the sequent calculus. The entailments of the latter can be interpreted as two-valued, but a more insightful interpretation is as a set, the elements of which can be understood as abstract proofs organized as the morphisms of a category. In this interpretation the cut rule of the sequent calculus corresponds to composition in the category. Boolean and Heyting algebras enter this picture as special categories having at most one morphism per homset, i.e. one proof per entailment, corresponding to the idea that existence of proofs is all that matters: any proof will do and there is no point in distinguishing them.

**Graphical calculi**

It is possible to generalize the definition of a formal language from a set of finite sequences over a finite basis to include many other sets of mathematical structures, so long as they are built up by finitary means from finite materials. What's more, many of these families of formal structures are especially well-suited for use in logic.

For example, there are many families of graphs that are close enough analogues of formal languages that the concept of a calculus is quite easily and naturally extended to them. Indeed, many species of graphs arise as *parse graphs* in the syntactic analysis of the corresponding families of text structures. The exigencies of practical computation on formal languages frequently demand that text strings be converted into pointer structure renditions of parse graphs, simply as a matter of checking whether strings are wffs or not. Once this is done, there are many advantages to be gained from developing the graphical analogue of the calculus on strings. The mapping from strings to parse graphs is called *parsing* and the inverse mapping from parse graphs to strings is achieved by an operation that is called *traversing* the graph.

**Other logical calculi**

Propositional calculus is about the simplest kind of logical calculus in any current use. (Aristotelian "syllogistic" calculus, which is largely supplanted in modern logic, is in some ways simpler — but in other ways more complex — than propositional calculus.) It can be extended in several ways.

The most immediate way to develop a more complex logical calculus is to introduce rules that are sensitive to more fine-grained details of the sentences being used. When the "atomic sentences" of propositional logic are broken up into terms, variables, predicates, and quantifiers, they yield first-order logic, or first-order predicate logic, which keeps all the rules of propositional logic and adds some new ones. (For example, from "All dogs are
mammals" we may infer "If Rover is a dog then Rover is a mammal".) It makes sense to refer to propositional logic as "zeroth-order logic", when comparing it with first-order logic and second-order logic.

With the tools of first-order logic it is possible to formulate a number of theories, either with explicit axioms or by rules of inference, that can themselves be treated as logical calculi. Arithmetic is the best known of these; others include set theory and mereology.

→ Modal logic also offers a variety of inferences that cannot be captured in propositional calculus. For example, from "Necessarily p" we may infer that p. From p we may infer "It is possible that p". The translation between modal logics and algebraic logics is as for classical and intuitionistic logics but with the introduction of a unary operator on Boolean or Heyting algebras, different from the Boolean operations, interpreting the possibility modality, and in the case of Heyting algebra a second operator interpreting necessity (for Boolean algebra this is redundant since necessity is the De Morgan dual of possibility). The first operator preserves 0 and disjunction while the second preserves 1 and conjunction.

Many-valued logics are those allowing sentences to have values other than true and false. (For example, neither and both are standard "extra values"; "continuum logic" allows each sentence to have any of an infinite number of "degrees of truth" between true and false.) These logics often require calculational devices quite distinct from propositional calculus. When the values form a Boolean algebra (which may have more than two or even infinitely many values), many-valued logic reduces to classical logic; many-valued logics are therefore only of independent interest when the values form an algebra that is not Boolean.

**Solvers**

Finding solutions to propositional logic formulas is an NP-complete problem. However, recent breakthroughs (Chaff algorithm, 2001) have led to the development of small, efficient SAT solvers, which are very fast for most cases. Recent work has extended the SAT solver algorithms to work with propositions containing arithmetic expressions; these are the SMT solvers.

**References**


See also

Higher logical levels

• First-order logic
• Second-order logic
• Higher-order logic

Related topics

• Ampheck
• Boolean algebra (logic)
• Boolean algebra (structure)
• Boolean algebra topics
• Boolean domain
• Boolean function
• Boolean-valued function
• Categorical logic
• Combinational logic
• Logical graph
• Logical value
• Minimal negation operator
• Multigrade operator
• Operation
• Parametric operator
• Peirce's law
• Propositional formula
• Symmetric difference
• Truth table
• Combinatory logic
• Conceptual graph
• Disjunctive syllogism
• Entitative graph
• Existential graph
• Frege's propositional calculus
• Implicational propositional calculus
• Intuitionistic propositional calculus
• Laws of Form

Related works


External links

• Introduction to Mathematical Logic (http://www.ltn.lv/~podnieks/mlog/ml2.htm)
• Elements of Propositional Calculus (http://www.visualstatistics.net/Scaling/Propositional Calculus/Elements of Propositional Calculus.htm)
• forall x: an introduction to formal logic (http://www.fecundity.com/logic/), by P.D. Magnus, covers formal semantics and proof theory for sentential logic.
Predicate logic

In mathematical logic, **predicate logic** is the generic term for symbolic formal systems like first-order logic, second-order logic, many-sorted logic or infinitary logic. This formal system is distinguished from other systems in that its formulas contain variables which can be quantified. Two common quantifiers are the existential $\exists$ and universal $\forall$ quantifiers. The variables could be elements in the universe, or perhaps relations or functions over the universe. For instance, an existential quantifier over a function symbol would be interpreted as modifier "there is a function".

In informal usage, the term "predicate logic" occasionally refers to first-order logic. Some authors consider the **predicate calculus** to be an axiomatized form of **predicate logic**, and the predicate logic to be derived from an informal, more intuitive development.\[^1\]

Footnotes

\[^1\] Among these authors is Stolyar, p. 166. Hamilton considers both to be calculi but divides them into an informal calculus and a formal calculus.

References

Modal logic

A modal logic is any system of formal logic that attempts to deal with modalities. Modals qualify the truth of a judgment. Traditionally, there are three "modes" or "moods" or "modalities" represented by modal logic, namely, possibility, probability, and necessity. Logics for dealing with a number of related terms, such as eventually, formerly, can, could, might, may, must, are by extension also called modal logics, since it turns out that these can be treated in similar ways.

A formal modal logic represents modalities using modal operators. For example, "It might rain today" and "It is possible that rain will fall today" both contain the notion of possibility. In a modal logic this is represented as an operator, Possibly, attached to the sentence It will rain today.

The basic unary (1-place) modal operators are usually written □ (or L) for Necessarily and ◊ (or M) for Possibly. In a classical modal logic, each can be expressed by the other and negation:

\[ ◊P \iff \neg □ \neg P; \]
\[ □P \iff \neg ◊ \neg P. \]

Thus it is possible that it will rain today if and only if it is not necessary that it will not rain today. For the standard formal semantics of the basic modal language, see Kripke semantics.

Brief history

In 1918 C.I. Lewis introduced a formal axiomatic modal logic system, acknowledging the use of the ideas of Hugh Brown from the 1890s. In 1932 C.I. Lewis and Cooper H. Langford introduced the systems S1 through S5 (with S3 being the original system proposed in C.I. Lewis's 1918 work). By the late 1930s many systems were known. Major changes in how the systems were viewed were later provided by Kurt Gödel (use of "Necessity" as primitive, and PC+axioms) and Saul Kripke (a reasonable semantics for most, but not all, modal logics).

Alethic modalities

Modalities of necessity and possibility are called alethic modalities. They are also sometimes called special modalities, from the Latin species. Modal logic was first developed to deal with these concepts, and only afterward was extended to others. For this reason, or perhaps for their familiarity and simplicity, necessity and possibility are often casually treated as the subject matter of modal logic. Moreover it is easier to make sense of relativizing necessity, e.g. to legal, physical, nomological, epistemic, and so on, than it is to make sense of relativizing other notions.

In classical modal logic, a proposition is said to be

• possible if and only if it is not necessarily false (regardless of whether it is actually true or actually false);
• necessary if and only if it is not possibly false; and
• contingent if and only if it is not necessarily false and not necessarily true (ie. possible but not necessary).
In classical modal logic, therefore, either the notion of possibility or necessity may be taken to be basic, where these other notions are defined in terms of it in the manner of De Morgan duality. Intuitionistic modal logic treats possibility and necessity as not perfectly symmetric.

For those with difficulty with the concept of something being possible but not true, the meaning of these terms may be made more comprehensible by thinking of multiple "possible worlds" (in the sense of Leibniz) or "alternate universes"; something "necessary" is true in all possible worlds, something "possible" is true in at least one possible world. These "possible world semantics" are formalized with Kripke semantics.

**Physical possibility**

Something is physically possible if it is permitted by the laws of physics. For example, current theory allows for there to be an atom with an atomic number of 150, though there may not in fact be one. On the other hand, it is not possible for there to be an atom whose nucleus contains cheese. While it is logically possible to accelerate beyond the speed of light, that is not, according to modern science, physically possible for material particles or information.

**Metaphysical possibility**

Philosophers ponder the properties that objects have independently of those dictated by scientific laws. For example, it might be metaphysically necessary, as some have thought, that all thinking beings have bodies and can experience the passage of time, or that God exists. Saul Kripke has argued that every person necessarily has the parents they do have: anyone with different parents would not be the same person.

Metaphysical possibility is generally thought to be more restricting than bare logical possibility (i.e., fewer things are metaphysically possible than are logically possible). Its exact relation to physical possibility is a matter of some dispute. Philosophers also disagree over whether metaphysical truths are necessary merely "by definition", or whether they reflect some underlying deep facts about the world, or something else entirely.

**Confusion with epistemic modalities**

Alethic modalities and epistemic modalities (see below) are often expressed in English using the same words. "It is possible that bigfoot exists" can mean either "Bigfoot could exist, whether or not bigfoot does in fact exist" (alethic), or more likely, "For all I know, bigfoot exists" (epistemic).

**Epistemic logic**

**Epistemic modalities** (from the Greek *episteme*, knowledge), deal with the certainty of sentences. The □ operator is translated as "x knows that..."., and the ◇ operator is translated as "For all x knows, it may be true that...". In ordinary speech both metaphysical and epistemic modalities are often expressed in similar words; the following contrasts may help:

A person, Jones, might reasonably say both: (1) "No, it is not possible that Bigfoot exists; I am quite certain of that"; and, (2) "Sure, Bigfoot possibly could exist". What Jones means by (1) is that given all the available information, there is no question remaining as to whether
Bigfoot exists. This is an epistemic claim. By (2) he makes the *metaphysical* claim that it is *possible for* Bigfoot to exist, *even though he does not* (which is not equivalent to "it is *possible that* Bigfoot exists - for all I know," which contradicts (1)).

From the other direction, Jones might say, (3) "It is *possible* that Goldbach's conjecture is true; but also *possible* that it is false", and also (4) "if it *is* true, then it is necessarily true, and not possibly false". Here Jones means that it is *epistemically possible* that it is true or false, for all he knows (Goldbach's conjecture has not been proven either true or false), but if there *is* a proof (heretofore undiscovered), then it would show that it is not *logically* possible for Goldbach's conjecture to be false—there could be no set of numbers that violated it. Logical possibility is a form of *alethic* possibility; (4) makes a claim about whether it is possible (ie, logically speaking) that a mathematical truth to have been false, but (3) only makes a claim about whether it is possible, for all Jones knows, (ie, speaking of certitude) that the mathematical claim is specifically either true or false, and so again Jones does not contradict himself. It is worthwhile to observe that Jones is not necessarily correct: It is possible (epistemically) that Goldbach's conjecture is both true and unprovable.[1]

Epistemic possibilities also bear on the actual world in a way that metaphysical possibilities do not. Metaphysical possibilities bear on ways the world *might have been*, but epistemic possibilities bear on the way the world *may be* (for all we know). Suppose, for example, that I want to know whether or not to take an umbrella before I leave. If you tell me "it is *possible* that it is raining outside" - in the sense of epistemic possibility - then that would weigh on whether or not I take the umbrella. But if you just tell me that "it is *possible for* it to rain outside" - in the sense of *metaphysical possibility* - then I am no better off for this bit of modal enlightenment.

Some features of epistemic modal logic are in debate. For example, if x knows that p, does x know that it knows that p? That is to say, should □P → □□P be an axiom in these systems? While the answer to this question is unclear, there is at least one axiom that *must* be included in epistemic modal logic, because it is minimally true of all modal logics (see the section on axiomatic systems):

• **K, Distribution Axiom**: □(p → q) → (□p → □q).

But this is disconcerting, because with K, we can prove that we know all the logical consequences of our beliefs: If q is a logical consequence of p, then □(p → q). And if so, then we can deduce that (□p → □q) using K. When we translate this into epistemic terms, this says that if q is a logical consequence of p, then we know that it is, and if we know p, we know q. That is to say, we know all the logical consequences of our beliefs. This must be true for all possible modal interpretations of epistemic cases where □ is translated as "knows that." But then, for example, if x knows that prime numbers are divisible only by themselves and the number one, then x knows that 8683317618811886495518194401279999999 is prime (since this number is only divisible by itself and the number one). That is to say, under the modal interpretation of knowledge, anyone who knows the definition of a prime number knows that this number is prime. This shows that epistemic modal logic is an idealized account of knowledge, and explains objective, rather than subjective knowledge (if anything).
**Temporal logic**

Temporal logic is an approach to the semantics of expressions with tense, that is, expressions with qualifications of when. Some expressions, such as '2 + 2 = 4', are true at all times, while tensed expressions such as 'John is happy' is only true sometimes.

In temporal logic, tense constructions are treated in terms of modalities, where a standard method for formalizing talk of time is to use two pairs of operators, one for the past and one for the future (P will just mean 'it is presently the case that P'). For example:

- \( FP \) : It will sometime be the case that \( P \)
- \( GP \) : It will always be the case that \( P \)
- \( PP \) : It was sometime the case that \( P \)
- \( HP \) : It has always been the case that \( P \)

There are then at least three modal logics that we can develop. For example, we can stipulate that,

- \( \Diamond P \) = \( P \) is the case at some time \( t \)
- \( \Box P \) = \( P \) is the case at every time \( t \)

and we can add two other operators to talk about the future by itself (these would have to be distinguished from the first set by some subscript). Either,

- \( \Diamond P = FP \)
- \( \Box P = GP \)

or,

- \( \Diamond P = P \) and/or \( FP \)
- \( \Box P = P \) and \( GP \)

The second and third interpretations may seem odd, but such assignments do create modal systems. Note that \( FP \) is the same as \( \neg G \neg P \). We can combine the above operators to form complex statements. For example, \( PP \rightarrow \Box PP \) says (effectively), *Everything that is past and true is necessary.*

It seems reasonable to say that possibly it will rain tomorrow, and possibly it won't; on the other hand, seeing as how we can't change the past, if it rained yesterday, it probably isn't true that it may not have rained yesterday. It seems the past is "fixed", or necessary, in a way the future is not. This is sometimes referred to as accidental necessity. But if the past is "fixed", and everything that is in the future will eventually be in the past, then it seems plausible to say that future events are necessary too.

Similarly, the problem of future contingents considers the semantics of assertions about the future: is either of the propositions 'There will be a sea battle tomorrow', or 'There will not be a sea battle tomorrow' now true? Considering this thesis led Aristotle to reject the → principle of bivalence for assertions concerning the future.

Additional binary operators are also relevant to temporal logics, q.v. Linear Temporal Logic.

Versions of temporal logic can be used in computer science to model computer operations and prove theorems about them. In one version, \( \Diamond P \) means "at a future time in the computation it is possible that the computer state will be such that \( P \) is true"; \( \Box P \) means "at all future times in the computation \( P \) will be true". In another version, \( \Diamond P \) means "at the immediate next state of the computation, \( P \) might be true"; \( \Box P \) means "at the immediate
next state of the computation, P will be true". These differ in the choice of Accessibility relation. (P always means "P is true at the current computer state".) These two examples involve nondeterministic or not-fully-understood computations; there are many other modal logics specialized to different types of program analysis. Each one naturally leads to slightly different axioms.

**Deontic logic**

Likewise talk of morality, or of obligation and norms generally, seems to have a modal structure. The difference between "You must do this" and "You may do this" looks a lot like the difference between "This is necessary" and "This is possible". Such logics are called deontic, from the Greek for "duty". One characteristic feature of deontic logics is that they lack the axiom $T$ semantically corresponding to the reflexivity of the accessibility relation in Kripke semantics: in symbols, $\square \phi \rightarrow \phi$. Interpreting $\square$ as "it is obligatory that", $T$ informally says that every obligation is true. For example, if it is obligatory not to kill others (i.e. killing is morally forbidden), then $T$ implies that people actually do not kill others. This consequence is obviously false.

However, in Kripke semantics for deontic logic, $T$ is supposed to hold at accessible worlds (relative to the actual world $w$). These worlds are to be thought of as idealized in the sense that all obligations (in $w$) are fulfilled there. Hence a sentence $A$ is obligatory just in case $A$ holds at all idealized worlds. So in order to discuss obligations under Kripke semantics, there must be some world where $\square \phi \rightarrow \phi$, where everything that ought to be the case, is the case. Though this was one of the first interpretations of the formal semantics, it has recently come under criticism. See e.g. Sven Hansson, "Ideal Worlds--Wishful Thinking in Deontic Logic", Studia Logica, Vol. 82 (3), pp. 329-336, 2006.

One other principle that is often (at least traditionally) accepted as a deontic principle is $D$, $\square \phi \rightarrow \Diamond \phi$, which corresponds to the seriality (or extendability or unboundedness) of the accessibility relation. It is an embodiment of the Kantian idea that "ought implies can". (Clearly the "can" can be interpreted in various senses, e.g. in a moral or alethic sense.)

**Intuitive problems with deontic logic**

When we try and formalize ethics with standard modal logic, we run into some problems. Suppose that we have a proposition $K$: you kill the victim, and another, $Q$: you kill the victim quickly. Now suppose we want to express the thought that "if you do kill the victim, you ought to kill him quickly." There are two likely candidates,

(1) $\langle K \rightarrow \square Q \rangle$

(2) $\square \langle K \rightarrow Q \rangle$

But (1) says that if you kill the victim, then it ought to be the case that you kill him quickly. This surely isn't right, because you ought not to have killed him at all. And (2) doesn't work either. If the right representation of "if you kill the victim then you ought to kill him quickly" is (2), then the right representation of (3) "if you kill the victim then you ought to kill him slowly" is $\square \langle K \rightarrow \neg Q \rangle$. Now suppose (as seems reasonable) that you should not kill the victim, or $\neg K$. Then we can deduce $\square \langle K \rightarrow \neg Q \rangle$ using the principle of explosion and the minimally true $K$ axiom, which would express sentence (3). So if you should not kill the victim, then if you kill him, you should kill him slowly. But that can't be right, and is not right when we use natural language. Telling someone they should not kill the victim
certainly does not imply that they should kill the victim slowly if they do kill him.[2]

**Doxastic logic**

*Doxastic logic* concerns the logic of belief (of some set of agents). The term doxastic is derived from the ancient Greek *doxa* which means "belief." Typically, a doxastic logic uses □, often written "B", to mean "It is believed that", or when relativized to a particular agent s, "It is believed by s that".

**Other modal logics**

Significantly, modal logics can be developed to accommodate most of these idioms; it is the fact of their common logical structure (the use of "intensional" sentential operators) that make them all varieties of the same thing.

**Interpretations of modal logic**

In the most common interpretation of modal logic, one considers "all logically possible worlds". If a statement is true in all possible worlds, then it is a necessary truth. If a statement happens to be true in our world, but is not true in all possible worlds, then it is a contingent truth. A statement that is true in some possible world (not necessarily our own) is called a possible truth.

Whether this "possible worlds idiom" is the best way to interpret modal logic, and how literally this idiom can be taken, is a live issue for metaphysicians. For example, the possible worlds idiom would translate the claim about Bigfoot as "There is some possible world in which Bigfoot exists". To maintain that Bigfoot's existence is possible, but not actual, one could say, "There is some possible world in which Bigfoot exists; but in the actual world, Bigfoot does not exist". But it is unclear what it is that making modal claims commits us to. Are we really alleging the existence of possible worlds, every bit as real as our actual world, just not actual? David Lewis made himself notorious by biting the bullet, asserting that all merely possible worlds are as real as our own, and that what distinguishes our world as *actual* is simply that it *is* indeed our world - *this* world (see Indexicality). That position is a major tenet of "modal realism". Most philosophers decline to endorse such a view, considering it ontologically extravagant, and preferring to seek various ways to paraphrase away these ontological commitments.

Computer scientists will generally pick a highly specific interpretation of the modal operators specialized to the particular sort of computation being analysed. In place of "all worlds", you may have "all possible next states of the computer", or "all possible future states of the computer".
Axiomatic Systems

Many systems of modal logic, with widely varying properties, have been proposed since C. I. Lewis began working in the area in 1910. Hughes and Cresswell (1996), for example, describe 42 normal and 25 non-normal modal logics. Zeman (1973) describes some systems Hughes and Cresswell omit.

Modern treatments of modal logic begin by augmenting the → propositional calculus with two unary operations, one denoting "necessity" and the other "possibility". The notation of Lewis, much employed since, denotes "necessarily p" by a prefixed "box" (□p) whose scope is established by parentheses. Likewise, a prefixed "diamond" (◊p) denotes "possibly p". Regardless of notation, each of these operators is definable in terms of the other:

- □p (necessarily p) is equivalent to ¬◊¬p ("not possible that not-p")
- ◊p (possibly p) is equivalent to ¬□¬p ("not necessarily not-p")

Hence □ and ◊ form a dual pair of operators.

In many modal logics, the necessity and possibility operators satisfy the following analogs of de Morgan’s laws from Boolean algebra:

"It is not necessary that X" is logically equivalent to "It is possible that not X".

"It is not possible that X" is logically equivalent to "It is necessary that not X".

Precisely what axioms and rules must be added to the → propositional calculus to create a usable system of modal logic is a matter of philosophical opinion, often driven by the theorems one wishes to prove; or, in computer science, it is a matter of what sort of computational or deductive system one wishes to model. Many modal logics, known collectively as normal modal logics, include the following rule and axiom:

- N, Necessitation Rule: If p is a theorem (of any system invoking N), then □p is likewise a theorem.
- K, Distribution Axiom: □(p → q) → (□p → □q).

The weakest normal modal logic, named K in honor of Saul Kripke, is simply the → propositional calculus augmented by □, the rule N, and the axiom K. K is weak in that it fails to determine whether a proposition can be necessary but only contingently necessary. That is, it is not a theorem of K that if □p is true then □□p is true, i.e., that necessary truths are "necessarily necessary". If such perplexities are deemed forced and artificial, this defect of K is not a great one. In any case, different answers to such questions yield different systems of modal logic.

Adding axioms to K gives rise to other well-known modal systems. One cannot prove in K that if "p is necessary" then p is true. The axiom T remedies this defect:

- T, Reflexivity Axiom: □p → p (If p is necessary, then p is the case.) T holds in most but not all modal logics. Zeman (1973) describes a few exceptions, such as S50.

Other well-known elementary axioms are:

- 4: □p → □□p
- B: p → □◊p
- D: □p → ◊p
- E: ◊p → □◊p.

These axioms yield the systems:

- K := K + N
• \( T := K + T \)
• \( S4 := T + 4 \)
• \( S5 := S4 + B \) or \( T + E \)
• \( D := K + D \).

K through S5 form a nested hierarchy of systems, making up the core of normal modal logic. But specific rules or sets of rules may be appropriate for specific systems. For example, in deontic logic, \( \Box p \to \Diamond p \) (If it ought to be that \( P \), then it is permitted that \( P \)) seems appropriate, but we should probably not include that \( p \to \Box \Diamond p \).

The commonly employed system S5 simply makes all modal truths necessary. For example, if \( p \) is possible, then it is "necessary" that \( p \) is possible. Also, if \( p \) is necessary, then it is necessary that \( p \) is necessary. Although controversial, this is commonly justified on the grounds that S5 is the system obtained if every possible world is possible relative to every other world. Other systems of modal logic have been formulated, in part because S5 does not describe every kind of metaphysical modality of interest. This suggests that talk of possible worlds and their semantics may not do justice to all modalities.

**Development of modal logic**

Although Aristotle's logic is almost entirely concerned with the theory of the categorical syllogism, there are passages in his work, such as the famous sea-battle argument in *De Interpretatione* § 9, that are now seen as anticipations of modal logic and its connection with potentiality and time. Modal logic as a self-aware subject owes much to the writings of the Scholastics, in particular William of Ockham and John Duns Scotus, who reasoned informally in a modal manner, mainly to analyze statements about essence and accident.

C. I. Lewis founded modern modal logic in his 1910 Harvard thesis and in a series of scholarly articles beginning in 1912. This work culminated in his 1932 book *Symbolic Logic* (with C. H. Langford), which introduced the five systems S1 through S5. The contemporary era in modal logic began in 1959, when Saul Kripke (then only a 19 year old Harvard University undergraduate) introduced the now-standard Kripke semantics for modal logics. These are commonly referred to as "possible worlds" semantics. Kripke and A. N. Prior had previously corresponded at some length.

A. N. Prior created temporal logic, closely related to modal logic, in 1957 by adding modal operators \([F]\) and \([P]\) meaning "henceforth" and "hitherto". Vaughan Pratt introduced dynamic logic in 1976. In 1977, Amir Pnueli proposed using temporal logic to formalise the behaviour of continually operating concurrent programs. Flavors of temporal logic include propositional dynamic logic (PDL), propositional linear temporal logic (PLTL), linear temporal logic (LTL), computational tree logic (CTL), Hennessy-Milner logic, and \( T \).

The mathematical structure of modal logic, namely Boolean algebras augmented with unary operations (often called "modal algebras"), began to emerge with J. C. C. McKinsey's 1941 proof that S2 and S4 are decidable, and reached full flower in the work of Alfred Tarski and his student Bjarni Jonsson (Jonsson and Tarski 1951-52). This work revealed that S4 and S5 are models of interior algebra, a proper extension of Boolean algebra originally designed to capture the properties of the interior and closure operators of topology. Texts on modal logic typically do little more than mention its connections with the study of Boolean algebras and topology. For a thorough survey of the history of formal modal logic and of the associated mathematics, see Goldblatt (2006). [3]
References


Free and online:

See also

- Accessibility relation
- Counterpart theory
- De dicto and de re
- Description logic
- Doxastic logic
- Dynamic logic
- Possible worlds
- Provability logic
- Two dimensionalism
- Modal verb
- Epistemic logic
- Hybrid logic
- Interior algebra
- Interpretability logic
- Kripke semantics

Notes

[1] See Goldbach's conjecture > Origins

External links

- Stanford Encyclopedia of Philosophy:
- Molle (http://molle.sourceforge.net/) a Java prover for experimenting with modal logics
- Suber, Peter, 2002, "Bibliography of Modal Logic. (http://www.earlham.edu/~peters/courses/logsys/nonstbib.htm#modal)"
Acknowledgements

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Informal logic

The precise nature and definition of informal logic are matters of some dispute.[1] Ralph H. Johnson and J. Anthony Blair define informal logic as "a branch of logic whose task is to develop non-formal standards, criteria, procedures for the analysis, interpretation, evaluation, criticism and construction of argumentation."[2] This definition reflects what had been implicit in their practice and what others[3] [4] [5] were doing in their informal logic texts.

Informal logic is associated with (informal) fallacies, critical thinking, the Thinking Skills Movement[6] and the interdisciplinary inquiry known as Argumentation theory.

Informal logic, formal logic and (informal) fallacies

Logic is the study of inference. In formal logic, the form of an argument either matches or does not match one of the forms of proper inference (or, the conclusion can be derived from the premises using accepted rules of derivation, or by some other formal method.) Informal logic, by contrast, invites us to think about the inference without formalizing it to any (great) extent.

As an example of the difference between formal and informal logic, consider how each would evaluate the argument "The (U. S.) President says it will rain this afternoon at the White House. So, it will rain this afternoon at the White House." In evaluating the inference informally, we think about how the premise ("The President says it will rain this afternoon at the White House.") could be true and yet the conclusion ("It will rain this afternoon at the White House.") could be false. The basic problem with this argument is that we doubt that the President is a reliable indicator of the weather, but we accept what people say only if we think that they are experts on the subject in question and trustworthy. Thus, in addition to the fact that he says it will rain, we would also need to be assured of claims such as "The President is an expert on weather." and "The President is unbiased." and perhaps various others. We are then led to think about the meaning and truth of these additional propositions, in addition to the original premise and conclusion.

To evaluate this argument using formal logic, we would abstract (at least some of) the concrete information from the premises and conclusion and compare the form generated against the forms (or: patterns) of inference which we have predetermined as having a strong connection between premise(s) and conclusion. If we abstract away as much as possible we get (perhaps) "Assembler-A asserts "Proposition-p". So, Proposition-p."[7] We then check to see if this form is included in our catalogue of accepted inference forms. If it is not, we reject the argument. This particular argument's form would not be found, and so the conclusion would not be accepted (on the basis of this argument, at least.)

Informal logic is often taught as (or at least importantly includes) instruction in a series of "fallacies". These "fallacies" particularly concern modes of argumentation which are not generally covered in formal logic, (though those treated formally can also be treated informally.) These are sometimes expressed in English and sometimes in quasi-formal
language, but in all expressions the "fallacies" require further work of the type described in the example above. Govier (1987) writes, "The informal fallacies, historically a central topic for informal logic, involve mistakes in reasoning which are relatively common, but neither formal nor formally characterizable in any useful way."

Although formally these are all fallacies, these so-called "fallacies" are not always fallacious in the context of informal logic. For example, the WP page on informal fallacies defines Argument from Authority as an argument in which "an assertion is deemed true because of the position or authority of the person asserting it." It could also be expressed in quasi-formal language as an argument which moves from the premise "Authority-A asserts "Proposition-p"" to the conclusion "Proposition-p". Even when understood using the quasi-formal expression, however, the intent is that we evaluate the inference in light of the concrete content of the argument. For example, the argument "The President says it will rain. So, it will rain." might be said to be a fallacious appeal to authority. On the other hand, the argument "The President says his dog is sad. So, his dog is sad." might be said to be a correct appeal to authority. The precise content of the argument (in this case, whether it concerns the weather or the mood of his dog) is important, since the President will be understood as an expert about the latter but not the former.

**A theoretical background**

To understand the definition above, one must understand "informal" which takes its meaning in contrast to its counterpart "formal." (This point was not made for a very long time, hence the nature of informal logic remained opaque, even to those involved in it, for a period of time.) Here it is helpful to have recourse[^8] to Barth and Krabbe (1982:14f) where they distinguish three senses of the term "form." By "form₁," Barth and Krabbe mean the sense of the term which derives from the Platonic idea of form—the ultimate metaphysical unit. Barth and Krabbe claim that most traditional logic is formal in this sense. That is, syllogistic logic is a logic of terms where the terms could naturally be understood as place-holders for Platonic (or Aristotelian) forms. In this first sense of "form," almost all logic is informal (not-formal). Understanding informal logic this way would be much too broad to be useful.

By "form₂," Barth and Krabbe mean the form of sentences and statements as these are understood in modern systems of logic. Here validity is the focus: if the premises are true, the conclusion must then also be true also. Now validity has to do with the logical form of the statement that makes up the argument. In this sense of "formal," most modern and contemporary logic is "formal." That is, such logics canonize the notion of logical form, and the notion of validity plays the central normative role. In this second sense of form, informal logic is not-formal, because it abandons the notion of logical form as the key to understanding the structure of arguments, and likewise retires validity as normative for the purposes of the evaluation of argument. It seems to many that validity is too stringent a requirement, that there are good arguments in which the conclusion is supported by the premises even though it does not follow necessarily from them (as validity requires). An argument in which the conclusion is thought to be "beyond reasonable doubt, given the premises" is sufficient in law to cause a person to be sentenced to death, even though it does not meet the standard of logical validity.

By "form₃," Barth and Krabbe mean to refer to "procedures which are somehow regulated or regimented, which take place according to some set of rules." Barth and Krabbe say that
"we do not defend formality\(^3\) of all kinds and under all circumstances." Rather "we defend the thesis that verbal dialectics must have a certain form (i.e., must proceed according to certain rules) in order that one can speak of the discussion as being won or lost" (19). In this third sense of "form", informal logic can be formal, for there is nothing in the informal logic enterprise that stands opposed to the idea that argumentative discourse should be subject to norms, i.e., subject to rules, criteria, standards or procedures. Informal logic does present standards for the evaluation of argument, procedures for detecting missing premises etc.

**Criticism**

Some hold the view that informal logic is not a branch or subdiscipline of logic, or even the view that there cannot be such a thing as informal logic.\(^9\) \(^10\) \(^11\) Massey criticizes informal logic on the grounds that it has no theory underpinning it. Informal logic, he says, requires detailed classification schemes to organize it, which in other disciplines is provided by the underlying theory. He maintains that there is no method of establishing the invalidity of an argument aside from the formal method, and that the study of fallacies may be of more interest to other disciplines, like psychology, than to philosophy and logic.\(^9\)

**Relation to critical thinking**

Since the 1980s, informal logic has been partnered and even equated,\(^12\) in the minds of many, with critical thinking. The precise definition of "critical thinking" is a subject of much dispute.\(^13\) Critical thinking, according to Johnson, is the evaluation of an intellectual product (an argument, an explanation, a theory) in terms of its strengths and weaknesses.\(^13\) While critical thinking will include evaluation of arguments and hence require skills of argumentation including informal logic, critical thinking requires additional abilities not supplied by informal logic, such as the ability to obtain and assess information and to clarify meaning. Also, many believe that critical thinking requires certain dispositions.\(^14\) Understood in this way, "critical thinking" is a broad term for the attitudes and skills that are involved in analyzing and evaluating arguments. The critical thinking movement promotes critical thinking as an educational ideal. The movement emerged with great force in the 80s in North America as part of an ongoing critique of education as regards the thinking skills not being taught.

**Relation to argumentation theory**

The social, communicative practice of argumentation can and should be distinguished from implication (or entailment)—a relationship between propositions; and from inference—a mental activity typically thought of as the drawing of a conclusion from premises. Informal logic may thus be said to be a logic of argumentation, as distinguished from implication and inference.\(^15\)

Argumentation theory (or the theory of argumentation) has come to be the term that designates the theoretical study of argumentation. This study is interdisciplinary in the sense that no one discipline will be able to provide a complete account. A full appreciation of argumentation requires insights from logic (both formal and informal), rhetoric, communication theory, linguistics, psychology, and, increasingly, computer science. Since the 1970s, there has been significant agreement that there are three basic approaches to
argumentation theory: the logical, the rhetorical and the dialectical. According to Wenzel,[16] the logical approach deals with the product, the dialectical with the process, and the rhetorical with the procedure. Thus, informal logic is one contributor to this inquiry, being most especially concerned with the norms of argument.

See also

- Argument map
- Informal fallacy
- Inference objection
- Lemma

Footnotes

[7] For discussion of the problem of logical constants, see MacFarlane (2005). Intermediate abstractions might include abstracting from this argument that it is about the President and about rain, to get the form "A President says "Proposition-p". So, Proposition-p." Then, abstracting from "President" to Person, to give "Person-A says "Proposition-p". So, Proposition-p." And if other beings can say (that is, speak), we might arrive at "Speaker-A says "Proposition-p". So, Proposition-p." and then, since speaking is only one form of assertion, and speakers only one form of asserters, to the version above, "Asserter-A asserts "Proposition-p". So, Proposition-p."
[12] Johnson (2000) takes the conflation to be part of the Network Problem and holds that settling the issue will require a theory of reasoning.

References

2000

External links
• An informal fallacy primer (http://www.napoletano.net/front/node/350) -- Introduces informal logic and reviews the major fallacies
• BadArguments.org (http://www.badarguments.org)

Mathematical logic

Mathematical logic is a subfield of mathematics with close connections to computer science and → philosophical logic.[1] The field includes the mathematical study of → logic and the applications of formal logic to other areas of mathematics. The unifying themes in mathematical logic include the study of the expressive power of formal systems and the deductive power of formal proof systems.

Mathematical logic is often divided into the subfields of set theory, model theory, recursion theory, and proof theory and constructive mathematics. These areas share basic results on logic, particularly first-order logic, and definability.

Since its inception, mathematical logic has contributed to, and has been motivated by, the study of foundations of mathematics. This study began in the late 19th century with the development of axiomatic frameworks for geometry, arithmetic, and analysis. In the early 20th century it was shaped by David Hilbert’s program to prove the consistency of foundational theories. Results of Kurt Gödel, Gerhard Gentzen, and others provided partial resolution to the program, and clarified the issues involved in proving consistency. Work in set theory showed that almost all ordinary mathematics can be formalized in terms of sets, although there are some theorems that cannot be proven in common axiom systems for set theory. Contemporary work in the foundations of mathematics often focuses on establishing which parts of mathematics can be formalized in particular formal systems, rather than trying to find theories in which all of mathematics can be developed.

History

Mathematical logic emerged in the mid-19th century as a subfield of mathematics independent of the traditional study of logic (Ferreirós 2001, p. 443). Before this emergence, logic was studied with rhetoric, through the syllogism, and with philosophy. The first half of the 20th century saw an explosion of fundamental results, accompanied by vigorous debate over the foundations of mathematics.
Early history
Sophisticated theories of logic were developed in many cultures, including China, India, Greece and the Islamic world. In the 18th century, attempts to treat the operations of formal logic in a symbolic or algebraic way had been made by philosophical mathematicians including Leibniz and Lambert, but their labors remained isolated and little known.

19th century
In the middle of the nineteenth century, George Boole and then Augustus De Morgan presented systematic mathematical treatments of logic. Their work, building on work by algebraists such as George Peacock, extended the traditional Aristotelian doctrine of logic into a sufficient framework for the study of foundations of mathematics (Katz 1998, p. 686). Charles Sanders Peirce built upon the work of Boole to develop a logical system for relations and quantifiers, which he published in several papers from 1870 to 1885. Gottlob Frege presented an independent development of logic with quantifiers in his Begriffsschrift, published in 1879. Frege's work remained obscure, however, until Bertrand Russell began to promote it near the turn of the century. The two-dimensional notation Frege developed was never widely adopted and is unused in contemporary texts.
From 1890 to 1905, Ernst Schröder published Vorlesungen über die Algebra der Logik in three volumes. This work summarized and extended the work of Boole, De Morgan, and Peirce, and was a comprehensive reference to symbolic logic as it was understood at the end of the 19th century.

Foundational theories
Some concerns that mathematics had not been built on a proper foundation led to the development of axiomatic systems for fundamental areas of mathematics such as arithmetic, analysis, and geometry.
In logic, the term arithmetic refers to the theory of the natural numbers. Giuseppe Peano (1888) published a set of axioms for arithmetic that came to bear his name, using a variation of the logical system of Boole and Schröder but adding quantifiers. Peano was unaware of Frege's work at the time. Around the same time Richard Dedekind showed that the natural numbers are uniquely characterized by their induction properties. Dedekind (1888) proposed a different characterization, which lacked the formal logical character of Peano's axioms. Dedekind's work, however, proved theorems inaccessible in Peano's system, including the uniqueness of the set of natural numbers (up to isomorphism) and the recursive definitions of addition and multiplication from the successor function and mathematical induction.
In the mid-19th century, flaws in Euclid's axioms for geometry became known (Katz 1998, p. 774). In addition to the independence of the parallel postulate, established by Nikolai Lobachevsky in 1826 (Lobachevsky 1840), mathematicians discovered that certain theorems taken for granted by Euclid were not in fact provable from his axioms. Among these is the theorem that a line contains at least two points, or that circles of the same radius whose centers are separated by that radius must intersect. Hilbert (1899) developed a complete set of axioms for geometry, building on previous work by Pasch (1882). The success in axiomatizing geometry motivated Hilbert to seek complete axiomatizations of other areas of mathematics, such as the natural numbers and the real line. This would prove to be a major area of research in the first half of the 20th century.
The 19th century saw great advances in the theory of real analysis, including theories of convergence of functions and Fourier series. Mathematicians such as Karl Weierstrass began to construct functions that stretched intuition, such as nowhere-differentiable continuous functions. Previous conceptions of a function as a rule for computation, or a smooth graph, were no longer adequate. Weierstrass began to advocate the arithmetization of analysis, which sought to axiomatize analysis using properties of the natural numbers. The modern "ε-δ" definition of limits and continuous functions was developed by Bolzano and Cauchy between 1817 and 1823 (Felscher 2000). In 1858, Dedekind proposed a definition of the real numbers in terms of Dedekind cuts of rational numbers (Dedekind 1872), a definition still employed in contemporary texts.

Georg Cantor developed the fundamental concepts of infinite set theory. His early results developed the theory of cardinality and proved that the reals and the natural numbers have different cardinalities (Cantor 1874). Over the next twenty years, Cantor developed a theory of transfinite numbers in a series of publications. In 1891, he published a new proof of the uncountability of the real numbers that introduced the diagonal argument, and used this method to prove Cantor's theorem that no set can have the same cardinality as its powerset. Cantor believed that every set could be well-ordered, but was unable to produce a proof for this result, leaving it as an open problem in 1895 (Katz 1998, p. 807).

20th century

In the early decades of the 20th century, the main areas of study were set theory and formal logic. The discovery of paradoxes in informal set theory caused some to wonder whether mathematics itself is inconsistent, and to look for proofs of consistency.

In 1900, Hilbert posed a famous list of 23 problems for the next century. The first two of these were to resolve the continuum hypothesis and prove the consistency of elementary arithmetic, respectively; the tenth was to produce a method that could decide whether a multivariate polynomial equation over the integers has a solution. Subsequent work to resolve these problems shaped the direction of mathematical logic, as did the effort to resolve Hilbert's Entscheidungsproblem, posed in 1928. This problem asked for a procedure that would decide, given a formalized mathematical statement, whether the statement is true or false.

Set theory and paradoxes

Ernst Zermelo (1904) gave a proof that every set could be well-ordered, a result Georg Cantor had been unable to obtain. To achieve the proof, Zermelo introduced the axiom of choice, which drew heated debate and research among mathematicians and the pioneers of set theory. The immediate criticism of the method led Zermelo to publish a second exposition of his result, directly addressing criticisms of his proof (Zermelo 1908). This paper led to the general acceptance of the axiom of choice in the mathematics community.

Skepticism about the axiom of choice was reinforced by recently discovered paradoxes in naive set theory. Cesare Burali-Forti (1897) was the first to state a paradox: the Burali-Forti paradox shows that the collection of all ordinal numbers cannot form a set. Very soon thereafter, Bertrand Russell discovered Russell's paradox in 1901, and Jules Richard (1905) discovered Richard's paradox.

Zermelo (1908) provided the first set of axioms for set theory. These axioms, together with the additional axiom of replacement proposed by Abraham Fraenkel, are now called
Zermelo–Fraenkel set theory (ZF). Zermelo's axioms incorporated the principle of limitation of size to avoid Russell's paradox.

In 1910, the first volume of *Principia Mathematica* by Russell and Alfred North Whitehead was published. This seminal work developed the theory of functions and cardinality in a completely formal framework of type theory, which Russell and Whitehead developed in an effort to avoid the paradoxes. *Principia Mathematica* is considered one of the most influential works of the 20th century, although the framework of type theory did not prove popular as a foundational theory for mathematics (Ferreirós 2001, p. 445).

Fraenkel (1922) proved that the axiom of choice cannot be proved from the remaining axioms of Zermelo's set theory with urelements. Later work by Paul Cohen (1966) showed that the addition of urelements is not needed, and the axiom of choice is unprovable in ZF. Cohen's proof developed the method of forcing, which is now an important tool for establishing independence results in set theory.

**Symbolic logic**

Leopold Löwenheim (1918) and Thoralf Skolem (1919) obtained the Löwenheim–Skolem theorem, which says that first-order logic cannot control the cardinalities of infinite structures. Skolem realized that this theorem would apply to first-order formalizations of set theory, and that it implies any such formalization has a countable model. This counterintuitive fact became known as Skolem's paradox.

In his doctoral thesis, Kurt Gödel (1929) proved the completeness theorem, which establishes a correspondence between syntax and semantics in first-order logic. Gödel used the completeness theorem to prove the compactness theorem, demonstrating the finitary nature of first-order logical consequence. These results helped establish first-order logic as the dominant logic used by mathematicians.

In 1931, Gödel published *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, which proved the incompleteness (in a different meaning of the word) of all sufficiently strong, effective first-order theories. This result, known as Gödel's incompleteness theorem, establishes severe limitations on axiomatic foundations for mathematics, striking a strong blow to Hilbert's program. It showed the impossibility of providing a consistency proof of arithmetic within any formal theory of arithmetic. Hilbert, however, did not acknowledge the importance of the incompleteness theorem for some time.

Gödel's theorem shows that a consistency proof of any sufficiently strong, effective axiom system cannot be obtained in the system itself, if the system is consistent, nor in any weaker system. This leaves open the possibility of consistency proofs that cannot be formalized within the system they consider. Gentzen (1936) proved the consistency of arithmetic using a finitistic system together with a principle of transfinite induction. Gentzen's result introduced the ideas of cut elimination and proof-theoretic ordinals, which became key tools in proof theory. Gödel (1958) gave a different consistency proof, which reduces the consistency of classical arithmetic to that of intuitionistic arithmetic in higher types.
Beginnings of the other branches
Alfred Tarski developed the basics of model theory. Beginning in 1935, a group of prominent mathematicians collaborated under the pseudonym Nicolas Bourbaki to publish a series of encyclopedic mathematics texts. These texts, written in an austere and axiomatic style, emphasized rigorous presentation and set-theoretic foundations. Terminology coined by these texts, such as the words bijection, injection, and surjection, and the set-theoretic foundations the texts employed, were widely adopted throughout mathematics.

The study of computability came to be known as recursion theory, because early formalizations by Gödel and Kleene relied on recursive definitions of functions. When these definitions were shown equivalent to Turing's formalization involving Turing machines, it became clear that a new concept - the computable function - had been discovered, and that this definition was robust enough to admit numerous independent characterizations. In his work on the incompleteness theorems in 1931, Gödel lacked a rigorous concept of an effective formal system; he immediately realized that the new definitions of computability could be used for this purpose, allowing him to state the incompleteness theorems in generality that could only be implied in the original paper.

Numerous results in recursion theory were obtained in the 1940s by Stephen Cole Kleene and Emil Leon Post. Kleene (1943) introduced the concepts of relative computability, foreshadowed by Turing (1939), and the arithmetical hierarchy. Kleene later generalized recursion theory to higher-order functionals. Kleene and Kreisel studied formal versions of intuitionistic mathematics, particularly in the context of proof theory.

Subfields and scope
Contemporary mathematical logic is roughly divided into four areas: set theory, model theory, recursion theory, and proof theory and constructive mathematics. Each area has a distinct focus, although many techniques and results are shared between multiple areas. The border lines between these fields, and the lines between mathematical logic and other fields of mathematics, are not always sharp. Gödel's incompleteness theorem marks not only a milestone in recursion theory and proof theory, but has also led to Loeb's theorem in modal logic. The method of forcing is employed in set theory, model theory, and recursion theory, as well as in the study of intuitionistic mathematics.

The mathematical field of category theory uses many formal axiomatic methods, but category theory is not ordinarily considered a subfield of mathematical logic. Because of its applicability in diverse fields of mathematics, mathematicians including Saunders Mac Lane have proposed category theory as a foundational system for mathematics, independent of set theory. These foundations use toposes, which resemble generalized models of set theory that may employ classical or nonclassical logic.

Formal logic
At its core, mathematical logic deals with mathematical concepts expressed using formal logical systems. These systems, though they differ in many details, share the common property of considering only expressions in a fixed formal language, or signature. The system of first-order logic is the most widely studied today, because of its applicability to foundations of mathematics and because of its desirable proof-theoretic properties.\[3\]
Stronger classical logics such as second-order logic or infinitary logic are also studied, along with nonclassical logics such as intuitionistic logic.

**First-order logic**

First-order logic is a particular formal system of logic. Its syntax involves only finite expressions as well-formed formulas, while its semantics are characterized by the limitation of all quantifiers to a fixed domain of discourse.

Early results about formal logic established limitations of first-order logic. The Löwenheim–Skolem theorem (1919) showed that if a set of sentences in a countable first-order language has an infinite model then it has at least one model of each infinite cardinality. This shows that it is impossible for a set of first-order axioms to characterize the natural numbers, the real numbers, or any other infinite structure up to isomorphism. As the goal of early foundational studies was to produce axiomatic theories for all parts of mathematics, this limitation was particularly stark.

Gödel's completeness theorem (Gödel 1929) established the equivalence between semantic and syntactic definitions of logical consequence in first-order logic. It shows that if a particular sentence is true in every model that satisfies a particular set of axioms, then there must be a finite deduction of the sentence from the axioms. The compactness theorem first appeared as a lemma in Gödel's proof of the completeness theorem, and it took many years before logicians grasped its significance and began to apply it routinely. It says that a set of sentences has a model if and only if every finite subset has a model, or in other words that an inconsistent set of formulas must have a finite inconsistent subset. The completeness and compactness theorems allow for sophisticated analysis of logical consequence in first-order logic and the development of model theory, and they are a key reason for the prominence of first-order logic in mathematics.

Gödel's incompleteness theorems (Gödel 1931) establish additional limits on first-order axiomatizations. The first incompleteness theorem states that no sufficiently strong, effectively given logical system can prove its own consistency unless it is actually inconsistent. Here a logical system is effectively given if it is possible to decide, given any formula in the language of the system, whether the formula is an axiom. When applied to first-order logic, the first incompleteness theorem implies that any sufficiently strong, consistent, effective first-order theory has models that are not elementarily equivalent, a stronger limitation than the one established by the Löwenheim–Skolem theorem. The second incompleteness theorem states that no sufficiently strong, consistent, effective axiom system for arithmetic can prove its own consistency, which has been interpreted to show that Hilbert's program cannot be completed.

**Other classical logics**

Many logics besides first-order logic are studied. These include infinitary logics, which allow for formulas to provide an infinite amount of information, and higher-order logics, which include a portion of set theory directly in their semantics.

The most well studied infinitary logic is $L_{\omega_1, \omega}$. In this logic, quantifiers may only be nested to finite depths, as in first order logic, but formulas may have finite or countably infinite conjunctions and disjunctions within them. Thus, for example, it is possible to say that an object is a natural number using a formula of $L_{\omega_1, \omega}$ such as

$$(x = 0) \lor (x = 1) \lor (x = 2) \lor \cdots.$$
Higher-order logics allow for quantification not only of elements of the domain of discourse, but subsets of the domain of discourse, sets of such subsets, and other objects of higher type. The semantics are defined so that, rather than having a separate domain for each higher-type quantifier to range over, the quantifiers instead range over all objects of the appropriate type. The logics studied before the development of first-order logic, for example Frege's logic, had similar set-theoretic aspects. Although higher-order logics are more expressive, allowing complete axiomatizations of structures such as the natural numbers, they do not satisfy analogues of the completeness and compactness theorems from first-order logic, and are thus less amenable to proof-theoretic analysis.

Nonclassical and modal logic

Modal logics include additional modal operators, such as an operator which states that a particular formula is not only true, but necessarily true. Although modal logic is not often used to axiomatize mathematics, it has been used to study the properties of first-order provability (Solovay 1976) and set-theoretic forcing (Hamkins and Löwe 2007).

Intuitionistic logic was developed by Heyting to study Brouwer's program of intuitionism, in which Brouwer himself avoided formalization. Intuitionistic logic specifically does not include the law of the excluded middle, which states that each sentence is either true or its negation is true. Kleene's work with the proof theory of intuitionistic logic showed that constructive information can be recovered from intuitionistic proofs. For example, any provably total function in intuitionistic arithmetic is computable; this is not true in classical theories of arithmetic such as Peano arithmetic.

Set theory

Set theory is the study of sets, which are abstract collections of objects. Many of the basic notions, such as ordinal and cardinal numbers, were developed informally by Cantor before formal axiomatizations of set theory were developed. The first such axiomatization, due to Zermelo (1908), was extended slightly to become Zermelo–Fraenkel set theory (ZF), which is now the most widely-used foundational theory for mathematics.

Other formalizations of set theory have been proposed, including von Neumann–Bernays–Gödel set theory (NBG), Morse–Kelley set theory (MK), and New Foundations (NF). Of these, ZF, NBG, and MK are similar in describing a cumulative hierarchy of sets. New Foundations takes a different approach; it allows objects such as the set of all sets at the cost of restrictions on its set-existence axioms. The system of Kripke-Platek set theory is closely related to generalized recursion theory.

Two famous statements in set theory are the axiom of choice and the continuum hypothesis. The axiom of choice, first stated by Zermelo (1904), was proved independent of ZF by Fraenkel (1922), but has come to be widely accepted by mathematicians. It states that given a collection of nonempty sets there is a single set \( C \) that contains exactly one element from each set in the collection. The set \( C \) is said to "choose" one element from each set in the collection. While the ability to make such a choice is considered obvious by some, since each set in the collection is nonempty, the lack of a general, concrete rule by which the choice can be made renders the axiom nonconstructive. Stefan Banach and Alfred Tarski (1924) showed that the axiom of choice can be used to decompose a solid ball into a finite number of pieces which can then be rearranged, with no scaling, to make two solid balls of the original size. This theorem, known as the Banach-Tarski paradox, is one of many
counterintuitive results of the axiom of choice.

The continuum hypothesis, first proposed as a conjecture by Cantor, was listed by David Hilbert as one of his 23 problems in 1900. Gödel showed that the continuum hypothesis cannot be disproven from the axioms of Zermelo-Frankel set theory (with or without the axiom of choice), by developing the constructible universe of set theory in which the continuum hypothesis must hold. In 1963, Paul Cohen showed that the continuum hypothesis cannot be proven from the axioms of Zermelo-Frankel set theory (Cohen 1966). This independence result did not completely settle Hilbert's question, however, as it is possible that new axioms for set theory could resolve the hypothesis. Recent work along these lines has been conducted by W. Hugh Woodin, although its importance is not yet clear (Woodin 2001).

Contemporary research in set theory includes the study of large cardinals and determinacy. Large cardinals are cardinal numbers with particular properties so strong that the existence of such cardinals cannot be proved in ZFC. The existence of the smallest large cardinal typically studied, an inaccessible cardinal, already implies the consistency of ZFC. Despite the fact that large cardinals have extremely high cardinality, their existence has many ramifications for the structure of the real line. Determinacy refers to the possible existence of winning strategies for certain two-player games (the games are said to be determined). The existence of these strategies implies structural properties of the real line and other Polish spaces.

Model theory

Model theory studies the models of various formal theories. Here a theory is a set of formulas in a particular formal logic and signature, while a model is a structure that gives a concrete interpretation of the theory. Model theory is closely related to universal algebra and algebraic geometry, although the methods of model theory focus more on logical considerations than those fields.

The set of all models of a particular theory is called an elementary class; classical model theory seeks to determine the properties of models in a particular elementary class, or determine whether certain classes of structures form elementary classes.

The method of quantifier elimination can be used to show that definable sets in particular theories cannot be too complicated. Tarski (1948) established quantifier elimination for real-closed fields, a result which also shows the theory of the field of real numbers is decidable. (He also noted that his methods were equally applicable to algebraically closed fields of arbitrary characteristic.) A modern subfield developing from this is concerned with o-minimal structures.

Morley's categoricity theorem, proved by Michael D. Morley (1965), states that if a first-order theory in a countable language is categorical in some uncountable cardinality, i.e. all models of this cardinality are isomorphic, then it is categorical in all uncountable cardinalities.

A trivial consequence of the continuum hypothesis is that a complete theory with less than continuum many nonisomorphic countable models can have only countably many. Vaught's conjecture, named after Robert Lawson Vaught, says that this is true even independently of the continuum hypothesis. Many special cases of this conjecture have been established.
Recursion theory

Recursion theory, also called computability theory, studies the properties of computable functions and the Turing degrees, which divide the uncomputable functions into sets which have the same level of uncomputability. Recursion theory also includes the study of generalized computability and definability. Recursion theory grew from the work of Alonzo Church and Alan Turing in the 1930s, which was greatly extended by Kleene and Post in the 1940s.

Classical recursion theory focuses on the computability of functions from the natural numbers to the natural numbers. The fundamental results establish a robust, canonical class of computable functions with numerous independent, equivalent characterizations using Turing machines, \( \lambda \) calculus, and other systems. More advanced results concern the structure of the Turing degrees and the lattice of recursively enumerable sets.

Generalized recursion theory extends the ideas of recursion theory to computations that are no longer necessarily finite. It includes the study of computability in higher types as well as areas such as hyperarithmetical theory and \( \alpha \)-recursion theory.

Contemporary research in recursion theory includes the study of applications such as algorithmic randomness and computable model theory as well as new results in pure recursion theory.

Algorithmically unsolvable problems

An important subfield of recursion theory studies algorithmic unsolvability; a problem is algorithmically unsolvable if there is no computable function which, given any [code for an] instance of the problem, returns the correct answer. The first results about unsolvability, obtained independently by Church and Turing in 1936, showed that the Entscheidungsproblem is algorithmically unsolvable. Turing proved this by establishing the unsolvability of the halting problem, a result with far-ranging implications in both recursion theory and computer science.

There are many known examples of undecidable problems from ordinary mathematics. The word problem for groups was proved algorithmically unsolvable by Pyotr Sergeyevich Novikov in 1955 and independently by W. Boone in 1959. The busy beaver problem, developed by Tibor Radó in 1962, is another well-known example.

Hilbert’s tenth problem asked for an algorithm to determine whether a multivariate polynomial equation with integer coefficients has a solution in the integers. Partial progress was made by Julia Robinson, Martin Davis, and Hilary Putnam. The algorithmic unsolvability of the problem was proved by Yuri Matiyasevich in 1970 (Davis 1973).

Proof theory and constructive mathematics

Proof theory is the study of formal proofs in various logical deduction systems. These proofs are represented as formal mathematical objects, facilitating their analysis by mathematical techniques. Several deduction systems are commonly considered, including Hilbert-style deduction systems, systems of natural deduction, and the sequent calculus developed by Gentzen.

The study of constructive mathematics, in the context of mathematical logic, includes the study of systems in non-classical logic such as intuitionistic logic, as well as the study of predicative systems. An early proponent of predicativism was Hermann Weyl, who showed...
it is possible to develop a large part of real analysis using only predicative methods (Weyl 1918).

Because proofs are entirely finitary, whereas truth in a structure is not, it is common for work in constructive mathematics to emphasize provability. The relationship between provability in classical (or nonconstructive) systems and provability in intuitionistic (or constructive, respectively) systems is of particular interest. Results such as the Gödel-Gentzen negative translation show that it is possible to embed (or translate) classical logic into intuitionistic logic, allowing some properties about intuitionistic proofs to be transferred back to classical proofs.

Recent developments in proof theory include the study of proof mining by Ulrich Kohlenbach and the study of proof-theoretic ordinals by Michael Rathjen.

Connections with computer science

The study of computability theory in computer science is closely related to the study of computability in mathematical logic. There is a difference of emphasis, however. Computer scientists often focus on concrete programming languages and feasible computability, while researchers in mathematical logic often focus on computability as a theoretical concept and on noncomputability.

The study of programming language semantics is related to model theory, as is program verification (in particular, model checking). The Curry-Howard isomorphism between proofs and programs relates to proof theory, especially intuitionistic logic. Formal calculi such as the lambda calculus and combinatory logic are now studied as idealized programming languages.

Computer science also contributes to mathematics by developing techniques for the automatic checking or even finding of proofs, such as automated theorem proving and logic programming.

Foundations of mathematics

In the 19th century, mathematicians became aware of logical gaps and inconsistencies in their field. It was shown that Euclid's axioms for geometry, which had been taught for centuries as an example of the axiomatic method, were incomplete. The use of infinitesimals, and the very definition of function, came into question in analysis, as pathological examples such as Weierstrass' nowhere-differentiable continuous function were discovered.

Cantor's study of arbitrary infinite sets also drew criticism. Leopold Kronecker famously stated "God made the integers; all else is the work of man," endorsing a return to the study of finite, concrete objects in mathematics. Although Kronecker's argument was carried forward by constructivists in the 20th century, the mathematical community as a whole rejected them. David Hilbert argued in favor of the study of the infinite, saying "No one shall expel us from the Paradise that Cantor has created."

Mathematicians began to search for axiom systems that could be used to formalize large parts of mathematics. In addition to removing ambiguity from previously-naïve terms such as function, it was hoped that this axiomatization would allow for consistency proofs. In the 19th century, the main method of proving the consistency of a set of axioms was to provide a model for it. Thus, for example, non-Euclidean geometry can be proved consistent by
defining point to mean a point on a fixed sphere and line to mean a great circle on the sphere. The resulting structure, a model of elliptic geometry, satisfies the axioms of plane geometry except the parallel postulate.

With the development of formal logic, Hilbert asked whether it would be possible to prove that an axiom system is consistent by analyzing the structure of possible proofs in the system, and showing through this analysis that it is impossible to prove a contradiction. This idea led to the study of proof theory. Moreover, Hilbert proposed that the analysis should be entirely concrete, using the term finitary to refer to the methods he would allow but not precisely defining them. This project, known as Hilbert's program, was seriously affected by Gödel's incompleteness theorems, which show that the consistency of formal theories of arithmetic cannot be established using methods formalizable in those theories. Gentzen showed that it is possible to produce a proof of the consistency of arithmetic in a finitary system augmented with axioms of transfinite induction, and the techniques he developed to do so were seminal in proof theory.

A second thread in the history of foundations of mathematics involves nonclassical logics and constructive mathematics. The study of constructive mathematics includes many different programs with various definitions of constructive. At the most accommodating end, proofs in ZF set theory that do not use the axiom of choice are called constructive by many mathematicians. More limited versions of constructivism limit themselves to natural numbers, number-theoretic functions, and sets of natural numbers (which can be used to represent real numbers, facilitating the study of mathematical analysis). A common idea is that in order to assert that a number-theoretic function exists, a concrete means of computing the values of the function must be known.

In the early 20th century, Luitzen Egbertus Jan Brouwer founded intuitionism as a philosophy of mathematics. This philosophy, poorly understood at first, stated that in order for a mathematical statement to be true to a mathematician, that person must be able to intuit the statement, to not only believe its truth but understand the reason for its truth. A consequence of this definition of truth was the rejection of the law of the excluded middle, for there are statements that, according to Brouwer, could not be claimed to be true while their negations also could not be claimed true. Brouwer's philosophy was influential, and the cause of bitter disputes among prominent mathematicians. Later, Kleene and Kreisel would study formalized versions of intuitionistic logic (Brouwer rejected formalization, and presented his work in unformalized natural language). With the advent of the BHK interpretation and Kripke models, intuitionism became easier to reconcile with classical mathematics.

See also

- List of mathematical logic topics
- List of computability and complexity topics
- List of set theory topics
- List of first-order theories
- Knowledge representation
- → Metalogic
Notes


References

Undergraduate texts


Graduate texts


**Research papers, monographs, texts, and surveys**


**Classical papers, texts, and collections**

- Dedekind, Richard (1872), *Stetigkeit und irrationale Zahlen*. English translation of title: "Consistency and irrational numbers".
- Dedekind, Richard (1888), *Was sind und was sollen die Zahlen?" Two English translations:
- Fraenkel, Abraham A. (1922), "Der Begriff 'definit' und die Unabhängigkeit des Auswahlsaxioms", *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse*, pp. 253–257 (German), reprinted in English translation as "The notion of 'definite' and the independence of the axiom of choice", van


- Leopold Löwenheim (1918)


Mathematical logic

1976, pp. 142–144.

• Thoralf Skolem (1919)
• Tarski, Alfred (1948), A decision method for elementary algebra and geometry, Santa Monica, California: RAND Corporation
• Zermelo, Ernst (1904), "Beweis, daß jede Menge wohlgeordnet werden kann", Mathematische Annalen 59: 514–516, doi: 10.1007/BF01445300 (German), reprinted in English translation as "Proof that every set can be well-ordered", van Heijenoort 1976, pp. 139–141.

External links

• Mathematical Logic around the world (http://world.logic.at/)
• Polyvalued logic (http://home.swipnet.se/~w-33552/logic/home/index.htm)
• forall x: an introduction to formal logic (http://www.fecundity.com/logic/), by P.D. Magnus, is a free textbook.
• A Problem Course in Mathematical Logic (http://euclid.trentu.ca/math/sb/pcml/), by Stefan Bilaniuk, is another free textbook.
• The London Philosophy Study Guide (http://www.ucl.ac.uk/philosophy/LPSG/) offers many suggestions on what to read, depending on the student's familiarity with the subject:
  • Mathematical Logic (http://www.ucl.ac.uk/philosophy/LPSG/MathLogic.htm)
  • Set Theory & Further Logic (http://www.ucl.ac.uk/philosophy/LPSG/SetTheory.htm)
  • Philosophy of Mathematics (http://www.ucl.ac.uk/philosophy/LPSG/PhilMath.htm)
Algebraic logic

In mathematical logic, **algebraic logic** formalizes logic using the methods of abstract algebra.

**Algebras as models of logics**

Algebraic logic treats algebraic structures, often bounded lattices, as models (interpretations) of certain logics, making logic a branch of order theory.

In algebraic logic:
- Variables are tacitly universally quantified over some universe of discourse. There are no existentially quantified variables or open formulas;
- Terms are built up from variables using primitive and defined operations. There are no connectives;
- Formulas, built from terms in the usual way, can be equated if they are logically equivalent. To express a tautology, equate a formula with a truth value;
- The rules of proof are the substitution of equals for equals, and uniform replacement.
  Modus ponens remains valid, but is seldom employed.

In the table below, the left column contains one or more logical or mathematical systems, and the algebraic structure which are its models are shown on the right in the same row. Some of these structures are either Boolean algebras or proper extensions thereof. Modal and other nonclassical logics are typically modeled by what are called "Boolean algebras with operators."

Algebraic formalisms going beyond first-order logic in at least some respects include:
- Combinatory logic, having the expressive power of set theory;
- Relation algebra, arguably the paradigmatic algebraic logic, can express Peano arithmetic and most axiomatic set theories, including the canonical ZFC.

<table>
<thead>
<tr>
<th>logical system</th>
<th>its models</th>
</tr>
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<tbody>
<tr>
<td>Classical sentential logic</td>
<td>Lindenbaum-Tarski algebra Two-element Boolean algebra</td>
</tr>
<tr>
<td>Intuitionistic propositional logic</td>
<td>Heyting algebra</td>
</tr>
<tr>
<td>Lukasiewicz logic</td>
<td>MV-algebra</td>
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<tr>
<td>Modal logic K</td>
<td>Modal algebra</td>
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<tr>
<td>Lewis's S4</td>
<td>Interior algebra</td>
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<tr>
<td>Lewis's S5; Monadic predicate logic</td>
<td>Monadic Boolean algebra</td>
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<tr>
<td>First-order logic</td>
<td>Cylindric algebra Polyadic algebra</td>
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<td></td>
<td>Predicate functor logic</td>
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<tr>
<td>Set theory</td>
<td>Combinatory logic Relation algebra</td>
</tr>
</tbody>
</table>
History
On the history of algebraic logic before World War II, see Brady (2000) and Grattan-Guinness (2000) and their ample references. On the postwar history, see Maddux (1991) and Quine (1976).

Algebraic logic has at least two meanings:
• The study of Boolean algebra, begun by George Boole, and of relation algebra, begun by Augustus DeMorgan, extended by Charles Sanders Peirce, and taking definitive form in the work of Ernst Schröder;
• Abstract algebraic logic, a branch of contemporary mathematical logic.

Perhaps surprisingly, algebraic logic is the oldest approach to formal logic, arguably beginning with a number of memoranda Leibniz wrote in the 1680s, some of which were published in the 19th century and translated into English by Clarence Lewis in 1918. But nearly all of Leibniz's known work on algebraic logic was published only in 1903, after Louis Couturat discovered it in Leibniz's Nachlass. Parkinson (1966) and Loemker (1969) translated selections from Couturat's volume into English.

Brady (2000) discusses the rich historical connections between algebraic logic and model theory. The founders of model theory, Ernst Schroder and Leopold Loewenheim, were logicians in the algebraic tradition. Alfred Tarski, the founder of set theoretic model theory as a major branch of contemporary mathematical logic, also:
• Co-discovered Lindenbaum-Tarski algebra;
• Invented cylindric algebra;
• Wrote the 1940 paper that revived relation algebra, and that can be seen as the starting point of abstract algebraic logic.

Modern mathematical logic began in 1847, with two pamphlets whose respective authors were Augustus DeMorgan and George Boole. They, and later C.S. Peirce, Hugh MacColl, Frege, Peano, Bertrand Russell, and A. N. Whitehead all shared Leibniz's dream of combining symbolic logic, mathematics, and philosophy. Relation algebra is arguably the culmination of Leibniz's approach to logic. With the exception of some writings by Leopold Loewenheim and Thoralf Skolem, algebraic logic went into eclipse soon after the 1910-13 publication of Principia Mathematica, not to revive until Tarski's 1940 reexposition of relation algebra.

Leibniz had no influence on the rise of algebraic logic because his logical writings were little studied before the Parkinson and Loemker translations. Our present understanding of Leibniz the logician stems mainly from the work of Wolfgang Lenzen, summarized in Lenzen (2004). [1] To see how present-day work in logic and metaphysics can draw inspiration from, and shed light on, Leibniz's thought, see Zalta (2000). [2]
See also

- Abstract algebraic logic
- Algebraic structure
- Boolean algebra (logic)
- Boolean algebra (structure)
- Cylindric algebra
- Lindenbaum-Tarski algebra
- Mathematical logic
- Model theory
- Monadic Boolean algebra
- Predicate functor logic
- Relation algebra
- Universal algebra

References


External links


References

Multi-valued logic

Multi-valued logics are ‘→ logical calculi’ in which there are more than two truth values. Traditionally, in ‘logical calculi’ - invented by Aristotle - there were only two possible values (i.e. TRUE and FALSE) for any proposition to take. An obvious extension to classical two-valued logic is an n-valued logic for n > 2. Those most popular in the literature are three-valued (e.g. Łukasiewicz’s and Kleene’s) which accept the values TRUE, FALSE, UNKNOWN, the finite-valued with more than 3 values, and infinite-valued (e.g. → fuzzy logic) ones.

Relation to classical logic

Logics are usually systems intended to codify rules for preserving some semantic property of propositions across transformations. In classical → logic, this property is “truth.” In a valid argument, the truth of the derived proposition is guaranteed if the premises are jointly true, because the application of valid steps preserves the property. However, that property doesn’t have to be that of "truth"; instead, it can be some other concept.

Multi-valued logics are intended to preserve the property of designationhood (or being designated). Since there are more than two truth values, rules of inference may be intended to preserve more than just whichever corresponds (in the relevant sense) to truth. For example, in a three-valued logic, sometimes the two greatest truth-values (when they are represented as e.g. positive integers) are designated and the rules of inference preserve these values. Precisely, a valid argument will be such that the value of the premises taken jointly will always be less than or equal to the conclusion.

For example, the preserved property could be justification, the foundational concept of intuitionistic logic. Thus, a proposition is not true or false; instead, it is justified or flawed. A key difference between justification and truth, in this case, is that the law of the excluded middle doesn't hold: a proposition that is not flawed is not necessarily justified; instead, it's only not proven that it's flawed. The key difference is the determinacy of the preserved property: One may prove that P is justified, that P is flawed, or be unable to prove either. A valid argument preserves justification across transformations, so a proposition derived from justified propositions is still justified. However, there are proofs in classical logic that depend upon the law of excluded middle; since that law is not usable under this scheme, there are propositions that cannot be proven that way.

Relation to fuzzy logic

Multi-valued logic is strictly related with Fuzzy set theory and → fuzzy logic. The notion of fuzzy subset was introduced by Lotfi Zadeh as a formalization of vagueness; i.e., the phenomenon that a predicate may apply to an object not absolutely, but to a certain degree, and that there may be borderline cases. Indeed, as in multi-valued logic, fuzzy logic admits truth values different from "true" and "false". As an example, usually the set of possible truth values is the whole interval [0,1]. Nevertheless, the main difference between fuzzy logic and multi-valued logic is in the aims. In fact, in spite of its philosophical interest (it can be used to deal with the sorites paradox), fuzzy logic is devoted mainly to the applications. More precisely, there are two approaches to → Fuzzy logic. The first one is very closely linked with multi-valued logic tradition (Hajek school). So a set of designed
values is fixed and this enables us to define an entailment relation. The deduction apparatus is defined by a suitable set of logical axioms and suitable inference rules. Another approach (Goguen, Pavelka and others) is devoted to defining a deduction apparatus in which approximate reasonings are admitted. Such an apparatus is defined by a suitable fuzzy subset of logical axioms and by a suitable set of fuzzy inference rules. In the first case the logical consequence operator gives the set of logical consequence of a given set of axioms. In the latter the logical consequence operator gives the fuzzy subset of logical consequence of a given fuzzy subset of hypotheses.

Another example of an infinitely-valued logic is probability logic.

History

The first known classical logician who didn't fully accept the law of the excluded middle was Aristotle (who, ironically, is also generally considered to be the first classical logician and the "father of logic"[1]), who admitted that his laws did not all apply to future events (De Interpretatione, ch. IX). But he didn't create a system of multi-valued logic to explain this isolated remark. The later logicians until the coming of the 20th century followed Aristotelian logic, which includes or implies the law of the excluded middle.

The 20th century brought the idea of multi-valued logic back. The Polish logician and philosopher Jan Łukasiewicz began to create systems of many-valued logic in 1920, using a third value "possible" to deal with Aristotle's paradox of the sea battle. Meanwhile, the American mathematician Emil L. Post (1921) also introduced the formulation of additional truth degrees with n>=2, where n are the truth values. Later Jan Łukasiewicz and Alfred Tarski together formulated a logic on n truth values where n>=2 and in 1932 Hans Reichenbach formulated a logic of many truth values where n→infinity. Kurt Gödel in 1932 showed that intuitionistic logic is not a finitely-many valued logic, and defined a system of Gödel logics intermediate between classical and intuitionistic logic; such logics are known as intermediate logics.

See also

- Fuzzy Logic
- Degrees of truth
- False dilemma
- Logical value
- MV-algebra
- IEEE 1164
- Perspectivism
- Rhizome (philosophy)
- Anekantavada
- Principle of Bivalence
Multi-valued logic

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External links


Notes

Fuzzy logic

Fuzzy logic is a form of → multi-valued logic derived from fuzzy set theory to deal with reasoning that is approximate rather than precise. In contrast with binary sets having binary logic, also known as crisp logic, the fuzzy logic variables may have a membership value of only 0 or 1. Just as in fuzzy set theory with fuzzy logic the set membership values can range ( inclusively) between 0 and 1, in fuzzy logic the degree of truth of a statement can range between 0 and 1 and is not constrained to the two truth values {true (1), false (0)} as in classic propositional logic.[1] And when linguistic variables are used, these degrees may be managed by specific functions, as discussed below.

The term "fuzzy logic" emerged as a consequence of the development of the theory of fuzzy sets by Lotfi Zadeh[2].

In 1965 Lotfi Zadeh proposed fuzzy set theory[3], and later established fuzzy logic based on fuzzy sets. Fuzzy logic has been applied to diverse fields, from control theory to artificial intelligence, yet still remains controversial among most statisticians, who prefer Bayesian logic, and some control engineers, who prefer traditional two-valued logic.

Earlier than Zadeh, a paper introducing the concept without using the term "fuzzy" was published by R.H. Wilkinson in 1963[4] and thus preceded fuzzy set theory. Wilkinson was the first one to redefine and generalize the earlier multivalued logics in terms of set theory. The main purpose of his paper, following his first proposals in his 1961 Electrical Engineering master thesis, was to show how any mathematical function could be simulated using hardwired analog electronic circuits. He did this by first creating various linear voltage ramps which were then selected in a "logic block" using diodes and resistor circuits which implemented the maximum and minimum Fuzzy Logic rules of the INCLUSIVE OR and the AND operations respectively. He called his logic "analog logic". Some say that the idea of fuzzy logic is set-theoretical equivalent of the "analog logic" of Wilkinson (without recourse to electrical circuits), but he never received any credit.

Degrees of truth

Both degrees of truth and probabilities range between 0 and 1 and hence may seem similar at first. However, they are distinct conceptually; truth represents membership in vaguely defined sets, not likelihood of some event or condition as in probability theory. For example, let a 100-ml glass contain 30 ml of water. Then we may consider two concepts: Empty and Full. The meaning of his paper, following his first proposals in his 1961 Electrical Engineering master thesis, was to show how any mathematical function could be simulated using hardwired analog electronic circuits. He did this by first creating various linear voltage ramps which were then selected in a "logic block" using diodes and resistor circuits which implemented the maximum and minimum Fuzzy Logic rules of the INCLUSIVE OR and the AND operations respectively. He called his logic "analog logic". Some say that the idea of fuzzy logic is set-theoretical equivalent of the "analog logic" of Wilkinson (without recourse to electrical circuits), but he never received any credit.

A probabilistic setting would first define a scalar variable for the fullness of the glass, and second, conditional distributions describing the probability that someone would call the glass full given a specific fullness level. This model, however, has no sense without accepting occurrence of some event, e.g. that after a few minutes, the glass will be half empty. Note that the conditioning can be achieved by having a specific observer that
randomly selects the label for the glass, a distribution over deterministic observers, or both. Consequently, probability has nothing in common with fuzziness, these are simply different concepts which superficially seem similar because of using the same interval of real numbers [0, 1]. Still, since theorems such as De Morgan's have dual applicability and properties of random variables are analogous to properties of binary logic states, one can see where the confusion might arise.

**Applying truth values**

A basic application might characterize subrange of a continuous variable. For instance, a temperature measurement for anti-lock brakes might have several separate membership functions defining particular temperature ranges needed to control the brakes properly. Each function maps the same temperature value to a truth value in the 0 to 1 range. These truth values can then be used to determine how the brakes should be controlled.

In this image, the meaning of the expressions *cold*, *warm*, and *hot* is represented by functions mapping a temperature scale. A point on that scale has three "truth values" — one for each of the three functions. The vertical line in the image represents a particular temperature that the three arrows (truth values) gauge. Since the red arrow points to zero, this temperature may be interpreted as "not hot". The orange arrow (pointing at 0.2) may describe it as "slightly warm" and the blue arrow (pointing at 0.8) "fairly cold".

**Linguistic variables**

While variables in mathematics usually take numerical values, in fuzzy logic applications, the non-numeric *linguistic variables* are often used to facilitate the expression of rules and facts.\(^5\)

A linguistic variable such as *age* may have a value such as *young* or its antonym *old*. However, the great utility of linguistic variables is that they can be modified via linguistic hedges applied to primary terms. The linguistic hedges can be associated with certain functions. For example, L. A. Zadeh proposed to take the square of the membership function. This model, however, does not work properly. For more details, see the references.
An example of fuzzy reasoning

Fuzzy Set Theory defines Fuzzy Operators on Fuzzy Sets. The problem in applying this is that the appropriate Fuzzy Operator may not be known. For this reason, Fuzzy logic usually uses IF-THEN rules, or constructs that are equivalent, such as fuzzy associative matrices.

Rules are usually expressed in the form:

IF variable IS property THEN action

For example, an extremely simple temperature regulator that uses a fan might look like this:

IF temperature IS very cold THEN stop fan
IF temperature IS cold THEN turn down fan
IF temperature IS normal THEN maintain level
IF temperature IS hot THEN speed up fan

Notice there is no "ELSE". All of the rules are evaluated, because the temperature might be "cold" and "normal" at the same time to different degrees.

The AND, OR, and NOT operators of boolean logic exist in fuzzy logic, usually defined as the minimum, maximum, and complement; when they are defined this way, they are called the Zadeh operators, because they were first defined as such in Zadeh's original papers. So for the fuzzy variables x and y:

\[
\begin{align*}
\text{NOT } x &= (1 - \text{truth}(x)) \\
\text{x AND y} &= \text{minimum}(\text{truth}(x), \text{truth}(y)) \\
\text{x OR y} &= \text{maximum}(\text{truth}(x), \text{truth}(y))
\end{align*}
\]

There are also other operators, more linguistic in nature, called hedges that can be applied. These are generally adverbs such as "very", or "somewhat", which modify the meaning of a set using a mathematical formula.

In application, the programming language Prolog is well geared to implementing fuzzy logic with its facilities to set up a database of "rules" which are queried to deduct logic. This sort of programming is known as logic programming.

Once fuzzy relations are defined, it is possible to develop fuzzy relational databases. The first fuzzy relational database, FRDB, appeared in Maria Zemankova's dissertation. Later, some other models arose like the Buckles-Petry model, the Prade-Testemale Model, the Umano-Fukami model or the GEFRED model by J.M. Medina, M.A. Vila et al. In the context of fuzzy databases, some fuzzy querying languages have been defined, highlighting the SQLf by P. Bosc et al. and the FSQL by J. Galindo et al. These languages define some structures in order to include fuzzy aspects in the SQL statements, like fuzzy conditions, fuzzy comparators, fuzzy constants, fuzzy constraints, fuzzy thresholds, linguistic labels and so on.
Other examples

- If a male is 1.8 meters, consider him as tall:
  
  IF male IS true AND height >= 1.8 THEN is_tall IS true; is_short IS false

- The fuzzy rules do not make sharp distinction between tall and short:
  
  IF height <= medium male THEN is_short IS agree somewhat
  IF height >= medium male THEN is_tall IS agree somewhat

In the fuzzy case, there are no such heights as 1.83 meters, but there are fuzzy values, like the following assignments:

  - dwarf male = [0, 1.3] m
  - short male = [1.3, 1.5] m
  - medium male = [1.5, 1.8] m
  - tall male = [1.8, 2.0] m
  - giant male > 2.0 m

For the consequent, there are may also be more than two values:

  - agree not = 0
  - agree little = 1
  - agree somewhat = 2
  - agree a lot = 3
  - agree fully = 4

In the binary (or "crisp") case, a person of 1.79 meters is considered of medium height, while another person who is 1.8 meters or 2.25 meters tall is considered tall.

The crisp example differs deliberately from the fuzzy one. The antecedent was not given fuzzy values:

  IF male >= agree somewhat AND ...

as gender is often considered binary information.

Mathematical fuzzy logic

In → mathematical logic, there are several formal systems of "fuzzy logic"; most of them belong among so-called t-norm fuzzy logics.

Propositional fuzzy logics

The most important propositional fuzzy logics are:

- Monoidal t-norm-based propositional fuzzy logic MTL is an axiomatization of logic where conjunction is defined by a left continuous t-norm, and implication is defined as the residuum of the t-norm. Its models correspond to MTL-algebras that are prelinear commutative bounded integral residuated lattices.

- Basic propositional fuzzy logic BL is an extension of MTL logic where conjunction is defined by a continuous t-norm, and implication is also defined as the residuum of the t-norm. Its models correspond to BL-algebras.

- Łukasiewicz fuzzy logic is the extension of basic fuzzy logic BL where standard conjunction is the Łukasiewicz t-norm. It has the axioms of basic fuzzy logic plus an axiom of double negation, and its models correspond to MV-algebras.

- Gödel fuzzy logic is the extension of basic fuzzy logic BL where conjunction is Gödel t-norm. It has the axioms of BL plus an axiom of idempotence of conjunction, and its
models are called G-algebras.

- Product fuzzy logic is the extension of basic fuzzy logic BL where conjunction is product t-norm. It has the axioms of BL plus another axiom for cancellativity of conjunction, and its models are called product algebras.

- Fuzzy logic with evaluated syntax (sometimes also called Pavelka's logic), denoted by EVŁ, is a further generalization of mathematical fuzzy logic. While the above kinds of fuzzy logic have traditional syntax and many-valued semantics, in EVŁ is evaluated also syntax. This means that each formula has an evaluation. Axiomatization of EVŁ stems from Łukasiewicz fuzzy logic. A generalization of classical Gödel completeness theorem is provable in EVŁ.

**Predicate fuzzy logics**

These extend the above-mentioned fuzzy logics by adding universal and existential quantifiers in a manner similar to the way that → predicate logic is created from propositional logic. The semantics of the universal resp. existential quantifier in t-norm fuzzy logics is the infimum resp. supremum of the truth degrees of the instances of the quantified subformula.

**Higher-order fuzzy logics**

These logics, called fuzzy type theories, extend predicate fuzzy logics to be able to quantify also predicates and higher order objects. A fuzzy type theory is a generalization of classical simple type theory introduced by B. Russell [6] and mathematically elaborated by A. Church [7] and L. Henkin[8].

**Decidability issues for fuzzy logic**

The notions of a "decidable subset" and "recursively enumerable subset" are basic ones for classical mathematics and classical logic. Then, the question of a suitable extension of such concepts to fuzzy set theory arises. A first proposal in such a direction was made by E.S. Santos by the notions of fuzzy Turing machine, Markov normal fuzzy algorithm and fuzzy program. Successively, L. Biacino and G. Gerla showed that such a definition is not adequate and therefore proposed the following one. \( \mathbb{U} \) denotes the set of rational numbers in \([0,1]\). A fuzzy subset \( s : S \rightarrow [0,1] \) of a set \( S \) is recursively enumerable if a recursive map \( h : S \times \mathbb{N} \rightarrow \mathbb{U} \) exists such that, for every \( x \) in \( S \), the function \( h(x,n) \) is increasing with respect to \( n \) and \( s(x) = \lim h(x,n) \). We say that \( s \) is decidable if both \( s \) and its complement \( \neg s \) are recursively enumerable. An extension of such a theory to the general case of the L-subsets is proposed in Gerla 2006. The proposed definitions are well related with fuzzy logic. Indeed, the following theorem holds true (provided that the deduction apparatus of the fuzzy logic satisfies some obvious effectiveness property).

**Theorem.** Any axiomatizable fuzzy theory is recursively enumerable. In particular, the fuzzy set of logically true formulas is recursively enumerable in spite of the fact that the crisp set of valid formulas is not recursively enumerable, in general. Moreover, any axiomatizable and complete theory is decidable.

It is an open question to give supports for a Church thesis for fuzzy logic claiming that the proposed notion of recursive enumerability for fuzzy subsets is the adequate one. To this aim, further investigations on the notions of fuzzy grammar and fuzzy Turing machine should be necessary (see for example Wiedermann's paper). Another open question is to
start from this notion to find an extension of Gödel’s theorems to fuzzy logic.

**Application areas**

- Automobile and other vehicle subsystems, such as automatic transmissions, ABS and cruise control (e.g. Tokyo monorail)
- Air conditioners
- The Massive engine used in the *Lord of the Rings* films, which helped huge scale armies create random, yet orderly movements
- Cameras
- Digital image processing, such as edge detection
- Rice cookers
- Dishwashers
- Elevators
- Washing machines and other home appliances
- Video game artificial intelligence
- Language filters on message boards and chat rooms for filtering out offensive text
- Pattern recognition in Remote Sensing
- Hydrometeor classification algorithms for polarimetric weather radar
- Fuzzy logic has also been incorporated into some microcontrollers and microprocessors, for instance, the Freescale 68HC12.

Mineral Deposit estimation

**Controversies**

**Fuzzy logic is the same as "imprecise logic".**

Fuzzy logic is not any less precise than any other form of logic: it is an organized and mathematical method of handling *inherently* imprecise concepts. The concept of "coldness" cannot be expressed in an equation, because although temperature is a quantity, "coldness" is not. However, people have an idea of what "cold" is, and agree that there is no sharp cutoff between "cold" and "not cold", where something is "cold" at N degrees but "not cold" at N+1 degrees — a concept classical logic cannot easily handle due to the → principle of bivalence. The result has no set answer so it is believed to be a 'fuzzy' answer. Fuzzy logic simply provides a mathematical model of the vagueness which is manifested in the above example.

**Fuzzy logic is a new way of expressing probability.**

Fuzzy logic and probability are different ways of expressing uncertainty. While both fuzzy logic and probability theory can be used to represent subjective belief, fuzzy set theory uses the concept of fuzzy set membership (i.e. *how much* a variable is in a set), probability theory uses the concept of subjective probability (i.e. *how probable* do I think that a variable is in a set). While this distinction is mostly philosophical, the fuzzy-logic-derived possibility measure is inherently different from the probability measure, hence they are not *directly* equivalent. However, many statisticians are persuaded by the work of Bruno de Finetti that only one kind of mathematical uncertainty is needed and thus fuzzy logic is unnecessary. On the other hand, Bart Kosko argues that probability is a subtheory of fuzzy logic, as probability only handles one kind of uncertainty. He also claims to have proven a derivation of Bayes' theorem
from the concept of fuzzy subsethood. Lotfi Zadeh argues that fuzzy logic is different in character from probability, and is not a replacement for it. He fuzzified probability to fuzzy probability and also generalized it to what is called possibility theory. Other approaches to uncertainty include Dempster-Shafer theory and rough sets.

Note, however, that fuzzy logic is not controversial to probability but rather complementary (cf. [9]).

**Fuzzy logic will be difficult to scale to larger problems.**

This criticism is mainly because there exist problems with conditional possibility, the fuzzy set theory equivalent of conditional probability (see Halpern (2003), Section 3.8). This makes it difficult to perform inference. However there have not been many studies comparing fuzzy-based systems with probabilistic ones.

**See also**

- Artificial intelligence
- Artificial neural network
- Biologically-inspired computing
- Cloud computing
- Combs method
- Concept mining
- Contextualism
- Control system
- Defuzzification
- Dynamic logic
- Expert system
- FuzzyCLIPS expert system
- Fuzzy associative matrix
- Fuzzy concept
- Fuzzy Control System
- Fuzzy Control Language
- False dilemma
- Fuzzy electronics
- Fuzzy mathematics
- Fuzzy set
- Fuzzy subalgebra
- Machine learning
- Multi-valued logic
- Neuro-fuzzy
- Paradox of the heap
- Perspectivism
- Pattern recognition
- Petr Hájek
- Rough set
- Type-2 fuzzy sets and systems
- Vagueness
Notes


Bibliography


External links

Additional articles
- Formal fuzzy logic (http://en.citizendium.org/wiki/Formal_fuzzy_logic) - article at Citizendium
- Fuzzy Logic (http://www.scholarpedia.org/article/Fuzzy_Logic) - article at Scholarpedia
- Modeling With Words (http://www.scholarpedia.org/article/Modeling_with_words) - article at Scholarpedia
- Fuzzy logic (http://plato.stanford.edu/entries/logic-fuzzy/) - article at Stanford Encyclopedia of Philosophy
- Fuzzy Logic and the Internet of Things: I-o-T (http://www.i-o-t.org/post/WEB_3)

Links pages
- Web page about FSQL (http://www.lcc.uma.es/~ppgg/FSQL/): References and links about FSQL

Software & tools
- DotFuzzy: Open Source Fuzzy Logic Library (http://www.havana7.com/dotfuzzy)
- JFuzzyLogic: Open Source Fuzzy Logic Package + FCL (sourceforge, java) (http://jfuzzylogic.sourceforge.net/)
- pyFuzzyLib: Open Source Library to write software with fuzzy logic (Python) (http://sourceforge.net/projects/pyfuzzylib)
- RockOn Fuzzy: Open Source Fuzzy Control And Simulation Tool (Java) (http://www.timtomtam.de/rockonfuzzy)
- Free Educational Software and Application Notes (http://www.fuzzytech.com)
- InrecoLAN FuzzyMath (http://www.openfuzzymath.org), Fuzzy logic add-in for OpenOffice.org Calc
- Open fuzzy logic based inference engine and data mining web service based on Metarule (http://www.metarule.com)
- Open Source Software "mbFuzzIT" (Java) (http://mbfuzzit.sourceforge.net)

Tutorials
- Fuzzy Logic Tutorial (http://www.jimbrule.com/fuzzytutorial.html)
- Another Fuzzy Logic Tutorial (http://www.calvin.edu/~pribeiro/othrlinks/Fuzzy/home.htm) with MATLAB/Simulink Tutorial
- Simple test to check how well you understand it (http://www.answermath.com/fuzzymath.htm)

Applications
- Research article that describes how industrial foresight could be integrated into capital budgeting with intelligent agents and Fuzzy Logic (http://econpapers.repec.org/paper/amrwpaper/398.htm)
- A doctoral dissertation describing how Fuzzy Logic can be applied in profitability analysis of very large industrial investments (http://econpapers.repec.org/paper/pramprapa/4328.htm)
Research Centres

- Institute for Research and Applications of Fuzzy Modeling (http://irafm.osu.cz/)

Metatheory

A **metatheory** or **meta-theory** is a theory whose subject matter is some other theory. In other words it is a theory about a theory. Statements made in the metatheory about the theory are called metatheorems.

According to the systemic TOGA meta-theory[^1], a meta-theory may refer to the specific point of view on a theory and to its subjective meta-properties, but not to its application domain. In the above sense, a theory $T$ of the domain $D$ is a meta-theory if $D$ is a theory or a set of theories. A general theory is not a meta-theory because its domain $D$ are not theories.

The following is an example of a meta-theoretical statement:[^2]

> Any physical theory is always provisional, in the sense that it is only a hypothesis; you can never prove it. No matter how many times the results of experiments agree with some theory, you can never be sure that the next time the result will not contradict the theory. On the other hand, you can disprove a theory by finding even a single observation that disagrees with the predictions of the theory.

Meta-theory belongs to the philosophical specialty of epistemology and metamathematics, as well as being an object of concern to the area in which the individual theory is conceived. An emerging domain of meta-theories is systemics.

Taxonomy

Examining groups of related theories, a first finding may be to identify classes of theories, thus specifying a taxonomy of theories. A proof engendered by a metatheory is called a **metatheorem**.

History

The concept burst upon the scene of twentieth-century philosophy as a result of the work of the German mathematician David Hilbert, who in 1905 published a proposal for proof of the consistency of mathematics, creating the field of metamathematics. His hopes for the success of this proof were dashed by the work of Kurt Gödel who in 1931 proved this to be unattainable by his incompleteness theorems. Nevertheless, his program of unsolved mathematical problems, out of which grew this metamathematical proposal, continued to influence the direction of mathematics for the rest of the twentieth century.

The study of metatheory became widespread during the rest of that century by its application in other fields, notably scientific linguistics and its concept of metalanguage.
References

[1] * Meta-Knowledge Unified Framework (http://hid.casaccia.enea.it/Meta-know-1.htm) - the TOGA meta-theory
[2] Stephen Hawking in A Brief History of Time

See also

- meta-
- meta-knowledge
- Metalogic
- Metamathematics

External links


Metalogic

Metalogic is the study of the metatheory of logic. While logic is the study of the manner in which logical systems can be used to decide the correctness of arguments, metalogic studies the properties of the logical systems themselves. According to Geoffrey Hunter, while logic concerns itself with the "truths of logic," metalogic concerns itself with the theory of "sentences used to express truths of logic." The basic objects of study in metalogic are formal languages, formal systems, and their interpretations. The study of interpretation of formal systems is the branch of mathematical logic known as model theory, while the study of deductive apparatus is the branch known as proof theory.

History

Metalogical questions have been asked since the time of Aristotle. However, it was only with the rise of formal languages in the late 19th and early 20th century that investigations into the foundations of logic began to flourish. In 1904, David Hilbert observed that in investigating the foundations of mathematics that logical notions are presupposed, and therefore a simultaneous account of metalogical and metamathematical principles was required. Today, metalogic and metamathematics are largely synonymous with each other, and both have been substantially subsumed by mathematical logic in academia.

Important distinctions in metalogic

Metalanguage-Object language

In metalogic, formal languages are sometimes called object languages. The language used to make statements about an object language is called a metalanguage. This distinction is a key difference between logic and metalogic. While logic deals with proofs in a formal system, expressed in some formal language, metalogic deals with proofs about a formal system which are expressed in a metalanguage about some object language.
**Syntax-semantics**

In metalogic, 'syntax' has to do with formal languages or formal systems without regard to any interpretation of them, whereas, 'semantics' has to do with interpretations of formal languages. The term 'syntactic' has a slightly wider scope than 'proof-theoretic', since it may be applied to properties of formal languages without any deductive systems, as well as to formal systems. 'Semantic' is synonymous with 'model-theoretic'.

**Use-mention**

In metalogic, the words 'use' and 'mention', in both their noun and verb forms, take on a technical sense in order to identify an important distinction.[1] The *use-mention distinction* (sometimes referred to as the *words-as-words distinction*) is the distinction between using a word (or phrase) and mentioning it. Usually it is indicated that an expression is being mentioned rather than used by enclosing it in quotation marks, printing it in italics, or setting the expression by itself on a line. The enclosing in quotes of an expression gives us the name of an expression, for example:

- 'Metalogic' is the name of this article.

This article is about metalogic.

**Type-token**

The *type-token distinction* is a distinction in metalogic, that separates an abstract concept from the objects which are particular instances of the concept. For example, the particular bicycle in your garage is a token of the type of thing known as "The bicycle." Whereas, the bicycle in your garage is in a particular place at a particular time, that is not true of "the bicycle" as used in the sentence: "The bicycle has become more popular recently." This distinction is used to clarify the meaning of symbols of formal languages.

**Overview**

**Formal language**

A *formal language* is an organized set of symbols the essential feature of which is that it can be precisely defined in terms of just the shapes and locations of those symbols. Such a language can be defined, then, without any reference to any meanings of any of its expressions; it can exist before any interpretation is assigned to it -- that is, before it has any meaning. First order logic is expressed in some formal language. A formal grammar determines which symbols and sets of symbols are formulas in a formal language.

A formal language can be defined formally as a set \( A \) of strings (finite sequences) on a fixed alphabet \( \alpha \). Some authors, including Carnap, define the language as the ordered pair \(<\alpha, A>\).[2] Carnap also requires that each element of \( \alpha \) must occur in at least one string in \( A \).
**Formal grammar**

A *formal grammar* (also called *formation rules*) is a precise description of the well-formed formulas of a formal language. It is synonymous with the set of strings over the alphabet of the formal language which constitute well-formed formulas. However, it does not describe their semantics (i.e., what they mean).

**Formal systems**

A *formal system* (also called a *logical calculus*, or a *logical system*) consists of a formal language together with a deductive apparatus (also called a *deductive system*). The deductive apparatus may consist of a set of transformation rules (also called *inference rules*) or a set of axioms, or have both. A formal system is used to derive one expression from one or more other expressions.

A *formal system* can be formally defined as an ordered triple \(<\alpha, \mathcal{T}, \mathcal{D}\ d>\), where \(\mathcal{D} \ d\) is the relation of direct derivability. This relation is understood in a comprehensive sense such that the primitive sentences of the formal system are taken as directly derivable from the empty set of sentences. Direct derivability is a relation between a sentence and a finite, possibly empty set of sentences. Axioms are laid down in such a way that every first place member of \(\mathcal{D} \ d\) is a member of \(\mathcal{T}\) and every second place member is a finite subset of \(\mathcal{T}\).

It is also possible to define a *formal system* using only the relation \(\mathcal{D} \ d\). In this way we can omit \(\mathcal{T}\), and \(\alpha\) in the definitions of *interpreted formal language*, and *interpreted formal system*. However, this method can be more difficult to understand and work with.\(^2\)

**Formal proofs**

A *formal proof* is a sequence of well-formed formulas of a formal language, the last one of which is a theorem of a formal system. The theorem is a syntactic consequence of all the well-formed formulae preceding it in the proof. For a well-formed formula to qualify as part of a proof, it must be the result of applying a rule of the deductive apparatus of some formal system to the previous well-formed formulae in the proof sequence.

**Interpretations**

An *interpretation* of a formal system is the assignment of meanings, to the symbols, and truth-values to the sentences of the formal system. The study of interpretations is called Formal semantics. *Giving an interpretation* is synonymous with *constructing a model*.

**Results in metalogic**

Results in metalogic consist of such things as formal proofs demonstrating the consistency, completeness, and decidability of particular formal systems.

Major results in metalogic include:

- Proof of the uncountability of the set of all subsets of the set of natural numbers (Cantor's theorem 1891)
- Löwenheim-Skolem theorem (Leopold Löwenheim 1915 and Thoralf Skolem 1919)
- Proof of the consistency of truth-functional → propositional logic (Emil Post 1920)
- Proof of the semantic completeness of truth-functional propositional logic (Paul Bernays 1918)\(^3\), (Emil Post 1920)\(^1\)
• Proof of the syntactic completeness of truth-functional propositional logic (Emil Post 1920)[1]
• Proof of the decidability of truth-functional propositional logic (Emil Post 1920)[1]
• Proof of the consistency of first order monadic predicate logic (Leopold Löwenheim 1915)
• Proof of the semantic completeness of first order monadic predicate logic (Leopold Löwenheim 1915)
• Proof of the decidability of first order monadic predicate logic (Leopold Löwenheim 1915)
• Proof of the semantic completeness of first order → predicate logic (Gödel's completeness theorem 1930)
• Proof of the consistency of first order predicate logic (David Hilbert and Wilhelm Ackermann 1928)
• Proof of the semantic completeness of first order predicate logic (Kurt Gödel 1930)
• Proof of the undecidability of first order predicate logic (Alonzo Church 1936)
• Gödel's first incompleteness theorem 1931
• Gödel's second incompleteness theorem 1931

See also
• Metamathematics
• Formal semantics

References
Quantum logic

In quantum mechanics, **quantum logic** is a set of rules for reasoning about propositions which takes the principles of quantum theory into account. This research area and its name originated in the 1936 paper by Garrett Birkhoff and John von Neumann, who were attempting to reconcile the apparent inconsistency of classical boolean logic with the facts concerning the measurement of complementary variables in quantum mechanics, such as position and momentum.

Quantum logic can be formulated either as a modified version of propositional logic or as a non-commutative and non-associative many-valued (MV) logic. It has some properties which clearly distinguish it from classical logic, most notably, the failure of the distributive law of propositional logic:

\[ p \land (q \lor r) = (p \land q) \lor (p \land r), \]

where the symbols \( p, q \) and \( r \) are propositional variables. To illustrate why the distributive law fails, consider a particle moving on a line and let

\[ p = "\text{the particle is moving to the right}" \]
\[ q = "\text{the particle is in the interval [-1,1]}" \]
\[ r = "\text{the particle is not in the interval [-1,1]}" \]

then the proposition "\( q \lor r \)" is true, so

\[ p \land (q \lor r) = p \]

On the other hand, the propositions "\( p \land q \)" and "\( p \land r \)" are both false, since they assert tighter restrictions on simultaneous values of position and momentum than is allowed by the uncertainty principle. So,

\[ (p \land q) \lor (p \land r) = \text{false} \]

Thus the distributive law fails.

Quantum logic has been proposed as the correct logic for propositional inference generally, most notably by the philosopher Hilary Putnam, at least at one point in his career. This thesis was an important ingredient in Putnam's paper → Is Logic Empirical? in which he analysed the epistemological status of the rules of propositional logic. Putnam attributes the idea that anomalies associated to quantum measurements originate with anomalies in the logic of physics itself to the physicist David Finkelstein. It should be noted, however, that this idea had been around for some time and had been revived several years earlier by George Mackey's work on group representations and symmetry.

The more common view regarding quantum logic, however, is that it provides a formalism for relating observables, system preparation filters and states. In this view, the quantum logic approach resembles more closely the C*-algebraic approach to quantum mechanics; in fact with some minor technical assumptions it can be subsumed by it. The similarities of the quantum logic formalism to a system of deductive logic may then be regarded more as a curiosity than as a fact of fundamental philosophical importance.
Introduction

In his classic treatise *Mathematical Foundations of Quantum Mechanics*, John von Neumann noted that projections on a Hilbert space can be viewed as propositions about physical observables. The set of principles for manipulating these quantum propositions was called *quantum logic* by von Neumann and Birkhoff. In his book (also called *Mathematical Foundations of Quantum Mechanics*) G. Mackey attempted to provide a set of axioms for this propositional system as an orthocomplemented lattice. Mackey viewed elements of this set as potential *yes or no questions* an observer might ask about the state of a physical system, questions that would be settled by some measurement. Moreover Mackey defined a physical observable in terms of these basic questions. Mackey’s axiom system is somewhat unsatisfactory though, since it assumes that the partially ordered set is actually given as the orthocomplemented closed subspace lattice of a separable Hilbert space. Piron, Ludwig and others have attempted to give axiomatizations which do not require such explicit relations to the lattice of subsaces.

The remainder of this article assumes the reader is familiar with the spectral theory of self-adjoint operators on a Hilbert space. However, the main ideas can be understood using the finite-dimensional spectral theorem.

Projections as propositions

The so-called *Hamiltonian* formulations of classical mechanics have three ingredients: *states, observables* and *dynamics*. In the simplest case of a single particle moving in $\mathbb{R}^3$, the state space is the position-momentum space $\mathbb{R}^6$. We will merely note here that an observable is some real-valued function $f$ on the state space. Examples of observables are position, momentum or energy of a particle. For classical systems, the value $f(x)$, that is the value of $f$ for some particular system state $x$, is obtained by a process of measurement of $f$.

The propositions concerning a classical system are generated from basic statements of the form

- Measurement of $f$ yields a value in the interval $[a, b]$ for some real numbers $a, b$.

It follows easily from this characterization of propositions in classical systems that the corresponding logic is identical to that of some Boolean algebra of subsets of the state space. By logic in this context we mean the rules that relate set operations and ordering relations, such as de Morgan's laws. These are analogous to the rules relating boolean conjunctives and material implication in classical propositional logic. For technical reasons, we will also assume that the algebra of subsets of the state space is that of all Borel sets.

The set of propositions is ordered by the natural ordering of sets and has a complementation operation. In terms of observables, the complement of the proposition $\{f \geq a\}$ is $\{f < a\}$.

We summarize these remarks as follows:

- The proposition system of a classical system is a lattice with a distinguished orthocomplementation operation: The lattice operations of *meet* and *join* are respectively set intersection and set union. The orthocomplementation operation is set complement. Moreover this lattice is *sequentially complete*, in the sense that any sequence $\{E_i\}$ of elements of the lattice has a least upper bound, specifically the set-theoretic union:

$$\text{LUB}(\{E_i\}) = \bigcup_{i=1}^{\infty} E_i.$$
In the Hilbert space formulation of quantum mechanics as presented by von Neumann, a physical observable is represented by some (possibly unbounded) densely-defined self-adjoint operator \( A \) on a Hilbert space \( H \). \( A \) has a spectral decomposition, which is a projection-valued measure \( E \) defined on the Borel subsets of \( \mathbb{R} \). In particular, for any bounded Borel function \( f \), the following equation holds:

\[
f(A) = \int_{\mathbb{R}} f(\lambda) \, dE(\lambda).
\]

In case \( f \) is the indicator function of an interval \([a, b]\), the operator \( f(A) \) is a self-adjoint projection, and can be interpreted as the quantum analogue of the classical proposition

- Measurement of \( A \) yields a value in the interval \([a, b]\).

### The propositional lattice of a quantum mechanical system

This suggests the following quantum mechanical replacement for the orthocomplemented lattice of propositions in classical mechanics. This is essentially Mackey's Axiom VII:

- The orthocomplemented lattice \( Q \) of propositions of a quantum mechanical system is the lattice of closed subspaces of a complex Hilbert space \( H \) where orthocomplementation of \( V \) is the orthogonal complement \( V^\perp \).

\( Q \) is also sequentially complete: any pairwise disjoint sequence \( \{V_i\}_i \) of elements of \( Q \) has a least upper bound. Here disjointness of \( W_1 \) and \( W_2 \) means \( W_2 \) is a subspace of \( W_1^\perp \). The least upper bound of \( \{V_i\}_i \) is the closed internal direct sum.

Henceforth we identify elements of \( Q \) with self-adjoint projections on the Hilbert space \( H \).

The structure of \( Q \) immediately points to a difference with the partial order structure of a classical proposition system. In the classical case, given a proposition \( p \), the equations

\[
I = p \lor q \\
0 = p \land q
\]

have exactly one solution, namely the set-theoretic complement of \( p \). In these equations \( I \) refers to the atomic proposition which is identically true and \( 0 \) the atomic proposition which is identically false. In the case of the lattice of projections there are infinitely many solutions to the above equations.

Having made these preliminary remarks, we turn everything around and attempt to define observables within the projection lattice framework and using this definition establish the correspondence between self-adjoint operators and observables: A Mackey observable is a countably additive homomorphism from the orthocomplemented lattice of the Borel subsets of \( \mathbb{R} \) to \( Q \). To say the mapping \( \varphi \) is a countably additive homomorphism means that for any sequence \( \{S_i\}_i \) of pairwise disjoint Borel subsets of \( \mathbb{R} \), \( \{\varphi(S_i)\}_i \) are pairwise orthogonal projections and

\[
\varphi \left( \bigcup_{i=1}^{\infty} S_i \right) = \sum_{i=1}^{\infty} \varphi(S_i).
\]

**Theorem.** There is a bijective correspondence between Mackey observables and densely-defined self-adjoint operators on \( H \).

This is the content of the spectral theorem as stated in terms of spectral measures.
Statistical structure

Imagine a forensics lab which has some apparatus to measure the speed of a bullet fired from a gun. Under carefully controlled conditions of temperature, humidity, pressure and so on the same gun is fired repeatedly and speed measurements taken. This produces some distribution of speeds. Though we will not get exactly the same value for each individual measurement, for each cluster of measurements, we would expect the experiment to lead to the same distribution of speeds. In particular, we can expect to assign probability distributions to propositions such as \( \{ a \leq \text{speed} \leq b \} \). This leads naturally to propose that under controlled conditions of preparation, the measurement of a classical system can be described by a probability measure on the state space. This same statistical structure is also present in quantum mechanics.

A quantum probability measure is a function \( P \) defined on \( Q \) with values in \([0,1]\) such that \( P(0)=0, P(I)=1 \) and if \( \{ E_i \} \) is a sequence of pairwise orthogonal elements of \( Q \) then

\[
P \left( \sum_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i).
\]

The following highly non-trivial theorem is due to Andrew Gleason:

**Theorem.** Suppose \( H \) is a separable Hilbert space of complex dimension at least 3. Then for any quantum probability measure on \( Q \) there exists a unique trace class operator \( S \) such that

\[
P(E) = \text{Tr}(SE)
\]

for any self-adjoint projection \( E \).

The operator \( S \) is necessarily non-negative (that is all eigenvalues are non-negative) and of trace 1. Such an operator is often called a density operator.

Physicists commonly regard a density operator as being represented by a (possibly infinite) density matrix relative to some orthonormal basis.

For more information on statistics of quantum systems, see quantum statistical mechanics.

**Automorphisms**

An automorphism of \( Q \) is a bijective mapping \( \alpha:Q \to Q \) which preserves the orthocomplemented structure of \( Q \), that is

\[
\alpha \left( \sum_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \alpha(E_i)
\]

for any sequence \( \{ E_i \} \) of pairwise orthogonal self-adjoint projections. Note that this property implies monotonicity of \( \alpha \). If \( P \) is a quantum probability measure on \( Q \), then \( E \to \alpha(E) \) is also a quantum probability measure on \( Q \). By the Gleason theorem characterizing quantum probability measures quoted above, any automorphism \( \alpha \) induces a mapping \( \alpha^* \) on the density operators by the following formula:

\[
\text{Tr}(\alpha^*(S)E) = \text{Tr}(S\alpha(E)).
\]

The mapping \( \alpha^* \) is bijective and preserves convex combinations of density operators. This means

\[
\alpha^*(r_1 S_1 + r_2 S_2) = r_1 \alpha^*(S_1) + r_2 \alpha^*(S_2)
\]

whenever \( 1 = r_1 + r_2 \) and \( r_1, r_2 \) are non-negative real numbers. Now we use a theorem of Richard Kadison:
Quantum logic

**Theorem.** Suppose $\beta$ is a bijective map from density operators to density operators which is convexity preserving. Then there is an operator $U$ on the Hilbert space which is either linear or conjugate-linear, preserves the inner product and is such that

$$\beta(S) = USU^*$$

for every density operator $S$. In the first case we say $U$ is unitary, in the second case $U$ is anti-unitary.

**Remark.** This note is included for technical accuracy only, and should not concern most readers. The result quoted above is not directly stated in Kadison’s paper, but can be reduced to it by noting first that $\beta$ extends to a positive trace preserving map on the trace class operators, then applying duality and finally applying a result of Kadison’s paper.

The operator $U$ is not quite unique; if $r$ is a complex scalar of modulus 1, then $rU$ will be unitary or anti-unitary if $U$ is and will implement the same automorphism. In fact, this is the only ambiguity possible.

It follows that automorphisms of $Q$ are in bijective correspondence to unitary or anti-unitary operators modulo multiplication by scalars of modulus 1. Moreover, we can regard automorphisms in two equivalent ways: as operating on states (represented as density operators) or as operating on $Q$.

**Non-relativistic dynamics**

In non-relativistic physical systems, there is no ambiguity in referring to time evolution since there is a global time parameter. Moreover an isolated quantum system evolves in a deterministic way: if the system is in a state $S$ at time $t$ then at time $s > t$, the system is in a state $F_{s,t}(S)$. Moreover, we assume

- The dependence is reversible: The operators $F_{s,t}$ are bijective.
- The dependence is homogeneous: $F_{s,t} = F_{s-t,0}$.
- The dependence is convexity preserving: That is, each $F_{s,t}(S)$ is convexity preserving.
- The dependence is weakly continuous: The mapping $R \rightarrow R$ given by $t \rightarrow \text{Tr}(F_{s,t}(S)E)$ is continuous for every $E$ in $Q$.

By Kadison’s theorem, there is a 1-parameter family of unitary or anti-unitary operators $\{U_t\}_t$ such that

$$F_{s,t}(S) = U_{s-t}SU_{s-t}^*$$

In fact,

**Theorem.** Under the above assumptions, there is a strongly continuous 1-parameter group of unitary operators $\{U_t\}_t$ such that the above equation holds.

Note that it easily from uniqueness from Kadison’s theorem that

$$U_{t+s} = \sigma(t,s)U_tU_s$$

where $\sigma(t,s)$ has modulus 1. Now the square of an anti-unitary is a unitary, so that all the $U_t$ are unitary. The remainder of the argument shows that $\sigma(t,s)$ can be chosen to be 1 (by modifying each $U_t$ by a scalar of modulus 1.)
Pure states

A convex combinations of statistical states $S_1$ and $S_2$ is a state of the form $S = p_1 S_1 + p_2 S_2$ where $p_1, p_2$ are non-negative and $p_1 + p_2 = 1$. Considering the statistical state of system as specified by lab conditions used for its preparation, the convex combination $S$ can be regarded as the state formed in the following way: toss a biased coin with outcome probabilities $p_1, p_2$ and depending on outcome choose system prepared to $S_1$ or $S_2$.

Density operators form a convex set. The convex set of density operators has extreme points; these are the density operators given by a projection onto a one-dimensional space. To see that any extreme point is such a projection, note that by the spectral theorem $S$ can be represented by a diagonal matrix; since $S$ is non-negative all the entries are non-negative and since $S$ has trace 1, the diagonal entries must add up to 1. Now if it happens that the diagonal matrix has more than one non-zero entry it is clear that we can express it as a convex combination of other density operators.

The extreme points of the set of density operators are called pure states. If $S$ is the projection on the 1-dimensional space generated by a vector $\psi$ of norm 1 then

$$\text{Tr}(SE) = \langle E \psi | \psi \rangle$$

for any $E$ in $Q$. In physics jargon, if

$$S = |\psi\rangle\langle\psi|,$$

where $\psi$ has norm 1, then

$$\text{Tr}(SE) = \langle \psi | E | \psi \rangle.$$ 

Thus pure states can be identified with rays in the Hilbert space $H$.

The measurement process

Consider a quantum mechanical system with lattice $Q$ which is in some statistical state given by a density operator $S$. This essentially means an ensemble of systems specified by a repeatable lab preparation process. The result of a cluster of measurements intended to determine the truth value of proposition $E$, is just as in the classical case, a probability distribution of truth values $T$ and $F$. Say the probabilities are $p$ for $T$ and $q = 1 - p$ for $F$. By the previous section $p = \text{Tr}(S E)$ and $q = \text{Tr}(S (I - E))$.

Perhaps the most fundamental difference between classical and quantum systems is the following: regardless of what process is used to determine $E$ immediately after the measurement the system will be in one of two statistical states:

- If the result of the measurement is $T$

  $$\frac{1}{\text{Tr}(ES)} ESE.$$

- If the result of the measurement is $F$

  $$\frac{1}{\text{Tr}((I - E)S)} (I - E)S(I - E).$$

(We leave to the reader the handling of the degenerate cases in which the denominators may be 0.) We now form the convex combination of these two ensembles using the relative frequencies $p$ and $q$. We thus obtain the result that the measurement process applied to a statistical ensemble in state $S$ yields another ensemble in statistical state:

$$M_E(S) = ESE + (I - E)S(I - E).$$
We see that a pure ensemble becomes a mixed ensemble after measurement. Measurement, as described above, is a special case of quantum operations.

**Limitations**

Quantum logic derived from propositional logic provides a satisfactory foundation for a theory of reversible quantum processes. Examples of such processes are the covariance transformations relating two frames of reference, such as change of time parameter or the transformations of special relativity. Quantum logic also provides a satisfactory understanding of density matrices. Quantum logic can be stretched to account for some kinds of measurement processes corresponding to answering yes-no questions about the state of a quantum system. However, for more general kinds of measurement operations (that is quantum operations), a more complete theory of filtering processes is necessary. Such an approach is provided by the consistent histories formalism. On the other hand, quantum logics derived from MV-logic extend its range of applicability to irreversible quantum processes and/or ‘open’ quantum systems.

In any case, these quantum logic formalisms must be generalized in order to deal with super-geometry (which is needed to handle Fermi-fields) and non-commutative geometry (which is needed in string theory and quantum gravity theory). Both of these theories use a partial algebra with an "integral" or "trace". The elements of the partial algebra are not observables; instead the "trace" yields "greens functions" which generate scattering amplitudes. One thus obtains a local S-matrix theory (see D. Edwards).

Since around 1978 the Flato school (see F. Bayen) has been developing an alternative to the quantum logics approach called deformation quantization (see Weyl quantization).

In 2004, Prakash Panangaden described how to capture the kinematics of quantum causal evolution using System BV, a deep inference logic originally developed for use in structural proof theory.[1] Alessio Guglielmi, Lutz Straßburger, and Richard Blute have also done work in this area.[2]

**See also**

- Quasi-set theory
- HPO formalism (An approach to temporal quantum logic)

**References**

- D. Cohen, *An Introduction to Hilbert Space and Quantum Logic*, Springer-Verlag, 1989. This is a thorough but elementary and well-illustrated introduction, suitable for advanced undergraduates.
Quantum logic


**External links**
- Stanford Encyclopedia of Philosophy entry on Quantum Logic and Probability Theory[^3]

**References**
[2] [http://alessio.guglielmi.name/res/cos/crt.html#CQE](http://alessio.guglielmi.name/res/cos/crt.html#CQE)
Philosophical logic

Philosophical logic is the study of the more specifically philosophical aspects of logic. The term contrasts with philosophy of logic, → metalogic, and → mathematical logic; and since the development of mathematical logic in the late nineteenth century, it has come to include most of those topics traditionally treated by → logic in general. It is concerned with characterizing notions like inference, rational thought, truth, and contents of thoughts, in the most fundamental ways possible, and trying to model them using modern formal logic.

The notions in question include reference, predication, identity, truth, negation, quantification, existence, necessity, definition and entailment.

Philosophical logic is not concerned with the psychological processes connected with thought, or with emotions, images and the like. It is concerned only with those entities — thoughts, sentences, or propositions — that are capable of being true and false. To this extent, though, it does intersect with philosophy of mind and philosophy of language.

Gottlob Frege is regarded by many as the founder of modern philosophical logic.

Not all philosophical logic, however, applies formal logical techniques. A good amount of it (including Grayling’s and Colin McGinn’s books cited below) is written in natural language. One definition, popular in Britain, is that philosophical logic is the attempt to solve general philosophical problems that arise when we use or think about formal logic: problems about existence, necessity, analyticity, a prioricity, propositions, identity, predication, truth. Philosophy of logic, on the other hand, would tackle metaphysical and epistemological problems about entailment, validity, and proof. So it could be said that philosophy of logic is a branch of philosophy but philosophical logic belongs to the domain of logic (though logic is itself a branch of philosophy).
Truth
Theories of truth
Truthbearers
Semantics

Definition

Necessity and analyticity

Presuppositions

Formal and Natural language

Consequence

Conditionals

Probability

Literature

• Journal of Philosophical Logic [5], Springer SBM
Logical in computer science describes topics where → logic is applied to computer science and artificial intelligence. These include:

• Investigations into logic that are guided by applications in computer science. For example: Combinatory logic and Abstract interpretation;
• Boolean logic, for the circuits used in computer processors.
• Fundamental concepts in computer science that are naturally expressible in formal logic. For example: Formal semantics of programming languages, Hoare logic, and Logic programming;
• Aspects of the theory of computation that cast light on fundamental questions of formal logic. For example: Curry-Howard correspondence and Game semantics;
• Tools for logicians considered as computer science. For example: Automated theorem proving and Model checking;
• Logics of knowledge and beliefs (of human and artificial agents);
• Logics for spatial reasoning, e.g. about moving in Euclidean space (which should not be confused with spatial logics used for concurrent systems);
• Formal methods and logics for reasoning about computation. For example → predicate logic and logical frameworks are used for proving programs correct, and logics such as temporal logic and spatial logics are used for reasoning about interaction between concurrent and distributed processes. Program logics often are → modal logics, e.g. dynamic logic or Hennessy-Milner logic;
• Specification languages provide a basis for formal software development; in this context, the notion of institution has been developed as an abstract formalization of the notion of logical system, with the goal of handling the "population explosion" of logics used in computer science.

The study of basic → mathematical logic such as propositional logic and → predicate logic (normally in conjunction with set theory) is considered an important theoretical underpinning to any undergraduate computer science course. Higher order logic is not normally taught, but is important in theorem proving tools like HOL.
Books


External links


References

Controversies in logic

Principle of bivalence

In logic, the semantic **principle of bivalence** states that every proposition takes exactly one of two truth values (e.g. *truth* or *falsehood*). The laws of bivalence, excluded middle, and non-contradiction are related, but they refer to the calculus of logic, not its semantics, and are hence not the same. The law of bivalence is compatible with classical logic, but not intuitionistic logic, linear logic, or multi-valued logic.

**The laws**

For any proposition P, at a given time, in a given respect, there are three related laws:

- **Law of bivalence:**
  For any proposition P, P is either true or false.
- **Law of the excluded middle:**
  For any proposition P, P is true or 'not-P' is true.
- **Law of non-contradiction:**
  For any proposition P, it is not the case that both P is true and 'not-P' is true.

**Bivalence is deepest**

Through the use of propositional variables, it is possible to formulate analogues of the laws of non-contradiction and the excluded middle in the formal manner of the traditional propositional logic:

- **Excluded middle:** $P \lor \neg P$
- **Non-contradiction:** $\neg(P \land \neg P)$

In second-order logic, second-order quantifiers are available to bind the propositional variables, allowing one to formulate closer analogues:

- **Excluded middle:** $\forall P(P \lor \neg P)$
- **Non-contradiction:** $\forall P\neg(P \land \neg P)$

Note that both the aforementioned logics assume the law of bivalence. The law of bivalence itself has no analogue in either of these logics: on pain of paradox, it can be stated only in the metalanguage used to study the aforementioned formal logics.

Analogues of excluded middle are not valid in intuitionistic logic; this rejection is founded in intuitionists' constructivist as opposed to Platonist conception of truth and falsity. On the other hand, in linear logic, analogues of both excluded middle and non-contradiction are valid,[^1] and yet it is not a two-valued (i.e., bivalent) logic.
Why these distinctions might matter

These different principles are closely related, but there are certain cases where we might wish to affirm that they do not all go together. Specifically, the link between bivalence and the law of excluded middle is sometimes challenged.

Future contingents

A famous example is the contingent sea battle case found in Aristotle's work, De Interpretatione, chapter 9:

Imagine P refers to the statement "There will be a sea battle tomorrow."

The law of the excluded middle clearly holds:

There will be a sea battle tomorrow, or there won't be.

However, some philosophers wish to claim that P is neither true nor false today, since the matter has not been decided yet. So, they would say that the principle of bivalence does not hold in such a case: P is neither true nor false. (But that does not necessarily mean that it has some other truth-value, e.g. indeterminate, as it may be truth-valueless). This view is controversial, however.

Vagueness

→ Multi-valued logics and → fuzzy logic have been considered better alternatives to bivalent systems for handling vagueness. Truth (and falsity) in fuzzy logic, for example, comes in varying degrees. Consider the following statement.

The apple on the desk is red.

Upon observation, the apple is a pale shade of red. We might say it is "50% red". This could be rephrased: it is 50% true that the apple is red. Therefore, P is 50% true, and 50% false. Now consider:

The apple on the desk is red and it is not red.

In other words, P and not-P. This violates the law of noncontradiction and, by extension, bivalence. However, this is only a partial rejection of these laws because P is only partially true. If P were 100% true, not-P would be 100% false, and there is no contradiction because P and not-P no longer holds.

However, the law of the excluded middle is retained, because P and not-P implies P or not-P, since "or" is inclusive. The only two cases where P and not-P is false (when P is 100% true or false) are the same cases considered by two-valued logic, and the same rules apply.

Of course, it may be stated that bivalence must always be true, and that multi-valued logic is simply by definition a vague state of perception. That is, multi-valued logic is a convenient way of saying, "This instance has not been observed in enough detail to determine the truth value of P." In other words, if a pale apple is 50% red (where red is noted as P), then P can be said to be 100% true, noting that bivalence makes little delineation as to the nature of not-P aside from the given, meaning that the apple might very well be 50% white as well (when white is noted as not-P), meaning that P and not-P can both be true, but in separate instances, as they both exist as separate colours, which combine in a larger instance set in perhaps an unobservable, exceedingly subtle way to create the shade of pale red. In this case, the apple might be set S, which consisted of P and not-P to greater or lesser or equal respective degrees, as long as it is acknowledged that P
and not-P are separate instances within a set instance. In this way, bivalence simply states that white cannot be red, and makes no claim about the colour of the set instance, to which is applied multi-value logic, in which case multi-value logic is simply derivative of bivalence as well.

**See also**

- Exclusive disjunction
- Degrees of truth
- Anekantavada
- False dilemma
- Fuzzy logic
- Logical disjunction
- Logical equality
- Logical value
- Multivalued logic
- Propositional logic
- Relativism
- Perspectivism
- Rhizome (philosophy)

**Notes**

[1] using linear logic's "multiplicative" conjunction and disjunction

**External links**

- The distinction between the three laws is described by Douglas Groothuis (http://www.ivpress.com/groothuis/doug/), in the Philosophical Dictionary (here (http://www.philosophypages.com/dy/b2.htm#biva) and here (http://www.philosophypages.com/dy/e9.htm#exmid)).
Paraconsistent logic

A paraconsistent logic is a logical system that attempts to deal with contradictions in a discriminating way. Alternatively, paraconsistent logic is the subfield of → logic that is concerned with studying and developing paraconsistent (or “inconsistency-tolerant”) systems of logic.

Inconsistency-tolerant logics have been discussed since at least 1910 (and arguably much earlier, for example in the writings of Aristotle); however, the term paraconsistent (“beside the consistent”) was not coined until 1976, by the Peruvian philosopher Francisco Miró Quesada.\(^1\)

Definition

In classical logic (as well as intuitionistic logic and most other logics), contradictions entail everything. This curious feature, known as the principle of explosion or ex contradictione sequitur quodlibet (“from a contradiction, anything follows”), can be expressed formally as

\[
\begin{array}{c|c}
\text{Premise} & \text{Consequent} \\
\hline
P \land \neg P & \text{conjunctive elimination} \\
P & \text{weakening} \\
P \lor A & \text{conjunctive elimination} \\
\neg P & \text{disjunctive syllogism} \\
\text{therefore } A & \text{Conclusion}
\end{array}
\]

Which means: if \(P\) and its negation \(\neg P\) are both assumed to be true, then \(P\) is assumed to be true, from which it follows that at least one of the claims \(P\) and some other (arbitrary) claim \(A\) is true. However, if we know that either \(P\) or \(A\) is true, and also that \(\neg P\) is not true (that \(\neg P\) is true) we can conclude that \(A\), which could be anything, is true. Thus if a theory contains a single inconsistency, it is trivial—that is, it has every sentence as a theorem. The characteristic or defining feature of a paraconsistent logic is that it rejects the principle of explosion. As a result, paraconsistent logics, unlike classical and other logics, can be used to formalize inconsistent but non-trivial theories.

Paraconsistent logics are propositionally weaker than classical logic

It should be emphasized that paraconsistent logics are \(\rightarrow\) propositionally weaker than classical logic; that is, they deem fewer propositional inferences valid. The point is that a paraconsistent logic can never be a propositional extension of classical logic, that is, propositionally validate everything that classical logic does. In that sense, then, paraconsistent logic is more conservative or cautious than classical logic. It is due to such conservativeness that paraconsistent languages can be more expressive than their classical
counterparts including the hierarchy of metalanguages due to Tarski et al. According to Feferman [1984]: “...natural language abounds with directly or indirectly self-referential yet apparently harmless expressions—all of which are excluded from the Tarskian framework.” This expressive limitation can be overcome in paraconsistent logic.

**Motivation**

The primary motivation for paraconsistent logic is the conviction that it ought to be possible to reason with inconsistent information in a controlled and discriminating way. The principle of explosion precludes this, and so must be abandoned. In non-paraconsistent logics, there is only one inconsistent theory: the trivial theory that has every sentence as a theorem. Paraconsistent logic makes it possible to distinguish between inconsistent theories and to reason with them. Sometimes it is possible to revise a theory to make it consistent. In other cases (e.g., large software systems) it is currently impossible to attain consistency.

Some philosophers take a more radical approach, holding that some contradictions are true, and thus a theory’s being inconsistent is not always an indication that it is incorrect. This view, known as dialetheism, is motivated by several considerations, most notably an inclination to take certain paradoxes such as the Liar and Russell’s paradox at face value. Not all advocates of paraconsistent logic are dialetheists. On the other hand, being a dialetheist rationally commits one to some form of paraconsistent logic, on pain of otherwise having to accept everything as true (i.e. trivialism). The most prominent contemporary defender of dialetheism (and hence paraconsistent logic) is Graham Priest, a philosopher at the University of Melbourne.

**Tradeoff**

Paraconsistency does not come for free: it involves a tradeoff. In particular, abandoning the principle of explosion requires one to abandon at least one of the following four very intuitive principles:[2]

<table>
<thead>
<tr>
<th>Principle</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disjunction introduction</td>
<td>A ⊃ A ∨ B</td>
</tr>
<tr>
<td>Disjunctive syllogism</td>
<td>A ∨ B, ¬A ⊃ B</td>
</tr>
<tr>
<td>Transitivity or “cut”</td>
<td>Γ ⊃ A; A ⊃ B ⊃ Γ ⊃ B</td>
</tr>
<tr>
<td>Double negation elimination</td>
<td>¬¬A ⊃ A</td>
</tr>
</tbody>
</table>

Though each of these principles has been challenged, the most popular approach among logicians is to reject disjunctive syllogism. If one is a dialetheist, it makes perfect sense that disjunctive syllogism should fail. The idea behind this syllogism is that, if ¬ A, then A is excluded, so the only way A ∨ B could be true would be if B were true. However, if A and ¬ A can both be true at the same time, then this reasoning fails.

Another approach is to reject disjunction introduction but keep disjunctive syllogism, transitivity, and double negation elimination. The disjunction (A ∨ B) is defined as ¬(¬A ∧ ¬B). In this approach all of the rules of natural deduction hold except for proof by contradiction and disjunction introduction. Also, the following usual Boolean properties hold: excluded middle and (for conjunction and disjunction) associativity, commutativity, distributivity, De Morgan’s laws, and idempotence. Furthermore, by defining the implication (A → B) as ¬(A ∧ ¬B), there is a Two-Way Deduction Theorem allowing
Paraconsistent logic

implications to be easily proved. Carl Hewitt favours this approach, claiming that having the usual Boolean properties, Natural Deduction, and Deduction Theorem are huge advantages in software engineering. Yet another approach is to do both simultaneously. In many systems of relevant logic, as well as linear logic, there are two separate disjunctive connectives. One allows disjunction introduction, and one allows disjunctive syllogism. Of course, this has the disadvantages entailed by separate disjunctive connectives including confusion between them and complexity in relating them.

The three principles below, when taken together, also entail explosion, so at least one must be abandoned:

<table>
<thead>
<tr>
<th>Principle</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reductio ad absurdum</td>
<td>$A \rightarrow (B \land \neg B) \vdash \neg A$</td>
</tr>
<tr>
<td>Rule of weakening</td>
<td>$A \Rightarrow B \vdash A$</td>
</tr>
<tr>
<td>Double negation elimination</td>
<td>$\neg \neg A \vdash A$</td>
</tr>
</tbody>
</table>

Both reductio ad absurdum and the rule of weakening have been challenged in this respect. Double negation elimination is challenged, but for unrelated reasons. Removing it alone would still allow all negative propositions to be proven from a contradiction.

A simple paraconsistent logic

Perhaps the most well-known system of paraconsistent logic is the simple system known as LP ("Logic of Paradox"), first proposed by the Argentinian logician F. G. Asenjo in 1966 and later popularized by Priest and others. One way of presenting the semantics for LP is to replace the usual functional valuation with a relational one. The binary relation relates a formula to a truth value: $V(A, 1)$ means that $A$ is true, and $V(A, 0)$ means that $A$ is false. A formula must be assigned at least one truth value, but there is no requirement that it be assigned at most one truth value. The semantic clauses for negation and disjunction are given as follows:

- $V(\neg A, 1) \iff V(A, 0)$
- $V(\neg A, 0) \iff V(A, 1)$
- $V(A \lor B, 1) \iff V(A, 1) \lor V(B, 1)$
- $V(A \lor B, 0) \iff V(A, 0) \land V(B, 0)$

(The other logical connectives are defined in terms of negation and disjunction as usual.) Or to put the same point less symbolically:

- not $A$ is true if and only if $A$ is false
- not $A$ is false if and only if $A$ is true
- $A$ or $B$ is true if and only if $A$ is true or $B$ is true
- $A$ or $B$ is false if and only if $A$ is false and $B$ is false

(Semantic) logical consequence is then defined as truth-preservation:

\[ \Gamma \models A \text{ if and only if } A \text{ is true whenever every element of } \Gamma \text{ is true.} \]

Now consider a valuation $V$ such that $V(A, 1)$ and $V(A, 0)$ but it is not the case that $V(B, 0)$. It is easy to check that this valuation constitutes a counterexample to both explosion and disjunctive syllogism. However, it is also a counterexample to modus ponens for the material conditional of LP. For this reason, proponents of LP usually advocate expanding the system to include a stronger conditional connective that is not definable in
Paraconsistent logic

terms of negation and disjunction.\[7\]
As one can verify, LP preserves most other inference patterns that one would expect to be valid, such as De Morgan’s laws and the usual introduction and elimination rules for negation, conjunction, and disjunction. Surprisingly, the logical truths (or tautologies) of LP are precisely those of classical propositional logic.\[8\] (LP and classical logic differ only in the inferences they deem valid.) Relaxing the requirement that every formula be either true or false yields the weaker paraconsistent logic commonly known as FDE (“First-Degree Entailment”). Unlike LP, FDE contains no logical truths.
It must be emphasized that LP is but one of many paraconsistent logics that have been proposed.\[9\] It is presented here merely as an illustration of how a paraconsistent logic can work.

**Relation to other logics**

One important type of paraconsistent logic is relevance logic. A logic is *relevant* iff it satisfies the following condition:

\[
\text{if } A \rightarrow B \text{ is a theorem, then } A \text{ and } B \text{ share a non-logical constant.}
\]

It follows that a relevance logic cannot have \( p \land \neg p \rightarrow q \) as a theorem, and thus (on reasonable assumptions) cannot validate the inference from \{p, \neg p\} to q.
Paraconsistent logic has significant overlap with many-valued logic; however, not all paraconsistent logics are many-valued (and, of course, not all many-valued logics are paraconsistent).

Intuitionistic logic allows \( A \lor \neg A \) not to be equivalent to true, while paraconsistent logic allows \( A \land \neg A \) not to be equivalent to false. Thus it seems natural to regard paraconsistent logic as the “dual” of intuitionistic logic. However, intuitionistic logic is a specific logical system whereas paraconsistent logic encompasses a large class of systems. Accordingly, the dual notion to paraconsistency is called paracompleteness, and the “dual” of intuitionistic logic (a specific paracomplete logic) is a specific paraconsistent system called anti-intuitionistic or dual-intuitionistic logic (sometimes referred to as Brazilian logic, for historical reasons).\[10\] The duality between the two systems is best seen within a sequent calculus framework. While in intuitionistic logic the sequent

\[ \vdash A \lor \neg A \]

is not derivable, in dual-intuitionistic logic

\[ A \land \neg A \vdash \]

is not derivable. Similarly, in intuitionistic logic the sequent

\[ \neg \neg A \vdash A \]

is not derivable, while in dual-intuitionistic logic

\[ A \vdash \neg \neg A \]

is not derivable. Dual-intuitionistic logic contains a connective \# known as pseudo-difference which is the dual of intuitionistic implication. Very loosely, \( A \# B \) can be read as “\( A \) but not \( B \)”. However, \# is not truth-functional as one might expect a ‘but not’ operator to be; similarly, the intuitionistic implication operator cannot be treated like “\( \neg (A \land \neg B) \)”.
Dual-intuitionistic logic also features a basic connective \( \top \) which is the dual of intuitionistic \( \bot \): negation may be defined as \( \neg A = (\top \# A) \)
A full account of the duality between paraconsistent and intuitionistic logic, including an explanation on why dual-intuitionistic and paraconsistent logics do not coincide, can be found in Brunner and Carnielli (2005).

Applications

Paraconsistent logic has been applied as a means of managing inconsistency in numerous domains, including:[11]

- Semantics. Paraconsistent logic has been proposed as means of providing a simple and intuitive formal account of truth that does not fall prey to paradoxes such as the Liar. However, such systems must also avoid Curry’s paradox, which is much more difficult as it does not essentially involve negation.
- Set theory and the foundations of mathematics (see paraconsistent mathematics). Some believe that paraconsistent logic has significant ramifications with respect to the significance of Russell’s paradox and Gödel’s incompleteness theorems.
- Epistemology and belief revision. Paraconsistent logic has been proposed as a means of reasoning with and revising inconsistent theories and belief systems.
- Knowledge management and artificial intelligence. Some computer scientists have utilized paraconsistent logic as a means of coping gracefully with inconsistent information.[12]
- Deontic logic and metaethics. Paraconsistent logic has been proposed as a means of dealing with ethical and other normative conflicts.
- Software engineering. Paraconsistent logic has been proposed as a means for dealing with the pervasive inconsistencies among the documentation, use cases, and code of large software systems.[3] [4]
- Electronics design routinely uses a four valued logic, with “hi-impedence (z)” and “don’t care (x)” playing similar roles to “don’t know” and “both true and false” respectively, in addition to True and False. This logic was developed independently of Philosophical logics.

Criticism

Some philosophers have argued against paraconsistent logic on the ground that the counterintuitiveness of giving up any of the three principles above outweighs any counterintuitiveness that the principle of explosion might have.

Others, such as David Lewis, have objected to paraconsistent logic on the ground that it is simply impossible for a statement and its negation to be jointly true.[13] A related objection is that “negation” in paraconsistent logic is not really negation; it is merely a subcontrary-forming operator.[14]

Alternatives

Approaches exist that allow for resolution of inconsistent beliefs without violating any of the intuitive logical principles. Most such systems use multivalued logic with Bayesian inference and the Dempster-Shafer theory, allowing that no non-tautological belief is completely (100%) irrefutable because it must be based upon incomplete, abstracted, interpreted, likely unconfirmed, potentially uninformed, and possibly incorrect knowledge. These systems effectively give up several logical principles in practice without rejecting
them in theory.
See also: Probability logic

See also
• Table of logic symbols
• Formal logic
• Deviant logic

Notable figures
Notable figures in the history and/or modern development of paraconsistent logic include:
• Alan Ross Anderson (USA, 1925–1973). One of the founders of relevance logic, a kind of paraconsistent logic.
• F. G. Asenjo (Argentina)
• Diderik Batens (Belgium)
• Nuel Belnap (USA, b. 1930). Worked with Anderson on relevance logic.
• Jean-Yves Béziau (France/Switzerland, b. 1965). Has written extensively on the general structural features and philosophical foundations of paraconsistent logics.
• Ross Brady (Australia)
• Bryson Brown (Canada)
• Walter Carnielli (Brazil). The developer of the possible-translations semantics, a new semantics which makes paraconsistent logics applicable and philosophically understood.
• Newton da Costa (Brazil, b. 1929). One of the first to develop formal systems of paraconsistent logic.
• Itala M. L. D’Ottaviano (Brazil)
• J. Michael Dunn (USA). An important figure in relevance logic.
• Stanisław Jaśkowski (Poland). One of the first to develop formal systems of paraconsistent logic.
• R. E. Jennings (Canada)
• David Kellogg Lewis (USA, 1941–2001). Articulate critic of paraconsistent logic.
• Jan Łukasiewicz (Poland, 1878–1956)
• Robert K. Meyer (USA/Australia)
• Chris Mortensen (Australia). Has written extensively on paraconsistent mathematics.
• Lorenzo Peña (Spain, b. 1944). Has developed an original line of paraconsistent logic, gradualistic logic (also known as transitive logic, TL), akin to Fuzzy Logic.
• Val Plumwood [formerly Routley] (Australia, b. 1939). Frequent collaborator with Sylvan.
• Graham Priest (Australia). Perhaps the most prominent advocate of paraconsistent logic in the world today.
• Francisco Miró Quesada (Peru). Coined the term paraconsistent logic.
• Peter Schotch (Canada)
• B. H. Slater (Australia). Another articulate critic of paraconsistent logic.
• Nicolai A. Vasiliev (Russia, 1880–1940). First to construct logic tolerant to contradiction (1910).
Notes

[2] See the article on the principle of explosion for more on this.
[6] LP is also commonly presented as a many-valued logic with three truth values (true, false, and both).
[7] See, for example, Priest (2002), §5.
[9] Surveys of various approaches to paraconsistent logic can be found in Bremer (2005) and Priest (2002), and a large family of paraconsistent logics is developed in detail in Carnielli, Congilio and Marcos (2007).
[12] See, for example, the articles in Bertossi et al. (2004).

Resources

Paraconsistent logic


**External links**

- Stanford Encyclopedia of Philosophy “Paraconsistent Logic” (http://plato.stanford.edu/entries/logic-paraconsistent/)

Is logic empirical?

"Is logic empirical?" is the title of two articles that discuss the idea that the algebraic properties of logic may, or should, be empirically determined; in particular, they deal with the question of whether empirical facts about quantum phenomena may provide grounds for revising classical logic as a consistent logical rendering of reality. The replacement derives from the work of Garrett Birkhoff and John von Neumann on → quantum logic. In their work, they showed that the outcomes of quantum measurements can be represented as binary propositions and that these quantum mechanical propositions can be combined in much the same way as propositions in classical logic. However, the algebraic properties of this structure are somewhat different from those of classical propositional logic in that the principle of distributivity fails.

The idea that the principles of logic might be susceptible to revision on empirical grounds has many roots, including the work of W.V. Quine and the foundational studies of Hans Reichenbach [1].

W.V. Quine

What is the epistemological status of the laws of logic? What sort of arguments are appropriate for criticising purported principles of logic? In his seminal paper "Two Dogmas of Empiricism," the logician and philosopher W.V. Quine argued that all beliefs are in principle subject to revision in the face of empirical data, including the so-called analytic propositions. Thus the laws of logic, being paradigmatic cases of analytic propositions, are not immune to revision.

To justify this claim he cited the so-called paradoxes of quantum mechanics. Birkhoff and von Neumann proposed to resolve those paradoxes by abandoning the principle of distributivity, thus substituting their quantum logic for classical logic.

Quine did not at first seriously pursue this argument, providing no sustained argument for the claim in that paper. In Philosophy of Logic (the chapter titled "Deviant Logics"), Quine rejects the idea case that classical logic should be revised in response to the paradoxes, being concerned with "a serious loss of simplicity", and "the handicap of having to think within a deviant logic". Quine, though, stood by his claim that logic is in principle not immune to revision.

Hans Reichenbach

Reichenbach considered one of the anomalies associated with quantum mechanics, the problem of complementary properties. A pair of properties of a system is said to be complementary if each one of them can be assigned a truth value in some experimental setup, but there is no setup which assigns a truth value to both properties. The classic example of complementarity is illustrated by the double-slit experiment in which a photon can be made to exhibit particle-like properties or wave-like properties, depending on the experimental setup used to detect its presence. Another example of complementary properties is that of having a precisely observed position or momentum.

Reichenbach approached the problem within the philosophical program of the logical positivists, wherein the choice of an appropriate language was not a matter of the truth or falsity of a given language – in this case, the language used to describe quantum mechanics
but a matter of "technical advantages of language systems". His solution to the problem was a logic of properties with a three-valued semantics; each property could have one of three possible truth-values: true, false, or indeterminate. The formal properties of such a logical system can be given by a set of fairly simple rules, certainly far simpler than the "projection algebra" that Birkhoff and von Neumann had introduced a few years earlier. However, because of this simplicity, the intended semantics of Reichenbach’s three-valued logic is unsuited to provide a foundation for quantum mechanics that can account for observables.

**First article: Hilary Putnam**

In his paper "Is logic empirical?" [2] Hilary Putnam, whose PhD studies were supervised by Reichenbach, pursued Quine’s idea systematically. In the first place, he made an analogy between laws of logic and laws of geometry: once Euclid's postulates were believed to be truths about the physical space in which we live, but modern physical theories are based around non-Euclidean geometries, with a different and fundamentally incompatible notion of straight line.

In particular, he claimed that what physicists have learned about quantum mechanics provides a compelling case for abandoning certain familiar principles of classical logic for this reason: realism about the physical world, which Putnam generally maintains, demands that we square up to the anomalies associated with quantum phenomena. Putnam understands realism about physical objects as involving that the properties of momentum and position exist for quanta. Since the uncertainty principle says that either of them can be determined, but both cannot be determined at the same time, he faces a paradox. He sees the only possible resolution of the paradox as lying in the embrace of quantum logic, in which he believes this is not inconsistent.

**Quantum logic**

The formal laws of a physical theory are justified by a process of repeated controlled observations. This from a physicist's point of view is the meaning of the empirical nature of these laws.

The idea of a propositional logic with rules radically different from Boolean logic in itself was not new. Indeed a sort of analogy had been established in the mid-nineteen thirties by Garrett Birkhoff and John von Neumann between a non-classical propositional logic and some aspects of the measurement process in quantum mechanics. Putnam and the physicist David Finkelstein proposed that there was more to this correspondence than a loose analogy: that in fact there was a logical system whose semantics was given by a lattice of projection operators on a Hilbert space. This, actually, was the correct logic for reasoning about the microscopic world.

In this view, classical logic was merely a limiting case of this new logic. If this were the case, then our "preconceived" Boolean logic would have to be rejected by empirical evidence in the same way Euclidean geometry (taken as the correct geometry of physical space) was rejected on the basis of (the facts supporting the theory of) general relativity. This argument is in favour of the view that the rules of logic are empirical.

That logic came to be known as quantum logic. There are, however, few philosophers today who regard this logic as a replacement for classical logic; Putnam may no longer hold
that view. Quantum logic is still used as a foundational formalism for quantum mechanics: but in a way in which primitive events are not interpreted as atomic sentences but rather in operational terms as possible outcomes of observations. As such, quantum logic provides a unified and consistent mathematical theory of physical observables and quantum measurement.

**Second article: Michael Dummett**

In an article also titled "Is logic empirical?," Michael Dummett argues that Putnam’s desire for realism mandates distributivity: the principle of distributivity is essential for the realist’s understanding of how propositions are true of the world, in just the same way as he argues the → principle of bivalence is. To grasp why: consider why truth tables work for classical logic: firstly, it must be the case that the variable parts of the proposition are either true or false: if they could be other values, or fail to have truth values at all, then the truth table analysis of logical connectives would not exhaust the possible ways these could be applied; for example intutionistic logic respects the classical truth tables, but not the laws of classical logic, because intuitionistic logic allows propositions to be other than true or false. Second, to be able to apply truth tables to describe a connective depends upon distributivity: a truth table is a disjunction of conjunctive possibilities, and the validity of the exercise depends upon the truth of the whole being a consequence of the bivalence of the propositions, which is true only if the principle of distributivity applies.

Hence Putnam cannot embrace realism without embracing classical logic, and hence his argument to endorse quantum logic because of realism about quanta is a hopeless case.

Dummett’s argument is all the more interesting because he is not a proponent of classical logic. His argument for the connection between realism and classical logic is part of a wider argument to suggest that, just as the existence of particular class of entities may be a matter of dispute, so disputation about the objective existence of such entities is question begging if use is made of classical logic. Consequently intuitionistic logic is privileged over classical logic, when it comes to disputation concerning phenomena whose objective existence is a matter of controversy.

Thus the question, "Is logic empirical?," for Dummett, leads naturally into the dispute over bivalence and anti-realism, one of the deepest issues in modern metaphysics.

**Notes**


Is logic empirical?
Is logic empirical?

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