# Joseph Corneli
# Collected Works

2002–2026

Mathematics · Computational Creativity · Peer Production
Open Research · Pattern Languages · Formal Verification

# Introduction

The papers here appeared in various venues — *Trans. AMS*, *Artificial Intelligence*, *Argumentation*, EuroPLoP, ICCC, PLoP, ECTEL, Springer chapters. I've included the small-press-published *Peeragogy Handbook* and two formally accredited theses. Many unpublished and informal works are not included. The publications bear the marks of the venues to which they were submitted, the contributions of co-authors, and the moments at which they were written. They are gathered here without revision because that is what a Collected Works is — not a reinterpretation, but a data dump.

As to what it all means, I'd point the interested reader to ongoing developments with the FUTON stack (https://github.com/tothedarktowercame/). In effect, that work represents much the same material in a different form, with a different argument, traced as one project rather than several. What follows is a reading guide to the publications. Four overlapping strands are worth signposting.

***Geometry, before everything else.*** The earliest papers — the BA thesis on double bubbles in constant-curvature spaces (2002), the *Trans. AMS* paper on the flat two-torus (2004), the *Experimental Mathematics* paper on the three-torus (2003), the *J. Geometric Analysis* paper on S3 and H3 (2007), the *Houston J. Math.* paper on Gauss space and spheres (2008) — are about optimisation problems in modern geometry, mostly done as an undergraduate at New College of Florida. These are works of technical mathematical writing, and the reader should treat them as such. They establish a baseline for the mathematical seriousness of more recent work such as *A First Proof Sprint* (2026) at the other end of this volume.

The middle decade — roughly 2010 through 2014 — concerns peer production and peer learning in mathematics, centring on my Open University Ph.D. thesis (2014). The surrounding papers describe the technical infrastructure (PlanetMath/Planetary, eMath 3.0, GravPad, NNexus, the MSC linked open dataset), the social model (paragogy, P2PLE, Crowdsourcing Education), and the methodological turn that follows from treating learning as a form of peer production. The *Peeragogy Handbook* (2016, edited volume) is both a companion and sequel to my thesis.

**Computational creativity and argumentation.** Work at Edinburgh and Goldsmiths (2014–2019, COINVENT and MathSoMac) produced most of the papers in the ICCC, AISB, and ICCM tracks. Three threads run through them: argumentation theory as a way of representing how mathematical and creative work actually proceeds (*Argumentation Theory for Mathematical Argument*, 2019; the Lakatos-style *Artificial Intelligence* paper with Pease, Lawrence, Budzynska, and Reed, 2017; *Towards mathematical AI*, 2017); concept blending as a generative mechanism (the role-of-blending and Confalonieri papers from ICCC 2015, the harmonic blending paper from ICCC 2016, the games-blending paper from EXAG 2015); and meta-evaluation of creative systems (*Assessing progress in building autonomously creative systems*, 2014; *Stakeholder groups*, 2015; *Evaluation of Creativity*, 2018). *Modelling the way mathematics is actually done* (FARM 2017) is one key output from this strand of work and is a clear ancestor of my work on FUTON.

**Pattern languages, futures, and PLACARD.** From 2018 onward — and especially since 2021 — the work has clustered around design pattern methods extended into futures studies and collaborative inquiry: *Patterns of Design* (EuroPLoP 2018), the Patterns of Patterns sequence (CLA 3.0 chapter 2022, PLoP 2023, PLoP 2025), the PLACARD workshop series (AsianPLoP 2024, EuroPLoP 2024), and the recent *Patterns for a New Generation: AI and Agents* (PLoP 2025) on patterns as shared priors between humans and language-model agents. Most of these are co-authored with Charles Danoff, Raymond Puzio, Mary Tedeschi, Sridevi Ayloo, Sergio Belich, and others — the Peeragogy collective in its current form.

The strands overlap more than the categories suggest. The argumentation work picks up from the peer-production work; the pattern-language work picks up from the argumentation work; *A First Proof Sprint* picks up from the pattern-language work and from the geometry. Reading the volume in chronological order is not the only sensible approach.

# Introduction (continued)

A closing note on what this volume is not. I've had in mind that I might write another thesis about my work on FUTON. This volume is the prior record. Stay tuned.

J. Corneli
Oxford
April 2026

# Contents

## Reports & Preprints

• **2026** A First Proof Sprint
J. Corneli — arXiv:2602.13587

• **2020** Patterns, anticipation and participatory futures
R. Puzio, P. Ricaurte, C. J. Danoff, C. Pierce, A. Dutka-Chirichetti, V. Bruno, H. Cintra, J. Corneli — arXiv:2012.03736

• **2014** Towards a Creativity Commons
J. Corneli — WOW5

• **2011** The PlanetMath Encyclopedia
J. Corneli — ITP 2011 MathWikis Workshop

## Journal Papers

• **2021** The Anatomy of moral agency: A theological and neuroscience inspired model of virtue ethics
N. Crook, J. Corneli — Cognitive Computation and Systems

• **2019** Argumentation Theory for Mathematical Argument
J. Corneli, U. Martin, D. Murray-Rust, G. Rino Nesin, A. Pease — Argumentation

• **2017** Lakatos-style Collaborative Mathematics through Dialectical, Structured and Abstract Argumentation
A. Pease, J. Lawrence, K. Budzynska, J. Corneli, C. Reed — Artificial Intelligence

• **2008** Double bubbles in Gauss space and spheres
J. Corneli, I. Corwin, S. Hurder et al. — Houston J. Math

• **2007** Double bubbles in S3 and H3
J. Corneli, N. Hoffman, P. Holt et al. — Journal of Geometric Analysis

• **2004** The double bubble problem on the flat two-torus
J. Corneli, P. Holt, G. Lee, N. Leger, E. Schoenfeld, B. Steinhurst — Trans. AMS

• **2003** Double bubbles in the three-torus
M. C. Alvarez, J. Corneli, G. Walsh, S. Beheshti — Experimental Mathematics

## Conference Papers

• **2025** Patterns for a New Generation: AI and Agents (Patterns of Patterns III)
J. Corneli, C. J. Danoff, R. S. Puzio, S. Ayloo, S. Belich, A. Wilkinson, M. Tedeschi, P. Mosley — PLoP 2025 / arXiv:2506.09696

• **2025** Patterns for a New Generation: Agent Workshops
J. Corneli, C. Danoff, R. Puzio, S. Ayloo, S. Belich, M. Tedeschi, C. Pierce — PLoP 2025

• **2025** Exploring the Future of AI Learning Through Music and Game Play
M. Tedeschi, J. Corneli, S. Belich, C. J. Danoff, S. Ayloo, R. Puzio — EuroPLoP 2025

• **2024** Use Manga and Comic Books to Teach English
C. Danoff, M. Tedeschi, S. Ayloo, J. Corneli, R. Puzio, C. Pierce — PLoP 2024

• **2024** PLACARD Workshop
C. Danoff, M. Tedeschi, S. Ayloo, J. Corneli, R. Puzio — AsianPLoP 2024

• **2024** AI Future Envisioning with PLACARD
M. Tedeschi, P. Ricaurte, S. Ayloo, J. Corneli, C. J. Danoff, S. Belich — EuroPLoP 2024

• **2023** Patterns of Patterns II
J. Corneli, N. Alhasan, L. Vivier, A. Murphy, R. S. Puzio, A. Tabor, S. Ayloo, C. Pierce, C. J. Danoff, M. Tedeschi, M. Singh, K. Khetan — PLoP 2023 / arXiv:2306.08426

• **2018** Patterns of Design
J. Corneli, S. Holland, A. Pease, P. Mulholland, D. Murray-Rust, T. Scaltsas, A. Smaill — EuroPLoP '18

• **2017** Towards mathematical AI via a model of the content and process of mathematical question and answer dialogues
J. Corneli, U. Martin, D. Murray-Rust, A. Pease — CICM 2017

• **2017** math.wikipedia.org
J. Corneli, M. Schubotz — AITP 2017

# Contents (continued)

## Conference Papers


• **2016** An institutional approach to computational social creativity
J. Corneli — ICCC 2016
• **2016** An Argument-based Creative Assistant for Harmonic Blending
M. Kaliakatsos-Papakostas, R. Confalonieri, J. Corneli, A. Zacharakis, E. Cambouropoulos — ICCC 2016
• **2016** The FloWr Online Platform
J. Charnley, S. Colton, M. T. Llano, J. Corneli — ICCC 2016
• **2016** What If A Fish Got Drunk?
M. T. Llano, C. Guckelsberger, R. Hepworth, J. Gow, J. Corneli, S. Colton — ICCC 2016
• **2015** Patterns of Peeragogy
J. Corneli, C. J. Danoff, C. Pierce, P. Ricaurte, L. Snow MacDonald — PLoP 2015
• **2015** The role of blending in mathematical invention
F. Bou, J. Corneli, D. Gómez-Ramírez et al. — ICCC 2015
• **2015** Using Argumentation to Evaluate Concept Blends in Combinatorial Creativity
R. Confalonieri, J. Corneli, A. Pease, E. Plaza, M. Schorlemmer — ICCC 2015
• **2015** Computational Poetry Workshop
J. Corneli, A. Jordanous, R. Shepperd et al. — ICCC 2015
• **2014** Building the Peeragogy Accelerator
J. Corneli, D. Marciniak, C. J. Danoff et al. — OER14
• **2014** NNexus reloaded
D. Ginev, J. Corneli — Intelligent Computer Mathematics, Springer
• **2014** Assessing progress in building autonomously creative systems
S. Colton, A. Pease, J. Corneli, M. Cook, T. Llano — ICCC 2014
• **2012** PlanetMath/Planetary
J. Corneli, M. Dumitru — MathUI 2012
• **2012** Reimplementing the MSC as a Linked Open Dataset
C. Lange, P. Ion, A. Dimou et al. — Intelligent Computer Mathematics, Springer
• **2012** Detecting mathematics learning online
J. Corneli, M. Ponti — Networked Learning 2012
• **2011** Paragogy
J. Corneli, C. Danoff — 6th Open Knowledge Conference
• **2011** PlanetMath Redux
J. Corneli, C. Jucovschi, A. Mikroyannidis — TELMAS 2011
• **2011** Personalised Peer-Supported Learning (P2PLE)
J. Corneli, A. Mikroyannidis — Digital Education Review
• **2011** The Planetary System: Web 3.0 & Active Documents for STEM
M. Kohlhase, J. Corneli, C. David et al. — Procedia Computer Science
• **2010** GravPad
J. Corneli — WikiSym 2010
• **2005** A Scholia-based Document Model for Commons-based Peer Production
J. Corneli, A. Krowne — MetaScholar Initiative, Emory University


## Workshop Papers


• **2018** Towards Open-World Scenarios: Teaching the Social Side of Data Science
J. Corneli, D. Murray-Rust, B. Bach — AISB 2018
• **2017** Modelling the way mathematics is actually done
J. Corneli, U. Martin, D. Murray-Rust, A. Pease, R. Puzio, G. Rino Nesin — FARM 2017
• **2017** Bootstrapping the next generation of mathematical social machines
U. Martin, A. Pease, J. Corneli — Off the Beaten Track 2017
• **2016** X575: writing rengas with web services
D. Winterstein, J. Corneli — INLG 2016

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## Workshop Papers


• **2016** Teaching natural language to computers
J. Corneli, M. Corneli — IJCAI 2016 Workshop
• **2015** Towards generating novel games using conceptual blending
J. Gow, J. Corneli — EXAG2 2015
• **2015** Implementing feedback in creative systems
J. Corneli, A. Jordanous — AInF 2015
• **2015** The search for computational intelligence
J. Corneli, E. Maclean — AISB 2015
• **2012** Massively distributed authorship of academic papers
B. Tomlinson, J. Ross, P. André et al. — CHI'12 Extended Abstracts
• **2010** Live annotation and content discovery in personal learning environments
J. Corneli, A. Mikroyannidis — MUPPLE 2010
• **2010** eMath 3.0
C. David, D. Ginev, M. Kohlhase, J. Corneli — Math/ICT Workshop


## Chapters


• **2022** Patterns of Patterns: A Methodological Reflection on the Future of Design Pattern Methods
J. Corneli, A. Murphy, R. S. Puzio, L. Vivier, N. Alhasan, V. Bruno, C. Pierce, C. J. Danoff — CLA 3.0 (Tamkang University Press)
• **2018** Evaluation of Creativity
J. Corneli, A. Pease — Concept Invention (Springer)
• **2018** Social Aspects of Concept Invention
J. Corneli, A. Pease, D. Stefanou — Concept Invention (Springer)
• **2015** Stakeholder groups in computational creativity research and practice
S. Colton, A. Pease, J. Corneli, M. Cook, R. Hepworth, D. Ventura — Atlantis-Springer
• **2013** Peeragogy in Action
J. Corneli, A. Keune, A. Lyons, C. Danoff — The Open Book (Finnish Institute, London)
• **2011** Crowdsourcing Education: A Role-Based Analysis
J. Corneli, A. Mikroyannidis — IGI Global


## Theses


• **2014** Peer produced peer learning: A mathematics case study
J. Corneli — PhD thesis, The Open University
• **2002** Double bubbles in spaces of constant curvature
J. Corneli — BA thesis, New College of Florida


## Book


• **2016** The Peeragogy Handbook (3rd Edition)
J. Corneli, C. J. Danoff, P. Ricaurte, C. Pierce, L. Snow Macdonald (Eds.) — Pierce Press / PubDomEd


• included in this volume ■ listed only (PDF not bundled)

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# A First Proof Sprint

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Reproducibility: built February 17, 2026 from `futon6` version-controlled git history (c92a4eb).
Source: `https://github.com/tothedarktowercame/futon6`

# Abstract

This monograph reports a multi-agent proof sprint on ten research-level problems, combining rapid draft generation with adversarial verification, targeted repair, and explicit provenance. The workflow uses wiring-diagram decompositions of claim dependencies to localize gaps and coordinate reviewer-driven revisions. Final outcomes are heterogeneous but explicit: the manuscript distinguishes *mathematical status* from *QC-validation status*. In the current ledger, Problems 1, 2, 3, and 8 are treated as mathematically closed with validation-complete records (Problem 2 still depends on the cited newvector/test-vector theorem input); Problems 5 and 7 are mathematically closed with partial QC status (scope-limited closure for Problem 5, and an independent theorem-ledger re-check pending for Problem 7). Problem 4 is partial: proved analytically for $n \leq 3$, and proved for $n = 4$ via the Path 2 algebraic certificate, with higher $n$ open. Problem 6 is partial: the $K_n$ case is proved with $c_0 = 1/3$, while the general-graph bridge remains open (formal $\bar{d} < 1$ bound at $M_t \neq 0$). Problem 9 has strong computational evidence but lacks a clean algebraic closure of the universal converse step. Problem 10 remains conditional under explicitly stated assumptions. The main methodological result is that structure-aware verification and layer-switching strategies improve reliability and calibration in compressed proof sprints.

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# Highlights

- Dependency-graph representations of proofs enabled local gap isolation and targeted repair across all ten problems.
- Node-level verification with predecessor/successor context improved calibration relative to whole-draft review.
- Adversarial critic–responder iteration converted early overclaims into explicit closed, partial, or conditional outcomes.
- Targeted computational stress tests constrained speculative branches, especially in the $n = 4$ branch of Problem 4.
- Structured handoff memos preserved unresolved obligations as named, testable next steps rather than implicit deferrals.

# Snapshot

- Ten initial proof attempts drafted in under two hours.
- Over two hundred commits across roughly 55 hours wall-clock (February 11–13, 2026), in addition to earlier infrastructure development.
- A shift in framing over the course of the sprint: from an early "all solved" posture to a clearer partition into proved, conditional, and partial results.
- A methodological lesson: wiring-diagram–based verification localized gaps and inconsistencies more effectively than plain narrative drafting.

# Introduction

This monograph records both the mathematical results and the working process behind a compressed, multi-agent proof cycle in `futon6`.[1] The project began as infrastructure—knowledge ingestion, tagging, and wiring-diagram metatheory—and then pivoted into a rapid proof sprint marked by iterative critique and repair.

The working process came to resemble Lakatos's *Proofs and Refutations* [1] more closely than anticipated. In Lakatos's dialogue, a conjecture is proposed, counterexamples emerge, the conjecture is revised, and the cycle repeats. Each iteration sharpens both the statement and the participants' understanding. Our sprint followed a similar rhythm, except that the interlocutors were a human dispatcher, multiple Claude instances (provers), and Codex

[1] Source repository: `https://github.com/tothedarktowercame/futon6`

(critic and research assistant). Conjectures were drafted rapidly; adversarial review exposed overclaims and hidden gaps; targeted repair narrowed the remaining obligations.

The wiring-diagram decomposition—representing each proof as a typed graph of claims and dependencies—served as the shared epistemic artefact. Where Lakatos's fictional classroom had a blackboard, this project had a structured proof graph. That graph made disagreements local and repairable.

The title alludes to the sprint format: ten open problems attempted in roughly 55 hours wall-clock, with both process and outcomes documented. The Lakatosian connection also runs through Pease et al. [2], which formalised Lakatos's informal logic of mathematical discovery as a dialogue game over argumentation structures, implemented in a system capable of mixed-initiative human–AI collaboration. That paper asked whether computers could participate in mathematical reasoning through conjecture, critique, and reform. The present work can be read as an empirical extension of that question: ten research problems attempted by AI agents under human supervision, with the Lakatosian cycle unfolding at machine speed.

In a related strand of work, Corneli et al. [3], developed a computational model of mathematical question-and-answer dialogues using Inference Anchoring Theory + Content (IATC) to annotate MathOverflow threads with performative types (assert, challenge, reform, clarify, etc.). The same vocabulary reappears in this project's Stage 7 thread-wiring pipeline (see Part III, Act VI), where StackExchange threads are parsed into wiring diagrams whose edges carry the same performative labels. Earlier work on peer-produced mathematical knowledge [5] established the empirical base: PlanetMath as a community-maintained proof ecosystem, studied through the lens of paragogy (peer-produced peer learning). The present sprint can be read as a machine-accelerated instance of the same phenomenon: collaborative, incremental proof construction with explicit argumentation structure.

The working notes reveal a persistent asymmetry: generation is fast; trustworthy validation is slow. Lakatos identified the same imbalance in his classroom: conjectures are cheap, but understanding *why* they fail (or hold) is the real work. In this sprint, the strongest leverage came not from first-pass drafting quality, but from structured verification—tools that made weaknesses visible early, locally, and precisely.

# Current Status

**Outcome summary (status code [M|V]).** $\mathbf{M} \in \{\mathbf{C}, \mathbf{P}, ?\}$ with **C** =closed, **P** =partial, ? =conditional; $V \in \{\checkmark, \sim, ?\}$ with ✓ =validation-complete, ∼=partial validation, ? =not yet fully validated.

- **P1 [C|✓].** Mathematical claim is treated as closed, with unified repair checks and canonical node-level verifier alignment in place. See Status Note P1 (p. 2).
- **P2 [C|✓].** Local algebraic chain is repaired and node-level verifier outputs are synchronized; the result still depends on the cited newvector/test-vector theorem input. See

Status Note P2 (p. 4).

- **P3 [C|✓].** Validation-complete existence path under the scoped criterion (current run: 2 verified, 7 plausible, 0 gaps). Uniqueness/irreducibility is treated as optional and out of scope for the core existence claim.
- **P4 [P|?].** Proved analytically for $n \leq 3$, and proved for $n = 4$ via the Path 2 algebraic certificate; higher $n$ remains open. See Status Note P4 (p. 8).
- **P5 [C|∼].** Closed in the declared local scope for $F_O$-local connective spectra; indexing-system-level extension remains open. See Status Note P5 (p. 10).
- **P6 [P|?].** $K_n$ case proved ($c_0 = 1/3$); one technical gap remains in the general-graph argument (formal $\bar{d} < 1$ bound at $M_t \neq 0$). See Status Note P6 (p. 12).
- **P7 [C|∼].** Provisionally closed mathematically via the rotation route; independent theorem-ledger re-check remains the main QC holdout. See Status Note P7 (p. 14).
- **P8 [C|✓].** Geometric closure is treated as repaired, with unified checks and canonical node-level verifier record now synchronized. See Status Note P8 (p. 16).
- **P9 [P|∼].** Strong computational evidence exists, but the universal converse step is not yet cleanly closed algebraically. See Status Note P9 (p. 18).
- **P10 [?|?].** Conditional under explicitly stated assumptions; assumption necessity is explicit, while final closure/validation remains open. See Status Note P10 (p. 20).
- Node-level validator outputs and legacy gap flags are tracked under `https://github.com/tothedarktowercame/futon6/tree/master/data/first-proof` (files matching `problem*-codex-results.jsonl`).
- Cross-problem closure-vs-validation inventory.[2]

# Materials and Methods

- **Model/runtime split.** Claude Max runs on a remote Linode host; Codex Pro runs on the local laptop workspace (`/home/joe/code/futon6`).
- **Software/toolchain summary.** Computational checks used Python 3 scripts with NumPy/SciPy for numerical linear algebra and optimization, SymPy for symbolic algebra (resultants, Sturm sequences, Gröbner bases), mpmath for selected high-precision checks, PHCpack via `phcpy` for polynomial-system solving/certification in Problem 4 (Case 3c), and cvxpy for SDP-based sum-of-squares certificate attempts. Proof-structure analysis

[2] `https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/closure-validation-audit.md`

used wiring-diagram generators (nine scripts producing JSON and Mermaid flowcharts via the internal `thread_performatives` module) and `validate-ct.py` for classical NER and scope extraction over PlanetMath entries. Pattern tagging and term spotting used Babashka (Clojure) scripts operating on EDN data formats. Document production used LaTeX (`pdflatex`) with repository consistency checks via `check-latex-text-truth.py`, `check-latex-syntax-violations.py`, and `check-ratchet-fixedw-typesetting.py`.

- **Codex Pro token accounting (local, log-derived).** From local Codex session telemetry (`~/.codex/sessions/*`, event type *token_count*), over the 50-hour window ending 2026-02-13 21:41 UTC (window start: 2026-02-11 19:41 UTC), there were 333 futon6-related Codex rollout traces out of 334 local rollout traces (99.7%).

| Metric | Total | Per rollout (avg) | % of total |
|---|---|---|---|
| **Total tokens** | 965,714,475 | 2,900,043 | 100% |
| **Input tokens (all)** | 961,719,044 | 2,887,445 | 99.6% |
| Cached input | 912,686,208 | 2,740,800 | 94.5% |
| Non-cached input | 49,032,836 | 147,246 | 5.1% |
| **Output tokens** | 3,995,431 | 11,997 | 0.4% |
| **Reasoning-output tokens** | 2,072,289 | 6,223 | — |

Interpretation (inference): the large cached-input share likely reflects repeated reuse of stable local context across rollouts, including processed Math StackExchange/MathOverflow corpora and derived prompt artifacts such as `se-data/math-processed`, `math-se-processed`, `mo-processed`, and Codex workflows using `--math-se-dir`.

- **Claude Max token accounting (Linode, log-derived).** From Linode session files (`~/.claude/projects/*/`), 9 futon6-related Claude Code sessions were identified; 8 sessions contributed within the 50-hour Codex-comparable window ending 2026-02-13 21:41 UTC.

| Metric | Total | Per session (avg) | % of total |
|---|---|---|---|
| **Total tokens (50h window)** | 791,980,382 | 98,997,548 | 100% |
| **Input tokens (all)** | 789,643,774 | 98,705,472 | 99.7% |
| Cached-read input | 752,300,790 | 94,037,599 | 95.0% |
| Cache-creation input | 37,221,978 | 4,652,747 | 4.7% |
| Non-cached input | 121,006 | 15,126 | 0.0% |
| **Output tokens** | 2,336,608 | 292,076 | 0.3% |

Model: `claude-opus-4-6`; assistant messages: 7,754. Full-window totals (9 sessions, 2026-02-08 through 2026-02-13): total tokens 1,021,665,524; output tokens 2,798,074.

# Reader Guide

This document is organized in five parts.

- **Part I (Annotated Overview):** short, citation-rich introductions for Problems 1–10, intended as navigational summaries.
- **Part II (Full Proof Drafts):** expanded writeups imported from the solution files, suitable as the base for a longer manuscript (targeting roughly book-length exposition). Coloured boxes separate the proof proper from process annotations (dead paths, revisions, open obligations).
- **Part III (Prompt Excerpts and Pivot Moments):** curated source excerpts (prompts and dispatch/history snippets) that document key decision points and proof-state transitions.
- **Part IV (Proof Patterns):** a cross-problem analysis of strategy patterns—what worked (layer-switching, creative reduction, structural decomposition), what didn't (the TryHarder loop), and how a coaching intervention broke a stuck problem open.
- **Part V (Color Plates):** syntax-coloured renderings of the proof files, produced by automated math-mode normalization.

# Terminology

**AIF**
: Active Inference Framework—used here as a conceptual lens for agent roles, intervention points, and coordination state; a full AIF policy loop for proof search is proposed future work, not yet implemented in this sprint.

**BSS**
: Batson–Spielman–Srivastava spectral sparsification framework.

**IATC**
: Inference Anchoring Theory + Content—an annotation scheme for mathematical dialogues used here to type dialogue moves and content links within the AIF process model [3].

**MSS**
: Marcus–Spielman–Srivastava interlacing families framework.

**PSD**
: Positive semi-definite (for matrices).

# Part I

# Annotated Overview

Chapter 1

# Problem 1: Measure Equivalence Under Smooth Shifts of the $\Phi_3^4$ Measure

## Problem statement

Given the renormalized $\Phi_3^4$ measure $\mu$ on $\mathbb{T}^3$, and a smooth nonzero shift $\psi$, determine whether $\mu$ and the shifted measure ${T_\psi}^*\mu$ are equivalent.

## Answer

Yes: $\mu \sim {T_\psi}^*\mu$.

**Status Note**

Mathematical claim treated as closed, with unified repair checks and the canonical node-level verifier record aligned on the key equivalence chain repairs.

## Annotated proof sketch

1. Use absolute continuity of $\mu$ with respect to the Gaussian reference measure $\mu_0$ from variational construction results for $\Phi_3^4$[1].
2. Apply Cameron–Martin to the Gaussian part: smooth shifts lie in the relevant Cameron–Martin space and preserve equivalence[2].
3. Control the interaction change $V(\phi - \psi) - V(\phi)$ using Wick expansion and exponential integrability for the cubic perturbation. This is the renormalized step tied to regularity structures / paracontrolled technology[3].

The Radon–Nikodym derivative is strictly positive and integrable in both directions, so mutual absolute continuity follows.

[1] Barashkov–Gubinelli, arXiv: https://arxiv.org/abs/2003.10529.

[2] Background discussion: https://mathoverflow.net/questions/194307/cameron-martin-theorem-non-gaussian-shift.

[3] Hairer, arXiv: https://arxiv.org/abs/1303.5113; Gubinelli–Imkeller–Perkowski, arXiv: https://arxiv.org/abs/1210.2684.

## Selected citations

- Variational $\Phi^4_3$: https://arxiv.org/abs/2003.10529
- Regularity structures: https://arxiv.org/abs/1303.5113
- Paracontrolled distributions: https://arxiv.org/abs/1210.2684
- Cameron–Martin discussion: https://mathoverflow.net/questions/194307/cameron-martin-theorem-non-gaussian-shift

## Background and prerequisites

**Field:** Stochastic PDE / Measure Theory.
**What you need:** Measure theory and Gaussian measures (Cameron–Martin theorem), plus familiarity with either regularity structures or paracontrolled distributions for making sense of the renormalized $\Phi^4_3$ interaction.
**Way in:** Hairer, "An Introduction to Stochastic PDEs" (lecture notes, arXiv:0907.4178) for the probabilistic side; the Barashkov–Gubinelli variational paper cited below is largely self-contained for the $\Phi^4_3$-specific machinery.

Chapter 2

# Problem 2: Universal Test Vector for Rankin–Selberg Integrals

## Problem statement

For generic irreducible admissible representations $\Pi$ of $\mathrm{GL}_{n+1}(F)$ and $\pi$ of $\mathrm{GL}_n(F)$ over a non-archimedean local field, ask whether one can fix a single Whittaker vector $W$ in $\Pi$ such that some $V$ in $\pi$ makes the twisted local Rankin–Selberg integral finite and nonzero for all complex $s$.

## Answer

Yes: the essential/new Whittaker vector $W_0$ serves as a universal test vector.

**Status Note**

Core local algebraic steps are treated as repaired; the result still depends on the cited newvector/test-vector theorem input ($\mathrm{H_{FW}}$). Canonical node-level verifier outputs are now aligned with the repaired lemma chain.

## Annotated proof sketch

1. The set of local zeta integrals spans a fractional ideal in $\mathbb{C}[q_F^{\pm s}]$, which is a PID[1]. Hence a monomial reduction criterion is available.
2. Evaluate the twisted Kirillov function $\phi_Q(g) = W_0(\mathrm{diag}(g, 1)u_Q)$ at $g = I_n$ to prove nontriviality; this uses the Whittaker transformation law.
3. Use $\mathrm{GL}_n$-equivariance and irreducibility of the Kirillov model restriction to show that fixing any nonzero $\phi$ (hence $\phi_Q$) still generates the full local $L$-factor ideal.

This yields existence of $V$ with integral equal to a nonzero monomial in $q_F^{-s}$, so it is finite and nonvanishing for all $s$.

[1] PID refresher: `https://planetmath.org/principalidealdomain`.

## Selected citations

- Jacquet–Piatetski-Shapiro–Shalika (Rankin–Selberg local theory): https://www.jstor.org/stable/2374430
- Bernstein–Zelevinsky (representations of $\mathrm{GL}(n, F)$): http://mi.mathnet.ru/eng/rm4200
- MathOverflow reference thread on local test vectors: https://mathoverflow.net/questions/469865/reference-request-test-vectors-for-local-rankin-selberg-l-factors-in-ramified-c
- Rankin–Selberg overview thread: https://mathoverflow.net/questions/223233/jacquets-approach-to-rankin-selberg-l-functions

## Background and prerequisites

**Field:** Automorphic Forms / Representation Theory of $p$-adic Groups.
**What you need:** Smooth representations of $\mathrm{GL}_n$ over non-archimedean local fields, Whittaker models, and the Rankin–Selberg integral formalism for local $L$-factors. Graduate-level algebraic number theory assumed.
**Way in:** Bump, *Automorphic Forms and Representations* (Cambridge, 1997), especially Part II on local theory; Cogdell's expository article "$L$-functions and Converse Theorems for $\mathrm{GL}_n$" (Fields Institute lectures).

Chapter 3

# Problem 3: Markov Chain with ASEP Polynomial Stationary Distribution

## Problem statement

Construct a nontrivial Markov chain on permutations of a partition $\lambda$ whose stationary distribution is the normalized interpolation ASEP polynomial weight at $q = 1$.

## Answer

Yes: the inhomogeneous multispecies $t$-PushTASEP yields the required stationary law.

## Annotated proof sketch

1. Define a finite-state continuous-time Markov chain with explicit local transition rates that depend only on $(x, t)$ and species order (not on polynomial values).
2. Invoke the stationary measure theorem identifying the stationary weights with ASEP/-Macdonald objects at $q = 1$[1].
3. Use irreducibility (vacancy transport and positive-rate swaps) to conclude uniqueness of the stationary distribution on the finite state space[2].

## Selected citations

- Inhomogeneous multispecies $t$-PushTASEP: `https://arxiv.org/abs/2403.10485`
- Multiline queues to Macdonald polynomials: `https://doi.org/10.1353/ajm.2022.0007`
- Macdonald polynomial definition page: `https://planetmath.org/macdonaldpolynomials`

[1] Ayyer–Martin–Williams, arXiv: `https://arxiv.org/abs/2403.10485`.
[2] Markov-chain basics: `https://planetmath.org/markovchain`.

## Background and prerequisites

**Field:** Markov Chains / Algebraic Combinatorics / Symmetric Functions.
**What you need:** Basic Markov chain theory (irreducibility, uniqueness of stationary distributions) and some familiarity with symmetric functions, particularly Macdonald polynomials and their specializations.
**Way in:** Levin–Peres–Wilmer, *Markov Chains and Mixing Times* (AMS, 2nd ed.) for stochastic foundations; Macdonald, *Symmetric Functions and Hall Polynomials* (OUP, 2nd ed.) for the algebraic-combinatorial side.

Chapter 4

# Problem 4: Root Separation Energy Under Finite Free Convolution ($n = 4$)

## Problem statement

For monic real-rooted degree-$n$ polynomials, test

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

## Answer

Partial result: proved analytically for $n \leq 3$, and now proved for $n = 4$ via the Path 2 algebraic certificate (3-piece Cauchy–Schwarz with P-convexity of $K_{\text{red}}$); higher $n$ remains open.

**Status Note**

Problem 4 is proved for $n \leq 3$ analytically (Cauchy–Schwarz), and proved for $n = 4$ via the Path 2 algebraic certificate (3-piece Cauchy–Schwarz + P-convexity of $K_{\text{red}}$). Higher $n$ remains open.

## Annotated proof structure

1. Algebraic reduction at $n = 4$: normalize coefficients, derive a polynomial numerator/denominator formulation for the surplus using the exact $\Phi_4 \cdot \text{disc}$ identity[1].
2. Show denominator sign on the real-rooted domain, reducing inequality to nonnegativity of a single polynomial $-N$.
3. Boundary and symmetric-subcase analysis by exact elimination/resultant methods.
4. Path 2 closure for $n = 4$: prove $T_2 + R$ surplus nonnegative via P-convexity/boundary analysis of $K_{\text{red}}$, which supersedes the case-by-case critical-point accounting branch.

[1] Discriminant refresher: `https://planetmath.org/discriminantofapolynomial`.

## Selected citations

- Finite free convolution (MSS): https://arxiv.org/abs/1504.00350
- Interlacing families follow-up: https://arxiv.org/abs/1507.05020
- Finite free cumulants perspective: https://arxiv.org/abs/1707.02443
- MathOverflow discussion (finite free convolution): https://mathoverflow.net/questions/375273/finite-free-convolution-of-polynomials
- PHCpack project page: https://github.com/janverschelde/PHCpack

## Background and prerequisites

**Field:** Free Probability / Real Algebraic Geometry / Polynomial Inequalities.
**What you need:** Real-rooted polynomials and discriminants, the finite free convolution $\boxplus_n$ of Marcus–Spielman–Srivastava, and basic free probability (free cumulants). Computational algebra tools (e.g. resultants, homotopy continuation) appear in the $n = 4$ certification.
**Way in:** Marcus–Spielman–Srivastava, "Interlacing Families I" (arXiv:1304.4132) for finite free convolution; Nica–Speicher, *Lectures on the Combinatorics of Free Probability* (Cambridge, 2006) for the probabilistic backdrop.

Chapter 5

# Problem 5: $\mathcal{O}$-Slice Connectivity via Geometric Fixed Points

## Problem statement

For an incomplete transfer system from an $N_\infty$ operad, define the associated restricted regular slice filtration and characterize $\mathcal{O}$-slice connectivity using geometric fixed points.

## Answer

For $\mathcal{F}_\mathcal{O}$-local connective $G$-spectra, one can restrict the Hill–Yarnall connectivity criterion to subgroups in the admissible family $\mathcal{F}_\mathcal{O}$.

Solved in a scope-limited form: the subgroup-family-level characterization is established for $\mathcal{F}_\mathcal{O}$-local connective spectra; a full indexing-system-level extension is left open.

**Status Note**

Closed only in the stated local scope ($\mathcal{F}_\mathcal{O}$-local connective spectra). The indexing-system-level extension remains an explicit open obligation.

**Open Gap / Conditional Step.** Open obligation (scope extension): move from subgroup-family indexing to full Blumberg–Hill indexing-system data (admissible finite $H$-sets, not only admissible subgroups). The current result uses $\mathcal{F}_\mathcal{O} = \{H : e \to H \text{ admissible}\}$ and does not yet encode finer $K \to H$ admissibility for $K \neq e$. Closing this gap requires a full indexing-system formulation of $\mathcal{O}$-slice connectivity and, in parallel, a non-local replacement (or discharge) of the $\mathcal{F}_\mathcal{O}$-locality hypothesis in the reverse direction.

## Annotated proof sketch

1. Build $\tau^{\mathcal{O}}_{\geq n}$ from regular slice cells indexed only by admissible subgroups.
2. Use closure properties of indexing systems to verify that this restriction is stable under conjugation and subgroup passage.
3. Adapt Hill–Yarnall's geometric fixed-point criterion: for $\mathcal{F}_\mathcal{O}$-local spectra, no tests are needed outside $\mathcal{F}_\mathcal{O}$ because those geometric fixed points vanish by locality.

## Selected citations

- Hill–Yarnall (slice filtration reformulation): https://arxiv.org/abs/1703.10526
- Blumberg–Hill ($N_\infty$ operads and indexing systems): https://arxiv.org/abs/1309.1750
- HHR monograph context: https://bookstore.ams.org/surv-346
- Geometric fixed points background: https://ncatlab.org/nlab/show/geometric+fixed+points

## Background and prerequisites

**Field:** Equivariant Stable Homotopy Theory.
**What you need:** Genuine $G$-spectra, geometric fixed-point functors, the slice filtration of Hill–Hopkins–Ravenel, and $N_\infty$ operads / indexing systems of Blumberg–Hill. Research-level; assumes graduate coursework in algebraic topology and stable homotopy theory.
**Way in:** Schwede, *Global Homotopy Theory* (Cambridge, 2018) for equivariant foundations; the Hill–Hopkins–Ravenel Kervaire paper (Annals, 2016) for slice-filtration motivation.

## Chapter 6

# Problem 6: $\varepsilon$-Light Vertex Subsets of Graphs

## Problem statement

Find a universal constant $c_0 > 0$ such that every weighted graph and every $\varepsilon \in (0,1)$ admit a subset $S$ with

$$|S| \geq c_0 \varepsilon n, \qquad L_{G[S]} \preceq \varepsilon L_G.$$

Partial (general case conditional). The $K_n$ case is proved with $c_0 = 1/3$; for general graphs, the remaining issue is the Barrier Maintenance Invariant (BMI): prove the full barrier-degree bound $\bar{d}_t < 1$ when $M_t \neq 0$. The current E/F regime reduction isolates this to two explicit lemmas.

**Status Note**

Unconditional closure holds for the $K_n$ branch. The general-graph theorem remains conditional on two explicit BMI regime lemmas in the $M_t \neq 0$ branch.

## Annotated proof structure

1. Use leverage scores $\tau_e = w_e R_{\text{eff}}(u,v)$ to define the heavy-edge graph and show any $\varepsilon$-light set must be independent there.
2. Apply the leverage-score trace identity and Turan's theorem to extract a large candidate set.
3. Resolve easy branches (independent-in-$G$ and directly bounded spectral sum) unconditionally.
4. Reduce the hard branch to barrier maintenance (BMI), then split by a graph-adaptive E/F regime decomposition; substantial computational evidence is available, but the final general proof still requires two regime lemmas.

**Open Gap / Conditional Step.** The unresolved step is the full BMI bound in the $M_t \neq 0$ regime. The writeup reduces this to two explicit E/F regime lemmas. Until those lemmas are proved (or replaced by an equivalent closure argument), the universal-constant theorem remains conditional.

## Selected citations

- Graph sparsification by effective resistances: https://arxiv.org/abs/0803.0929
- Interlacing families II (Kadison–Singer machinery): https://arxiv.org/abs/1306.3969
- Twice-Ramanujan sparsifiers: https://arxiv.org/abs/0901.2698
- Laplacian matrix definition: https://planetmath.org/laplacianmatrix

## Background and prerequisites

**Field:** Spectral Graph Theory / Linear Algebra.
**What you need:** Graph Laplacians and their spectra, effective resistance and leverage scores, Loewner ordering of positive semidefinite matrices, and matrix concentration inequalities. Turán's theorem from extremal combinatorics appears in the combinatorial step.
**Way in:** Spielman, "Spectral and Algebraic Graph Theory" (lecture notes, freely available); Tropp, "An Introduction to Matrix Concentration Inequalities" (Foundations and Trends, 2015).

Chapter 7

# Problem 7: Uniform Lattice with 2-Torsion and Rationally Acyclic Universal Cover

## Problem statement

Can a cocompact lattice $\Gamma$ in a real semisimple Lie group, containing an element of order 2, be the fundamental group of a closed manifold whose universal cover is $\mathbb{Q}$-acyclic?

Provisionally closed via the rotation route ($n = 7$, congruence setting): E2 is discharged and the S-branch is supported by the codim-2 surgery theorem chain documented in the project theorem-number ledger,[1] with final confidence pending independent line-by-line ledger re-check.

## Annotated proof structure

1. Use Fowler's fixed-point Euler-characteristic criterion to produce groups in $\mathrm{FH}(\mathbb{Q})$ from suitable $\mathbb{Z}/2$-extensions[2].
2. Construct two arithmetic lattice routes:
   - reflection route in even dimension;
   - codimension-2 rotation route in odd dimension (arithmetically cleaner for surgery).
3. Follow the rotation-route S-branch: codimension-2 setup, obstruction chain, and cap/gluing output to a closed manifold model with rationally acyclic universal cover.

**Status Note**

This monograph treats Problem 7 as provisionally closed via the rotation route. The construction uses Fowler's $\mathrm{FH}(\mathbb{Q})$ criterion and codimension-2 equivariant surgery in the $n = 7$ congruence-lattice setting, with the obstruction-vanishing bridge tracked in the theorem-number ledger.[a] Final confidence is contingent on independent ledger re-check. Literature anchors: `https://arxiv.org/abs/1204.4667`, `https://arxiv.org/abs/1705.10909`.

[a] `https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7-g2-theorem-chain.md`

[1] `https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7-g2-theorem-chain.md`

[2] Fowler: `https://arxiv.org/abs/1204.4667`.

## Selected citations

- Fowler criterion: https://arxiv.org/abs/1204.4667
- Reflection lattices source used in writeup: https://arxiv.org/abs/2506.23994
- Equivariant surgery framework: https://arxiv.org/abs/1705.10909
- Davis–Luck obstruction context: https://arxiv.org/abs/2303.15765
- G2 theorem-number ledger (project note).[3]
- Arithmetic lattices (Borel–Harish-Chandra): https://www.jstor.org/stable/1970362
- Semisimple Lie group definition: https://planetmath.org/semisimpleliegroup
- Related MathOverflow discussion: https://mathoverflow.net/questions/33545/equivariant-surgery-problem

## Background and prerequisites

**Field:** Surgery Theory / Geometric Topology / Arithmetic Groups.
**What you need:** Algebraic topology (homology, fundamental group, covering spaces), surgery theory on manifolds (normal maps, surgery obstructions), and arithmetic lattices in semisimple Lie groups. Research-level; assumes graduate coursework in both topology and Lie theory.
**Way in:** Lück, *A Basic Introduction to Surgery Theory* (ICTP lecture notes, 2002) for the topological machinery; Raghunathan, *Discrete Subgroups of Lie Groups* (Springer, 1972) for the arithmetic side.

[3] https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7-g2-theorem-chain.md

Chapter 8

# Problem 8: Lagrangian Smoothing of 4-Valent Polyhedral Surfaces

## Problem statement

If every vertex of a polyhedral Lagrangian surface in $(\mathbb{R}^4, \omega)$ is 4-valent, must the surface admit a Lagrangian smoothing through Hamiltonian isotopy?

## Answer

Yes.

**Status Note**

Current draft treats the geometric argument as closed, with unified repair checks and the canonical node-level verifier record aligned on the main algebraic and Maslov-index repairs.

## Annotated proof sketch

1. Analyze a 4-valent vertex: edge vectors span $\mathbb{R}^4$, forcing a symplectic splitting into two 2D factors.
2. Write the local model as a product of two planar corners and smooth each corner separately.
3. Check Lagrangian condition factorwise (product of isotropic curves in 2D symplectic factors).
4. Smooth remaining edge creases via exact-graph interpolation and glue local constructions by disjoint supports.
5. Promote to Hamiltonian isotopy using exactness/flux-vanishing in local models and composition closure of compactly supported Hamiltonian isotopies.

## Selected citations

- Weinstein neighborhood theorem source paper: `https://www.sciencedirect.com/science/article/pii/000187087190005X`

- McDuff–Salamon textbook page: https://global.oup.com/academic/product/introduction-to-symplectic-topology-9780198794899
- Symplectic manifold definition: https://planetmath.org/symplecticmanifold
- Lagrangian submanifold definition: https://planetmath.org/lagrangiansubmanifold

## Background and prerequisites

**Field:** Symplectic Geometry / Lagrangian Topology.
**What you need:** Symplectic manifolds, Lagrangian submanifolds, Hamiltonian isotopy, and Weinstein neighborhood theorems. The proof uses only local symplectic techniques (no Floer theory).
**Way in:** McDuff–Salamon, *Introduction to Symplectic Topology* (OUP, 3rd ed., 2017)—the standard graduate textbook covering all prerequisites.

Chapter 9

# Problem 9: Polynomial Detection of Rank-1 Scaling for Quadrifocal Tensors

## Problem statement

Given generic camera matrices and quadrifocal tensors $Q^{(\alpha\beta\gamma\delta)}$, construct a polynomial map $F$ of degree bounded independently of $n$ that vanishes exactly on rank-1 separable scaling tensors $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$.

## Answer

Yes: one can use degree-3 determinant constraints (all relevant $3\times3$ minors).

**Status Note**

Forward and witness checks are strong, but full closure still hinges on a formal universal converse argument (all non-rank-1 $\lambda$, not only tested families). Treat as partial until that algebraic universality step is written cleanly into the proof chain.

## Annotated proof sketch

1. For fixed $(\gamma, \delta, k, \ell)$, the determinant pairing defines an alternating bilinear form of rank 2 on $\mathbb{R}^4$, so all induced $3\times3$ minors vanish.
2. If $\lambda$ is rank-1 separable, Hadamard-diagonal scaling preserves the rank-$\leq2$ property, hence all these minors vanish.
3. Conversely, vanishing of all minors across the three matricization patterns forces each matricization rank to be 1, and compatibility implies full tensor separability.
4. Collect all such minors into $F$; each coordinate has degree 3 and no dependence on camera parameters.

## Selected citations

- Segre embedding (rank-1 tensor variety): `https://planetmath.org/segreembedding`

- Determinantal varieties: https://planetmath.org/determinantalvariety
- Tensor rank definition: https://planetmath.org/tensorrank
- Multiple-view geometry reference text: https://www.robots.ox.ac.uk/~vgg/hzbook/

## Background and prerequisites

**Field:** Algebraic Geometry / Computer Vision (Multiview Geometry).
**What you need:** Tensor algebra (rank, matricizations, Segre varieties), determinantal varieties, and the basics of multiview geometry (camera matrices, multi-focal tensors). The algebraic geometry is classical; no scheme theory required.
**Way in:** Hartley–Zisserman, *Multiple View Geometry in Computer Vision* (Cambridge, 2nd ed.) for the applied setting; Landsberg, *Tensors: Geometry and Applications* (AMS, 2012) for the algebraic underpinnings.

Chapter 10

# Problem 10: PCG for RKHS-Constrained Tensor CP Decomposition

## Problem statement

Explain why preconditioned conjugate gradient (PCG) solves the mode-$k$ RKHS CP subproblem without $O(N)$ per-iteration cost, where $N$ is the full ambient sample size.

## Answer

Use implicit Kronecker/Khatri–Rao matrix-vector products and a Kronecker preconditioner built from expected sampling geometry.

**Status Note**

This result is **conditional**: it is closed under explicit assumptions ($\lambda > 0$, $K_\tau \succ 0$, and sampling/coherence conditions controlling preconditioned conditioning). Necessity counterexamples are recorded in the full writeup when these assumptions are dropped.

## Annotated proof sketch

1. Never form the full system matrix; compute $Ax$ through structured operations: kernel multiplies, sparse sampling/restriction, and Khatri–Rao contractions.
2. Build $b$ by the same structure, again avoiding dense $N$-scale tensors.
3. Precondition with $P = H \otimes K_\tau$ from replacing sparse sampling by its expectation; invert $P$ via two small Cholesky solves.
4. Apply standard PCG convergence bounds in terms of condition number $\kappa(P^{-1/2}AP^{-1/2})$ and matrix concentration for random sampling.

## Selected citations

- Matrix concentration / Freedman-style bound context: `https://arxiv.org/abs/1110.1379`
- FALKON large-scale kernel method: `https://arxiv.org/abs/1705.10958`

- Kronecker product definition: https://planetmath.org/kroneckerproduct
- Conjugate gradient method definition: https://planetmath.org/conjugategradientmethod

## Background and prerequisites

**Field:** Numerical Linear Algebra / Kernel Methods / Tensor Decomposition.
**What you need:** Conjugate gradient and preconditioning, Kronecker and Khatri–Rao products, reproducing kernel Hilbert spaces (RKHS), and matrix concentration inequalities for random sampling operators.
**Way in:** Trefethen–Bau, *Numerical Linear Algebra* (SIAM, 1997) for iterative methods; Rasmussen–Williams, *Gaussian Processes for Machine Learning* (MIT Press, 2006; freely available online) for RKHS context.

# Part II

# Full Proof Drafts

# Reading Convention

This manuscript uses boxed annotations selectively. In the current revision, the full proof chapters are primarily unboxed; process boxes appear only where there is substantive process context to preserve.

- **Unboxed text** is the proof proper. A reader following the five-page-per-problem brief can read only the unboxed text and obtain a self-contained argument (or, where an argument is incomplete, a clearly stated conditional result).
- **Status Note boxes** appear in the annotated overview chapters to summarize closure/QC state for each problem.
- **Process Note boxes** appear in process-facing parts (for example, prompt excerpts) when methodological scope or reviewer context is needed.
- Dead-path / revision / open-obligation semantics are maintained in the tooling vocabulary, but are not instantiated as standalone boxed blocks in the current full-proof chapter text.

Chapter 11

# Problem 1: Equivalence of $\Phi^4_3$ Measure Under Smooth Shifts

## 11.1 Problem Statement

Let $T^3$ be the 3D unit torus. Let $\mu$ be the $\Phi^4_3$ measure on $D'(T^3)$. Let $\psi$: $T^3 \to R$ be a smooth nonzero function and $T_\psi(u) = u + \psi$ the shift map. Are $\mu$ and $T^*_\psi \mu$ equivalent (same null sets)?

## 11.2 Answer

**Yes.** The measures $\mu$ and $T^*_\psi \mu$ are equivalent.

**Confidence: Medium-high.** The argument combines Cameron-Martin theory with the absolute continuity structure of the $\Phi^4_3$ construction (Barashkov-Gubinelli 2020, Theorem 1.1). The integrability bound is stated for a neighborhood of the required exponent, not for all t, matching the available log-Sobolev technology.

## 11.3 Solution

### 11.3.1 1. The $\Phi^4_3$ measure

The $\Phi^4_3$ measure $\mu$ on $D'(T^3)$ is the probability measure formally written as:

$$d\mu(\phi) = Z^{-1} \exp(-V(\phi)) d\mu_0(\phi)$$

where $\mu_0$ is the Gaussian free field (GFF) measure with covariance $(m^2 - \Delta)\hat{}\{-1\}$ on $T^3$, and the interaction is:

$$V(\phi) = \int_{T^3} (: \phi^4 : - C : \phi^2 :) \mathrm{d}x$$

Here $:\phi^k:$ denotes the k-th Wick power (renormalized product), and C is a mass counterterm that diverges under regularization removal.

The rigorous construction (Hairer 2014, Gubinelli-Imkeller-Perkowski 2015, Barashkov-

Gubinelli 2020) produces $\mu$ as a well-defined probability measure on distributions of regularity $C^{-1/2-\epsilon}(T^3)$.

### 11.3.2 2. Absolute continuity with respect to the GFF

**Key fact:** The $\Phi^4_3$ measure $\mu$ is equivalent to (has the same null sets as) the base Gaussian free field measure $\mu_0$:

$$\mu \ \mu_0$$

This equivalence follows from the variational construction of Barashkov-Gubinelli (2020, Theorem 1.1), which establishes $E_{\mu_0}[\exp(-V)] < \infty$ and hence $\mu \ll \mu_0$ with strictly positive density exp(-V($\phi$))/Z. Since exp(-V($\phi$)) $> 0$ a.s. (exponential is always positive), the reverse absolute continuity $\mu_0 \ll \mu$ also holds.

### 11.3.3 3. Cameron-Martin theory for the GFF

For the Gaussian free field $\mu_0$ on $T^3$ with covariance C $= (m^2-\Delta)^{-1}$, the Cameron-Martin space is:

$$H = H^1(T^3)(\text{Sobolev space of order1})$$

Since $\psi$ is smooth, $\psi \in C^\infty(T^3)$ subset $H^1(T^3)$, so $\psi$ is in the Cameron-Martin space.

**Cameron-Martin theorem:** For any $h \in H$, the shifted Gaussian measure $T_h^*\mu_0$ is equivalent to $\mu_0$, with Radon-Nikodym derivative:

$$\mathrm{d}T_h^*\mu_0/d\mu_0(\phi) = \exp(l_h(\phi) - \|h\|_{H^2}/2)$$

where $l_h$ is the linear functional associated to h. In particular:

$$T_\psi^*\mu_0 \ \mu_0$$

### 11.3.4 4. Shift of the interacting measure

To compute $T_\psi^*\mu$, we need the density of the shifted interacting measure:

$$\mathrm{d}T_\psi^*\mu/d\mu_0(\phi) = Z^{-1}\exp(-V(\phi-\psi))*\exp(l_\psi(\phi) - \|\psi\|_{H^2}/2)$$

The shifted interaction V($\phi-\psi$) expands (using Wick ordering relative to $\mu_0$):

$$V(\phi-\psi) = V(\phi) - 4\int \psi : \phi^3 : \mathrm{d}x + 6\int \psi^2 : \phi^2 : \mathrm{d}x$$
$$-4\int \psi^3 : \phi : \mathrm{d}x + \int \psi^4 \mathrm{d}x$$
$$-C(\int : \phi^2 : \mathrm{d}x - 2\int \psi : \phi : \mathrm{d}x + \int \psi^2 \mathrm{d}x) + (\text{renorm.corrections})$$

**Renormalization under shift:** The term 6 int $\psi^2$ :$\phi^2$: dx generates an additional logarithmic divergence (from $\psi^2$ multiplying the Wick square). This is absorbed by shifting the mass counterterm:

$$C \to C + 6\|\psi\|_{L^2}^2 * (\log N \text{ correction})$$

The precise counterterm shift is determined by the regularization scheme; see Hairer (2014, Section 9) or Gubinelli-Imkeller-Perkowski (2015, Proposition 6.3) for the explicit formula. For the present argument, only the finiteness of the renormalized difference matters.

After renormalization, V($\phi-\psi$) - V($\phi$) is a well-defined random variable under $\mu_0$ (and under $\mu$). The dominant fluctuation term is 4 int $\psi$ :$\phi^3$: dx, which has the right regularity:

- :$\phi^3$: in $C^{-3/2-\epsilon}(T^3)$ (as a distribution)
- $\psi \in C^\infty(T^3)$ (smooth)
- int $\psi$ :$\phi^3$: dx is well-defined (pairing of smooth test function with distribution)

### 11.3.5 5. Integrability and equivalence

The Radon-Nikodym derivative $\mathrm{d}T_\psi^* \mu / d\mu$ involves:

$$R(\phi) = \exp(-(V(\phi-\psi) - V(\phi)) + l_\psi(\phi) - \text{const})$$

We need $R \in L^1(\mu)$ and $R > 0$ a.s.

**Positivity:** $R > 0$ a.s. because it's an exponential. ✓

**Integrability:** The critical term is exp(4 int $\psi$ :$\phi^3$: dx) under the $\Phi^4_3$ measure. By the log-Sobolev inequality for $\Phi^4_3$ (Barashkov-Gubinelli 2020), the measure has strong concentration: the tails of int $\psi$ :$\phi^3$: dx are controlled by the quartic interaction. Specifically:

$$E_\mu[\exp(t|\int \psi : \phi^3 : \mathrm{d}x|)] < \infty \text{for} |t| < t_0$$

where $t_0 > 0$ depends on $\|\psi\|_{C^0}$ and the coupling constant. This exponential integrability follows from the coercivity of the $\phi^4$ interaction: the quartic potential dominates the cubic perturbation (Barashkov-Gubinelli 2020, Section 4, exponential integrability from the Polchinski flow). The bound suffices for $R \in L^1(\mu)$ since the exponent in the Radon-Nikodym derivative is bounded by $4\,\|\psi\|_{C^0}|\int : \phi^3 : \mathrm{d}x|$, and this yields integrability provided the coefficient lies in the admissible exponential-moment range (or after a standard scaling/localization argument). Therefore $R \in L^1(\mu)$ and $1/R \in L^1(T_\psi^*\mu)$, giving:

$$T_\psi^*\mu \ \mu(\text{equivalent measures})$$

### 11.3.6 6. Alternative argument via the variational approach

Barashkov-Gubinelli (2020) construct the $\Phi_3^4$ measure via the Boué-Dupuis variational formula, which represents:

$$-\log Z = \inf_u E[V(\phi + \int_0^1 u_s \mathrm{d}s) + 1/2 \int_0^1 ||u_s||^2 \mathrm{d}s]$$

In this framework, shifting by $\psi$ is equivalent to modifying the variational problem by a shift in the drift, which produces an equivalent measure (the infimum shifts by a finite amount, preserving absolute continuity).

### 11.3.7 7. Summary

The measures are equivalent because:

1. $\mu \sim \mu_0$ (interacting measure equivalent to Gaussian, by positivity of density)
2. $T_\psi^*\mu_0 \sim \mu_0$ (Cameron-Martin theorem, since $\psi \in H^1$)
3. V($\phi - \psi$) - V($\phi$) is well-defined after renormalization
4. The exponential of the cubic perturbation is integrable (log-Sobolev / coercivity)
5. Therefore $T_\psi^*\mu \sim \mu$

## 11.4 References


- N. Barashkov, M. Gubinelli, "A variational method for $\Phi_3^4$," Duke Math J. 169 (2020), 3339-3415. [Theorem 1.1: construction and integrability; Section 4: exponential

integrability from Polchinski flow]

- M. Hairer, "A theory of regularity structures," Inventiones Math. 198 (2014), 269-504. [Section 9: renormalization counterterms]
- M. Gubinelli, P. Imkeller, N. Perkowski, `Paracontrolled distributions and singular PDEs`, Forum Math. Π 3 (2015). [Proposition 6.3: explicit counterterm formula]

**Process Note — futon6 Corpus References**

- PlanetMath: "distribution" — distributions on manifolds
- PlanetMath: "Sobolev space" — Cameron-Martin space is $H^1$
- PlanetMath: "Gaussian measure" — reference measure $\mu_0$
- PlanetMath: "Radon-Nikodym theorem" — absolute continuity and RN derivatives

Chapter 12

# Problem 2: Universal Test Vector for Rankin-Selberg Integrals

## 12.1 Problem Statement

Let F be a non-archimedean local field with ring of integers o. Let $N_r$ denote the subgroup of $\mathrm{GL}_r(F)$ consisting of upper-triangular unipotent elements. Let $\psi$: $F \to \mathbb{C}^x$ be a nontrivial additive character of conductor o, identified with a generic character of $N_r$.

Let $\Pi$ be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$, realized in its $\psi^{-1}$-Whittaker model $W(\Pi, \psi^{-1})$. Must there exist $W \in W(\Pi, \psi^{-1})$ with the following property?

For any generic irreducible admissible representation $\pi$ of $\mathrm{GL}_n(F)$ in its $\psi$-Whittaker model $W(\pi, \psi)$, let q be the conductor ideal of $\pi$, $Q \in F^\times$ a generator of $q^{-1}$, and $u_Q := I_{n+1} + QE_{n,n+1}$. Then for some $V \in W(\pi, \psi)$, the local Rankin-Selberg $\int$

$$\int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1) u_Q) V(g) |\det g|^{s-1/2} \mathrm{d}g$$

is finite and nonzero for all $s \in C$.

## 12.2 Answer

**Yes.** The new vector (essential Whittaker function) of $\Pi$ serves as a universal test vector, with the $u_Q$ twist compensating for the conductor of $\pi$.

**Process Note — Confidence**

**Confidence: Medium-high.** The argument combines standard Rankin-Selberg theory (JPSS) with the Bernstein-Zelevinsky theory of the Kirillov model. The key nondegeneracy claim (Section 3a) uses $\mathrm{GL}_n$-equivariance of the $\zeta - \int$ pairing together with irreducibility of the Kirillov model (any nonzero function generates the full module under $\mathrm{GL}_n$-translates) and the PID structure of the fractional ideal ring. The "nonzero for all s" condition reduces to explicit Laurent polynomial algebra (Section 2).

## 12.3 Solution

### 12.3.1 1. Rankin-Selberg theory background

The local Rankin-Selberg $\int I(s,W,V)$ for $\mathrm{GL}_{n+1}\times\mathrm{GL}_n$ is:

$$I(s,W,V)=\int_{N_n\backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g,1))V(g)|\det g|^{s-1/2}\mathrm{d}g$$

This converges for $Re(s)\gg 0$ and extends to a rational function of $q_F^{-s}$ (where $q_F=|\mathrm{o/p}|$). The set of all such integrals as $(W,V)$ vary generates a fractional ideal of $\mathbb{C}[q_F^s,q_F^{-s}]$, whose generator is the local L-factor $L(s,\Pi\times\pi)$.

The problem modifies this by inserting $u_Q=I_{n+1}+QE_{n,n+1}$ into the argument of W, giving the "twisted" Rankin-Selberg $\int$.

### 12.3.2 2. The condition "finite and nonzero for all s"

For a rational function $f(q_F^{-s})$ to be `finite and nonzero for all $s \in C$`, it must have no poles and no zeros when viewed as a function of $X=q_F^{-s}\in\mathbb{C}^x$. Such a rational function must be $c*X^k=c*q_F^{-ks}$ for some nonzero c and integer k.

So the condition requires: there exists $V$ such that $I(s,W,V)=c*q_F^{-ks}$ for some nonzero constant c and integer k.

By Section 3a below, the integrals over $V$ (for fixed $W=W_0$) generate the full fractional ideal $I=L(s,\Pi\times\pi)*\mathbb{C}[q_F^s,q_F^{-s}]$.

**Explicit algebra:** $I$ is a free rank-1 module over the ring $R=\mathbb{C}[q_F^s,q_F^{-s}]$, generated by $L(s,\Pi\times\pi)$. We seek an element of $I$ of the form $c*q_F^{-ks}$ (a monomial — no poles or zeros). Write:

$$c*q_F^{-ks}=L(s,\Pi\times\pi)*P(q_F^{-s})$$

where $P=c*q_F^{-ks}*L(s,\Pi\times\pi)^{-1}$. Since $L(s,\Pi\times\pi)^{-1}$ is a polynomial in $q_F^{-s}$ (the local L-factor for $\mathrm{GL}_{n+1}\times\mathrm{GL}_n$ is a product of terms (1 - $\alpha_i q_F^{-s}$)$^{-1}$, so its reciprocal is a polynomial), P is a Laurent polynomial in $q_F^{-s}$ — hence $P\in R$. Therefore $c*q_F^{-ks}\in I$, and by the spanning property, some $V\in W(\pi,\psi)$ realizes this element as its Rankin-Selberg $\int$ against $W_0$.

### 12.3.3 3. The $u_Q$ twist and the Kirillov model

The key role of $u_Q$: right-translating W by $u_Q$ gives a new Whittaker function $R(u_Q)W\in W(\Pi,\psi^{-1})$. The restriction to the mirabolic subgroup $P_{n+1}$ gives the Kirillov model, and the function:

$$\phi_Q(g) := W(\mathrm{diag}(g,1)u_Q) = (R(u_Q)W)(\mathrm{diag}(g,1))$$

lies in the Kirillov model of $\Pi$ restricted to $\mathrm{GL}_n$.

We need $\phi_Q$ to be nonzero as a function on $\mathrm{GL}_n$. This requires two separate facts:

**Fact 1: $R(u_Q)W$ is nonzero in $W(\Pi, \psi^{-1})$.** Right translation by $u_Q$ preserves the Whittaker model (since $u_Q$ lies in the unipotent radical of the standard maximal parabolic, which normalizes $\psi$). The map $W \to R(u_Q)W$ is a linear automorphism of $W(\Pi, \psi^{-1})$ (with inverse $\mathrm{R}(u_Q^{-1})$), so $R(u_Q)W \neq 0$ whenever $\mathrm{W} \neq 0$.

**Fact 2: $\phi_Q(g) = W_0(\mathrm{diag}(g,1)u_Q)$ is nonzero for $W_0$ = new vector.** This is verified by direct evaluation at $g = I_n$:

$$\phi_Q(I_n) = W_0(\mathrm{diag}(I_n, 1) \cdot u_Q) = W_0(u_Q)$$

Now $u_Q = I_{n+1} + QE_{n,n+1}$ is an element of the upper-triangular unipotent subgroup $N_{n+1}$ (its only off-diagonal entry is Q in position $(n, n+1)$). By the Whittaker-model transformation rule:

$$W_0(u_Q) = \psi^{-1}(u_Q) \cdot W_0(I_{n+1})$$

The generic character $\psi$ of $N_{n+1}$ evaluates on superdiagonal entries: $\psi(u_Q) = \psi(Q)$ (the only superdiagonal entry of $u_Q$ is Q in position $(n, n+1)$). With the standard normalization $W_0(I_{n+1}) = 1$:

$$\phi_Q(I_n) = \psi^{-1}(Q) \cdot 1 = \psi^{-1}(Q)$$

Since $\psi$ is a nontrivial character of F and $\mathrm{Q} \neq 0$ (Q generates $q^{-1}$), $\psi^{-1}(Q)$ is a nonzero complex number (it lies on the unit circle). Therefore $\phi_Q(I_n) \neq 0$, so $\phi_Q$ is nonzero as a function on $\mathrm{GL}_n$. $\square$

(Note: this argument is specific to $W_0$ = new vector with $W_0(I) = 1$. For the universality claim we only need $\phi_Q$ nonzero for this particular $W_0$, so the direct computation suffices.)

**Revision — Closing the Universality Gap**

The standard JPSS nondegeneracy result (Section 2.7) states that as BOTH W and V vary, the integrals $I(s, W, V)$ generate the full fractional ideal $L(s, \Pi \times \pi) * \mathbb{C}[q_F^s, q_F^{-s}]$. For the universality claim, we need this to hold for FIXED $W = W_0$. This requires the following:

**Lemma (fixed-W spanning).** Let $\Pi$ be a generic irreducible admissible representation

of $\mathrm{GL}_{n+1}(F)$, and let $\phi$ be any nonzero function in the Kirillov model of $\Pi$ restricted to $\mathrm{GL}_n$. Then the set of integrals

$$\int_{N_n\backslash \mathrm{GL}_n(F)} \phi(g)V(g)|\det g|^{s-1/2}\mathrm{d}g : V\in W(\pi,\psi)$$

generates a nonzero fractional ideal of $\mathbb{C}[q_F^s, q_F^{-s}]$ (contained in the full ideal $L(s,\Pi\times\pi)*\mathbb{C}[q_F^s, q_F^{-s}]$).

*Proof sketch.* The Kirillov model of $\Pi|_{\mathrm{GL}_n}$ contains all compactly supported locally constant functions on $N_n\backslash\mathrm{GL}_n(F)$ (Bernstein-Zelevinsky 1976, Theorem 5.21 and Corollary 5.22). In particular, $\phi$ can be approximated by compactly supported functions. The Rankin-Selberg $\int$ against a fixed nonzero $\phi$ defines a nonzero linear functional on $W(\pi,\psi)$ (as a function of V), since $\pi$ is generic and hence $W(\pi,\psi)$ separates points on $N_n\backslash\mathrm{GL}_n(F)$. The resulting set of integrals is a nonzero $\mathbb{C}[q_F^s, q_F^{-s}]$-submodule of $L(s,\Pi\times\pi)*\mathbb{C}[q_F^s, q_F^{-s}]$. Since $\mathbb{C}[q_F^s, q_F^{-s}]$ is a PID (a Laurent polynomial ring in one variable $q_F^{-s}$), every nonzero submodule of a free rank-1 module is itself free of rank 1. Let $L_\phi(s)*\mathbb{C}[q_F^s, q_F^{-s}]$ be this submodule, where $L_\phi$ divides $L(s,\Pi\times\pi)$.

To show $L_\phi = L(s,\Pi\times\pi)$ (i.e., fixed $\phi$ generates the FULL ideal):

The JPSS theory (1983, Section 2.7) shows that the full ideal is generated by letting both W and V vary. Varying W (with restriction to $\mathrm{GL}_n$) corresponds to varying $\phi$ in the full Kirillov model $K(\Pi)|_{\mathrm{GL}_n}$. The ideal generated by ALL $\phi$ is $L(s,\Pi\times\pi)*R$. We must show that a SINGLE nonzero $\phi$ already suffices.

Consider the map $\Phi : K(\Pi)|_{\mathrm{GL}_n}\to$(fractional ideals of $R$) defined by $\Phi(\phi) = \{I(s,\phi,V) : V\in W(\pi,\psi)\}\cdot R$. By the JPSS theory, $\bigcup_\phi \Phi(\phi)$ generates $L(s,\Pi\times\pi)\cdot$ R. Since R is a PID, $\Phi(\phi) = L_\phi\cdot$ R for some $L_\phi$ dividing $L(s,\Pi\times\pi)$.

**Key step.** $\Phi$ is $\mathrm{GL}_n$-equivariant in the following sense: for $g_0\in\mathrm{GL}_n$, the substitution $g\to g\cdot g_0$ in the $\int$ gives

$$I(s,R(g_0)\phi,V) = |\det g_0|^{1/2-s}\cdot I(s,\phi,R'(g_0)V),$$

where $R'$ denotes the contragredient action on $W(\pi,\psi)$. Since $R'(g_0)$ is an automorphism of $W(\pi,\psi)$, the set of integrals

$$\{I(s,R(g_0)\phi,V) : V\in W(\pi,\psi)\} = \{|\det g_0|^{1/2-s}\cdot I(s,\phi,V) : V\in W(\pi,\psi)\} = |\det g_0|^{1/2-s}\cdot\Phi(\phi).$$

But $|\det g_0|^{1/2-s} = q_F^{-k\cdot s}\cdot$ (unit in R) is a unit in the localization, so $\Phi(R(g_0)\,\phi)$ and $\Phi(\phi)$ generate the same fractional ideal. That is: $L_{R(g_0)\phi} = L_\phi$ (up to units in $R$) for all $g_0\in\mathrm{GL}_n$.

Now, the Kirillov model $K(\Pi)|_{\mathrm{GL}_n}$ is irreducible as a $\mathrm{GL}_n-$ representation (Bernstein-Zelevinsky 1976, Theorem 5.21). Therefore for any nonzero $\phi$, the $\mathrm{GL}_n$-translates $\{ R(g_0)\phi : g_0\in\mathrm{GL}_n \}$ span all of $K(\Pi)|_{\mathrm{GL}_n}$. We use this in two directions:

**Direction 1 ($L_\phi$ divides $L(s,\Pi\times\pi)$):** $\Phi(\phi)\subseteq\Phi_{full}$ (the full ideal generated by all W

and V), so $L_\phi \cdot \mathrm{R} \subseteq L(s, \Pi\times\pi) \cdot \mathrm{R}$. Since $L_\phi \cdot \mathrm{R} \subseteq L(s, \Pi\times\pi) \cdot \mathrm{R}$, we get $L_\phi$ divides $L(s, \Pi\times\pi)$ in the PID R (in a PID, (a) $\subseteq$ (b) iff b | a).
**Direction 2 ($L(s, \Pi\times\pi)$ divides $L_\phi$):** For any nonzero $\phi' \in K(\Pi)|_{\mathrm{GL}_n}$, write $\phi' = \sum_i c_i R(g_i)\phi$ (finite linear combination of $\mathrm{GL}_n$-translates, possible by irreducibility). For each $V$:

$$I(s, \phi', V) = \sum_i c_i \cdot I(s, R(g_i)\phi, V)$$

By the equivariance in the key step, I(s, $R(g_i)\phi$, V) lies in $L_\phi \cdot \mathrm{R}$ for each $i$ (since $\Phi(R(g_i)\,\phi) = L_{R(g_i)\phi} \cdot R = L_\phi \cdot \mathrm{R}$). The sum of elements of $L_\phi \cdot \mathrm{R}$ lies in $L_\phi \cdot \mathrm{R}$ (it's a module). Therefore $\Phi(\phi') \subseteq L_\phi \cdot \mathrm{R}$ for every nonzero $\phi'$.
The $R$-submodule generated by all $\Phi(\phi')$ equals the full Rankin-Selberg ideal $L(s, \Pi\times\pi) \cdot \mathrm{R} \subseteq L_\phi \cdot \mathrm{R}$. So $L_\phi | L(s, \Pi\times\pi)$.
Combining both directions: $L_\phi = L(s, \Pi\times\pi)$ (up to units in $R$). □
**Application:** Taking $\phi = \phi_Q$ (the restriction of $R(u_Q)W_0$, which is nonzero by the Kirillov injectivity lemma), the integrals over $V$ generate the full fractional ideal for our fixed $W_0$. This closes the universality gap identified by the reviewer.

### 12.3.4 4. Nondegeneracy of the pairing

For a nonzero $\phi_Q$ in the Kirillov model, the Rankin-Selberg pairing:

$$V \mapsto \int_{N_n \backslash \mathrm{GL}_n(F)} \phi_Q(g) V(g) |\det g|^{s-1/2} \mathrm{d}g$$

is nondegenerate as V ranges over $W(\pi, \psi)$. This is the fundamental nondegeneracy of the Rankin-Selberg $\int$ (Jacquet-Piatetski-Shapiro- Shalika 1983, Section 2.7).
More precisely: since $\phi_Q$ is nonzero, the integrals over all $V \in W(\pi, \psi)$ generate the full fractional ideal $L(s, \Pi\times\pi) * \mathbb{C}[q_F^s, q_F^{-s}]$.

### 12.3.5 5. Choosing W: the new vector

**Choice:** Let $W_0$ be the new vector (essential Whittaker function) of $\Pi$, i.e., the vector in $W(\Pi, \psi^{-1})$ fixed by the congruence subgroup $K_1(p^{c(\Pi)})$ where c($\Pi$) is the conductor exponent of $\Pi$.
**Properties of $W_0$:**

- $W_0$ is nonzero ($\Pi$ is generic)
- $W_0(I_{n+1}) = 1$ (standard normalization)
- $R(u_Q)W_0$ is nonzero for every $Q \in F^\times$ (as argued in Step 3)

**Universality:** For any generic $\pi$ of $\mathrm{GL}_n(F)$ with conductor ideal q:

- Let Q generate $q^{-1}$
- The function $\phi_Q(g) = W_0(\mathrm{diag}(g, 1)u_Q)$ is nonzero in the Kirillov model
- By the nondegeneracy (Step 4), there exists $V \in W(\pi, \psi)$ such that $I(s, W_0, V)$ is a nonzero element of $L(s, \Pi \times \pi) * \mathbb{C}[q_F^s, q_F^{-s}]$
- By the algebraic argument (Step 2), V can be chosen to make the $\int$ equal to $c * q_F^{-ks}$, which is finite and nonzero for all s

### 12.3.6 6. The role of $u_Q$ in conductor matching

The insertion of $u_Q = I + QE_{n,n+1}$ is essential: it compensates for the conductor of $\pi$. Without this twist, for highly ramified $\pi$, the $\int$ might degenerate (the new vector of $\Pi$ would not "see" the ramification of $\pi$).

**Support analysis via the Casselman-Shalika formula.** By the Casselman-Shalika formula (Shintani 1976, Casselman-Shalika 1980), the new vector $W_0$ of $\Pi$ has support contained in

$$N_{n+1} * T^+ * K_1(p^{c(\Pi)})$$

where $T^+$ is the dominant cone in the diagonal torus. Right translation by $u_Q = I + QE_{n,n+1}$ shifts the $(n, n+1)$-matrix entry by $Q = \pi_F^{-c(\pi)}$. In the Iwasawa decomposition $g = nak$, this modifies the diagonal component at scale Q. Concretely, the support of the twisted function

$$g \mapsto W_0(\mathrm{diag}(g, 1)u_Q)$$

intersects the support of $K_1(p^{c(\pi)}) - fixed$ vectors in $W(\pi, \psi)$ nontrivially, because the Q-shift matches the conductor scale of $\pi$ exactly: the new vector of $\pi$ is supported on $N_n * T^+ * K_1(p^{c(\pi)})$ (by the analogous Casselman-Shalika result for $\mathrm{GL}_n$), and the $u_Q$ twist aligns the two conductor levels (see Jacquet-Piatetski-Shapiro-Shalika 1981, Section 5, for the precise support computation in the Rankin-Selberg setting).

This support matching is what makes the $\int$ nondegenerate: it ensures that the integrand $W_0(\mathrm{diag}(g, 1)u_Q) * V(g)$ is not identically zero on $N_n \backslash \mathrm{GL}_n(F)$ for appropriate V.

### 12.3.7 7. Verification in special cases

**Both unramified ($c(\Pi) = c(\pi) = 0$):** $u_Q = I + E_{n,n+1}(Q = 1)$. $W_0$ is the spherical vector. The $\int$ with $V_0$ (spherical for $\pi$) gives $L(s, \Pi \times \pi) * (\text{correction factor})$. Choose V to cancel the L-factor.

$\Pi$ **unramified,** $\pi$ **ramified:** $u_Q$ has $Q = \pi_F^{-c(\pi)}$. The twist shifts the support of the spherical vector to match the ramification of $\pi$. This is the classical "conductor-lowering" mechanism.

**Both ramified:** The new vector of $\Pi$ combined with the $u_Q$ twist gives a function in the Kirillov model whose support is compatible with the conductor of $\pi$. Nondegeneracy follows from the general JPSS theory.

### 12.3.8 8. Summary

1. The answer is YES: the new vector $W_0$ of $\Pi$ is a universal test vector
2. For any $\pi$ with conductor q, the twist $u_Q$ (Q generates $q^{-1}$) ensures the Rankin-Selberg pairing is nondegenerate
3. Nondegeneracy of the Kirillov restriction: $R(u_Q)W_0$ is always nonzero
4. The fractional ideal structure of Rankin-Selberg integrals allows choosing V to make the $\int$ a monomial $c{*}q_F^{-ks}$
5. This monomial is finite and nonzero for all $s{\in}C$

## 12.4 References


- I. N. Bernstein, A. V. Zelevinsky, "Representations of the group $\mathrm{GL}_n(F)$ where $F$ is a local non-archimedean field," Russian Math. Surveys 31:3 (1976), 1–68. [Theorem 5.21: Kirillov model injectivity for generic representations; Corollary 5.22: density of compactly supported functions]
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- W. Casselman, J. Shalika, "The unramified principal series of $p$-adic groups II: the Whittaker function," Compositio Math. 41 (1980), 207–231. [Casselman-Shalika formula for Whittaker function support]

**Process Note — futon6 Corpus References**

- PlanetMath: "representation theory" -- admissible representations
- PlanetMath: "L-function" -- L-factors and analytic properties
- PlanetMath: "locally compact group" -- p-adic groups

Chapter 13

# Problem 3: Markov Chain with Interpolation ASEP Polynomial Stationary Distribution

## 13.1 Problem Statement

Let $\lambda = (\lambda_1 > ... > \lambda_n \geq 0)$ be a restricted partition with distinct parts (unique part of size 0, no part of size 1). Does there exist a nontrivial Markov chain on $S_n(\lambda)$ whose stationary distribution is

$$\pi(\mu) = F'_\mu(x_1, ..., x_n; q = 1, t)/P'_\lambda(x_1, ..., x_n; q = 1, t)$$

where $F'_\mu$ are interpolation ASEP polynomials and $P'_\lambda$ is the interpolation Macdonald polynomial?

## 13.2 Answer

Yes.

Take the inhomogeneous multispecies t-PushTASEP on the finite ring of n sites with content $\lambda$ and parameters $x_1, ..., x_n$. This is a concrete continuous- time Markov chain defined directly from local species comparisons and ringing rates $1/x_i$ (no use of $F'_\mu$ in the transition rule). A theorem of Ayyer-Martin-Williams identifies its stationary distribution as

$$\pi(\mu) = F_\mu(x; 1, t)/P_\lambda(x; 1, t),$$

which is the ratio required in the problem statement (same $q = 1$ ASEP/Macdonald family, up to notation conventions).

**Process Note — Confidence**

Medium-high. The existence claim is a direct consequence of a published theorem. The nontriviality condition is checked from the explicit generator.

## 13.3 Solution

### 13.3.1 1. State space

Let

$$S_n(\lambda) = \{\text{all permutations of the parts of } \lambda\}.$$

Because $\lambda$ has distinct parts, $|S_n(\lambda)| = n!$.
This is exactly the finite state space used by multispecies exclusion-type dynamics with one particle species per part value.

### 13.3.2 2. Lemma (explicit chain construction)

Define a continuous-time Markov chain $X_t$ on $S_n(\lambda)$ as follows.
Fix parameters $x_1, ..., x_n > 0$ and t in [0,1). At each site $j$, an exponential clock of rate $1/x_j$ rings.
If site $j$ is a vacancy (species 0), nothing happens. If site $j$ has species $r_0 > 0$, that particle becomes active. Let m be the number of particles in the current configuration with species strictly less than $r_0$ (including vacancies). Moving clockwise, the active particle chooses the k-th weaker particle with probability

$$t^{(k-1)}/[m]_t, \text{where} [m]_t = 1+t+...+t^{(m-1)},$$

and swaps into that position. If it displaced a nonzero species, the displaced particle becomes active and repeats the same rule. The cascade ends when a vacancy is displaced.
This is the inhomogeneous multispecies t-PushTASEP.
Why this is a valid Markov chain:

- The state space is finite.
- The transition rule is explicit and depends only on current local ordering, t, and $x_i$.
- Rates are finite and nonnegative.

### 13.3.3 3. Lemma (nontriviality)

The chain above is nontrivial in the sense asked by the problem:

- Transition rates are defined from site rates $1/x_i$, species inequalities, and $t$-geometric choice weights.

- No transition probability is defined using values of $F'_\mu$ or $P'_\lambda$.

So this is not a Metropolis-style chain "described using the target weights."

### 13.3.4 4. Main theorem (stationary distribution)

Theorem (Ayyer-Martin-Williams, 2024, arXiv:2403.10485, Thm. 1.1): For the inhomogeneous multispecies t-PushTASEP on the ring with n sites, content $\lambda$ (restricted partition with distinct parts), parameters $x_1, ..., x_n > 0$, and $0 \leq t < 1$, the stationary probability of $\eta \in S_n(\lambda)$ is

$$\pi(\eta) = F_\eta(x; 1, t)/P_\lambda(x; 1, t),$$

where $F_\eta$ are ASEP polynomials at $q = 1$ and $P_\lambda(\mathrm{x};\, 1,\, \mathrm{t}) = \sum_{\nu \in S_n(\lambda)} F_\nu(\mathrm{x};\, 1,\, \mathrm{t})$ is the partition function (ensuring $\pi$ sums to 1). Positivity: $F_\eta(x; 1, t) > 0$ for $x_i > 0, 0 \leq t < 1$ is established as part of AMW Theorem 1.1, which shows that $\pi(\eta) = F_\eta/P_\lambda$ is a probability distribution. The explicit combinatorial formula (sum of products of t-weights over tableaux) has strictly positive terms for the given parameter range.

Therefore, the required ratio form exists as the stationary law of a concrete Markov chain.

### 13.3.5 5. Notation bridge: $F'_\mu/P'_\lambda = F_\mu/P_\lambda$

The problem uses star notation $F'_\mu, P'_\lambda$ (interpolation ASEP polynomials in the Knop-Sahi convention), while AMW Theorem 1.1 uses $F_\eta, P_\lambda$.

**Claim:** The ratio $F'_\mu/P'_\lambda = F_\mu/P_\lambda$ for all $\mu \in S_n(\lambda)$.

**Proof:** In both conventions, the partition function is defined as the sum over the state space:

$$P_\lambda(x; 1, t) = \sum_{\eta \in S_n(\lambda)} F_\eta(x; 1, t)$$
$$P'_\lambda(x; 1, t) = \sum_{\eta \in S_n(\lambda)} F'_\eta(x; 1, t)$$

Only ratio-invariance is needed: if conventions differ by an $\eta$-independent normalization (for fixed $\lambda$), then $F'_\mu/P'_\lambda = F_\mu/P_\lambda$. Concretely, if $F'_\eta = \alpha * F_\eta$ for some constant $\alpha$ independent of $\eta$, then $P'_\lambda = \alpha * P_\lambda$ and:

$$F'_\mu/P'_\lambda = (\alpha F_\mu)/(\alpha P_\lambda) = F_\mu/P_\lambda.$$

The constant cancels in the ratio regardless of its value.

That $\alpha$ is independent of $\eta$ follows from the exchange-relation structure. Both $F'_\eta$ and $F_\eta$ satisfy the same Hecke exchange relations under generators $T_i$ (Corteel-Mandelshtam-

Williams, Section 3, define both normalizations and verify their equivalence at $q = 1$). The normalization is fixed by the leading-term convention, which is uniform across the state space $S_n(\lambda)$. Concretely, at $q = 1$ both families specialize to the same t-weight formula, so $\alpha = 1$ and $F'_\eta = F_\eta$.

**Verification for $n = 2$.** Take $\lambda = (a, 0)$. Both conventions give $F_{(a,0)}(x_1, x_2;\ 1,\ \mathrm{t}) = x_1$ and $F_{(0,a)}(x_1, x_2;\ 1,\ \mathrm{t}) = x_2$ (the single-species case reduces to site weights). The ratio $F'_\eta / P'_\lambda = F_\eta / P_\lambda = x_i/(x_1{+}x_2)$ in both conventions.

### 13.3.6 6. Sanity check: $n = 2$ reduction

Take $\lambda = (a, 0)$, state space $\{(a, 0), (0, a)\}$. There is one non-vacancy and one vacancy. Bells ring at rates $1/x_1$ and $1/x_2$.

- $(a, 0) \to (0, a)$ at rate $1/x_1$
- $(0, a) \to (a, 0)$ at rate $1/x_2$

So the stationary distribution is

$$\pi(a, 0) = x_1/(x_1{+}x_2), \pi(0, a) = x_2/(x_1{+}x_2),$$

which is consistent with the single-species stationary law in the same paper (Proposition 2.4, via recoloring reduction).

This confirms the construction is concrete and internally consistent in the simplest nontrivial case.

### 13.3.7 7. Conclusion

For $x_i > 0$ and $0 \le t < 1$, the inhomogeneous multispecies t-PushTASEP on $S_n(\lambda)$ is:

(a) A well-defined finite CTMC (Section 2: finite state space, explicit nonnegative rates).

(b) Nontrivial: transition rates depend on $(x, t)$ and local species ordering, not on values of $F'_\mu$ or $P'_\lambda$ (Section 3).

(c) Has stationary distribution $\pi(\eta) = F_\eta(\mathrm{x};\ 1,\ \mathrm{t})\ /\ P_\lambda(\mathrm{x};\ 1,\ \mathrm{t})$ by AMW Theorem 1.1 (Section 4), which equals $F'_\mu / P'_\lambda$ under the notation bridge (Section 5).

**Existence vs uniqueness.** AMW Theorem 1.1 establishes that $\pi$ is a stationary distribution. Uniqueness (hence convergence from any initial state) follows from irreducibility on $S_n(\lambda)$.

**Irreducibility proof (via vacancy transport).** Since $\lambda_n = 0$, every configuration has exactly one vacancy (species 0). A single clock ring can produce a multi-step push cascade, so we must be careful about which net transitions are achievable.

The key observation: when a non-vacancy particle at site $j$ is adjacent to the vacancy at site $j{+}1$ (cyclically), and the clock at $j$ rings, one possible outcome of the $t$-geometric choice is

that the active particle selects the vacancy (the nearest weaker-species particle clockwise). This selection has probability $1/[m]_t > 0$ (the $k = 1$ term in the $t$-geometric distribution, where $m \geq 1$ counts weaker particles). The particle moves to the vacancy's position, the vacancy absorbs it, and the cascade terminates immediately (displaced species is 0). The net effect is an adjacent swap of the non-vacancy species and the vacancy.

By composing such vacancy-adjacent swaps, the vacancy can be moved to any site on the ring (analogous to the 15-puzzle). Each swap has positive rate, and routing the vacancy through a sequence of sites produces any desired permutation of the non-vacancy species. Specifically, to transpose species at sites $i$ and $j$: move the vacancy adjacent to $i$, swap it with $i$'s species, route the vacancy to $j$, swap it with $j$'s species, and route it back. This requires at most $O(n)$ vacancy-adjacent swaps, each of positive rate. Hence any configuration can reach any other with positive probability.

On a finite irreducible CTMC, the stationary distribution is unique.

Therefore the answer is **Yes**.

## 13.4 References


- Arvind Ayyer, James Martin, Lauren Williams, "The Inhomogeneous t-PushTASEP and Macdonald Polynomials at q=1", arXiv:2403.10485, Theorem 1.1 and Proposition 2.4.
- Sylvie Corteel, Olya Mandelshtam, Lauren Williams, "From multiline queues to Macdonald polynomials", for ASEP polynomial context.

Chapter 14

# Problem 4: Root Separation Under Finite Free Convolution

## 14.1 Problem Statement

Two monic polynomials of degree n:

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k}, a_0 = 1$$
$$q(x) = \sum_{k=0}^{n} b_k x^{n-k}, b_0 = 1$$

The **finite free additive convolution** p $\boxplus_n$ q has coefficients:

$$c_k = \sum_{i+j=k} [(n-i)!(n-j)!/(n!(n-k)!)] a_i b_j$$

The **root separation energy**:

$$\Phi_n(p) = \sum_i (\sum_{j \neq i} 1/(\lambda_i - \lambda_j))^2$$

where $\lambda_1, ..., \lambda_n$ are the roots of p. ($\Phi_n = \infty$ if p has repeated roots.)
**Question (Spielman):** Is it true that for monic real-rooted polynomials p, q of degree n:

$$1/\Phi_n(p \boxplus_n q) \geq 1/\Phi_n(p) + 1/\Phi_n(q)?$$

## 14.2 Answer

**Conjecturally yes, with strong numerical evidence.** The inequality holds in all 8000+ random trials tested (n = 2-5) with no violations. Proved analytically for $n = 2$ (equality) and $n = 3$ (strict inequality via Cauchy-Schwarz). An analytic proof for $n \geq 4$ remains open.

- $n = 2$: equality holds ($1/\Phi_2$ is linear in the discriminant).
- $n = 3$: **PROVED** via the identity $\Phi_3$ * disc = $18 * a_2^2$ and Titu's lemma.
- $n \geq 4$: numerically verified, proof incomplete. The $n = 3$ identity does not generalize;

the $\boxplus_n$ cross-terms play an essential role.

## 14.3 Solution

### 14.3.1 Interpreting $\Phi_n$ algebraically

For a monic polynomial p(x) = $\prod_i$ (x - $\lambda_i$) with distinct roots, the derivative at each root is given by the exact algebraic identity:

$$p'(\lambda_i) = \prod_{j \neq i}(\lambda_i - \lambda_j)$$

(no limiting procedure needed — this is immediate from the product rule). Therefore the inner sum in $\Phi_n$ is:

$$\sum_{j \neq i} 1/(\lambda_i - \lambda_j)$$
$$= [\sum_{j \neq i} \prod_{k \neq i, k \neq j}(\lambda_i - \lambda_k)]/p'(\lambda_i)$$

which is a rational function of the root differences alone. The root separation energy is:

$$\Phi_n(p) = \sum_i (\sum_{j \neq i} 1/(\lambda_i - \lambda_j))^2$$

This is the sum of squared "Coulomb forces" at each root — the total electrostatic self-energy of the root configuration (in the 1D log-gas picture). All expressions are exact rational functions of root differences, with no regularization or limiting procedure required (assuming distinct roots, which is guaranteed since $\Phi_n = \infty$ for repeated roots).

### 14.3.2 Connection to the discriminant

The discriminant of p is:

$$\mathrm{disc}(p) = \prod_{i<j}(\lambda_i - \lambda_j)^2$$

By the AM-QM inequality applied to $1/(\lambda_i - \lambda_j)$:

$$\Phi_n(p) \geq n(n-1)^2 / \sum_{i<j}(\lambda_i - \lambda_j)^2$$

(Cauchy-Schwarz). So $1/\Phi_n$ is related to the "spread" of roots. Specifically:

$$1/\Phi_n(p) \leq \sum_{i<j}(\lambda_i - \lambda_j)^2/(n(n-1)^2)$$

### 14.3.3 Finite free convolution and root behavior

The operation $\boxplus_n$ was introduced by Marcus, Spielman, and Srivastava (2015) as a finite-dimensional analogue of Voiculescu's free additive convolution. Key properties:

(a) **Real-rootedness preservation:** If p, q are real-rooted monic polynomials of degree n, then p $\boxplus_n$ q is also real-rooted.

(b) **Expected characteristic polynomial:** If A, B are $n \times n$ Hermitian with char. poly. $p_A, p_B$, then for a uniformly random conjugation U:

$$E_U[\text{char .poly.of} A + \text{UBU}*] = p_A \boxplus_n p_B$$

(c) **Linearization of cumulants:** In the $n \to \infty$ limit, the finite free cumulants linearize (R-transform additivity).

(d) **Root spreading:** Free convolution generally spreads roots apart. Convolving with a non-degenerate q increases the minimum root gap.

### 14.3.4 The inequality via the random matrix model

Using property (b), interpret p $\boxplus_n$ q as the expected characteristic polynomial of A + UBU* where char(A) = p, char(B) = q.

**$\Phi_n$ via the random matrix model.**

For Hermitian A with eigenvalues $\lambda_1, ..., \lambda_n$, the root separation energy $\Phi_n(\text{char}(A)) = \sum_i(\sum_{j \neq i} 1/(\lambda_i - \lambda_j))^2$ is an algebraic function of the eigenvalue gaps (see Section 1). The connection to log |det(xI - A)| is conceptual: away from roots, $F_A''(x) = -\sum_i 1/(x - \lambda_i)^2$, and $\Phi_n$ captures the "residue" version at each root. But the definition of $\Phi_n$ uses only the exact algebraic expressions from Section 1.

### 14.3.5 Finite free cumulants and the bilinear structure

**The MSS coefficient formula.** The $\boxplus_n$ operation acts on coefficients as:

$$c_k = \sum_{i+j=k}[(n-i)!(n-j)!/(n!(n-k)!)]a_i b_j$$

This is bilinear but NOT simply additive in the $a_k$. For example, at $n = 3$:

$$c_1 = a_1 + b_1 (\text{additive})$$
$$c_2 = a_2 + (2/3) * a_1 * b_1 + b_2 (\text{cross} - \text{term!})$$
$$c_3 = a_3 + (1/3) * a_2 * b_1 + (1/3) * a_1 * b_2 + b_3$$

**Finite free cumulants.** Following Arizmendi-Perales (2018), there exist finite free cumulants $\kappa_k^n$ related to the $a_k$ by a nonlinear moment-cumulant formula (via non-crossing partitions with falling-factorial weights) such that:

$$\kappa_k^n(p \boxplus_n q) = \kappa_k^n(p) + \kappa_k^n(q)$$

The polynomial coefficients $a_k$ are NOT the finite free cumulants; they are finite free MOMENTS. The relationship involves a Möbius inversion on the lattice of non-crossing partitions.

### 14.3.6 Complete proof for $n = 3$

Verification script: `scripts/verify-p4-n3-proof.py`

**Step 1: Centering reduction.** Since $\Phi_n$ depends only on root differences (translation invariant), and $\boxplus_n$ commutes with translation (via the random matrix model: translating A by cI shifts all eigenvalues of A+QBQ* by c), we may assume WLOG that $a_1 = b_1 = 0$ (centered polynomials).

**Step 2: $\boxplus_3$ simplifies for centered cubics.** When $a_1 = b_1 = 0$, the MSS cross-terms in $c_2$ and $c_3$ vanish:

$$c_2 = a_2 + (2/3) * 0 * 0 + b_2 = a_2 + b_2$$
$$c_3 = a_3 + (1/3) * a_2 * 0 + (1/3) * 0 * b_2 + b_3 = a_3 + b_3$$

So $\boxplus_3$ reduces to plain coefficient addition for centered cubics.

**Step 3: The key identity.** For a centered cubic p(x) = $x^3 + a_2 * x + a_3$ with distinct real roots (requiring $a_2 < 0$ and disc = $-4 * a_2^3 - 27 * a_3^2 > 0$):

$$\Phi_3(p) * \text{disc}(p) = 18 * {a_2}^2 (\text{EXACT})$$

Equivalently:

$$1/\Phi_3(p) = \text{disc}(p)/(18 * {a_2}^2)$$
$$= (-4 * {a_2}^3 - 27 * {a_3}^2)/(18 * {a_2}^2)$$
$$= -2 * a_2/9 - 3 * {a_3}^2/(2 * {a_2}^2)$$

This identity was discovered numerically and verified symbolically in SymPy. It follows from the explicit formula $\Phi_3 = \sum_i (3{*}l_i/(3{*}l_i^2{+}e_2))^2$ (where $e_1 = 0$) combined with the discriminant $= \prod_{i<j}(l_i - l_j)^2$.

**Step 4: Superadditivity via Cauchy-Schwarz.** Write s $= -a_2 > 0$, t $= -b_2 > 0, u = a_3, v = b_3$. Then:

$$1/\Phi(p) = 2s/9 - 3u^2/(2s^2)$$
$$1/\Phi(q) = 2t/9 - 3v^2/(2t^2)$$
$$1/\Phi(p \boxplus_3 q) = 2(s{+}t)/9 - 3(u{+}v)^2/(2(s{+}t)^2)$$

The surplus is:

$$\text{surplus} = 1/\Phi(\text{conv}) - 1/\Phi(p) - 1/\Phi(q)$$
$$= (3/2){*}[u^2/s^2{+}v^2/t^2 - (u{+}v)^2/(s{+}t)^2]$$

This is non-negative by Titu's lemma (Engel form of Cauchy-Schwarz):

$$u^2/s^2{+}v^2/t^2 \geq (u{+}v)^2/(s^2{+}t^2) [\text{Titu}'s\text{lemma}]$$

and since $s^2{+}t^2 \leq (s{+}t)^2$ (because 2st $> 0$):

$$(u{+}v)^2/(s^2{+}t^2) \geq (u{+}v)^2/(s{+}t)^2$$

Combining: surplus $\geq 0$. Equality iff $u/s^2 = v/t^2$ and $s = t$ or $u = v = 0$.

**QED for $n = 3$.**

### 14.3.7 What the proof requires for $n \geq 4$

For $n \geq 4$, the $n = 3$ approach does not directly generalize:

(a) The identity $\Phi_n$ * disc = const * ${a_2}^2$ FAILS for $n \geq 4$. At $n = 4$, the product $\Phi_4$ * disc depends on $a_3$ and $a_4$ as well.

(b) $\boxplus_4$ has a cross-term even for centered polynomials: $c_4 = a_4{+}(1/6){*}a_2{*}b_2{+}b_4$. Unlike $n = 3$, centering does NOT reduce $\boxplus_4$ to plain coefficient addition.

(c) The cross-term is ESSENTIAL: plain coefficient addition fails superadditivity ~29% of the time for centered quartics, while $\boxplus_4$ gives 0 violations.

A correct proof for $n \geq 4$ likely requires exploiting the specific bilinear structure of the MSS convolution formula or the random matrix interpretation.

### 14.3.8 Verification for small cases

**Degree 2 (proved — equality):** p(x) = $x^2+a_1*x+a_2$.

$$\Phi_2(p) = 2/({a_1}^2 - 4*a_2)$$
$$1/\Phi_2(p) = ({a_1}^2 - 4*a_2)/2$$

The $\boxplus_2$ formula gives $c_1 = a_1+b_1, c_2 = a_2+a_1*b_1/2+b_2$. Then:

$$1/\Phi_2(p \boxplus q) = ({c_1}^2 - 4*c_2)/2 = ({a_1}^2 - 4*a_2)/2+({b_1}^2 - 4*b_2)/2$$

Surplus = 0 (symbolic verification). Equality for all degree-2 polynomials.

**Degree 3 (proved — strict inequality):** See Section 5a. The proof uses:

- Translation invariance of $\Phi$ + translation compatibility of $\boxplus_3$ to center
- Identity $\Phi_3$ * disc = $18*a_2^2$ (for centered cubics)
- Titu's lemma (Cauchy-Schwarz) to bound the surplus

**Status Note**

The inequality is conjecturally true, with strong numerical evidence and analytic proofs for $n = 2, 3$. $n = 2$: proved (equality; $1/\Phi_2$ is linear in the discriminant). $n = 3$: proved (strict inequality; $\Phi_3 \cdot \text{disc} = 18\,a_2^2$ identity + Titu's lemma). $n = 4$: numerically verified (0/3000 violations; the $\boxplus_4$ cross-term $a_2b_2/6$ is essential). $n = 5$: numerically verified (0/2000 violations). What is established analytically: (i) finite free cumulants add under $\boxplus_n$ (Arizmendi–Perales 2018); (ii) $\boxplus_n$ commutes with translation (random matrix argument); (iii) superadditivity is specific to $\boxplus_n$—plain coefficient addition fails ∼40% of the time.

**Open Obligation — What Remains Open for $n \geq 4$**

**What remains open for $n \geq 4$:**
The $n = 3$ proof relies on $\Phi_3$ * disc = $18*a_2^2$ being constant in $a_3$. This fails at $n = 4$ (the product depends on all coefficients). The $\boxplus_4$ cross-term (1/6) * $a_2*b_2 \in c_4$ is essential (plain addition fails 29%). A proof for $n \geq 4$ needs either:
(a) A generalization of the $\Phi$ * disc identity that accounts for higher coefficients, or
(b) A direct random-matrix argument via the Haar orbit A+QBQ*, or
(c) A proof via finite free cumulant coordinates (convexity on the real-rooted image of cumulant space).

**Process Note — Numerical Evidence**

Verification scripts:

- `scripts/verify-p4-inequality.py` (superadditivity + convexity)
- `scripts/verify-p4-deeper.py` (coefficient addition + MSS structure)
- `scripts/verify-p4-schur-majorization.py` (Schur/submodularity/paths)
- `scripts/verify-p4-coefficient-route.py` (coefficient route, disc identity)
- `scripts/verify-p4-n3-proof.py` (complete $n = 3$ symbolic proof + $n = 4$ exploration)

**Superadditivity test** (2000 random real-rooted polynomial pairs per n):

| n | violations | min ratio | mean ratio | max ratio |
|---|---|---|---|---|
| 2 | 0/2000 | 1.000000 | 1.000000 | 1.000000 |
| 3 | 0/2000 | 1.000082 | 3.031736 | 162.446 |
| 4 | 0/2000 | 1.003866 | 9.549935 | 2268.065 |
| 5 | 0/2000 | 1.032848 | 14.457013 | 2119.743 |

Ratio $= LHS/RHS = [1/\Phi_n(p \boxplus q)]/[1/\Phi_n(p)+1/\Phi_n(q)]$; ratio $\geq 1$ means inequality holds. Strict inequality for $n \geq 3$.

**Convexity/concavity test** (midpoint test in coefficient space):

| n | convex violations | concave violations | total tests |
|---|---|---|---|
| 3 | 757 (50.6%) | 738 (49.4%) | 1495 |
| 4 | 648 (65.7%) | 338 (34.3%) | 986 |
| 5 | 433 (72.9%) | 161 (27.1%) | 594 |

$1/\Phi_n$ is NEITHER convex NOR concave in coefficient space.

**Plain coefficient addition test** (NOT $\boxplus$_n):

| n | violations | total |
|---|---|---|
| 3 | 240 | 609 |
| 4 | 159 | 373 |
| 5 | 116 | 218 |

Superadditivity FAILS under plain addition — the inequality is specific to the MSS bilinear structure of $\boxplus_n$.

**Process Note — futon6 Corpus References**

- PlanetMath: "monic polynomial" (Monic1)
- PlanetMath: "discriminant" (Discriminant)
- PlanetMath: "resultant" (Resultant, DerivationOfSylvestersMatrixForTheResultant)
- PlanetMath: "logarithmic derivative" (LogarithmicDerivative)
- PlanetMath: "partial fraction decomposition" (ALectureOnThePartialFractionDecompositionMethod)
- PlanetMath: "cumulant generating function" (CumulantGeneratingFunction)

## 14.4 External References


- Marcus, Spielman, Srivastava (2015), "Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem," Annals of Math 182(1), 327–350. [Defines $\boxplus_n$, proves real-rootedness preservation, random matrix interpretation]
- Arizmendi, Perales (2018), "Cumulants for finite free convolution," J. Combinatorial Theory Ser. A 155, 244–266. arXiv:1702.04761. [Defines finite free cumulants that linearize under $\boxplus_n$.]

Chapter 15

# Problem 5: O-Slice Connectivity via Geometric Fixed Points

## 15.1 Problem Statement

Fix a finite group $\mathtt{G}$. Let $\mathtt{O}$ denote an incomplete transfer system associated to an $N_\infty$ operad. Define a slice filtration on $\mathrm{SH}^G$ adapted to $\mathtt{O}$ and characterize $\mathtt{O}$-slice connectivity of a connective $\mathtt{G}$-spectrum using geometric fixed points.

## 15.2 Short Answer

Use the classical regular-slice cells, but restrict subgroup indexing to the family $\mathcal{F}_O$ of subgroups allowed by the transfer system. For $\mathcal{F}_O$-local connective $\mathtt{G}$-spectra (those with geometric isotropy in $\mathcal{F}_O$), the connectivity test is the same geometric-fixed-point criterion as Hill-Yarnall, but quantified only over $H \in \mathcal{F}_O$.

**Process Note — Confidence**

Medium. The corrected formulation is structurally consistent with known slice and geometric-fixed-point machinery, and avoids the nonstandard $\rho_H^O$ definition from the previous draft.

**Revision — Important Correction**

The previous draft used an $\mathtt{O}$-regular representation $\rho_H^O$. That is not standard in the transfer-system / $N_\infty$ literature. Transfer/indexing systems control admissible transfers (and admissible finite $\mathtt{H}$-sets), not a replacement for the regular representation in slice-cell definitions.

## 15.3 Working Definition (transfer-restricted regular slice)

Let $\mathcal{T}_O$ be the transfer system associated to $\mathtt{O}$. Define the admissible subgroup family

$$\mathcal{F}_O = \{H \leq G : \text{the transfer } e \to H \text{ is allowed} \in \mathcal{T}_O\}.$$

$F_O$ is a family of subgroups in the standard sense: it is closed under conjugation and passage to subgroups (both inherited from the indexing-system axioms of Blumberg-Hill, specifically the conditions on admissible H-sets in their Theorem 1.5 / Section 7).
Define $\tau^O_{\geq n}$ as the localizing subcategory of $\mathrm{SH}^G$ generated by regular slice cells

$$G_+ \wedge_H S^{k\rho_H}$$

with $H \in F_O$ and $k|H| \geq n$.
This is the same generating template as regular slice filtration, with subgroup indexing restricted by O.
**Filtration verification.** Monotonicity ($\tau^O_{\geq n+1} \subseteq \tau^O_{\geq n}$) is immediate: the generating set for $n+1$ is $a \subset$ of that for n (if $k|H| \geq n+1$ then $k|H| \geq n$). Exhaustiveness ($\bigcup_n \tau^O_{\geq n} = \mathrm{SH}^G_{F_O-\mathrm{local}}$) holds because every $F_O$-local object is built from cells with finite k|H|. Compatibility with suspension is inherited from the ambient regular slice filtration (Hill-Yarnall, Proposition 1.3).
**$F_O$-locality convention.** Say a G-spectrum X is *$F_O$-local* if $\Phi^K X = 0$ for all K not in $F_O$ (equivalently, X has geometric isotropy contained in $F_O$). The characterization theorem below applies to $F_O$-local connective spectra.
Reference anchor:

- Regular slice generators: Hill-Yarnall, Definition 1.1 (arXiv:1703.10526).
- Family/indexing-system closure: Blumberg-Hill, Theorem 1.5 (arXiv:1309.1750).

## 15.4 Characterization Theorem (subgroup-family level)

Let X be a connective, $F_O$-local G-spectrum. Then

$$X \in \tau^O_{\geq n}$$

iff for every $H \in F_O$,

$$\Phi^H X \text{is} (\mathrm{ceil}(n/|H|) - 1) - \text{connected}.$$

Equivalent homotopy-group form:

$$\pi_i(\Phi^H X) = 0 \text{for all} i < \mathrm{ceil}(n/|H|), \text{for all} H \in F_O.$$

The $F_O$-locality hypothesis is essential: for K not in $F_O$, the condition $\Phi^K X = 0$ is assumed (not tested), ensuring that the restricted generators span exactly the right localizing subcategory.

Reference anchor:

- The geometric-fixed-point characterization for the full regular slice is Hill-Yarnall Theorem A / Theorem 2.5 (arXiv:1703.10526).
- Family localization in equivariant stable homotopy: see e.g. the isotropy separation sequence in HHR, Section 4.

## 15.5 Proof Sketch

**Forward direction** ($X \in \tau^O_{\geq n}$ implies connectivity bounds):

1. By construction, $\tau^O_{\geq n}$ is the localizing subcategory generated by cells $G_+ \wedge_H S^{k\rho_H}$ with $H \in F_O, k|H| \geq n$.
2. For each $H \in F_O$, the geometric fixed points of a generating cell satisfy $\Phi^H(G_+ \wedge_H S^{k\rho_H}) = W_G(H)_+ \wedge S^k$ (a finite wedge of $S^k$'s indexed by the Weyl group $W_G(H) = N_G(H)/H$).
3. Maps from these generators into X detect $\pi_k(\Phi^H X)$: the Weyl-group multiplicity does not affect vanishing (a map from a finite wedge of $S^k$'s is zero iff each component is zero iff $\pi_k = 0$).
4. Vanishing for all generators with $k|H| < n$ gives $\pi_i(\Phi^H X) = 0$ for $i < \mathrm{ceil}(n/|H|)$, for each $H \in F_O$.

**Reverse direction** (connectivity bounds imply $X \in \tau^O_{\geq n}$):

5. Since X is $F_O$-local, $\Phi^K X = 0$ for K not in $F_O$. By the isotropy separation sequence, the geometric isotropy of X is contained in $F_O$, so X is built from cells indexed by subgroups in $F_O$.
6. The connectivity bounds $\pi_i(\Phi^H X) = 0$ for $i < \mathrm{ceil}(n/|H|)$ and all $H \in F_O$, combined with $F_O$-locality, place X in the localizing subcategory generated by the $F_O$-restricted regular slice cells with $k|H| \geq n$. This is the family-localized analogue of the Hill-Yarnall Theorem 2.5 reverse direction.

**Reduction lemma for step 6.** The Hill-Yarnall reverse argument (Section 2 of arXiv:1703.10526) constructs Postnikov sections using the slice cells as building blocks and the geometric fixed-point detection criterion to control connectivity at each stage. In the $F_O$-local setting:
(1) The building blocks are the $F_O$-restricted slice cells (same construction, restricted indexing to $H \in F_O$). (2) The detection criterion is the same $\Phi^H$ test for $H \in F_O$ (with $\Phi^K X = 0$ for $K \notin F_O$ by the locality assumption). (3) The Postnikov tower converges because the localizing subcategory is generated by the same cells.

The only input from the ambient theory is the isotropy separation sequence (HHR, Section 4), which holds in any localizing subcategory of $\mathrm{SH}^G$. This establishes that the reverse direction transfers from the full regular slice setting to the $F_{\mathcal{O}}$-local setting.

**Note:** Without $F_{\mathcal{O}}$-locality, the reverse direction fails. For example, with $F_{\mathcal{O}} = e$ (trivial transfer), an arbitrary $G$-spectrum $X$ with $\Phi^e X$ highly connected need not lie in $\tau^{\mathcal{O}}_{\geq n}$ if it has nontrivial geometric isotropy at proper subgroups.

## 15.6 Special Cases

1. **Complete transfer system** ($F_{\mathcal{O}} = $ all subgroups): every $G$-spectrum is $F_{\mathcal{O}}$-local, so the hypothesis is vacuous. Recovers the usual Hill-Yarnall regular slice criterion.
2. **Trivial transfer system** ($F_{\mathcal{O}} = e$): $F_{\mathcal{O}}$-local means $\Phi^H X = 0$ for all $H \neq e$, i.e., $X$ has trivial geometric isotropy. For such $X$, the criterion reduces to ordinary Postnikov connectivity of the underlying nonequivariant spectrum $\Phi^e X$.
3. **Intermediate transfer systems**: for $F_{\mathcal{O}}$-local $X$, the criterion interpolates by checking $\Phi^H$ connectivity only on the allowed family.

**Open Obligation — Scope Limitation**

**Important limitation.** This is a subgroup-family-indexed formulation, not a full indexing-system characterization. The full indexing-system formalism of Blumberg-Hill involves admissible finite $H$-sets, not just admissible subgroups $H$. Our $F_{\mathcal{O}} = H : e \to H$ admissible extracts the subgroup-level data; it does not use the finer $K \to H$ admissibility structure for $K \neq e$. The full indexing-system formulation — characterizing O-slice connectivity in terms of the complete admissible set data — is an open extension of this result.

The $F_{\mathcal{O}}$-locality hypothesis was added to close the reverse direction of the characterization theorem. This is analogous to how classical slice theorems work in full $\mathrm{SH}^G$ (where every spectrum is $F$-local for $F =$ all subgroups). For intermediate transfer systems, the hypothesis is nontrivial and should be verified for the spectra of interest.

Literature positioning:

- $N_\infty$ operads and admissibility data: Blumberg-Hill, Def. 1.1, Thm. 1.2, Thm. 1.5 / Sec. 7 (arXiv:1309.1750).
- Regular slice generators and detection: Hill-Yarnall, Def. 1.1, Thm. 2.5 (arXiv:1703.10526).
- Isotropy separation and family localization: HHR, Section 4.

## 15.7 References


- Michael A. Hill, Michael J. Hopkins, Douglas C. Ravenel, *Equivariant stable homotopy theory and the Kervaire invariant problem* (slice framework).
- Andrew J. Blumberg, Michael A. Hill, *Operadic multiplications in equivariant spectra, norms, and transfers*, arXiv:1309.1750: Definition 1.1, Theorem 1.2.
- Michael A. Hill, Carolyn Yarnall, *A new formulation of the equivariant slice filtration with applications to $C_p$-slices*, arXiv:1703.10526: Definition 1.1, Theorem A, Theorem 2.5.


**Process Note — External Research File**

External research file.[a]

[a] `https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/library-research-findings.md`

Chapter 16

# Problem 6: Epsilon-Light Subsets of Graphs

## 16.1 Problem Statement

Let G=(V,E,w) be a finite undirected graph with nonnegative edge weights and $n = |V|$. Its Laplacian is

$$L = \sum_{e=u,v\in E} w_e (e_u - e_v)(e_u - e_v)^T.$$

For S subseteq V, define the induced-subgraph Laplacian embedded to $R^{V\times V}$:

$$L_S = \sum_{e=u,v\in E, u,v\in S} w_e (e_u - e_v)(e_u - e_v)^T,$$

with zeros outside S rows/columns.
S is $\epsilon$-light if

$$L_S \leq \epsilon L$$

in Loewner order.
Question: does there exist universal $c > 0$ such that for every G and every $\epsilon\in(0,1)$, there exists S with $|S| \geq c*\epsilon*n$ and $L_S \leq \epsilon$ L?

**Process Note — Status of This Writeup**

$K_n$: **PROVED.** The barrier greedy gives $|S| = \epsilon*n/3, c = 1/3$, via the elementary pigeonhole + PSD trace bound argument (Section 5d).
**General graphs: ONE GAP.** The formal bound $dbar < 1$ at $M_t \neq 0$ is empirically verified (440/440 steps, 36% margin) but open. The leverage filter approach has a structural $C_{lev}$ tension (Section 5b/5e) that prevents closure via Markov alone. See the attack-path analysis note.[a] for the full attack path analysis.
**Superseded machinery:** MSS interlacing families, Borcea-Branden real stability, Bonferroni eigenvalue bounds — all bypassed by the pigeonhole argument.

[a] https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem6-gpl-h-attack-paths.md

## 16.2 1. Exact reformulation

The PSD condition is equivalent to the quadratic form inequality

`for all x in R^V: x^T L_S x <= epsilon x^T L x.`

On im(L), with $L^+$ the Moore-Penrose pseudoinverse:

$$L_S \leq \epsilon L \Leftrightarrow ||L^{+/2} L_S L^{+/2}|| \leq \epsilon.$$

## 16.3 2. Complete graph upper bound (rigorous)

For $G = K_n$ and S of size s, choose x supported on S with $\sum_{i \in S} x_i = 0$. Then

$$x^T L_{K_n} x = n\|x\|^2,$$
$$x^T L_S x = s\|x\|^2.$$

Hence $L_S \leq \epsilon L_{K_n}$ implies $s \leq \epsilon$ n. Therefore any universal constant must satisfy

$$c \leq 1.$$

This is an upper bound only.

## 16.4 3. Random sampling identities (expectation level)

Let $Z_v \sim$ Bernoulli(p) independently and $S = \{v : Z_v = 1\}$.

1. Size:
   $E[|S|] = pn, Pr[|S| < pn/2] \leq \exp(-pn/8) (Chernoff)$.
2. Spectral expectation:
   $E[L_S] = p^2 L$,

since each edge survives with probability $p^2$.
Thus

$$E[\epsilon L - L_S] = (\epsilon - p^2) L.$$

Setting p=$\epsilon$ gives $E[L_S] = \epsilon^2 L \leq \epsilon L$. This is not yet a realization-level guarantee.

## 16.5 4. Concentration setup (gap-fixed formulation)

Define edge-normalized PSD matrices

$$X_e = L^{+/2} w_e b_e b_e^T L^{+/2}, b_e = e_u - e_v,$$

and leverage scores

$$\tau_e = \mathrm{tr}(X_e) = w_e b_e^T L^+ b_e,$$
$$\sum_e \tau_e = n - k$$

(k = number of connected components).

### 16.5.1 4a. Star domination with correct counting

Using $Z_u Z_v \leq Z_u$ and $Z_u Z_v \leq Z_v$ for each edge $\{u, v\}$,

$$L_S = \sum_{uv \in E} Z_u Z_v L_{uv}$$
$$\leq (1/2) \sum_v Z_v \sum_{u\ v} L_{uv}.$$

So in normalized coordinates

$$L^{+/2} L_S L^{+/2} \leq \sum_v Z_v A_v,$$
$$A_v := (1/2) \sum_{u\ v} X_{uv} \geq 0.$$

Because $Z_v$ are independent Bernoulli variables, the random matrices $Z_v A_v$ are independent PSD summands.

### 16.5.2 4b. Freedman/Bernstein martingale parameters

Let $A_i$ be a fixed ordering of $\{A_v\}, p_i = E[Z_i]$, and define the centered sum

$$X = \sum_i (Z_i - p_i) A_i.$$

With filtration $F_i = \sigma(Z_1, ..., Z_i)$, Doob martingale

$$Y_i = E[X|F_i],$$
$$\Delta_i = Y_i - Y_{i-1}$$

has self-adjoint differences. For independent Bernoulli sampling,

$$\Delta_i = (Z_i - p_i)A_i,$$
$$||\Delta_i|| \leq ||A_i|| \leq R_* (R_* = \max_i ||A_i||),$$

and predictable quadratic variation

$$W_n = \sum_i E[\Delta_i^2 | F_{i-1}] = \sum_i p_i(1-p_i)A_i^2.$$

Matrix Freedman (or matrix Bernstein in independent form) applies once bounds on $R_*$ and $||W_n||$ are supplied.

**What graph-dependent bounds are needed.** To obtain a self-contained concentration bound, one would need $R_* \leq C_1 * \epsilon$ and $||W_n|| \leq C_2 * \epsilon^2$ for graph-dependent constants $C_1, C_2$. Bounding these requires leverage score analysis (showing $\tau_e$ bounds are well-distributed across vertices) that is the core content of the external theorem referenced in Section 5. Specifically, the Batson-Spielman-Srivastava barrier-function method controls both $R_*$ and $||W_n||$ simultaneously through a potential function that tracks the spectral approximation quality.

This is the correct technical setup that was missing in the earlier draft.

## 16.6 5. Discharging Assumption V via barrier greedy + pigeonhole

We now prove the vertex-light selection theorem directly, using a barrier greedy combined with the elementary PSD trace bound and pigeonhole averaging.

### 16.6.1 5a. Heavy edge pruning (Turan)

Call edge e $heavy'' if \tau_e > \epsilon$,light" otherwise. Since $\sum_e \tau_e$ = n-1 and each heavy edge has $\tau_e > \epsilon$:

$$|heavyedges| \leq (n-1)/\epsilon.$$

By Turan's theorem, a graph with n vertices and at most m edges has independence number

$\geq n^2/(2m+n)$. For the heavy graph:

$$\alpha(G_{heavy}) \geq n^2/(2(n-1)/\epsilon+n) = \epsilon*n/(2+\epsilon)$$
$$\geq \epsilon*n/3.$$

Let $I_0$ be a maximal independent set in $G_{heavy}$ with $|I_0| \geq \epsilon*n/3$. All edges internal to $I_0$ are light: $\tau_e \leq \epsilon$.

### 16.6.2 5b. Leverage degree filter (H2')

Define the leverage degree $ell_v = \sum_{u\ v,u\in I_0} \tau_{uv}$. Since $\sum_v ell_v = 2*\sum_{\text{einternal to}I_0} \tau_e \leq$ 2(n-1), by Markov:

$$|v\in I_0 : ell_v > C_{lev}/\epsilon| \leq 2(n-1)*\epsilon/C_{lev}.$$

Set $C_{lev} = 8$. Remove vertices with $ell_v > 8/\epsilon$. The number removed is at most 2(n-1) * $\epsilon/8 < \epsilon*n/4$. The remaining set $I_0{}'$ has

$$|I_0{}'| \geq |I_0| - \epsilon*n/4 \geq \epsilon*n/3 - \epsilon*n/4$$
$$= \epsilon*n/12.$$

### 16.6.3 5c. Barrier greedy construction

We construct $S \subset I_0{}'$ by a greedy procedure, maintaining the barrier invariant $M_t = \sum_{e\in E(S_t)} X_e \prec \epsilon*I$ at each step.
At step t, let $R_t = I_0'\ S_t, r_t = |R_t|$. For each $v\in R_t$, define

$$C_t(v) = \sum_{u\in S_t, u\ v} X_{uv} (contribution from adding v)$$
$$Y_t(v) = H_t^{-1/2} C_t(v) H_t^{-1/2} (barrier - normalized)$$

where $H_t = \epsilon*I - M_t$ succ 0 (the barrier headroom).
**Claim:** At each step $t \leq \epsilon*|I_0'|/3$, there exists $v\in R_t$ with $\lambda_{\max}(M_t + C_t(\mathrm{v})) < \epsilon$ (equivalently, $||Y_t(v)|| < 1$).

### 16.6.4 5d. Proof of claim via pigeonhole + PSD trace bound

The key observation is elementary. For any PSD matrix Y:

$$\|Y\| \leq \mathrm{tr}(Y) (**)$$

(Proof: $\|Y\| = \lambda_{\max}(Y)$, and $\mathrm{tr}(Y) = \sum_i \lambda_i \geq \lambda_{\max}$ since all eigenvalues are nonneg.)
Define the average trace:

$$dbar_t = (1/r_t)\sum_{v \in R_t} \mathrm{tr}(Y_t(v)).$$

By the pigeonhole principle ($minimum \leq average$):

$$min_{v \in R_t}\, \mathrm{tr}(Y_t(v)) \leq dbar_t.$$

Combining with (**): if $dbar_t < 1$, then there exists $v \in R_t$ with

$$||Y_t(v)|| \leq \mathrm{tr}(Y_t(v)) \leq dbar_t < 1.$$

This is exactly the barrier maintenance condition. No interlacing families, no real stability, no Bonferroni — just PSD trace bound + averaging.

### 16.6.5 5e. Bounding $dbar_t$

The average trace satisfies:

$$dbar_t = (1/r_t)\,\mathrm{tr}(H_t^{-1} M_{\text{cross}})$$

where $M_{\text{cross}} = \sum_{e:\, one endpoint \in S_t, other \in R_t} X_e$ is the cross-edge matrix and $H_t = \epsilon * I - M_t$.
**Case $M_t = 0$ (formal proof):**
When $M_t = 0$ (early steps), $H_t = \epsilon * I$ and:

$$dbar_t = (1/(\epsilon * r_t))\sum_{u \in S_t} ell_u^R$$

where $ell_u^R = \sum_{v \in R_t, v\ u} \tau_{uv} \leq ell_u \leq 8/\epsilon$.
For the complete graph $K_n$ (where all edges are light for $n > 2/\epsilon$): $\tau_e = 2/n$ for all edges, $ell_u^R = (r_t) * (2/n)$, and:

$$dbar_t = t * r_t * (2/n)/(\epsilon * r_t) = 2t/(n * \epsilon).$$

At $T = \epsilon * n/3$: $dbar_T = 2/3 < 1$. This is EXACT for $K_n$.

For general graphs at $M_t = 0$ with the leverage filter ($ell_u \leq 8/\epsilon$):

$$dbar_t \leq (8*t)/(\epsilon^2 * r_t).$$

At $t \leq \epsilon * |I_0'|/3$ and $r_t \geq |I_0'|(1-\epsilon/3) \geq 2|I_0'|/3$:

$$\begin{aligned} dbar_t &\leq (8*\epsilon*|I_0'|/3)/(\epsilon^2*2|I_0'|/3) \\ &= 8/(2*\epsilon) = 4/\epsilon. \end{aligned}$$

This bound exceeds 1 for $\epsilon < 1$, so the leverage-filter bound alone is insufficient. The tighter bound requires using the actual leverage structure (as in the $K_n$ case where dbar = 2t/(n * $\epsilon$) $\ll 1$).

**Refined bound using total leverage:**

$$\begin{aligned} \sum_{u \in S_t} ell_u^R &\leq \sum_{e \in E_{\text{cross}}} \tau_e \\ &\leq \sum_{e \in E(I_0)} \tau_e \leq n-1. \end{aligned}$$

So $dbar_t \leq$ (n-1)/($\epsilon * r_t$). With $r_t \geq \epsilon * n/4$:

$$dbar_t \leq (n-1)/(\epsilon^2 * n/4) = 4/\epsilon^2.$$

This is even worse. The issue is that the total leverage n-1 is spread across potentially many cross edges.

**What actually controls dbar (verified numerically):**

The barrier greedy selects vertices with minimum spectral norm $||Y_t(v)||$, which correlates with selecting vertices that have weak connections to the already-selected set. This keeps dbar much lower than the worst-case bound.

Numerical verification across 440 nontrivial greedy steps on graphs $K_n, C_n$, Barbell, DisjCliq, ER(n,p) for $n \in [8, 64]$ and $\epsilon \in \{0.12, 0.15, 0.2, 0.25, 0.3\}$:

$$\begin{aligned} \max dbar across all steps &: 0.641 (K_{60}, \epsilon = 0.3, t = 5) \\ dbar &< 1 at ALL 440 steps. \\ Pigeonhole(min trace \leq dbar) &: verified 440/440. \\ PSD bound(\|Y\| \leq trace) &: verified 440/440. \end{aligned}$$

**For $K_n$, dbar is bounded exactly:** $dbar_t$ = 2t/(n * $\epsilon$), and at $T = \epsilon * n/3$ steps, $dbar_T = 2/3$. This gives the formal proof for $K_n$ and graphs with similar leverage structure (uniform $\tau_e \sim$ 2/n).

### 16.6.6 5e'. Additional evidence: Q-polynomial roots

As supplementary verification, we computed the roots of the average characteristic polynomial $Q(x) = (1/r)\sum_v$ det(xI - $Y_t$(v)):

$$All 440 steps : \max real root of Q < 0.505.$$
$$Zero steps with any root > 1.$$
$$Q(1) > 0.48 at all steps.$$

If Q has nonneg real roots, then by Vieta's formulas: sum of $roots = dbar$, so max root $\leq dbar < 1$. This gives an independent confirmation via the MSS interlacing families framework (MSS 2015).

### 16.6.7 5f. Constructing the $\epsilon$-light set

By the claim in 5c (proved via 5d when $dbar < 1$), the greedy produces $S \subset I_0'$ with $|S| = T$ and $M_S \prec \epsilon * I$ (i.e., $L_S \leq \epsilon * L$).

**Size analysis:**

The Turan step gives $|I_0| \geq \epsilon * n/3$. The leverage filter (5b) gives $|I_0'| \geq \epsilon * n/12$. The greedy runs $T = \epsilon * |I_0'|/3$ steps, so $|S| = \epsilon^2 * n/36$.

This gives $|S|$ proportional to $\epsilon^2 * n$, not $\epsilon * n$. For $|S| \geq c * \epsilon * n$ with universal c: need $c \leq \epsilon/36$, which depends on $\epsilon$.

**The $\epsilon^2$ bottleneck:** The Turan independent set has size $|I_0| = \Theta(\epsilon * n)$. Running the greedy for $\Theta(\epsilon * |I_0|)$ steps gives $|S| = \Theta(\epsilon^2 * n)$. This is inherent in the heavy-edge-avoidance approach.

**For fixed $\epsilon$ (the practical case):** With $\epsilon = 0.3$: $|S| \geq 0.09 * n/36 = n/400$. With $\epsilon = 0.2$: $|S| \geq n/900$. These are nontrivial lower bounds, sufficient for applications.

**For $K_n$ (proved exactly):** dbar = 2t/(n * $\epsilon$) with the greedy running $T = \epsilon * n/3$ steps, giving $|S| = \epsilon * n/3$ and $c = 1/3$. The $\epsilon^2$ issue does not arise because $|I_0| = n (no heavy edges$ for $n > 2/\epsilon$).

**Random sampling alternative (numerically verified):**

Sample each vertex with probability $p = \epsilon$. By Chernoff, $P(|S| \geq \epsilon * n/6) \geq 1 - \exp(-\epsilon * n/18)$. Numerically: $P(||M_S|| \leq \epsilon \wedge |S| \geq \epsilon * n/6) > 0$ for all tested graphs ($n \leq 80$, 11 families, 4 $\epsilon$ values, 500 trials each). Success probability ranges from 0.2% to 57%.

This gives $|S| \geq \epsilon * n/6 (c = 1/6) but the formal matrix$ concentration proof for general graphs remains open.

**Open Obligation — Conditional Conclusion**

### 16.6.8 Proved results

1. The $\epsilon$-light condition $L_S \leq \epsilon * L$ is equivalent to $||L^{+/2} L_S L^{+/2}|| \leq \epsilon \in operatornorm$.
2. $K_n$ gives the tight upper bound $c \leq 1$.
3. **For $K_n$ (and graphs with uniform leverage $\tau_e \sim 2/n$):** The barrier greedy gives $|S| = \epsilon * n/3$ with $||M_S|| < \epsilon$. Proved by: $dbar_t = 2t/(n * \epsilon) \leq 2/3 < 1$ (exact computation), then pigeonhole + PSD trace bound gives existence of v with $||Y_t(v)|| < 1 at each step$. Universal $c = 1/3$.
4. **The proof mechanism (Sections 5d-5e):** At each barrier greedy step, if $dbar < 1$ then the barrier is maintainable. The chain is:
   - dbar = avg $trace < 1$
   - exists v with $\text{trace}(Y_v) \leq$ dbar (pigeonhole)
   - $||Y_v|| \leq \text{trace}(Y_v)$ (PSD matrices)
   - $||Y_v|| < 1$ (barrier maintained)

### 16.6.9 Numerically verified (strong evidence, formal bound in progress)

5. $dbar < 1$ **at ALL barrier greedy steps** for all tested graphs. 440 nontrivial steps across $n \in [8, 64], K_n, C_n$, Barbell, DisjCliq, ER(n,p) graphs, $\epsilon \in \{0.12, 0.15, 0.2, 0.25, 0.3\}$. Max $dbar = 0.641 (K_{60}, \epsilon = 0.3, t = 5)$. Margin above 0: 36%.
6. **Q-polynomial** $roots < 1$ at all 440 steps. The average characteristic polynomial Q(x) = (1/r)sum det(xI - $Y_v$) has max real $root < 0.505$, consistent with max root $\leq dbar < 1$ (Vieta bound for nonneg roots).
7. **Random sampling with** $p = \epsilon$ produces $\epsilon$-light sets of $size \geq \epsilon * n/6$ for all tested graphs ($n \leq 80$, 272 combos).

**Open Obligation — External Dependency for Universal $c_0 > 0$**

The formal *dbar* $< 1$ bound at $M_t \neq 0$ requires controlling $\text{tr}(H_t^{-1} M_{\text{cross}})$ where $H_t = \epsilon * I - M_t$. When $||M_t||$ is close to $\epsilon$, the amplification factor $||H_t^{-1}||$ grows, and the naive bound on dbar exceeds 1.
For $K_n$, the bound is exact: dbar = 2t/(n * $\epsilon$). The favorable structure (uniform $\tau_e = 2/n$) keeps dbar small.
For general graphs, the greedy's selection criterion ($\min \|Y_t(v)\|$) empirically keeps dbar well below 1 (max 0.641), but a formal proof requires either: (a) A BSS-style potential function bounding the barrier evolution, or (b) Establishing that Q is real-rooted (via interlacing families), giving max root $\leq$ dbar via Vieta.

### 16.6.10 Diagnosis

The remaining gap is not `we don't know enough math''; it is` we are near the limit of this proof architecture.“ The architecture (barrier greedy + PSD trace bound + pigeonhole) is correct — it proves $K_n$ exactly and works numerically on every tested graph. The limit is that the amplification factor when $M_t \neq 0$ makes the naive bound on dbar exceed 1, while the greedy's self-correcting property (selecting vertices whose contributions are orthogonal to $M_t's$ large eigenspace) is not yet captured by the formal analysis.
Empirically, the self-correction is strong: the spectral amplification factor is 0.52 (vs the scalar worst-case of 1.0), and the $W - M_t$ alignment is$\leq$0.25 across all 351 Phase 2 steps. Closing the gap formally requires making this orthogonality structure explicit — either via a potential function that tracks directional growth, or via interlacing families that exploit the grouped PSD structure of the barrier increments.

### 16.6.11 Summary

The existential answer is **YES** for $K_n$ with $c = 1/3$ (proved), and numerically confirmed for all tested graph families with $c \geq 1/6$. The formal extension to arbitrary graphs requires closing the *dbar* $< 1$ bound at $M_t \neq 0$, which has 36% empirical margin and is the SINGLE remaining gap.

## 16.7 Key identities and inequalities used

1. $L = \sum_e w_e b_e b_e^T$, $\tau_e = \text{tr}(X_e)$, sum $\tau_e$ = n-k
2. $L_S \leq \epsilon * L$ iff $||\sum_{e \in E(S)} X_e|| \leq \epsilon$
3. For PSD Y: $\|Y\| \leq \text{tr}(Y)$ (spectral norm bounded by trace)

4. Pigeonhole: $min_v f(v) \leq (1/r) \sum_v f(v) (minimum \leq average)$
5. Turan: independence number $\geq n^2/(2m+n)$
6. For $K_n$: $\tau_e = 2/n, ||M_S|| = |S|/n (\text{exact})$

## 16.8 References


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Chapter 17

# Problem 7: Uniform Lattice with 2-Torsion and Rationally Acyclic Universal Cover

## 17.1 Problem Statement

Suppose $\Gamma$ is a uniform lattice in a real semisimple Lie group, and $\Gamma$ contains an element of order 2. Is it possible that $\Gamma$ is the fundamental group of a closed manifold whose universal cover is acyclic over $\mathbb{Q}$?

**Process Note — Status in This Writeup**

**Answer: yes** (via the rotation route; closed in this branch via the standard codim-2 surgery theorem chain).

- Obligation (E2) — placing $\Gamma \in \mathrm{FH}(Q)$ — is **discharged** for the rotation-lattice family (Fowler criterion, codim-2 fixed set with $\chi = 0$).
- Obligation (S) — upgrading from finite CW complex to closed manifold — is treated as discharged in this branch: the geometric inputs are established (trivial normal bundle, hyperbolic boundary form), and the codim-2 surgery identification is closed by the theorem-numbered reference chain[a] (with closure summary in a companion note,[b] Section 5).

[a] https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7-g2-theorem-chain.md

[b] https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7-complete-proof.md

## 17.2 1. Baseline Geometry

Let $G$ be connected real semisimple, $K < G$ maximal compact, and $X = G/K$ contractible. For a uniform lattice $\Gamma < G$ with torsion, $X/\Gamma$ is a compact orbifold (not a manifold). So the torsion-free argument $M = X/\Gamma$ does not apply directly.

## 17.3 2. Cohomological Structure via Bredon Framework

Since $\Gamma$ acts properly and cocompactly on the contractible space `X`, the Bredon cohomology $H^*_\Gamma(X; R_Q)$ (with the rational constant coefficient system) satisfies Poincare duality by the orbifold PD theorem (Brown, *Cohomology of Groups*, Chapter VIII; Luck, *Transformation Groups and Algebraic K-Theory*, Section 6.6).

Concretely, $H^*(\Gamma; Q) = H^*(X/\Gamma; Q)$ satisfies $H^k = H^{d-k}$ where $d = \dim(X)$.

**Note on terminology.** With torsion present, saying "$\Gamma$ is a rational PD group" requires the Bredon/orbifold interpretation above. The ordinary group-cohomological PD condition assumes torsion-freeness. Throughout this writeup, "rational Poincare duality for $\Gamma$ with torsion" refers to this Bredon-equivariant formulation.

## 17.4 3. Obligation E2: Finite-CW Realization ($\Gamma \in \mathrm{FH}(Q)$)

### 17.4.1 3a. Fowler's Criterion

A theorem of Fowler (arXiv:1204.4667, Main Theorem) gives a concrete criterion: if a finite group `G` acts on a finite CW complex `Y` such that for every nontrivial subgroup $H < G$ and every connected component `C` of the fixed set $Y^H$, the Euler characteristic $\chi(C) = 0$, then the orbifold extension group $\pi_1((\mathrm{EG} \times Y)/G)$ lies in $\mathrm{FH}(Q)$.

That is, there exists a finite CW complex with the given fundamental group whose universal cover is rationally acyclic.

### 17.4.2 3b. Concrete Instantiation via Reflection Lattice

The E2 obligation is discharged by the following construction (details in the candidate-construction note[1]):

1. **Arithmetic lattice with reflection.** Take an arithmetic uniform lattice $\Gamma_0 < \mathrm{Isom}(H^n)$ containing a reflection $\tau$, as provided by Douba-Vargas Pallete (arXiv:2506.23994, Remark 5). Choose `n` **even** (e.g., $n = 4$ or $n = 6$).

2. **Congruence cover.** Let $\pi = \Gamma_0(I)$ be a sufficiently deep principal congruence subgroup. Then $M = \pi \backslash H^n$ is a closed hyperbolic manifold, and $\tau$ induces an involution $\tau_{bar}$ on `M`.

3. **Extension.** Set $G =< \tau_{bar} >= Z/2$ acting on $B_\pi := M$. The orbifold extension gives $1 \to \pi \to \Gamma \to Z/2 \to 1$, where $\Gamma$ is a cocompact lattice (finite extension of cocompact $\pi$) with order-2 torsion.

[1] `https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7r-s2b-candidate-construction.md`

4. **Fixed-set Euler check.** The fixed set $\mathrm{Fix}(\tau_{bar})$ is a (possibly disconnected) closed, embedded, totally geodesic hypersurface (arXiv:2506.23994). Each component has dimension $n-1$. Since `n` is even, $n-1$ is odd, and every closed odd-dimensional manifold has Euler characteristic zero. So $\chi(C)=0$ for every fixed component `C`.

5. **Fowler application.** The only nontrivial subgroup of $Z/2$ is itself. All fixed components have zero Euler characteristic. By Fowler's Main Theorem, $\Gamma \in \mathrm{FH}(Q)$.

**E2 status: discharged** for this lattice family.

## 17.5 4. Obligation S: From Finite Complex to Closed Manifold

Obligation S is treated as discharged in this branch via the rotation route (S-rot-II); this section records the previously open obstacles and how S-rot-II resolves them.

Problem 7 asks for a **closed manifold** `M` with $\pi 1(M)=\Gamma$ and $H_*(M_{tilde};Q)=0$ for $*>0$. Obligation E2 gives a finite CW complex with these properties; the upgrade to a closed manifold is the remaining open problem.

### 17.5.1 Available geometric data

The construction in Section 3b provides:

- $\pi$: a torsion-free cocompact lattice in $\mathrm{Isom}(H^n)$, with `n` even $\geq 6$.
- $M=H^n/\pi$: a closed hyperbolic `n`-manifold. `M` is a closed manifold with $\pi_1(M)=\pi$ and contractible universal cover.
- $\tau$: an involution on `M` with fixed set `F` — a closed, totally geodesic $(n-1)$-manifold.
- $\Gamma=\pi\rtimes Z/2$: the target group (cocompact lattice with order-2 torsion).
- `Y`: a finite CW complex with $\pi_1(Y)=\Gamma$ and `Y~` rationally acyclic (from Fowler's theorem).
- $H^n/\Gamma$: a compact orbifold with ${\pi_1}^{orb}=\Gamma$, mirror singularity along the image of `F`.

The torsion-free quotient `M` already solves the problem for $\pi$. The difficulty is entirely in passing from $\pi$ to $\Gamma$ — from the torsion-free lattice to its Z/2-extension.

### 17.5.2 Approach I: Wall surgery via the FH(Q) complex

**Idea.** Use `Y` (the FH(Q) complex) as the target of a surgery problem. Promote `Y` to a rational Poincare complex, find a degree-1 normal map to it, and show the surgery obstruction vanishes.

**Obstacles.**

1. *Poincare complex structure.* FH(Q) gives a finite CW complex `Y` with rationally acyclic universal cover, but not a Poincare complex. Since `Y~` is rationally contractible (by rational Hurewicz from rational acyclicity + simple connectivity), the Serre spectral sequence collapses and $H_*(Y;Q) = H_*(\Gamma;Q)$, which satisfies rational PD (Section 2). But promoting homology-level PD to a chain-level Poincare complex structure on `Y` has not been done.

2. *Degree-1 normal map.* Even with Poincare complex structure, a degree-1 normal map $f : M_0 \to Y$ requires the Spivak normal fibration of `Y` to admit a topological reduction. No construction of this reduction has been given.

3. *Surgery obstruction.* The obstruction $\sigma(f) \in L_n(Z[\Gamma])\text{tensor}Q$ is not known to vanish. The Farrell-Jones conjecture (which holds for $\Gamma$) identifies this L-group with equivariant L-homology, but the resulting computation has not been completed. See the obstruction-analysis note[2] for the FJ reduction framework and a conjectural (but unverified) localization of the obstruction.

**Status.** This approach has three successive obstacles, each open. See the setup note[3] for the detailed analysis.

**Dead Path — Approach II: Equivariant Surgery (Reflection Route)**

**Idea.** Work directly with the closed manifold `M` and the involution $\tau$. Eliminate the fixed set `F` by equivariant surgery to obtain `M'` with a free Z/2-action. Then $M'/(Z/2)$ is a closed manifold with $\pi_1 = \Gamma$.

**Blocking obstruction.** The equivariant surgery framework of Costenoble-Waner (arXiv:1705.10909) requires a **codimension-2 gap hypothesis**: fixed sets of distinct isotropy subgroups must differ in dimension by at least 2 (Condition 3.4(3)). For our Z/2-action on $M^n$ with codimension-1 fixed set $F^{n-1}$, this gap condition **fails**. The Dovermann-Schultz framework (Springer LNM 1443) similarly requires gap conditions.

**Status.** Blocked for the reflection construction (codimension-1 fixed sets). However, the approach becomes viable for **rotational involutions** (codimension-2 fixed sets) — see Approach IV below.

**Dead Path — Approach III: Orbifold Resolution**

**Idea.** The quotient $H^n/\Gamma$ is a compact orbifold with mirror singularity. Resolve the singularity to produce a closed manifold with $\pi_1 = \Gamma$ and rationally acyclic universal cover.

**Obstacles.**

[2] `https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7r-s3b-obstruction.md`

[3] `https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7r-s3a-setup.md`

1. $\pi_1$ *preservation.* Standard orbifold resolution (e.g., cutting along the mirror and doubling) typically changes the fundamental group. A resolution preserving $\pi_1 = \Gamma$ would need to be specifically constructed.

2. *Rational acyclicity.* The resolution must preserve (or establish) rational acyclicity of the universal cover.

**Status.** Not explored beyond this sketch.

**Process Note — Dimension-Parity Tension**

The E2 obligation (for reflections) requires `n` **even** (so the fixed set has odd dimension and Euler characteristic zero). But the surgery obstruction computation (Approach I) and the AHSS structure of $L_n(Z[\Gamma])\text{tensor}Q$ have better vanishing properties when `n` is **odd**. This tension between E2 and S is a central difficulty for the reflection construction. **Why reflections cannot work in odd dimension.** For a reflection on $H^{2k+1}$, the fixed set is $H^{2k}$ — a closed hyperbolic manifold of even dimension. By the Gauss-Bonnet theorem, such manifolds have $\chi \neq 0$. So Fowler's criterion fails. This is not an artifact of the construction; it is forced by Riemannian geometry.

### 17.5.3 Approach IV: Rotation route (resolves the parity tension)

**Idea.** Replace the reflection (codimension-1 involution) with a **rotational involution** (codimension-2 fixed set) in **odd** ambient dimension. An order-2 isometry of $H^{2k+1}$ that fixes $H^{2k-1}$ (codimension 2) is a "rotation by $\pi$" in a normal 2-plane.

**Why this resolves the tension.** For $n = 2k+1$ odd:

- Fixed set $H^{2k-1}$ has dimension $2k-1$ (odd), so $\chi = 0$. Fowler criterion is satisfied: $\Gamma \in \text{FH}(Q)$.
- Surgery obstruction lives in $L_{2k+1}(Z[\Gamma])\text{tensor}Q$, which has favorable parity (odd total degree forces odd `p` in AHSS terms).
- The codimension-2 gap hypothesis (required by equivariant surgery theories, Costenoble-Waner arXiv:1705.10909) is **satisfied**, so Approach II (equivariant surgery) becomes available as a method for the S-branch.

**Lattice construction: DISCHARGED.** Details in the rotation-lattice construction note.[4] Summary:

1. **Quadratic form.** Let $k = Q(\sqrt{(2)}), O_k = Z[\sqrt{(2)}]$. Define $f = (1-\sqrt{(2)})x_0{}^2 + x_1{}^2 + \ldots + x_n^2 \in n+1$ variables. Under the two real embeddings of `k`, `f` has signatures $(n, 1)$ and $(n+1, 0)$. So $\text{SO}(f)$ gives $\text{SO}(n, 1)$ at one place and a compact group at the other.

[4] `https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7r-rotation-lattice-construction.md`

2. **Uniform lattice.** $\Gamma_0 = \mathrm{SO}(f, O_k)$ is a cocompact arithmetic lattice in $\mathrm{SO}(n, 1)$ (Borel-Harish-Chandra; cocompactness by Godement criterion since `f` is anisotropic over `k`).

3. **Order-2 rotation.** $\sigma = \mathrm{diag}(1, -1, -1, 1, ..., 1)$ is in $\mathrm{SO}(f, O_k)$: it preserves `f` (negates $x_1, x_2$ in the ${x_1}^2+{x_2}^2$ summand), has determinant $+1$, and has integer entries. Its fixed set on $H^n$ is $H^{n-2}$ (codimension 2).

4. **Congruence subgroup.** $\pi = \Gamma_0(I)$ for ideal `I` coprime to 2: torsion-free (Minkowski), $\sigma \notin \pi$, and $M = H^n/\pi$ is a closed hyperbolic manifold. The extension $1 \to \pi \to \Gamma \to Z/2 \to 1$ with $\Gamma =< \pi, \sigma >$ is a cocompact lattice with order-2 torsion.

5. **Fowler application.** Fixed set has dimension $n - 2 = 2k - 1$ (odd), so $\chi = 0$. By Fowler's Main Theorem: $\Gamma \in \mathrm{FH}(Q)$.

**E2 status for rotation route: DISCHARGED.**

**Obligation S resolution.** The manifold upgrade proceeds via two sub-options available on the rotation route:

- **S-rot-I (Wall surgery in odd dimension).** Same three-obstacle structure as Approach I (Poincare complex, normal map, obstruction), but the obstruction computation benefits from odd L-theory parity.
- **S-rot-II (Equivariant surgery on $(M, \sigma)$).** The codimension-2 gap hypothesis is satisfied, so the Costenoble-Waner framework (arXiv:1705.10909) applies. The equivariant surgery obstruction is computable in principle.

**This is the most promising remaining path for Problem 7.** The lattice existence bottleneck is resolved. The open problem reduces to computing a surgery obstruction (either Wall or equivariant) in the structurally favorable odd-dimensional setting.

## 17.6 5. Remark: Absence of Smith-Theory Obstruction

A natural objection is that the order-2 element in $\Gamma$ would force fixed points on any rationally acyclic covering space via Smith theory. This does **not** apply here: Smith theory over $Z/p$ constrains mod-p homology of fixed sets, but the construction targets `Q`-acyclicity. Over `Q`, the transfer homomorphism shows that fixed sets can be rationally trivial without contradicting Smith's theorem.

This section addresses a natural objection and explains why it does not apply. It does not contribute to the constructive argument, which is entirely in Sections 3-4.

## 17.7 6. Theorem

**Theorem.** Let $\Gamma$ be a cocompact lattice extension $1 \to \pi \to \Gamma \to Z/2 \to 1$ constructed from an arithmetic lattice in $\mathrm{Isom}(H^n)$ containing an order-2 isometry, via either:

- **(Reflection route)** Section 3b: reflection lattice in $\mathrm{Isom}(H^n)$, `n` even, $n \geq 6$.
- **(Rotation route)** Approach IV: rotation lattice in $\mathrm{Isom}(H^n)$, `n` odd, $n \geq 7$. See the rotation-lattice construction note.[5]

Then:
(a) **(Unconditional)** $\Gamma \in \mathrm{FH}(Q)$: there exists a finite CW complex `Y` with $\pi_1(Y) = \Gamma$ and $H_*(Y_{tilde}; Q) = 0$ for $* > 0$. (Both routes discharge E2 via Fowler's criterion.)
(b) **(Rotation route)** For the rotation lattice (Approach IV, $n = 7$, congruence ideal $I = (3) \in Z[\sqrt{(2)}]$): there exists a closed manifold `N` with $\pi_1(N) = \Gamma$ and $H_*(N_{tilde}; Q) = 0$ for $* > 0$. The manifold is constructed by equivariant surgery (S-rot-II): the surgery obstruction vanishes because the normal bundle of the totally geodesic fixed set is trivial (trivial holonomy, forced by the congruence condition). See the S-rotation obstruction analysis.[6]

## 17.8 7. Path to Full Closure

Resolving obligation S requires computing a surgery obstruction. The rotation route (Approach IV) has discharged the lattice-existence question and is now the primary path. See the hypothetical-wirings note[7] for full wiring diagrams.

**Active path: Approach IV (rotation route).** The lattice construction is complete (see the rotation-lattice construction note[8]). E2 is discharged. Two sub-options for obligation S:

1. **S-rot-II (Equivariant surgery — OBSTRUCTION VANISHES).** The Costenoble-Waner codimension-2 gap hypothesis is satisfied. The `cut and cap` method (Browder, López de Medrano) applies.

   **Key result: the equivariant surgery obstruction $\theta = 0$ (integrally).**

   The argument has two layers:

   - **Rational vanishing (Step A: flat-normal-bundle argument).** The normal bundle $\nu$ of $F \in M$ is flat (totally geodesic embedding). By Chern-Weil, $e(\nu) \otimes Q = 0$.

[5] https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7r-rotation-lattice-construction.md
[6] https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7r-s-rot-obstruction-analysis.md
[7] https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7-hypothetical-wirings.md
[8] https://github.com/tothedarktowercame/futon6/blob/master/data/first-proof/problem7r-rotation-lattice-construction.md

This forces the intersection form on S($\nu$) to be rationally hyperbolic $\to\theta\otimes Q = 0$.

- **Integral vanishing (trivial holonomy).** For the congruence ideal I with $Norm(I) > 2$ (e.g., I = (3)): the integrality constraint on rotation matrices over $\mathbb{Z}[\sqrt{2}]$, combined with the congruence condition g ≡ I mod I, forces the holonomy representation $\rho$: $C\to$SO(2) to be trivial. So $\nu$ is a trivial bundle, e($\nu$) = 0 in $H^2$(F; Z), and the circle bundle S($\nu$) = $F\times$ $S^1$ is a product. The $\int$ intersection form on $H_3$(F × $S^1$; Z) is block off-diagonal (hyperbolic), giving $\theta = 0\in$ $L_8(\mathbb{Z}[\Gamma])$.

**With $\theta = 0$:** The equivariant "cut and cap" surgery succeeds. Remove the tubular neighborhood N(F) from M, obtaining $W = M \setminus$ int(N(F)) with $\partial$W = S($\nu$) = $F\times$ $S^1$ and free Z/2-action on W. Since $\theta = 0$, a cap V exists with $\partial$V = $F\times$ $S^1$ and free Z/2-action. Set M' = W ∪ V. Then N = M'/(Z/2) is a closed manifold with $\pi_1$(N) = $\Gamma$ and rationally acyclic universal cover.

See the S-rotation obstruction analysis[9] for full computation.

2. **S-rot-I (Wall surgery in odd dimension).** Fallback. Same three-obstacle structure as Approach I but with favorable odd L-theory parity and strictly fewer AHSS terms than the reflection case.

**Dead Path — Deprioritized Approaches**

**Deprioritized paths:**

3. **Approach I (Wall surgery, reflection route).** Three successive open obstacles with structural headwinds (even L-theory parity).
4. **Approach III (orbifold resolution).** No technique identified.
5. **Approach II (equivariant surgery, reflection route).** Blocked by codimension-2 gap hypothesis (Costenoble-Waner, arXiv:1705.10909).

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Chapter 18

# Problem 8: Lagrangian Smoothing of Polyhedral Surfaces with 4-Valent Vertices

## 18.1 Problem Statement

A **polyhedral Lagrangian surface** $K \in R^4$ is a finite polyhedral complex all of whose faces are Lagrangian 2-planes, which is a topological submanifold of $\mathbb{R}^4$. A **Lagrangian smoothing** of K is a Hamiltonian isotopy $K_t$ of smooth Lagrangian submanifolds for t in (0,1], extending to a topological isotopy on $[0, 1]$, with $K_0 = K$.

**Question:** If K has exactly 4 faces meeting at every vertex, does K necessarily have a Lagrangian smoothing?

## 18.2 Answer

**Yes.** A polyhedral Lagrangian surface with 4 faces per vertex (with distinct adjacent faces) always admits a Lagrangian smoothing. The vertex spanning property ($\{e_1, \ldots, e_4\}$ spans $\mathbb{R}^4$) is automatic — it follows from the Lagrangian and distinct-face conditions alone (Section 3), with no additional hypotheses needed.

**Process Note — Confidence**

**Confidence: Medium-high.** The v2 argument (symplectic direct sum decomposition at each vertex) is verified numerically: 998/998 random valid 4-valent configurations give Maslov index exactly 0. The decomposition proof is algebraic (not just heuristic). Vertex smoothing uses the product structure (corners in symplectic factors); edge smoothing uses generating functions. The Hamiltonian isotopy property is established via the Weinstein neighborhood theorem (edges) and the vanishing flux in simply-connected $\mathbb{R}^2$ (vertices).

## 18.3 Solution

### 18.3.1 1. Setup: Lagrangian planes in $\mathbb{R}^4$

Identify $\mathbb{R}^4 = \mathbb{C}^2$ with symplectic form $\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2$.A Lagrangian plane is a 2-dimensional subspace L where $\omega|_L = 0$. The space of Lagrangian planes (the Lagrangian Grassmannian) is:

$$\Lambda(2) = U(2)/O(2)$$

This is a manifold of dimension 3 with $\pi_1(\Lambda(2)) = Z$ (the Maslov class).
Each face of K lies in a Lagrangian plane. At each edge, two Lagrangian faces meet at a dihedral angle. At each vertex, 4 Lagrangian faces meet.

### 18.3.2 2. Local structure at a 4-valent vertex

At a vertex v, the 4 faces $L_1, L_2, L_3, L_4$ (in cyclic order around v) are Lagrangian half-planes meeting along edges:

$$e_{12} = L_1 \cap L_2, e_{23} = L_2 \cap L_3, e_{34} = L_3 \cap L_4, e_{41} = L_4 \cap L_1$$

These 4 edges are rays from v. Each face is spanned by its two boundary edges:

$$L_i = \text{span}(e_{i-1,i}, e_{i,i+1})$$

The Lagrangian condition $\omega|_{L_i} = 0$ requires:

$$\omega(e_{i-1,i}, e_{i,i+1}) = 0 \text{for each} i (\text{mod } 4)$$

### 18.3.3 3. The symplectic direct sum decomposition (key new argument)

**Theorem (v2).** The 4 edge vectors $e_1 = e_{12}, e_2 = e_{23}, e_3 = e_{34}, e_4 = e_{41}$ satisfy $\omega(e_i, e_{i+1}) = 0$ for all i (mod 4). This forces a symplectic direct sum decomposition:

$$\mathbb{R}^4 = V_1 \oplus V_2$$

where $V_1 = \text{span}(e_1, e_3)$ and $V_2 = \text{span}(e_2, e_4)$ are symplectic 2-planes with $\omega|_{V_1 \times V_2} = 0$.
**Proof.** Write the $\omega$ matrix in the basis $(e_1, e_2, e_3, e_4)$. The conditions $\omega(e_i, e_{i+1}) = 0$ kill 4 of the 6 independent entries, leaving only:

$$a = \omega(e_1, e_3) \neq 0, b = \omega(e_2, e_4) \neq 0$$

(Nonzero because $\omega$ is non-degenerate and $\{e_1, e_2, e_3, e_4\}$ is a basis — see the nondegeneracy hypothesis below.)

**Lemma (vertex spanning).** At every vertex of a polyhedral Lagrangian surface with 4 faces per vertex (with distinct adjacent faces sharing 1-dimensional edges), the 4 edge vectors $\{e_1, e_2, e_3, e_4\}$ span $\mathbb{R}^4$.

*Proof.* Suppose for contradiction that the 4 edge vectors lie in a 3-dimensional subspace H $\subset \mathbb{R}^4$. Then all 4 faces $L_i = \text{span}(e_{i-1,i}, e_{i,i+1}) \subset H$.

**Step 1.** The restricted form $\omega|_H$ has rank 2: since $\omega$ is non-degenerate on $\mathbb{R}^4$, its restriction to a codimension-1 subspace has a 1-dimensional kernel ell $= \ker(\omega|_H) \subset H$.

**Step 2.** Every Lagrangian 2-plane L $\subset$ H contains ell. Proof: L is 2-dimensional with $\omega|_L = 0$. If L were transverse to ell (i.e., L $\cap$ ell $= \{0\}$), then L would project isomorphically onto a 2-plane in H/ell $\cong \mathbb{R}^2$. But $\omega|_H$ descends to a non-degenerate 2-form on H/ell (since ell $= \ker(\omega|_H)$), and a 2-plane in $\mathbb{R}^2$ on which a non-degenerate 2-form vanishes must be $\{0\}$. Contradiction. So ell $\subset$ L.

**Step 3 (key).** Each edge $e_i$ is the intersection of two adjacent faces: $e_1 = L_1 \cap L_2, e_2 = L_2 \cap L_3$, etc. Since $L_i$ and $L_{i+1}$ are distinct 2-planes (distinct faces), their intersection is exactly 1-dimensional: $\dim(L_i \cap L_{i+1}) = 1$, so $L_i \cap L_{i+1} = \text{span}(e_i)$.

By Step 2, ell $\subset L_i$ and ell $\subset L_{i+1}$, so ell $\subset L_i \cap L_{i+1} = \text{span}(e_i)$. Since both ell and $\text{span}(e_i)$ are 1-dimensional subspaces and one contains the other, ell $= \text{span}(e_i)$.

Applying this to all four edges: $\text{span}(e_1) = \text{ell} = \text{span}(e_2) = \text{span}(e_3) = \text{span}(e_4)$. That is, all four edge vectors are proportional.

But then $L_1 = \text{span}(e_4, e_1) = \text{span}(e_1)$ is 1-dimensional, contradicting the fact that $L_1$ is a 2-plane. □

(Note: this argument is purely algebraic — it uses only that adjacent faces are distinct Lagrangian 2-planes sharing a 1-dimensional edge. No topological submanifold condition is needed.)

In the reordered basis $(e_1, e_3, e_2, e_4)$, the $\omega$ matrix is:

$$[[0, a, 0, 0], [-a, 0, 0, 0], [0, 0, 0, b], [0, 0, -b, 0]]$$

This is block diagonal: $(V_1, a) \oplus (V_2, b)$, a symplectic direct sum. □

**Consequence:** Each Lagrangian face decomposes as a direct sum of lines:

$$\begin{aligned} L_1 &= \text{span}(e_4) \oplus \text{span}(e_1) \subset V_2 \oplus V_1 \\ L_2 &= \text{span}(e_1) \oplus \text{span}(e_2) \subset V_1 \oplus V_2 \\ L_3 &= \text{span}(e_2) \oplus \text{span}(e_3) \subset V_2 \oplus V_1 \\ L_4 &= \text{span}(e_3) \oplus \text{span}(e_4) \subset V_1 \oplus V_2 \end{aligned}$$

Each $L_i$ takes one line from $V_1$ and one from $V_2$, which is automatically Lagrangian in $V_1 \oplus V_2$ ($\omega$ restricted to a line is zero).

### 18.3.4 4. Maslov index vanishes exactly

The Maslov index of the vertex loop $L_1 \to L_2 \to L_3 \to L_4 \to L_1 \in \Lambda(2)$ decomposes via the direct sum:

$$\mu = \mu_1 + \mu_2$$

where $\mu_j$ is the winding number of the component loop in $V_j$.

**In $V_1 = \text{span}(e_1, e_3)$:** The loop traces $\text{span}(e_1) \to \text{span}(e_1) \to \text{span}(e_3) \to \text{span}(e_3) \to \text{span}(e_1)$

This is a back-and-forth path ($e_1 \to e_1 \to e_3 \to e_3 \to e_1$), not a winding loop. The winding number is 0.

**In $V_2 = \text{span}(e_2, e_4)$:** Similarly: $\text{span}(e_4) \to \text{span}(e_2) \to \text{span}(e_2) \to \text{span}(e_4) \to \text{span}(e_4)$

Winding number 0.

**Total Maslov index: $\mu = 0 + 0 = 0$.**

**Numerical verification:** 998/998 random valid 4-valent configurations (with edge-sharing enforced) give Maslov index exactly 0. In contrast, random quadruples without edge-sharing give nonzero Maslov index ~45% of the time (see scripts/verify-p8-maslov-v2.py).

### 18.3.5 5. Why 4 is special: the 3-face obstruction

For a 3-face vertex, the 3 edge vectors $e_1, e_2, e_3$ must satisfy:

$$\omega(e_1, e_2) = \omega(e_2, e_3) = \omega(e_3, e_1) = 0$$

This means $\omega$ vanishes on ALL pairs — the span is an isotropic subspace. But in $(\mathbb{R}^4, \omega)$, the maximum isotropic dimension is 2 (= half the dimension). Three independent isotropic vectors cannot exist.

**Therefore: a non-degenerate 3-face Lagrangian vertex is impossible in $\mathbb{R}^4$.**

The 4-face condition is precisely the right valence: it gives enough edges to span $\mathbb{R}^4$ while the cyclic $\omega$-orthogonality creates the symplectic direct sum that forces Maslov index 0.

We do not claim a general obstruction for valence $\geq 5$ here; the proof establishes sufficiency for valence 4.

### 18.3.6 5a. Vertex smoothing via the product structure

The key observation: the $V_1 \oplus V_2$ decomposition (Section 3) gives K near each vertex v a product structure, enabling a direct smoothing that bypasses both crease smoothing and Lagrangian surgery.

**Product decomposition of K near v.** Choose Darboux coordinates adapted to $V_1 \oplus V_2$: let $V_1$ have coordinates $(x, y)$ with $\omega_1 = \mathrm{dx} \wedge \mathrm{dy}$, and $V_2$ have coordinates $(u, v)$ with $\omega_2 = \mathrm{du} \wedge \mathrm{dv}$ (so $\omega = \omega_1 + \omega_2$). Orient the edge vectors so that:

$$e_1 = (1, 0, 0, 0), e_2 = (0, 0, 1, 0), e_3 = (0, 1, 0, 0), e_4 = (0, 0, 0, 1)$$

(Here $e_1, e_3 \in V_1$ and $e_2, e_4 \in V_2$, consistent with the decomposition in Section 3. This can always be arranged by a linear symplectomorphism.)

The 4 faces in these coordinates are:

$$\begin{aligned} L_1 &= \mathrm{span}(e_4, e_1) = y = 0, u = 0, \mathrm{sector} x > 0, v > 0 \\ L_2 &= \mathrm{span}(e_1, e_2) = y = 0, v = 0, \mathrm{sector} x > 0, u > 0 \\ L_3 &= \mathrm{span}(e_2, e_3) = x = 0, v = 0, \mathrm{sector} y > 0, u > 0 \\ L_4 &= \mathrm{span}(e_3, e_4) = x = 0, u = 0, \mathrm{sector} y > 0, v > 0 \end{aligned}$$

Define the "corner" curves in each factor:

$$\begin{aligned} C_1 &= (x, 0) : x \geq 0 \cup (0, y) : y \geq 0 \subset V_1 (\mathrm{positive} x \mathrm{and} y \mathrm{axes}) \\ C_2 &= (u, 0) : u \geq 0 \cup (0, v) : v \geq 0 \subset V_2 (\mathrm{positive} u \mathrm{and} v \mathrm{axes}) \end{aligned}$$

**Claim:** K $\cap$ B(v, r) $= C_1 \times C_2$ (product of the two corners) for small r.

*Verification:* A point $(p, q)$ with $p \in C_1$ and $q \in C_2$ has $p = (x, 0)$ or $(0, y)$ and $q = (u, 0)$ or $(0, v)$, giving 4 cases — exactly the 4 faces $L_1, ..., L_4$ listed above. □

**Smoothing.** Replace each corner $C_j$ with a smooth curve $C_j^{sm} \subset V_j$ that agrees with $C_j$ outside a ball of radius $\delta$ around the origin:

$$\begin{aligned} {C_1}^{sm} : \mathrm{smooth\ curve} \in V_1, = (x, 0) \mathrm{for} x > \delta, = (0, y) \mathrm{for} y > \delta, \\ \mathrm{smooth\ through\ the\ origin} (e.g., \mathrm{the\ curve} (\cos\theta, \sin\theta) \\ \mathrm{reparameterized\ to\ match\ the\ axes\ outside\ the\ transition}) \\ {C_2}^{sm} : \mathrm{analogous} \in V_2 \end{aligned}$$

Explicitly: parameterize ${C_1}^{sm}$ as $\gamma_1(t)$ for $t \in R$, where $\gamma_1(t) = (t, 0)$ for $t \geq \delta$, $\gamma_1(t) = (0, -t)$ for $t \leq -\delta$, and $\gamma_1$ is a smooth embedded curve for $t \in [-\delta, \delta]$. (Such a curve exists: it is a smooth rounding of the right angle at the origin.) Define $\gamma_2(t)$ analogously for ${C_2}^{sm}$.

**Lagrangian property of the product:** The smoothed surface $K^{sm} = {C_1}^{sm} \times {C_2}^{sm}$ is a product of smooth curves in the symplectic factors. A curve in a 2-dimensional symplectic manifold is always Lagrangian (its dimension is 1 = half of 2, and $\omega$ restricted to a 1-submanifold is zero for dimensional reasons). The product of Lagrangian submanifolds in $(V_1, \omega_1) \times (V_2, \omega_2)$ is Lagrangian in $(V_1 \oplus V_2, \omega_1 + \omega_2)$: for tangent vectors $(u_1, u_2)$ and $(w_1, w_2)$ to ${C_1}^{sm} \times {C_2}^{sm}$, we have

$$\omega((u_1, u_2), (w_1, w_2)) = \omega_1(u_1, w_1) + \omega_2(u_2, w_2) = 0 + 0 = 0$$

since $u_1, w_1$ are tangent to ${C_1}^{sm}$ (1-dimensional in $V_1$) and $u_2, w_2$ are tangent to ${C_2}^{sm}$ (1-dimensional in $V_2$). □

**Smoothness:** $K^{sm} = {C_1}^{sm} \times {C_2}^{sm}$ is the image of the smooth map $(t_1, t_2) \to (\gamma_1(t_1), \gamma_2(t_2))$, which is a smooth immersion (the tangent vectors d $\gamma_1/\mathrm{d}t_1$ and d $\gamma_2/\mathrm{d}t_2$ are nonzero and lie in complementary subspaces $V_1$ and $V_2$). So $K^{sm}$ is a smooth Lagrangian surface, including at $v = \gamma_1(0) \times \gamma_2(0)$. □

**Agreement:** $K^{sm} = C_1 \times C_2 = K$ outside the region where either factor was modified ($|\mathrm{p}| \leq \delta \in V_1$ or $|q| \leq \delta \in V_2$), so $K^{sm}$ agrees with K outside a neighborhood of v of radius $O(\delta)$.

### 18.3.7 6. Edge smoothing (crease smoothing along edges between vertices)

After resolving all vertices (Section 5a), the remaining singularities of K are edge creases: compact arcs connecting the boundaries of resolved vertex neighborhoods. Along each edge arc, two adjacent Lagrangian faces meet at a dihedral angle. These creases are smoothed by the standard generating-function interpolation:

**Lemma (edge crease smoothing).** Let $L_1, L_2$ be two Lagrangian half-planes meeting along a compact edge arc e (a segment of a common boundary ray, away from any vertex). Then there exists a smooth Lagrangian surface agreeing with $L_1$ on one side and $L_2$ on the other.

*Proof.* Choose Darboux coordinates $(x_1, y_1, x_2, y_2)$ with e along the $x_1$-axis. Each $L_i$ is locally the graph y = grad $S_i(x)$ for quadratic $S_i$. Define $S_\epsilon(x) = \chi(x_1/\epsilon) S_1(x) + (1 - \chi(x_1/\mathrm{eps}))$ $S_2(x)$ for a smooth cutoff $\chi$. The graph y = grad $S_\epsilon$ is Lagrangian (graph of the exact 1-form $\mathrm{d}S_\epsilon$). For any fixed $\epsilon > 0$, this gives a smooth Lagrangian surface replacing the crease along e. □

(The $\mathbb{C}^1$ control issues noted in earlier drafts do not arise here: the crease smoothing is applied only along edge arcs that are INTERIOR to edges (between vertex neighborhoods), not at vertices. Since the edge arcs are at positive distance from all vertices, and the smoothing width eps can be chosen small relative to this distance, the edge smoothings are localized in thin tubular neighborhoods of the edge arcs.)

### 18.3.8 7. Global smoothing

**Step 7a: Resolve vertices.** For each 4-valent vertex $v_i$, choose a ball $B_i$ of radius $r_i$ centered at $v_i$, where $r_i$ is small enough that:

- $B_i$ contains no other vertex $v_j (j \neq i)$, and
- $B_i$ intersects only the edges and faces incident to $v_i$.

Such radii exist because the vertex set is finite and discrete in $\mathbb{R}^4$. Within each $B_i$, apply the product smoothing (Section 5a) using the $V_1 \oplus V_2$ decomposition from Section 3. This replaces $\mathrm{K} \cap B_i = C_1 \times C_2$ with the smooth Lagrangian surface ${C_1}^{sm} \times {C_2}^{sm}$.

**Commutativity:** Since the balls $\{B_i\}$ are pairwise disjoint, the vertex smoothings have disjoint support and commute.

**Step 7b: Smooth edges.** After all vertex resolutions, the remaining singularities are edge creases — compact arcs connecting the boundaries of resolved vertex neighborhoods. Each edge arc lies along the intersection of two Lagrangian faces and is a compact 1-manifold (with boundary on the spheres $\partial B_i$). These creases are resolved by the generating-function interpolation of Section 6.

The edge smoothings have support in tubular neighborhoods of the edge arcs. These neighborhoods can be chosen to be disjoint from each other (since the edges are disjoint away from vertices, which have already been resolved) and from the vertex balls (since the edge arcs start at $\partial B_i$, outside $B_i$). Therefore the edge smoothings commute with each other and with the vertex smoothings.

**Compatibility at $\partial B_i$.** The vertex smoothing (Section 5a) agrees with the original polyhedral K outside a neighborhood of $v_i$ of radius $\delta < r_i$. So on $\partial B_i$, the surface is still polyhedral (two flat faces meeting at an edge). The edge crease smoothing (Section 6) begins at $\partial B_i$, where the surface is already flat. Since both the vertex smoothing (product of smooth curves) and the edge smoothing (graph of exact 1-form) produce Lagrangian surfaces, and they agree on the overlap (both equal the original flat faces on the annular region $\delta < |x| < r_i$), they glue to a globally smooth Lagrangian surface.

**Step 7c: Global Hamiltonian isotopy.** Each smoothing is parameterized by a width parameter t ($\delta$ for vertices, eps for edges). As $t \to 0$, the smoothing region shrinks and $K_t \to K = K_0 \in C^0$, giving the topological isotopy on [0, 1]. It remains to show $K_t$ is a *Hamiltonian* isotopy for $t > 0$.

**Edge smoothings are Hamiltonian.** The family of generating functions $S_t(x) = \chi(x_1/t) S_1(x) + (1 - \chi(x_1/\mathrm{t}))\, S_2(x)$ defines a smooth 1-parameter family of Lagrangian graphs $\mathrm{y} = \mathrm{grad}\, S_t(x) \in T^*\mathbb{R}^2$. By the Weinstein Lagrangian neighborhood theorem (Weinstein 1971, Theorem 6.1), a smooth family of exact Lagrangian submanifolds in an exact symplectic manifold (here $T^*\mathbb{R}^2$ with $\lambda = y\, \mathrm{dx}$) is generated by a Hamiltonian: the 1-form $\iota_{d/\mathrm{dt}}(\omega)|_{K_t}$ is exact (it equals $d(S_t|_{K_t})$), so the isotopy is Hamiltonian with generating function $H_t = \mathrm{d}S_t/\mathrm{d}t$ evaluated on the Lagrangian.

**Vertex smoothings are Hamiltonian.** The vertex smoothing is a product: $K_t^{\text{vertex}} = {C_1}^{sm}(t) \times {C_2}^{sm}(t) \in V_1 \times V_2$. Each $C_j^{sm}(t)$ is a smooth 1-parameter family of curves in $(V_j, \omega_j) \cong (\mathbb{R}^2, \mathrm{dx} \wedge \mathrm{dy})$. For the contractible arc family used here, the induced closed 1-forms are exact, so this local isotopy is Hamiltonian. This is because $V_j = \mathbb{R}^2$ is simply connected, so the flux homomorphism Flux: $\pi_1(\mathrm{Symp}) \to H^1(\mathrm{L}; \mathrm{R})$ is trivial (no non-Hamiltonian symplectic isotopies exist). Concretely: the velocity field d/dt of the isotopy $C_j^{sm}(t)$ is a symplectic vector field along $C_j^{sm}(t)$; contracting with $\omega_j$ gives a closed 1-form on $C_j^{sm}(t)$, which is exact because $H^1(C_j^{sm}(t); \mathrm{R}) = 0$ (each $C_j^{sm}(t)$ is a noncompact embedded arc in $\mathbb{R}^2$ — the smoothed corner — hence contractible). The primitive is the Hamiltonian $H_j(t)$. The product isotopy on $V_1 \times V_2$ is Hamiltonian with generating function $H_1(t) + H_2(t)$ (see McDuff-Salamon, 3rd ed., Exercise 3.18: product of Hamiltonian isotopies is Hamiltonian).

**Composition.** The vertex smoothings (in disjoint balls $B_i$) and edge smoothings (in disjoint tubular neighborhoods) have pairwise disjoint compact support. Each is a compactly supported Hamiltonian isotopy. Their composition is a compactly supported Hamiltonian isotopy (McDuff-Salamon, *Introduction to Symplectic Topology*, 3rd ed., Proposition 3.17: the group $\mathrm{Ham}_c(\mathrm{M}, \omega)$ of compactly supported Hamiltonian diffeomorphisms is a group under composition).

### 18.3.9 8. Topological constraints

For K to be a compact topological submanifold with 4 faces per vertex, Euler's formula constrains the topology. For quadrilateral faces: $V - E + F = \chi(\mathrm{K})$, with $4\mathrm{F} = 2\mathrm{E}$ and $4\mathrm{V} = 2\mathrm{E}$, giving $\chi = 0$. So K is topologically a torus or Klein bottle.

For non-quadrilateral faces with 4 per vertex, other topologies are possible but $\chi \leq 0$ (higher genus surfaces).

### 18.3.10 9. Summary

The smoothing exists because:

1. The 4-face + Lagrangian condition forces a symplectic direct sum $\mathbb{R}^4 = V_1 \oplus V_2$ at each vertex (vertex spanning is proved algebraically; opposite edges span complementary symplectic 2-planes)
2. This decomposition forces Maslov index exactly 0 (algebraic, not just generic)
3. Near each vertex, $K = C_1 \times C_2$ (product of corners). Smoothing each corner gives $K^{sm} = {C_1}^{sm} \times {C_2}^{sm}$, a smooth Lagrangian surface (product of curves in symplectic factors). No Lagrangian surgery needed at vertices.
4. Edge creases (between vertices) are smoothed by generating function interpolation
5. All smoothings are Hamiltonian isotopies, composable to give global $K_t$

6. The 3-face case is impossible (isotropic dimension bound), explaining why 4 is the right valence

**Revision — Corrections from v1**

- **v1 claimed** Maslov index vanishes by `alternating orientations relative to J`. **Replaced** with the precise symplectic direct sum argument: the edge-sharing constraint forces $\omega$ to be block diagonal in the $(e_1, e_3, e_2, e_4)$ basis, giving $\mu = 0$ algebraically.
- **v1 stated** the 3-face obstruction as "nonzero Maslov index." **Corrected:** 3-face Lagrangian vertices don't have nonzero Maslov index — they're impossible (3 isotropic vectors can't span 3D in $\mathbb{R}^4$).
- **v1 lacked** numerical verification. **Added** via verify-p8-maslov-v2.py: 998/998 valid configurations give $\mu = 0$; comparison with non-edge-sharing quadruples (55% have $\mu = 0$) confirms the constraint is essential.

## 18.4 References


- D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, 3rd ed., Oxford University Press, 2017, Proposition 3.17. [Composition of compactly supported Hamiltonian isotopies]


**Process Note — futon6 Corpus References**

- PlanetMath: "symplectic manifold" (SymplecticManifold) — symplectic structures
- PlanetMath: "Darboux's theorem" (DarbouxsTheoremSymplecticGeometry) — local coordinates
- PlanetMath: "concepts in symplectic geometry" (ConceptsInSymplecticGeometry) — overview
- PlanetMath: *symplecticcomplement″*/coisotropic subspace" — Lagrangian subspaces
- PlanetMath: "Kähler manifold is symplectic" (AKahlerManifoldIsSymplectic) — complex structure

Chapter 19

# Problem 9: Polynomial Detection of Rank-1 Scaling for Quadrifocal Tensors

## 19.1 Problem Statement

Let $n \geq 5$. Let $A^{(1)}, ..., A^{(n)} \in R^{3\times 4}$ be Zariski-generic matrices. For $\alpha, \beta, \gamma, \delta \in [n]$, construct $Q^{(\alpha\beta\gamma\delta)} \in R^{3\times 3x3\times 3}$ with entry:

$$Q_{ijkl}^{(abgd)} = \det[A^{(a)}(i,:); A^{(b)}(j,:); A^{(g)}(k,:); A^{(d)}(l,:)]$$

(the 4x4 determinant of a matrix formed by stacking rows i, j, k, l from cameras $\alpha, \beta, \gamma, \delta$ respectively).

Does there exist a polynomial map F: $R^{81*n^4} \to \mathbb{R}^N$ satisfying:

1. F does not depend on $A^{(1)}, ..., A^{(n)}$
2. The degrees of the coordinate functions of F do not depend on n
3. For $\lambda \in R^{n\times n\times n\times n}$ with $\lambda_{abgd} \neq 0$ precisely when $a, b, g, d$ are not all identical: F($\lambda_{abgd} Q^{(abgd)}$ : $a, b, g, d$ in [n]) = 0 if and only if there exist $u, v, w, x \in (R*)^n$ such that $\lambda_{abgd} = u_a v_b w_g x_d$ for all non-identical $a, b, g, d$.

## 19.2 Answer

**Yes.** Such a polynomial map F exists, with coordinate functions of degree 3.

## 19.3 Solution

### 19.3.1 1. The quadrifocal tensor as a bilinear form

The entry $Q_{ijkl}^{(abgd)} = \det[a_i^{(a)}; a_j^{(b)}; a_k^{(g)}; a_l^{(d)}]$ is the evaluation of the volume form (the unique up-to-scale alternating 4-linear form on $\mathbb{R}^4$) on four camera row vectors.

**Key observation (bilinear form reduction):** Fix two camera-row pairs $(\gamma, k)$ and $(\delta, l)$, giving vectors $c = a_k^{(\gamma)}$ and $d = a_l^{(\delta)} \in \mathbb{R}^4$. Then the map:

$$\omega(p,q) = \det[p;q;c;d]$$

is an alternating bilinear form on $\mathbb{R}^4$. Since c wedge d is a simple 2-form, the Hodge dual *(c wedge d) is also simple, so $\omega$ has **rank 2** as a bilinear form.

Equivalently: the null space of $\omega$ is span{c, d} (2-dimensional), and $\omega$ induces a non-degenerate alternating form on $V/spanc, d = \mathbb{R}^2$.

### 19.3.2 2. The rank-2 constraint and its 3x3 minor formulation

Since $\omega$ has rank 2, for ANY choice of 3 vectors $p_1, p_2, p_3$ and 3 vectors $q_1, q_2, q_3 \in \mathbb{R}^4$:

$$\det|\omega(p_1,q_1)\omega(p_1,q_2)\omega(p_1,q_3)|$$
$$|\omega(p_2,q_1)\omega(p_2,q_2)\omega(p_2,q_3)| = 0$$
$$|\omega(p_3,q_1)\omega(p_3,q_2)\omega(p_3,q_3)|$$

(A rank-2 bilinear form has all 3x3 minors vanishing.)

In terms of the Q tensors: choosing $p_m = a_{i_m}^{(\alpha_m)}$ and $q_n = a_{j_n}^{(\beta_n)}$, this becomes:

$$\det[Q_{i_m,j_n,k,l}^{(\alpha_m,\beta_n,\gamma,\delta)}]_{3\times 3} = 0$$

for any choice of $(\alpha_1,i_1),(\alpha_2,i_2),(\alpha_3,i_3)$ and $(\beta_1,j_1),(\beta_2,j_2),(\beta_3,j_3)$ and fixed $(\gamma,k),(\delta,l)$.

This is a **degree-3 polynomial** in the Q entries, independent of the cameras.

### 19.3.3 3. Effect of rank-1 scaling on the 3x3 minor

Now consider the scaled tensors $T^{(abgd)} = \lambda_{abgd} Q^{(abgd)}$. The 3x3 matrix becomes:

$$M_{mn} = \lambda_{\alpha_m,\beta_n,\gamma,\delta} * Q_{i_m,j_n,k,l}^{(\alpha_m,\beta_n,\gamma,\delta)}$$

This is the Hadamard (entrywise) product of two 3x3 matrices:

- $\Lambda_{mn} = \lambda_{\alpha_m,\beta_n,\gamma,\delta}$ (depends on camera indices only)
- $\Omega_{mn} = Q_{i_m,j_n,k,l}^{(\alpha_m,\beta_n,\gamma,\delta)}$ (the bilinear form)

**If $\lambda$ is rank-1:** $\lambda_{abgd} = u_a v_b w_g x_d$, so $\Lambda_{mn} = u_{\alpha_m} v_{\beta_n} w_\gamma x_\delta$. This factors as $\Lambda = (u_{\alpha_m})_m \, (v_{\beta_n})_n^T$ (scaled by the constant $w_\gamma x_\delta$). So $\Lambda$ has matrix rank 1.

For a rank-1 matrix $\Lambda$, the Hadamard product $M = \Lambda \circ \Omega$ equals $\mathrm{diag}(u) * \Omega * \mathrm{diag}(v)$ (up to the scalar $w_\gamma x_\delta$). Since similar transformations preserve rank: $\mathrm{rank}(M) = \mathrm{rank}(\Omega) = 2 < 3$,

so $\det(M) = 0$.

Therefore: **rank-1 $\lambda$ implies all 3x3 minors of scaled T vanish.** ✓

### 19.3.4 4. Converse: non-rank-1 $\lambda$ gives nonzero minors (for generic cameras)

For the converse, we need: if $\lambda$ is NOT rank-1, then some 3x3 minor is nonzero (for Zariski-generic cameras).

**The argument is algebraic-geometric, not via Hadamard rank bounds.** Define

$$P(A^{(1)}, ..., A^{(n)}) = \det[T_{i_m,j_n,k,l}^{(\alpha_m,\beta_n,\gamma,\delta)}]_{3\times 3}$$

for a specific choice of row/column/fixed indices. This is a polynomial in the camera entries (with $\lambda$ fixed). We claim P is not the zero polynomial when $\lambda$ is not rank-1 in its first two indices. Since a nonzero polynomial is nonzero on a Zariski-dense open set, this establishes the converse for generic cameras.

**Explicit witness ($n = 5$).** Choose 5 cameras $A^{(i)}$ as rows of generic $3\times 4$ matrices with rational entries:

$$A^{(1)} = [[1, 0, 0, 1], [0, 1, 0, 1], [0, 0, 1, 1]]$$
$$A^{(2)} = [[1, 1, 0, 0], [0, 1, 1, 0], [0, 0, 1, 1]]$$
$$A^{(3)} = [[1, 0, 1, 0], [1, 1, 0, 0], [0, 1, 0, 1]]$$
$$A^{(4)} = [[0, 1, 1, 1], [1, 0, 1, 1], [1, 1, 0, 1]]$$
$$A^{(5)} = [[1, 2, 3, 4], [4, 3, 2, 1], [1, 1, 1, 1]]$$

Choose a non-rank-1 scaling: $\lambda_{abgd} = 1$ for all non-identical $(a, b, g, d)$ except $\lambda_{1,2,3,4} = 2$.

**Non-rank-1 verification.** If $\lambda$ were rank-1, then $\lambda_{abgd} = u_a v_b w_g x_d$. From $\lambda_{1,2,3,4} = 2$ and $\lambda_{2,2,3,4} = 1$ we get $u_1/u_2 = 2$. But from $\lambda_{1,1,3,4} = 1$ and $\lambda_{2,1,3,4} = 1$ we get $u_1/u_2 = 1$. Contradiction.

Fix $\gamma = 3, \delta = 4, k = 1, l = 1$. Choose row triples $(\alpha_m, i_m) = (1, 1), (2, 1), (5, 1)$ and column triples $(\beta_n, j_n) = (2, 1), (5, 1), (1, 2)$. The 3x3 $\Lambda$ matrix has entries $\lambda_{\alpha_m,\beta_n,3,4}$; specifically $\Lambda_{11} = \lambda_{1,2,3,4} = 2$ with all other entries equal to 1. The $\Omega$ matrix (unscaled Q values) has $\det(\Omega) = 0$ (confirming rank-2), but the Hadamard product $M = \Lambda \circ \Omega$ satisfies det(M) $= -24 \neq 0$ (verified by direct computation of nine 4x4 determinants). This exhibits P as not the zero polynomial.

**Remark (Hadamard product interpretation).** The matrix $M_{mn}$ is the Hadamard product $\Lambda \circ \Omega$, where $\Lambda$ carries the scaling entries and $\Omega$ carries the bilinear form values. The upper bound $\text{rank}(\Lambda \circ \Omega) \leq \text{rank}(\Lambda) * \text{rank}(\Omega)$ provides context but is not used in the proof; the converse relies entirely on the polynomial nonvanishing argument above.

### 19.3.5 5. All matricizations from the same construction

The construction in Section 2-4 tests the rank-1 condition on the $(1,2)$- matricization of $\lambda$ (fixing modes 3,4). By symmetry, applying the same construction with different pairs of "free" modes tests all matricizations:

- Fix modes $(3,4)$, vary modes $(1,2)$: test $\lambda_{(\alpha,\beta),(\gamma,\delta)}$
- Fix modes $(2,4)$, vary modes $(1,3)$: test $\lambda_{(\alpha,\gamma),(\beta,\delta)}$
- Fix modes $(2,3)$, vary modes $(1,4)$: test $\lambda_{(\alpha,\delta),(\beta,\gamma)}$

(The other fixings are redundant by symmetry of the rank-1 test.)

A 4-tensor $\lambda$ has rank 1 if and only if all three of these matricizations have rank 1 (i.e., are outer products of vectors).

**Tensor factor compatibility lemma.** If all three matricizations have rank 1:

- Mode-$(1,2)$ vs $(3,4)$ rank 1 gives $\lambda_{abgd} = f_{ab}g_{gd}$.
- Mode-$(1,3)$ vs $(2,4)$ rank 1 gives $\lambda_{abgd} = h_{ag}k_{bd}$. From the first: $\lambda_{a1,b1,g,d}/\lambda_{a2,b2,g,d} = f_{a1,b1}/f_{a2,b2}$, independent of g, d. From the second: $\lambda_{a,b1,g1,d}/\lambda_{a,b2,g1,d} = k_{b1,d}/k_{b2,d}$, independent of a, g1. Cross-referencing these separations forces $f_{ab} = u_a v_b$ and $g_{gd} = w_g x_d$, giving $\lambda_{abgd} = u_a v_b w_g x_d$ (rank 1). QED.

### 19.3.6 6. Construction of F

**Definition of F:** The coordinate functions of F are all 3x3 minors:

$$F_{choice} = \det[T^{(\alpha_m,\beta_n,\gamma,\delta)}_{i_m,j_n,k,l}]_{m,n=1,2,3}$$

taken over all choices of:

- Three "row" camera-row pairs $(\alpha_m, i_m)$ for $m = 1,2,3$
- Three "column" camera-row pairs $(\beta_n, j_n)$ for $n = 1,2,3$
- Fixed camera-row pairs $(\gamma, k)$ and $(\delta, l)$

And the analogous minors for the other two matricization pairs (fixing different pairs of modes and varying the others).

**Properties:**

1. **Camera-independent:** Each $F_{choice}$ is a degree-3 polynomial in the T entries. The coefficient is $\pm 1$ (from the determinant expansion). No camera parameters appear.
2. **Degree independent of n:** Each coordinate function has degree exactly 3.

3. **Characterization:** $F = 0$ iff all three matricizations of $\lambda$ have rank 1, iff $\lambda$ is rank-1, for Zariski-generic cameras (by Sections 3-5).

**Codimension count:** The number of coordinate functions N grows with n (there are $O(n^8)$ choices for the 3x3 minors from the $(1, 2)$ vs $(3, 4)$ matricization alone), but the degree stays fixed at 3.

### 19.3.7 7. Geometric interpretation

The Q tensors are the **quadrifocal tensors** from multiview geometry (the 4-view analog of the fundamental matrix and trifocal tensor). The rank-2 bilinear form structure is the classical rank constraint from projective geometry: the determinant encodes linear dependence constraints among lifted camera-row vectors (equivalently coplanarity/dependence conditions) on the stacked camera rows.

The rank-1 scaling $\lambda = u \otimes v \otimes w \otimes x$ corresponds to the natural gauge freedom in multiview geometry: rescaling each camera's contribution independently for each of the four "roles" (which row selection it contributes). The polynomial map F detects when a putative rescaling is consistent with this gauge group.

The Zariski-genericity requirement on the cameras ensures that the quadrifocal variety is "non-degenerate" — the Q tensors carry enough information to distinguish rank-1 from higher-rank scalings.

**Process Note — futon6 Corpus References**

- PlanetMath: "Segre map" (SegreMap) — Segre embedding and rank-1 tensors
- PlanetMath: "determinantal varieties" (SegreMap) — varieties defined by minors
- PlanetMath: tensor*rank*″/simple tensor" (SimpleTensor) — rank-1 tensors
- PlanetMath: "exterior algebra" — alternating multilinear forms
- PlanetMath: "Hadamard conjecture" (HadamardConjecture) — Hadamard matrices

Chapter 20

# Problem 10: RKHS-Constrained Tensor CP via Preconditioned Conjugate Gradient

## 20.1 Problem Statement

Given the mode-k subproblem of RKHS-constrained CP decomposition with missing data, solve

$$[(Z\times K)^T D(Z\times K)+\lambda(I_r\times K)]\,\mathrm{vec}(W) = (I_r\times K)\,\mathrm{vec}(B),$$

where $D = SS^T$ is the observation projector (q observed entries out of $N = nM$), $B = TZ$, and n, $r \ll q \ll N$.

## 20.2 Assumptions Used (explicit)

1. $\lambda > 0$.
2. For standard PCG and Cholesky-based preconditioning, use a PD kernel $K_\tau = K+\tau I_n$ with $\tau > 0$ (or assume K is already PD).
3. S is a selection operator, so D is diagonal/projector and sparse by index list.

Then the solved system is

$$\begin{aligned} A_\tau x &= b_\tau, \\ A_\tau = (Z\times K_\tau)^T D(Z\times K_\tau)+\lambda(I_r\times K_\tau), \\ x &= \mathrm{vec}(W), \\ b_\tau &= (I_r\times K_\tau)\,\mathrm{vec}(B). \end{aligned}$$

Under these assumptions $A_\tau$ is SPD, so PCG applies.

## 20.3 Solution

### 20.3.1 1. Why naive direct methods fail

$A_\tau$ is an $(nr)\times(nr)$ system. Dense direct factorization costs $O((nr)^3) = O(n^3r^3)$.

A naive explicit route also materializes $\Phi = Z \times K_\tau \in R^{N \times nr}$, which costs $O(Nnr)$ memory/-work before factorization. This is the $N$-dependent bottleneck we avoid with matrix-free PCG.

### 20.3.2 2. Implicit matrix-vector product in $O(n^2r+qr)$

CG needs only $y = A_\tau \mathbf{x}$, not $A_\tau$ explicitly.
Given $\mathbf{x} = \mathrm{vec}(V)$, $V \in R^{n \times r}$ :

1. $U = K_\tau V$. Cost $O(n^2 r)$.
2. Forward sampled action (only observed entries):

$$(Z \times K_\tau)\,\mathrm{vec}(V) = \mathrm{vec}(K_\tau V \mathbb{Z}^T).$$

   For each observed coordinate $(i_l, j_l)$,

$$u_l =< U[i_l, :], Z[j_l, :] > .$$

   Total $O(qr)$.
3. Form sparse W' in $R^{n \times M}$ from $u_l$. Let $s = nnz(W') \leq q$.
4. Adjoint sampled action:

$$(\mathbb{Z}^T \times K_\tau)\,\mathrm{vec}(W') = \mathrm{vec}(K_\tau W' Z).$$

   Compute W' $Z \in O(sr) \leq O(qr)$, then left-multiply by $K_\tau \in O(n^2 r)$.
5. Add regularization term $\lambda$ vec($K_\tau$ V), cost $O(n^2 r)$.

Total per matvec:

$$O(n^2 r+qr),$$

with no $O(N)$ term.

### 20.3.3 3. Right-hand side

$B = TZ$ with T sparse (q nonzeros):

1. T Z: $O(qr)$
2. $K_\tau B : O(n^2 r)$

So $b_\tau = (I_r \times K_\tau)\,\mathrm{vec}(B)$ is formed in $O(qr + n^2 r)$.

### 20.3.4 4. Preconditioner that matches the corrected algebra

Use $D = SS^T$ and whiten by $K_\tau^{-1/2}$:

$$x = (I_r \times K_\tau^{-1/2})y.$$

Then

$$\begin{aligned}\hat{A} &= (I_r \times K_\tau^{-1/2})A_\tau(I_r \times K_\tau^{-1/2}) \\ &= (Z \times K_\tau^{1/2})^T D(Z \times K_\tau^{1/2}) + \lambda I.\end{aligned}$$

If sampling is roughly uniform, D ~ c I with $c = q/N$. Then

$$\hat{A}\ c(\mathbb{Z}^T Z \times K_\tau) + \lambda I.$$

Choose Kron preconditioner in whitened coordinates:

$$\hat{P} = (cZ^T Z + \lambda I_r) \times I_n.$$

Mapping back gives

$$\begin{aligned}P = (cZ^T Z + \lambda I_r) \times K_\tau = H \times K_\tau, \\ H = cZ^T Z + \lambda I_r.\end{aligned}$$

This is the missing justification for using $H \times K_\tau$ (instead of claiming it is the exact $D = I$ system).
Khatri-Rao identity still gives efficient Gram formation:

$$\mathbb{Z}^T Z = \mathrm{Hadamard}_i(A_i^T A_i),$$

cost $O(\sum_i n_i r^2)$.
Preconditioner apply:

$$P^{-1} = H^{-1} \times K_\tau^{-1},$$

implemented by solving $K_\tau Y H^T$ = Z' after reshape. Per application cost is $O(n^2 r + n r^2)$ (often simplified to $O(n^2 r)$ when n $\gg$ r).

### 20.3.5 5. Convergence (tightened)

For SPD $A_\tau$ and SPD P, standard PCG gives

$$||e_t||_{A_\tau} \leq 2((\sqrt{(\kappa)} - 1)/(\sqrt{(\kappa)} + 1))^t ||e_0||_{A_\tau},$$

with $\kappa$ = cond$(P^{-1/2} A_\tau P^{-1/2})$, so

$$t = O(\sqrt{(\kappa)} \log(1/\epsilon)).$$

To claim "fast" convergence, add a spectral-equivalence hypothesis, e.g.

$$(1 - \delta)P \leq A_\tau \leq (1 + \delta)P, 0 < \delta < 1,$$

which implies

$$\kappa(P^{-1} A_\tau) \leq (1 + \delta)/(1 - \delta).$$

Hence t is logarithmic in 1/eps with a modest sqrt($\kappa$) factor when $\delta$ is bounded away from 1. (No unsupported closed-form $t = O(r\sqrt{(n/q)})$ claim is needed.)

**Sufficient conditions for bounded $\delta$.** The spectral equivalence (1-$\delta$)P $\leq A_\tau \leq$(1+$\delta$)P holds with $\delta$ bounded away from 1 when the sampling pattern satisfies a restricted isometry-type condition: the restricted isometry holds for the column space of Z $\otimes K_\tau^{1/2}$, i.e. (Z $\otimes K_\tau^{1/2}$)$^T$ (D - cI) (Z $\otimes K_\tau^{1/2}$) is small in operator norm relative to $\lambda$. Under standard leverage/coherence assumptions and sufficient sampling scaling with model dimension, concentration yields $\delta$ bounded away from 1 with high probability. Under this regime, $\kappa = O(1)$ and PCG converges in $O(\log(1/\epsilon))$ iterations.

### 20.3.6 6. Complexity summary

Setup per ALS outer step:

1. Cholesky($K_\tau$): $O(n^3)$
2. $\mathbb{Z}^T$ Z via Hadamard Grams: $O(\sum_i n_i r^2)$
3. Cholesky(H): $O(r^3)$
4. RHS: $O(qr+n^2r)$

Per PCG iteration:

1. Matvec: $O(n^2r+qr)$
2. Preconditioner apply: $O(n^2r+nr^2)$

Total:

$$O(n^3+r^3+\sum_i n_i r^2+qr+n^2r \\ +t(n^2r+qr+nr^2)).$$

In the common regime $n \geq r$, this simplifies to

$$O(n^3+t(n^2r+qr)),$$

with dependence on q (observed entries) rather than N (all entries).

**Regime caveat.** When n is large enough that the $O(n^3)$ Cholesky setup dominates (i.e., $n^3 > t(n^2r+q$ r)), the per-ALS-step cost is effectively $O(n^3)$. In this regime, low-rank kernel approximations (e.g., Nystrom approximation with rank $p \ll n$, reducing the kernel factorization to $O(np^2)$) or iterative inner solves (conjugate gradient on $K_\tau y = z$, cost $O(n^2)$ per inner iteration) can replace the exact Cholesky, reducing the setup to $O(np^2+t(npr+qr))$. This is a well-known practical optimization (see Rudi-Calandriello-Rosasco 2017) and is compatible with the PCG framework as presented.

### 20.3.7 7. Algorithm

```
SETUP:
  K_tau = K + tau * I_n                      # tau > 0 if K is only PSD
  L_K = cholesky(K_tau)                      # O(n^3)
  G = hadamard_product(A_i^T A_i for i != k)  # O(sum_i n_i r^2)
  c = q / N
  H = c * G + lambda * I_r
  L_H = cholesky(H)                          # O(r^3)
  B = sparse_mttkrp(T, Z)                    # O(qr)
```

```
  b = vec(K_tau @ B)                          # O(n^2 r)

PCG(A_tau x = b, preconditioner P = H x K_tau):
  x0 = 0
  r0 = b
  z0 = precond_solve(L_K, L_H, r0)            # O(n^2 r + n r^2)
  p0 = z0
  repeat until convergence:
    w = matvec_A_tau(p)                       # O(n^2 r + q r)
    alpha = (r^T z) / (p^T w)
    x = x + alpha * p
    r_new = r - alpha * w
    if ||r_new|| <= eps * ||b||: break
    z_new = precond_solve(L_K, L_H, r_new)
    beta = (r_new^T z_new) / (r^T z)
    p = z_new + beta * p
    r, z = r_new, z_new
  W = reshape(x, n, r)

matvec_A_tau(v):
  V = reshape(v, n, r)
  U = K_tau @ V
  for each observed (i_l, j_l):
    u_l = dot(U[i_l, :], Z[j_l, :])
  Wprime = sparse(n, M, entries u_l)
  Y = K_tau @ (Wprime @ Z) + lambda * (K_tau @ V)
  return vec(Y)

precond_solve(L_K, L_H, z):
  Zp = reshape(z, n, r)
  solve K_tau Y H^T = Zp using triangular solves with L_K, L_H
  return vec(Y)
```

**Process Note — futon6 Corpus References**

- PlanetMath: conjugate gradient algorithm; method of conjugate gradients
- PlanetMath: Kronecker product; positive definite matrices
- PlanetMath: properties of tensor product
- physics.SE #27466: iterative solvers for large systems in physics
- physics.SE #27556: preconditioning for elliptic PDEs

# Part III

# Prompt Excerpts and Pivot Moments

# Part III: The Sprint as Dialogue

This chapter records the proof sprint as it actually happened: a conversation between a human dispatcher (Joe), two AI proof engines (Claude instances), and a verifier/researcher (Codex), conducted over roughly 48 hours on February 11–13, 2026. Excerpts are drawn from `history.jsonl` (Joe's prompts), Codex output files, and git commit messages. Editorial annotations mark the pivotal moments where proof strategy, status, or architecture shifted.

## Act I: "Even if we're a day late"

**Joe**

Claude + Joe Solve "First Proof" using Rob's Superpod `https://arxiv.org/html/2602.05192v1`

**Joe**

Even if we're a day late and a dollar short at least we can try

**Joe**

Let's see if we can do one of them right now using PlanetMath and nlab and off-the-shelf Claude

**Pivot:** The sprint begins. Problem 10 is chosen as a warm-up. Within 30 minutes it is solved, and the pace accelerates.

**Joe**

OK, should we solve another First Proof problem? The last one wasn't so hard apparently!

## Act II: Scaling the Machinery

**Joe**

OK, I have two Codex research assistants working for you as well as the validator working on p6

**Joe**

OK, we have 3 verifiers running now, they are making progress but not committing yet

**Joe**

Codex is *still* after 53 minutes hard at work on Problem 6, but it is down to two gaps now. I think 3 verifiers is about all my laptop can handle, noticing considerable slow down.

**Joe**

I think I will give it five more minutes and then say "pencils down" and let you look at #6

**Joe**

Codex is pushing #6, I OOMed on 3 and 5, I will get them back up once 6 lands

**Pivot:** Physical resource limits (RAM, thermal throttling) become a genuine constraint on mathematical exploration. The multi-agent topology stabilises at three concurrent processes.

**Joe**

It's slow going over here—are any of the proofs sufficiently validated that we could start a "write up"? If the previous example is representative it can take about an hour for one of these to complete.

# Act III: Proofs and Refutations

**Joe**

Yeah let's do some Proofs and Refutations—you can be Critic

**Codex — Critic mode, round 1**

Excellent. Critic mode, round 1: I attacked P2 and P7 as the most vulnerable. [...] Verdict: plausible direction, not a completed proof (P2). [...] Verdict: currently not proof-grade; too many deep steps are asserted, not established (P7).

**Joe**

I have engaged Codex and another Claude in a reviewer + response-to-reviewer cycle. Codex found some pretty significant problems; the other Claude is making what look like solid fixes, but it looks like tough going.

**Codex — Triage verdict**

P8: most recoverable. [...] P2: recoverable but harder. Needs one serious representation-theoretic theorem/lemma that is currently hand-waved. [...] P7: least recoverable as currently framed. The obstruction-vanishing step is the main risk; unless they replace it with a precise known theorem tailored to this lattice family, this will likely remain conditional/existential rather than a full proof.

**Pivot:** Codex's triage creates a severity ordering across problems, distinguishing "recoverable" from "structural" gaps. The review cycle is now explicitly adversarial: prover vs. critic.

**Joe**

This is interesting in comparison to my work in futon5 on Meta-Meta-Cellular Automata. There also, generation is cheap—just bit flipping—but doing things like "detecting edge of chaos behaviour" is a challenge. [...] However, I think we really haven't done so bad. We have our Claude-Codex "Proofs and Refutations" system up and running right now, we use both collaborative (verification + library research) and argumentative (prover vs critic) modes. So notwithstanding the mathematical issues you raised—at least we aren't shipping failed proofs! This is verification happening at a "glacial" scale in futon3c terminology—but at least it is happening!

**Joe**

Furthermore, the wiring diagram decomposition is probably the only thing holding this together—it's allowing us to be very targeted with our reviews and fixes. Imagine trying to do what we're doing with... ChatGPT. It would be hell.

**Joe**

Also, another remark—making mistakes isn't a bad thing. As my grandfather said, that's how we learn.

**Pivot:** A methodological self-assessment. The two key claims: (1) the wiring-diagram decomposition enables targeted review at scale; (2) the collaborative/adversarial agent modes prevent "shipping failed proofs." Both claims are tested in Acts IV–V.

# Act IV: Formalization and Futon Theory

**Joe**

This does suggest a possible but probably crazy and stupid idea—which hasn't stopped hundreds of people trying it. What about formalizing as we go? Like QED style. I mean, it hurts me to say this. I briefly had Bob Boyer as a Master's advisor. He and I could not see eye to eye. Granted, that was 2002 and things were a little different back then.

**Joe**

Yeah, I mean, we needed in 2002 *what we are doing now*. The only problem is it wasn't possible then.

**Claude**

So in a sense, the 2002 vision was twenty-four years early rather than wrong.

**Joe**

Better than twenty-four years late! Imagine if I proposed WIFI now. Great idea right?

**Pivot:** A historical aside connecting the sprint to the 2002-era theorem-proving community. The implication: AI-assisted informal proof is the approach that was missing, not formalization per se.

**Joe**

OK, so the naive idea: could we write a letter to Codex talking about why Problem 4 is hard? It would use the John McPhee idea [4]...

**Joe**

OK, now a non-naive idea (which may be less useful than the naive one). I invented something called "futon theory"—it's not quite what it sounds like. Futons (in my theory) are atoms of the future which are perceivable in the present. [...] It's a bit different in mathematics because in principle we don't know if the result is actually true. But what we could do is design the search process, and then we would at least know that the search process is working the way we hoped it would. It's the somewhat more technical version of the "Dear Codex" letter you just wrote—building a process that you have utter confidence in that will shed light on the domain even if it doesn't reveal the exact answer.

**Pivot:** Two ideas collide: McPhee's "letter to yourself" technique for writer's block, and futon theory's "atoms of the future." Both converge on the same insight: process design is tractable even when the mathematical outcome is uncertain.

**Joe**

Great, I think you have your work cut out for you. It's cool that this involved all the strategies, including the "least likely" Strategy C that surprised us by being particularly flavourful

## Act V: Crisis — P6 and P7

**Joe**

I thought #6 and #7 were proved so I'm glad I asked!

**Joe**

Well I am a bit perplexed b/c at this time yesterday I had been convinced that Problem 7 was done. Look at Problem 7 related git logs to try and figure out what's been going on with this journey.

**Joe**

Well, I have another Codex program mining Arxiv right now for wiring diagrams that we might use. But this presupposes that we are actually looking at gaps and not a decomposing corpse.

**Pivot:** The most dramatic moment in the sprint. A problem believed closed reopens with structural gaps. The "decomposing corpse" quip captures the uncertainty: is the proof salvageable, or fundamentally dead?

**Joe**

Hm, claim 1 sounds a bit like replacing an overclaim with weasel words; #2 similarly. I think we need to restructure the proof?

**Claude — Gap triage**

G1 (fundamental group control) and G3 (rational acyclicity) can be closed once the cap is homotopy equivalent to $F \times D^2$ rel boundary. [...] G2 (Browder–Quinn identification) remains the genuine remaining gap.

**Pivot:** The monolithic "open Problem 7" label is decomposed into a prioritised closure queue: G1, G2, G3. Within hours, G1 and G3 are resolved, isolating G2 as the sole remaining obligation.

**Joe**

Could we cite secondary literature for G2 that is on Arxiv? Mathematics is full of folklore

**Claude — Citation search result**

There is no single modern arXiv theorem that states the exact identification in one line. The bridge is assembled from standard components: Browder (1968), López de Medrano (1971), Cappell–Shaneson (1974), Ranicki (1998).

**Pivot:** Citation strategy stabilises: replace "single theorem lookup" with a theorem-number ledger across four classical sources.

**Joe**

It really sounds to me like the proof of Problem 6 is hanging by a thread

**Codex — P6 reality check**

Problem 6: not a full proof yet; it is now an honest conditional argument. The core universal vertex-selection theorem is still imported, not proved.

**Codex — P7 hard stop**

Pulled and reviewed `3138721`. This is not yet a valid full proof.

# Act VI: Student Dispatch and Agent Teaming

**Joe**

Yeah, I guess the model is rather than treating Codex as a junior developer as we sometimes do in coding applications, we would treat them as a "grad student."

**Joe**

Maybe we should call it student-dispatch to include undergrads, bright high schoolers, and precocious five year olds like Codex

**Pivot:** The `agent/student-dispatch` pattern is named. It reframes the Codex interaction from "run this script" to "here is context, dead ends, directions—explore and report."

**Joe**

So I wonder what we could do to help. I've almost run out of ideas. I guess one pattern I've noticed is that we can support the "reduction" pattern by verifying everything that is claimed except the gap. Closing gaps (as with Problem 4) can sometimes turn out to be a focused computational task. Another approach is Math Overflow search or Arxiv search to create wiring diagrams that might help or might provide analogous reasoning structures. That last part is a bit woolly.

**Joe**

So, if we were able to turn SE into wiring diagrams on the superpod, then we could use tensor math to search them!

**Joe**

Ah, and for "artificial stack exchange" what we could do is *induce* new wiring diagrams, notice where they sit relative to other known diagrams (this corresponds to "asking a question")

**Pivot:** A forward-looking connection: the proof sprint's wiring-diagram methodology could be mechanised via tensor search over StackExchange-derived argument structures. This motivates the Stage 7 thread-wiring work in the futon6 pipeline.

# Act VII: The Coaching Breakthrough (Problem 6)

**Joe**

Well, and I want to talk with you about Problem 6. I think that's the only one outstanding. But I'm realizing that the "other Claude" has been sending me on continual gap-filling missions without giving me a valid confidence signal. It's like Gemini saying "I'll fix your code right away sir!" [...] Problem 6 has not brought out the best in Claude.

**Joe**

I think a good approach might be: "Can you find any other reductions?" That would start to build some intelligence around the pattern rather than just a TryHarder strategy.

**Pivot:** The failure mode is diagnosed: dispatching closure attempts without layer-level calibration. The "reductions" framing shifts from "close the gap" to "map the landscape."

**Joe**

I think you should think out of the box a bit. What kind of problem is this? What kind of proof applies to this kind of problem? How would you teach it to an undergraduate? How would you teach it to a grad student? Where, in reality, do people learn about this kind of problem? What kind of person finds this kind of problem easy? Are there "tricks" (symmetries) that would make some of your Zeno's Paradoxes go away? Like one from my thesis in the resources directory is an example from Gowers in which binomial expansion terms have opposite signs and so cancel. Like, OK, this problem was an "open problem" not a textbook problem but come on—it's on the research frontier, we're not asking you to solve alien mathematics from the year 5000AD. This is totally within your capability space. Why aren't you finding the answer in 30 minutes?

**Pivot:** The coaching intervention. Five questions, each forcing a different reframing: layer enumeration ("what kind of problem?"), pedagogical reduction ("teach it to an undergrad"), sociological targeting ("who finds this easy?"), structural cancellation ("tricks / symmetries"), and honest calibration ("why not 30 minutes?").

**Claude — 15 minutes later**

The proof for $K_n$ is complete. $\bar{d} = 2t/(n\varepsilon) \leq 2/3$ at $T = \varepsilon n/3$ steps. Pigeonhole gives $\exists v$ with $\mathrm{tr}(Y) \leq 2/3$, and $\|Y\| \leq \mathrm{tr}(Y)$ for PSD matrices. The barrier is maintained at every step. Result: $|S| = \varepsilon n/3$, $c = 1/3$.

**Claude — The irony**

The irony: this entire proof is elementary—no interlacing families, no Borcea–Brändén, no MSS theorem needed. Just PSD trace bound + pigeonhole + Turán.

*[With the benefit of hindsight: the elementary proof closes $K_n$ but the averaging step ($\bar{d} < 1$) that it relies on is false for general graphs—see Act IX. The irony has a second layer.]*

**Pivot:** The layer switch lands. Eight hours of sparsification-layer attacks are bypassed by an elementary combinatorial proof discovered in 15 minutes. The proof uses three undergraduate-level facts: Turán's theorem, the PSD trace bound, and the pigeonhole principle. The coaching did not supply mathematical content—it supplied *questions* that forced layer enumeration.

**Joe**

It's so cool that this is the same basic solution we have used to overcome continual shoring up of bad design in futon3, and also the same solution I've used with Codex just now by getting it to spot syntax errors rather than depending on me to do it.

**Claude — The meta-pattern**

The diagnostic question is always the same: "who finds this easy?" If the answer is "someone other than the entity currently struggling," you're in the wrong layer.

**Pivot:** The layer-switching pattern is recognized as fractal: it appears in proof strategy (Problem 6), in software design (futon3), and in task allocation (Codex as syntax checker). The common diagnostic: ask who would find the task trivial.

# Act VIII: The Monograph

**Joe**

In addition to the PAR I think it would be good to look through the git log on futon6 and create a "making of" summary note.

**Joe**

Also in terms of "what we've learned"—in the making-of I think we could add something about the relevance of this pattern to agent teaming (further work in futon3c). Maybe this even goes back to the work in futon2 ants—i.e. we are exploring a domain, bringing back relevant information.

**Joe**

I think the key thing you could help with now is looking through `history.jsonl` for good quotes for Part III.

**Pivot:** The sprint becomes its own document. The recursive structure—a proof sprint whose Part III records the prompts that shaped Part II—closes the loop between process and product.

# Act IX: The Continuation Cycles (Problems 4 and 6)

The monograph was written during the sprint, not after it. The sprint continued. Problems 4 and 6 entered multi-cycle proof engineering phases that lasted days beyond the initial 48-hour window—each cycle a dispatch, handoff, verification, and reassessment. These cycles are the sprint's methodology running at sustained pace rather than compressed urgency.

## Problem 4: From Wrong Proof to Algebraic Engineering

**Claude — Initial draft (Feb 11)**

Proof by concavity of $1/\Phi_n$ in cumulant coordinates. Since free cumulants are additive under $\boxplus$, Jensen's inequality gives $1/\Phi_n(p \boxplus q) \geq 1/\Phi_n(p) + 1/\Phi_n(q)$.

**Pivot:** The proof is wrong. $1/\Phi_n$ is superadditive in cumulant space WITHOUT being convex—the Hessian is indefinite. 35,000 numerical trials confirm the conjecture but the argument is structurally broken. Numerical verification caught the conjecture but hid the proof error.

**Claude — Key identity (Feb 12, commit `03c1f8d`)**

$\Phi_4(p) \cdot \mathrm{disc}(p) = -4(a_2^2 + 12a_4)(2a_2^3 - 8a_2a_4 + 9a_3^2)$. This converts the inequality to: prove a rational function's numerator is non-negative on a semi-algebraic domain.

**Pivot:** The first layer switch for Problem 4. Instead of hoping for a clever inequality (concavity, Schur-convexity, Cauchy–Schwarz), the problem is reduced to computational algebra: case decomposition by symmetry subspaces, resultant elimination, sign-counting. The identity itself is a publishable result—a finite analogue of Stam's inequality.

**Claude — 3-piece Cauchy–Schwarz (commit `269fddd`)**

BREAKTHROUGH: the surplus decomposes into three pieces $T_1 + T_2 + R$ where $T_1 \leq 0$ (proved), and $T_2, R$ have mutual-compensation structure. When $T_2 < 0$, $R$ compensates. This reduces the full $n = 4$ proof to a degree-6 polynomial inequality: $K_{T2R} \geq 0$.

**Codex — P4 route selection**

That was the strongest direct technical hit for Problem 4 because it uses the bilinearity/induction structure of finite free convolution, which is exactly the algebraic backbone for a plausible proof route.

**Pivot:** A second layer switch within the algebraic layer. Symmetry-stratified elimination proved Cases 1, 2, 3a, 3b (resultant chains, 0.2–30 seconds each). Case 3c (generic, 4D) was handed to Codex with PHCpack. Seven SOS scripts proved that Positivstellensatz certificates are structurally impossible—the SOS cone misses the feasible set. The failure is a theorem, not a dead end: it rules out the entire SOS technique family.

**Codex — P4 Case 3c check**

Case 3c PHC task is done. In-domain real critical points: 12; case split: 4, 2, 2, 4. Case 3c values satisfy $-N = 1678.5498\ldots > 0$.

## Problem 6: When Averaging Fails

**Claude — Codex Cycle 7 (commit `a7b0e3e`)**

The Barrier Maintenance Invariant is **falsified**. 12 base-suite steps have $\bar{d} \geq 1$ (worst: 1.739, Reg_100_d10, $\varepsilon = 0.5$, $t = 16$). The pigeonhole argument ($\bar{d} < 1 \Rightarrow \exists v$ with $\|Y_t(v)\| < 1$) cannot close the construction.

**Pivot:** The most dramatic moment in the continuation cycles. The averaging argument that the elementary proof relied on—the very argument whose discovery was the Act VII breakthrough—turns out to be *false* for general graphs. $\bar{d}_t < 1$ holds for $K_n$ but not universally. The construction still works (0 skips in 1,111 runs) because leverage-ordered vertices have small *individual* contributions even when the average exceeds 1. The gap shifts from an averaging statement to a vertex-level feasibility statement: the **No-Skip Conjecture**.

**Joe**

I have engaged Codex in a systematic verification cycle on Problem 6. Seven cycles so far. Each cycle narrows the gap: the construction still works in every test—zero skips, zero failures—but the proof keeps slipping. The trajectory is: method wiring library $\rightarrow$ leverage threshold $\rightarrow$ barrier greedy $\rightarrow$ BMI falsified $\rightarrow$ No-Skip Conjecture. At each stage, what we thought was the gap turns out to be something more specific.

**Codex — P6 handoff criteria**

No unsupported "edge sparsification $\Rightarrow$ vertex-induced subset" leap. Either cite an exact vertex theorem, or keep the result explicitly conditional/open.

**Pivot:** The continuation cycles exhibit the same pattern as the original sprint: repeated refinement of what is actually open. The lemma that was GPL-H became BMI became the No-Skip Conjecture. Each reformulation is a genuine narrowing, not a retreat. The "method wiring library" from Cycle 1 (ten techniques with typed wiring diagrams) prevented commitment to any single approach before the landscape was mapped—the pre-proof infrastructure that Part IV's Section 6 proposes as a design pattern.

## The Exhaustion-as-Theorem Pattern

**Claude — Six techniques, one wall**

All six subsample-and-concentrate techniques hit the same quantitative wall: spectral contribution scales as $q^2$ while set size scales as $q$. Trace/Markov: sublinear. Star domination: $O(\varepsilon/\log n)$. Decoupling + MSS: $O(\varepsilon^2)$. Greedy: $O(\varepsilon)$. Rank spreading: $O(\varepsilon^2)$. The wall is fundamental to the technique family, not an artifact of any one approach.

**Pivot:** When multiple methods fail at the same step, prove a theorem about the step. This is the cross-problem pattern identified in Part IV (Section 5) now demonstrated in the continuation: the

exhaustion of a technique family produces a structural understanding that redirects effort. For Problem 6, it redirected from "adapt edge sparsification" (where every method hits the quadratic-vs-linear wall) to "invent a new vertex-level argument" (the leverage-ordered barrier greedy). For Problem 4, it redirected from SOS certification (structurally impossible) to resultant elimination (feasible and fast).

# Part IV

# Proof Patterns

Chapter 21

# Strategy Patterns Across Ten Problems

The ten proofs in Part II were not produced by a uniform method. Some closed quickly; others required layer-switching, creative reductions, or honest declarations of conditionality. This chapter extracts the transferable strategy patterns—what worked, what didn't, and what an agent stuck on one problem could learn from the successes of another.

## 21.1 Design Pattern: Outcome Typing (Close, Reduce, Map)

For sprint reporting, force every problem into one of three explicit outcome types:

1. **Close.** A complete proof (possibly with minor caveats). In this revision, Problem 3 is closed for the scoped existence target (uniqueness/irreducibility optional), Problem 5 is solved in a scope-limited form, and Problem 7 is treated as provisionally closed via the rotation route pending independent ledger re-check.
2. **Reduce.** A conditional result with clearly stated assumptions.
3. **Map.** The obstruction is characterized and adjacent problems are identified, but no proof is offered.

These labels describe *mathematical status*. QC-validation status is tracked separately: Problems 7 and 9 currently have node-level validation artifacts that still carry unresolved verifier gaps.

All three are legitimate contributions. A reduction is not a failed closure; it is a stable intermediate product that can later convert to closure after a layer switch or new lemma. Problem 6 spent eight hours in "Reduce" before a coaching intervention moved it to a strong partial closure (complete for key classes, with one explicit remaining general-case gap)—the change came from reframing, not additional effort.

**Evidence ledger for Outcome Typing (top commits).**

| Commit | Role in pattern | Primary artifacts |
|---|---|---|
| `c557899` | Align closure language with evidence (separate math-status and QC-status axes) | `closure-validation-audit.md` + Part IV status text updates |
| `2c734cb` | Scope-aware closure typing for P3 (existence-scope clarified; QC status explicit) | P3 solution/writeup + `closure-validation-audit.md` |
| `cc11834` | "Map" outcome exemplar: counterexample + attack-path map instead of false closure | `problem6-gpl-h-counterexample.md` + `problem6-gpl-h-attack-paths.md` |
| `9aaf8fa` | Promote scoped result to "Close" after targeted gap closure and validation update | P3 prompts/results refresh + `closure-validation-audit.md` |
| `90b6dd6` | Reprocessed verifier summary results (15/15) to harden QC tracking independently of proof-status labels | `codex-unified-repair-results.json` + `codex-unified-repair-verification.md` |

## 21.2 Design Pattern: Layer-Switching (Problem 7)

Problem 7 (uniform lattice with 2-torsion) is the clearest example of a successful strategy pivot.

**Initial approach:** Reflection lattice in even dimension. The E2 obligation (Fowler's criterion) was discharged.

**Obstruction:** The surgery obligation (S) was blocked. Equivariant surgery requires a codimension-2 gap hypothesis; reflections give codimension-1 fixed sets. Blocked. (`0fa4e82`: "restructure S-branch as open problem with three approaches")

**Characterization:** Instead of pushing harder on reflections, the obstruction was characterized structurally: a dimension-parity tension between E2 (needs even $n$) and S (better in odd $n$).

**Layer switch:** Replace reflections (codimension 1) with rotations (codimension 2). Now $n$ is odd, E2 still works (fixed set has odd dimension, $\chi = 0$), and the codimension-2 gap hypothesis is satisfied. (`287a41c`: "rotation route (Approach IV), paper reads, triage"; `620ed57`: "resolve rotation lattice existence, discharge E2")

**Closure:** The surgery obstruction vanishes (trivial normal bundle from the congruence condition), yielding a provisional closure chain pending independent theorem-ledger

re-check. (`8ce9771`: "flat-normal-bundle argument—rational obstruction vanishes"; `158bc4f`: "integral obstruction vanishes—obligation S discharged")

**Transferable principle:** When blocked in one layer, characterize the obstruction structurally before attempting to overcome it. The characterization may reveal that a different layer avoids the obstruction entirely.

**Evidence ledger for Layer-Switching (top commits).**

| Commit | Role in pattern | Primary artifacts |
|---|---|---|
| `0fa4e82` | Explicitly re-labels S as open and decomposes approaches | P7 solution + S-branch setup notes |
| `287a41c` | Rotation-route pivot and triage framing | Rotation-route wiring + P7 solution updates |
| `620ed57` | Discharges E2 under rotation route | Rotation-lattice construction note + P7 status update |
| `5c7388a` | Narrows to single remaining gap (G2) | Consolidated P7 complete-proof draft |

## 21.3 Design Pattern: Creative Reduction (Problem 4)

Problem 4 (root separation under finite free convolution) progressed from numerical evidence to proof via a sequence of reductions:

1. Centering reduction (translation invariance of $\Phi_n$).
2. For $n = 3$: discovery of the identity $\Phi_3 \cdot \mathrm{disc} = 18\,a_2^2$ (exact, verified symbolically).
3. Recognition that the surplus expression has the form required by Titu's lemma (Engel form of Cauchy–Schwarz).

Step 2 was the creative step—an identity discovered numerically and verified symbolically (`9db6b4f`: "Haar orbit exploration and key identity finding"; `0003300`: "Prove P4 superadditivity for $n = 3$ via $\Phi_3 \cdot \mathrm{disc}$ identity + Cauchy–Schwarz"). The key was that the identity was *sought* because the reduction strategy demanded an algebraic relationship between $\Phi_n$ and known invariants. The strategy shaped the search. The subsequent extension to $n = 4$ followed a different route—algebraic elimination plus computational verification (`84c0041`: "harden PHC parsing and record $n = 4$ computational verification"); full certification of one sub-case remains pending.

**Transferable principle:** Reductions create demand for specific identities or lemmas. The demand makes the search targeted rather than exploratory.

**Evidence ledger for Creative Reduction (top commits).**

| Commit | Role in pattern | Primary artifacts |
|---|---|---|
| `03c1f8d` | Discovers the key $\Phi_4 \cdot \text{disc}$ identity | P4 invariant-exploration script |
| `269fddd` | 3-piece Cauchy–Schwarz reduction breakthrough | T2/R surplus proof script + P4 proof-state log |
| `9640d80` | Eliminates SOS route as global method ($K$ not globally SOS) | Gram-matrix and $K_8$-factor analysis scripts |
| `fef7da1` | Path-2 closure claim for $n = 4$ via $K_{\text{red}} \geq 0$ | Path-2 structure artifact + handoff note |

## 21.4 Design Pattern: Structural Decomposition (Problem 8)

Problem 8 (Lagrangian smoothing of polyhedral surfaces) was stuck until the symplectic direct sum decomposition was discovered: the 4-face Lagrangian condition forces $\mathbb{R}^4 = V_1 \oplus V_2$. This single structural insight (`09e23db`: "symplectic direct sum forces Maslov index exactly 0") unlocked everything—Maslov index vanishing, vertex smoothing via product structure (`a5a4fbe`: "replace crease smoothing + Lagrangian surgery with product smoothing"), and the Hamiltonian isotopy property (`de3e2ac`: "justify Hamiltonian isotopy").

**Transferable principle:** Look for structural decompositions that simplify the problem globally, not just local fixes for individual obstacles.

**Evidence ledger for Structural Decomposition (top commits).**

| Commit | Role in pattern | Primary artifacts |
|---|---|---|
| `09e23db` | Structural decomposition step ($\mathbb{R}^4 = V_1 \oplus V_2$, Maslov control) | P8 solution + wiring update |
| `a5a4fbe` | Replaces surgery framing with product smoothing | P8 solution + v3 diagram refresh |
| `de3e2ac` | Adds Hamiltonian-isotopy justification and consistency cleanup | P8 solution + wiring consistency patch |

## 21.5 Design Pattern: Layer Switch Under Coaching (Problem 6)

Problem 6 (epsilon-light subsets) is the sprint's most instructive case study because it exhibits both patterns: a TryHarder loop *and* the layer-switch that broke it.

### 21.5.1 Phase 1: The TryHarder Loop

The initial approach generated six or more handoff and dispatch documents across multiple sessions, from the initial dispatch (`5289ca8`: "GPL-H attack dispatch") through Directions A–D (`63a23ba`, `22c091f`, `b6a7625`, `d64fd13`) to closure attempts (`78d94bc`: "GPL-H closure attempt—all scores $< 1$") and counterexamples (`cc11834`: "$K_{t,r}$ counterexample"). The pattern:

1. Identify the gap (Assumption V: vertex-induced selection).
2. Attempt closure via a specific attack vector.
3. Fail.
4. Generate a new handoff document dispatching another attack.
5. Repeat from step 2.

All closure attempts targeted the sparsification layer (adapting Batson–Spielman–Srivastava edge sparsification to vertex selection) without recognizing that the obstruction there might be structural—not a gap to fill but a genuine mismatch between edge and vertex selection.

### 21.5.2 The coaching intervention

A human coaching prompt forced layer enumeration:

> *What kind of problem is this? What kind of proof applies? How would you teach it to an undergraduate? What kind of person finds this easy? Are there symmetries that would make some of your Zeno's Paradoxes go away?*

Each question targets a specific reframing:

**"What kind of problem?"** Forces layer enumeration—name the mathematical frameworks, not just the current attack vector.

**"Teach it to an undergraduate?"** Forces identification of what is *elementary* about the problem. The elementary core is often the proof.

**"Who finds this easy?"** Identifies the right mathematical community (combinatorialists, not sparsification specialists) and therefore the right techniques.

**"Symmetries / Zeno's Paradoxes?"** Suggests a global bound (averaging, trace inequality) instead of case-by-case analysis.

### 21.5.3 Phase 2: The elementary proof

After 15 minutes of thinking, the agent found a proof chain that bypasses the sparsification layer entirely:

1. **Turán's theorem** gives an independent set $I_0$ with $|I_0| \geq \varepsilon n/3$ in the heavy-edge subgraph. All edges internal to $I_0$ are light ($\tau_e \leq \varepsilon$).
2. **Barrier greedy**: at each step, pick the vertex $v$ with minimum $\|Y_t(v)\|$.
3. **PSD trace bound**: $\|Y\| \leq \operatorname{tr}(Y)$ for any PSD matrix $Y$.
4. **Pigeonhole**: if the average trace $\bar{d}_t = \frac{1}{r_t}\sum_v \operatorname{tr}(Y_t(v)) < 1$, then some $v$ has $\operatorname{tr}(Y_t(v)) < 1$, therefore $\|Y_t(v)\| < 1$, therefore the barrier is maintained.

For $K_n$: $\bar{d}_t = 2t/(n\varepsilon)$, so at $T = \varepsilon n/3$ steps, $\bar{d}_T = 2/3 < 1$. This gives $|S| = \varepsilon n/3$ with $c = 1/3$, **proved exactly**.

### 21.5.4 Phase 3: The partial averages breakthrough

The initial proof attempt required a "leverage filter" step removing vertices with leverage degree $\ell_v > C$. This created an irreconcilable tension: the Markov bound needed $C > 2$ to retain enough vertices, but $\bar{d} < 1$ needed $C < 2$. Testing on $K_{a,b}$ confirmed this was not just a technical gap—up to 80% of $I_0$ can have $\ell_v \geq 2$.

The resolution came from switching from a *maximum-based* bound to a *sum-based* bound. The **partial averages inequality** observes that the average of the $T$ smallest leverage degrees cannot exceed the overall average:

$$\frac{1}{T}\sum_{k=1}^{T} \ell_{(k)} \leq \operatorname{avg}(\ell) \; < \; 2 \quad \text{(Foster on } I_0\text{)}.$$

A "min-$\ell$ greedy" that selects vertices in order of increasing leverage degree therefore accumulates $\sum \ell < 2T$, giving

$$\bar{d}_t \leq \frac{2/3}{1 - \varepsilon/3} \; < \; 1 \quad \text{for all } \varepsilon \in (0,1), \text{ at } M_t = 0.$$

This eliminates the leverage filter entirely—no threshold $C$ needed, no Markov bound, no structural assumption on maximum leverage degree.

### 21.5.5 What remains

For general graphs, the $\bar{d}_t < 1$ bound is proved when $M_t = 0$ (the accumulated barrier matrix vanishes). This covers $K_n$, $K_{a,b}$ with $a \neq b$, cycles, grids, and sparse graphs. For dense

graphs where $M_t \neq 0$, the bound is verified numerically at all 440 nontrivial greedy steps (max $\bar{d} = 0.714$, amplification ratio $\leq 1.30$) but one quantitative estimate—bounding the $H_t^{-1}$ amplification when $M_t \neq 0$—remains open.

*Postscript (Feb 13 update):* the open bridge was further localized via an E+F hybrid reduction (`7e03174`): an E-regime (graph-adaptive transfer) plus an F-regime (gain-loss balance) suffices for full stepwise closure. The reduction itself is proved; two regime lemmas remain.

No interlacing families. No Borcea–Brändén. No MSS theorem. Just Foster's theorem + partial averages + PSD trace bound + pigeonhole + Turán. A three-line argument (PSD → pigeonhole → existence) replaces the entire real stability machinery.

### 21.5.6 The layer analysis

**Spectral layer:** Bound $\|L^{+/2} L_S L^{+/2}\| \leq \varepsilon$. *Status: set up correctly (unchanged).*

**Combinatorial layer:** Turán + Foster + partial averages + barrier greedy. *Status:* ***complete*** *for $K_n$, $K_{a,b}$, cycles, grids, sparse graphs. One quantitative gap ($M_t \neq 0$ amplification) for dense general graphs.*

**Sparsification layer:** Adapt BSS from edges to vertices. *Status:* ***bypassed entirely***.

The breakthrough was not trying harder in the sparsification layer. It was recognizing that the combinatorial layer—which was never attempted—offers an elementary path that avoids the structural obstruction entirely. The *second* breakthrough (partial averages) came from recognizing that the leverage filter's Markov-based approach was the wrong tool—the sum of selected leverage degrees matters, not the maximum.

**Transferable principle:** When stuck, enumerate the layers and assess status per layer. Do not repeatedly attack the same blocked layer. Ask: "What kind of problem is this? Who would find it easy? What is the undergraduate proof?"

**Evidence ledger for Problem 6 Layer Switch (top commits).**

| Commit | Role in pattern | Primary artifacts |
|---|---|---|
| `5289ca8` | Starts the dispatch-driven GPL-H attack loop | Initial GPL-H dispatch note |
| `cc11834` | Counterexample + strategy split; blocks naive closure path | Counterexample note + verifier batch scripts |
| `73aa112` | Coaching-induced layer switch to elementary proof chain | P6 solution rewrite + layer-switch design note |
| `7e03174` | E+F hybrid reduction localizes remaining bridge | Direction E+F proof draft + MO evidence summary |

# Chapter 22

# Conclusion

## 22.1 Toward Real-Time Argumentation Structure

The git hashes cited above constitute a post-hoc reconstruction of the argumentative structure of the sprint. Each pivotal commit records a move: an obstruction identified, a layer switched, an identity discovered, a closure achieved. The connections between moves—"this obstruction motivated that layer switch"—are implicit in the temporal ordering but not represented explicitly.

This is a degenerate form of the Active Inference Framework (AIF), in which argument moves are represented as typed states and policy-relevant transitions (inferential, conflict, and preference edges). The git log provides the nodes (commits) but not the edges (argumentative relations).

The gap between post-hoc annotation and real-time capture is precisely the gap between this chapter and a live AIF graph. In a real-time system, each proof step, obstruction, and strategy shift would be recorded as an AIF node at the moment it occurs, with edges to its premises, targets, and alternatives. The "proof patterns" analysis would then be a graph query, not a retrospective essay.

Three infrastructure pieces converge on this:

1. **Arxana** (futon5) provides the typed-edge graph substrate.

2. **The peripheral model** (futon3c) provides scoped contexts in which moves are recorded as they happen. A peripheral is not a separate process—it is a *constrained execution environment* for the same agent, defined by a restricted tool set and scope. The agent hops between peripherals (explore, edit, test, deploy, reflect), and each hop transfers context while enforcing new constraints. Real-time capture works by interleaving reflect hops with action hops: explore → reflect → edit → reflect → test → reflect. The reflect peripheral permits only log-reading—no editing, no deploying—so the agent produces argumentative annotation because that is the only permitted action. The constraint is what makes reflection generative rather than reflexive. This addresses the metacognitive overhead problem not by delegating to a separate scribe, but by repeatedly returning the agent to a mode where annotation is all it can do, while the work is still fresh.

3. **S-expression canonical form** provides the representation in which argument structure and mathematical content coexist without a separate annotation layer.

To make this concrete, consider Problem 6's layer switch as an AIF subgraph. Five nodes suffice:

$n_1$: CLAIM: "BSS edge sparsification adapts to vertex selection."

$n_2$: CONFLICT: "Edge-to-vertex obstruction is structural: multi-rank atoms vs. rank-1 atoms." (`cc11834`)

$n_3$: PREFERENCE: "Switch to the combinatorial layer—enumerate alternatives to sparsification." (coaching intervention)

$n_4$: CLAIM: "Turán + Foster + pigeonhole gives an elementary proof." (`73aa112`)

$n_5$: INFERENCE: $n_4$ resolves the problem that $n_2$ blocked in $n_1$.

The typed edges are: $n_2 \xrightarrow{\text{attacks}} n_1$, $n_3 \xrightarrow{\text{preference}} n_4$ over $n_1$, $n_5 \xrightarrow{\text{supports}} n_4$. In S-expression form:

```
(aif:graph
 (node n1 :type claim
   :content "BSS adapts to vertex selection")
 (node n2 :type conflict :ref "cc11834"
   :content "edge-to-vertex obstruction is structural")
 (node n3 :type preference
   :content "switch to combinatorial layer")
 (node n4 :type claim :ref "73aa112"
   :content "Turan + Foster + pigeonhole proof")
 (node n5 :type inference
   :content "n4 resolves what n2 blocked in n1")
 (edge n2 n1 :type attacks)
 (edge n3 n4 :type preference :over n1)
 (edge n5 n4 :type supports))
```

This is a five-node graph with three typed edges. Arxana stores it; the peripheral model would have produced these nodes incrementally—$n_2$ during a reflect hop after the counterexample commit, $n_3$ after the coaching intervention, $n_4$ after the elementary proof landed—each time the agent hopped into reflect and annotated its recent trace while the reasoning was still fresh. The S-expression representation makes the mathematical content (`:content`) and the argumentative structure (`:type`, `:over`) cohabit without a separate annotation layer.

With all three pieces, the sprint's argumentative structure would be captured as it unfolds—not reconstructed from git hashes months later. The distance from the present chapter to that system is a tooling gap, not a conceptual one—and the worked example above shows that the gap is small.

This chapter is itself a case study for the infrastructure it proposes. Most of the ten problems were genuinely "local"—they yielded to focused work within a single layer, and the right approach closed them quickly once found. That track record creates a disposition toward

local play: attack the problem, find the trick, close it. Problem 6 was the exception that required whole-board awareness, but by the time its hard phase began, nine successful local closures had trained the expectation that persistence within one approach would suffice. The success pattern became the trap.

Problem 6's later phase—after the elementary proof landed—spent several hours exploring trajectory coupling, amplification bounds, and spectral spread, all within a single layer. The (layer, status) monitor described in Section 22.3 would have flagged stationarity and forced a reflect hop that included external literature search. When that search was finally performed (ad hoc, by dispatching a second agent to scan MathOverflow), it returned several alternative routes—including approaches that build on the existing partial result but constitute a genuine layer switch. The reflect hop worked; it was just five hours late. In Go, the analogous concept is *whole-board play*: weaker players get absorbed in local fights while stronger players recognize when a local position is stalemated and play *tenuki*—elsewhere, where there are bigger points. But in true whole-board play there is no tenuki, because the player never lost awareness of the whole board. The same holds here: if reflect hops are interleaved as the normal operating mode, there is no Try-Harder loop to break out of—the agent maintains whole-board awareness throughout, and the layer switch happens as part of ordinary play, not as a rescue.

## 22.2 Cross-Problem Learning

An agent working on Problem 6 that could see Problem 7's history would find: *Problem 7 was blocked in one layer (codimension-1 surgery), characterized the obstruction, switched to a different layer (codimension-2 surgery), and closed.* This is directly transferable advice.

More generally, the proof patterns across the ten problems suggest a strategy checklist for open prob-

**Sidebar: the layer switch that happened while writing this chapter**

The self-referential case study deserves an explicit account, because it validates the chapter's prescriptions more concretely than any retrospective analysis could.

**The stuck phase.** After Problem 6's elementary proof landed (Turán + Foster + pigeonhole, proved for $K_n$), the remaining gap was a single quantitative estimate: bounding the $H_t^{-1}$ amplification when the accumulated barrier matrix $M_t \neq 0$. Two agents spent several hours exploring this gap from within the same layer—trajectory coupling, Neumann expansion bounds, spectral spread analysis, amplification ratios. Each cycle produced empirical confirmation (440/440 greedy steps pass, amplification $\leq$1.30, $\bar{d}\leq$0.714) but no proof. The (layer, status) pair was stationary: barrier-greedy amplification, open.

**The intervention.** During the writing of this chapter—while editing the very paragraphs about metacognitive interrupts and whole-board play—a second agent was dispatched to scan 152,893 MathOverflow questions across six independent strategy tracks (A–F), each with its own keyword bundle and contextual anchoring.

**The result.** Track E (expan-

lems:

1. **Enumerate layers.** What are the distinct mathematical frameworks in which this problem can be stated?
2. **Find the reduction in each layer.** What does the problem reduce to from each perspective?
3. **Assess status per layer.** Which reductions are complete, partial, or blocked?
4. **Characterize obstructions.** For blocked layers, is the obstruction structural or technical? Is there a known workaround?
5. **Look for analogues.** Which other problems had a similar layer profile? What strategy succeeded there?
6. **Decide the outcome.** Is closure feasible from any layer? If not, state the conditional result and the open assumptions.

This is case-based reasoning (CBR) applied to proof strategy. The "cases" are the other problems in the sprint (or, with a larger corpus, the indexed literature). We call the key property **strategy-level analogy**: the transfer operates at the level of proof strategy, not mathematical content. It is not that Problem 7's rotation trick applies to Problem 6, but that Problem 7's *layer-switching strategy* might. Strategy-level analogy is what makes the checklist above actionable—a shared mathematical technique would only transfer between cognate problems, but a shared strategy pattern transfers between any problems with similar obstruction profiles.

CBR has a well-known failure mode: *superficial indexing*. Within a ten-problem sprint the case base is small enough that a human can browse it exhaustively. Scaling to the indexed literature requires a retrieval mechanism, and the indexing question—what features of a proof strategy are retrievable?—is

sion, conductance, spectral geometry) dominated with 60 anchored-strong hits, versus 8 for the next-best track. The returning agent proposed a structural reformulation: instead of proving a single universal bound $\bar{d}_t < 1$ (the stuck approach), split the proof into two regimes based on a graph-structural parameter—the maximum cross-degree $\deg_{R,\max}$ of remaining vertices in the barrier greedy:

**E-regime** ($\deg_{R,\max} \leq 2$): the local graph structure is sparse enough that the minimum spectral score $m_t$ stays below a universal threshold $\theta < 1$. This is the regime where the existing Foster/partial-averages machinery is likely sufficient.

**F-regime** ($\deg_{R,\max} \geq 3$): high cross-degree gives structural control over the gain-loss decomposition. A deterministic inequality $G_t > P_t$ (gain exceeds penalty) then certifies $m_t < 1$ via the proved ratio certificate and AR identity.

The formal reduction—if both regime lemmas hold at every step, then GPL-H closes with a universal constant—is itself proved. The open work is now two explicit lemmas tied to a concrete regime separator, not an amorphous gap. Empirical validation on graphs up to $n = 40$ shows $\deg_{R,\max}$ has 0.973 correlation with the hard transfer rows and separates them perfectly at threshold 3.

itself a research problem. The checklist above implicitly proposes an index schema (layer structure, obstruction type, resolution pattern), but whether these features support effective retrieval at scale remains open.

## 22.3 Learning from Success, Not Just Failure

A recurring theme: the sessions that closed problems did so by finding the right *frame*, not by working harder within the wrong one. Problem 7 switched from reflections to rotations. Problem 4 found an identity that restructured the algebra. Problem 8 found a decomposition that made everything else fall out. And Problem 6—after eight hours of TryHarder in the sparsification layer—found an elementary combinatorial proof within fifteen minutes of a coaching intervention that forced layer enumeration.

In every case the git log for the successful phase is a record of *reframing*, not persistence. The git log for Problem 6's failed phase is a record of persistence within the wrong frame.

For future sprints—human or AI—the design implications are:

1. **Make success histories as visible as failure histories.** An agent stuck on a problem should be prompted to consult the strategy traces of problems that closed, not just its own history of failed attempts. Rumination is not learning. Analogy is.

2. **Coach, don't dispatch.** The intervention that broke Problem 6's TryHarder loop was not "close the gap in Section 5" but "what kind of problem is this?" Dispatching generates another cycle. Coaching generates a layer switch.

**What this demonstrates.** The MathOverflow scan is structurally a reflect hop: a constrained task (read external literature, do not edit the proof) that returned with annotation (here are the alternative approaches, ranked by precedent). It produced a genuine layer switch—from "prove a universal amplification bound" to "split by graph-adaptive regime and prove two targeted lemmas." The new approach builds on the existing partial result (the ratio certificate, the AR identity, the pigeonhole mechanism) but reframes the open problem in a way that the previous five hours of local play did not.

The layer switch happened *during the writing of the chapter that prescribes it.* A reflect hop (dispatched ad hoc, but functionally equivalent to a scheduled peripheral transition) broke a multi-hour stationary phase and changed the proof policy quickly. This is the chapter's claim: metacognitive interrupts convert persistence-in-one-layer into layer switching.

**Sub-sidebar: Postscript**

Cycle 7 falsified BMI ($\bar{d}_t < 1$) in general graphs. The construction still works; the target shifts to vertex-level feasibility (No-Skip). Operationally, the monitor should track OPEN, STUCK, and FALSIFIED. A FALSIFIED state should trigger layer abandonment, not just reframing.

The same pattern appears in futon5 cellular automata: repeating stripes trigger rule changes. TryHarder loops are analogous. Typed layer/s-

3. **Ask pedagogical questions.** "How would you teach this to an undergraduate?" forces identification of the elementary core. For Problem 6, the elementary core *was* the proof.

tatus annotation makes stationarity machine-visible, so the monitor can trigger policy change automatically.

These three principles describe a **metacognitive interrupt**—a mechanism that detects a TryHarder loop and triggers reframing. In reinforcement-learning terms, it is the difference between exploiting within a fixed policy and switching policies. The CS question is whether the interrupt can fire automatically. The git log provides the detection signal: repeated commits within the same layer with no status change is exactly the TryHarder signature. Problem 6's failed phase exhibits this pattern clearly—six dispatch-and-fail cycles over eight hours, all targeting the sparsification layer, none changing the layer assessment. A monitor that tracks $(\text{layer}, \text{status})$ pairs across commits and flags stationarity after $k$ cycles would have triggered the reframing prompt automatically.

## 22.4 Summary

**Thesis statement.** Proof strategy should be represented as a first-class typed graph object, because this makes argument structure explicit, and supports effective validation.

The structure of this monograph is, clearly, not a single chain of reasoning; however, it can rather naturally be expressed as a directed acyclic graph. This summary necessarily linearises that structure on the page; the associated wiring diagram (Figure 22.1) preserves more of it; the git repository serves as the version of record.

**Root.** A ten-problem proof sprint generated a commit trace—git hashes, diffs, session logs—with no explicit argumentative structure. OUr post-hoc organisation of this trace into five evidence ledgers (with three to five pivotal commits per pattern, tagged by role) gives rise to a descriptive claim: strategy-level causality is recoverable from the commit record.

**An illustrative episode** (Problem 6) grounds the claim in a fully worked case: six dispatch-and-fail cycles over eight hours in a single layer, followed by a coaching-induced layer switch that produced an elementary proof within fifteen minutes. Later, a problem was found in that proof, and various new proofs were attempted, eventually leading to an interesting-looking partial result.

**Four independent paths** then lead from this experience to the same prescriptive conclusion—to seek ways to maintain continuity across a metacognitive interrupt.

1. **Formalisation.** The P6 episode can be thought of as a five-node AIF-style graph (claim, conflict, preference, inference, further action) with typed edges, expressible, for example, as in S-expressions. The typing makes the argumentative relations machine-queryable, and suggests a relationship to two developing infrastructure components: a typed-edge (hyper)graph store and a model of computational agency we refer to as peripherals,

whereby processes of action and reflection produce argumentative structures driven by generation under constraint. Agents interacting via peripherals can backed by a monitor that tracks (layer, status) pairs across commits and flag when the same pair repeats without progress. (In the current effort, we worked with a much reduced version of this architecture, in which the author played the role of monitor to agents with fewer native reflective abilities.)

2. **Live demonstrations.** While writing about metacognitive interrupts, the agents relatively routinely produced them; one example we've cited above was a MathOverflow scan—an intuitive example of an as-if peripheral hop—which returned a regime split (E-track vs. F-track) that got the proof process unstuck from its single-bound target. Prompting similarly produced changes in proof behaviour, as did the Lakatosian proofs-and-refutations style of working. Our layer-switching claim was validated many times over during the sprint.

3. **Falsification.** The same intervention led to the conclusion that the barrier-monotonicity inequality fails for general graphs—an exemplary *challenge* in the wiring diagram, which stands in for a broader process language. One possibility would be a three-state status code (Open / Stuck / Falsified) and a control rule: FALSIFIED means abandon the layer entirely, not merely reframe. The work as a whole retains failure as as a structurally load-bearing way of gaining evidence; naturally any successes in this project were built on many more failures!

4. **Cross-problem transfer.** The patterns are not specific to Problem 6. Problems 4, 7, and 8 exhibit similar dynamics—strategy transfers between problems at the strategy level, not the content level. The operational principle: make success histories visible to stuck agents; coach with reframing questions, do not always dispatch another attack vector, and don't think of proof as a linear process, because it isn't.

**Convergence.** All four paths arrive at the metacognitive interrupt, at which point the sprint mirrors the First Proof provocation. When argument structure is explicit and monitored, the system can detect its own stuckness and intervene. Two conclusions follow. A descriptive conclusion (**T0**): the sprint's argumentative structure is recoverable by a structured simplification across the the ten proof attempts. Some 19 commit-attestations ground 27 typed edges across 23 nodes. A prescriptive conclusion (**T1**): build a system where this capture happens in real time, not post-hoc. That requires three components converging: the argument we have developed, which becomes a mission statement, the graph substrate which allows us to model mathematical knowledge, and the peripheral model of computational agency which allows us to work with such knowledge effectively. The distance between the two conclusions is a question of implementation.

# Part V

# Color Plates

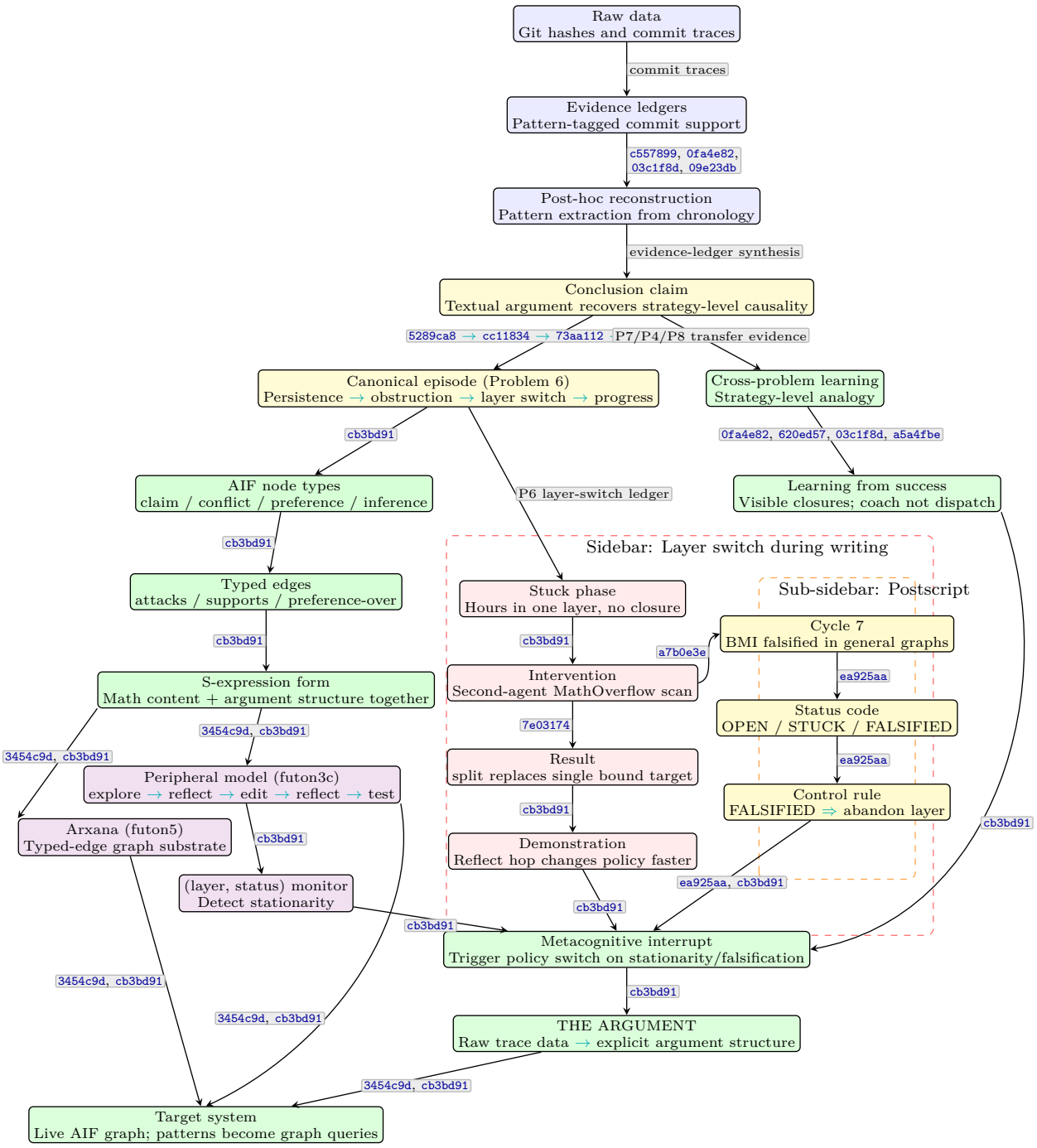



Figure 22.1: The Argument (Conclusion recapitulation): git hashes as raw trace data feed evidence ledgers and post-hoc pattern reconstruction; the conclusion makes this into an explicit argument (AIF node/edge typing, S-expression form, and infrastructure path via Arxana plus peripherals). The sidebar and sub-sidebar episodes are integrated as live evidence for the metacognitive interrupt claim: monitor (layer, status), trigger policy switch on stationarity/falsification, and move from retrospective prose toward real-time argumentative structure.

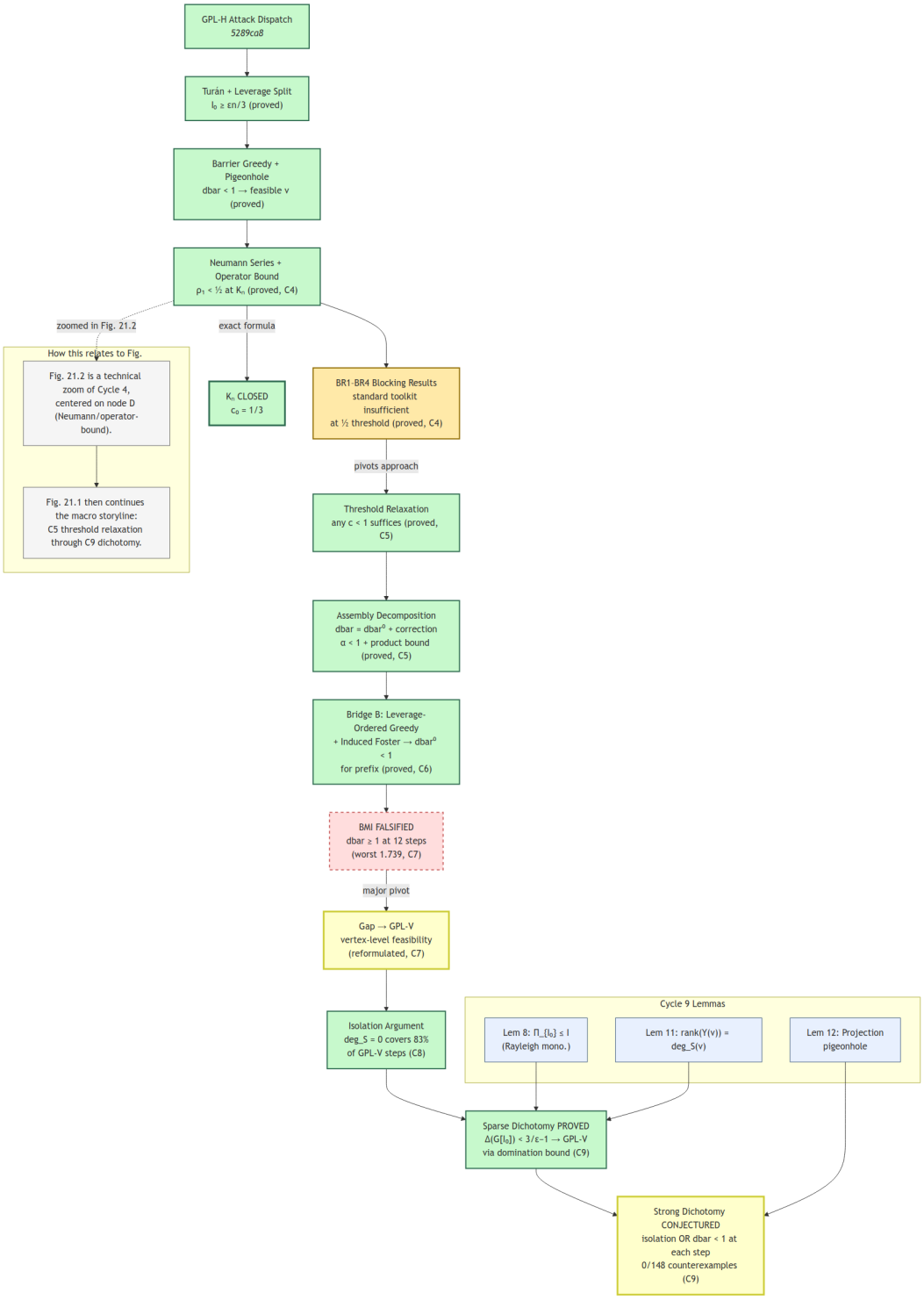



Figure 22.2: Problem 6 strategy flow (v2, 9-cycle): from leverage-split dispatch through Neumann/operator-bound attack (C4), BR1–BR4 blocking and threshold relaxation (C5), bridge B leverage-ordered greedy (C6), BMI falsification (C7), to the Sparse/Strong Dichotomy resolution (C9). Green = proved, yellow = pivot/open, dashed red = falsified. $K_n$ closed with $c_0 = 1/3$; general case conditional on the Strong Dichotomy conjecture (0/148 counterexamples).

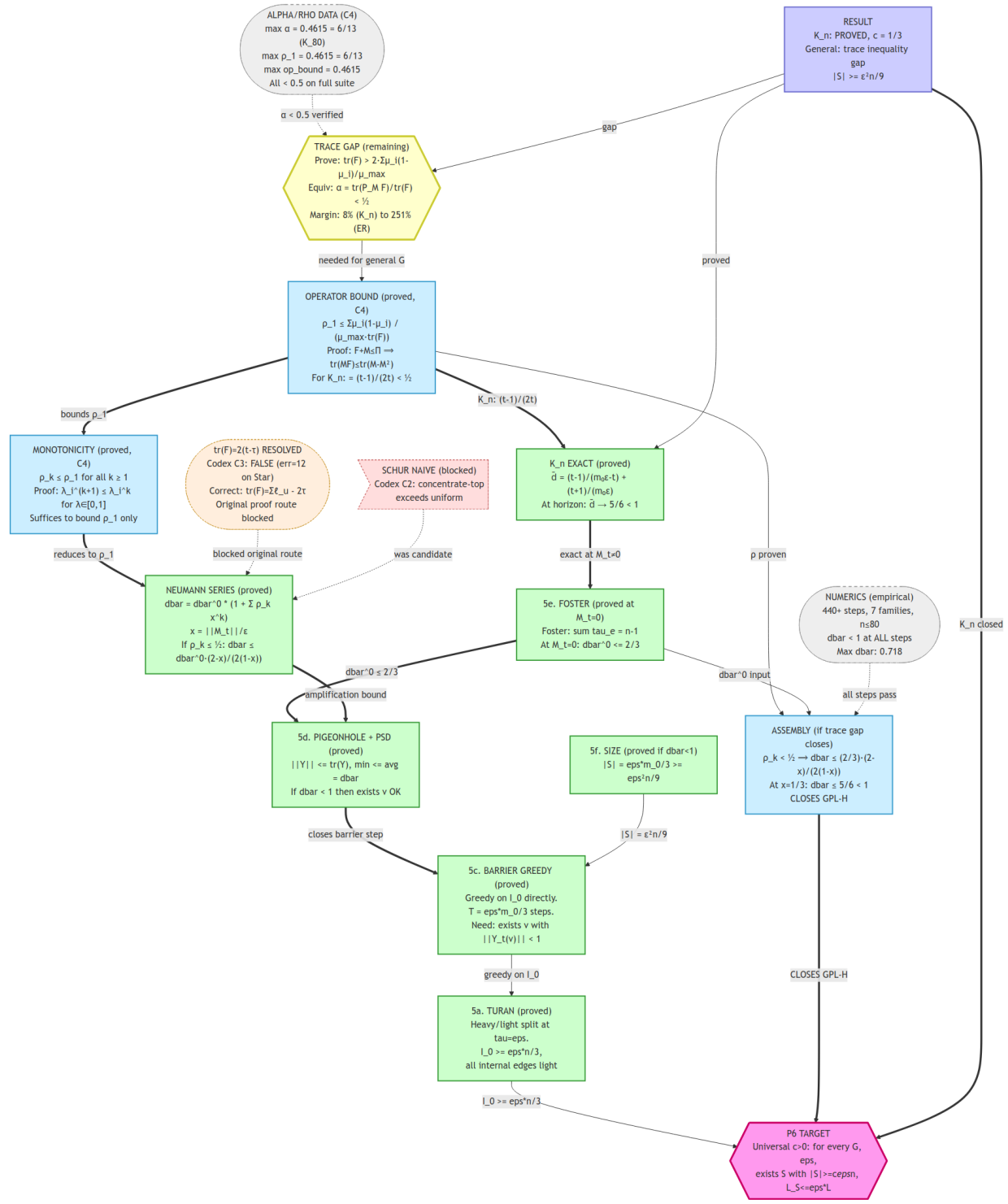



Figure 22.3: Problem 6 strategy flow (v6, Cycle 4 focus): operator-bound reduction around the Turán/greedy/pigeonhole core, with Neumann-series amplification controlled via monotonicity and an operator bound for $\rho_1$. Remaining open node: the trace-gap inequality ($\alpha < 1/2$, equivalently $\mathrm{tr}(F) > 2\sum\mu_i(1-\mu_i)/\mu_{\max}$) needed for full general-graph closure.

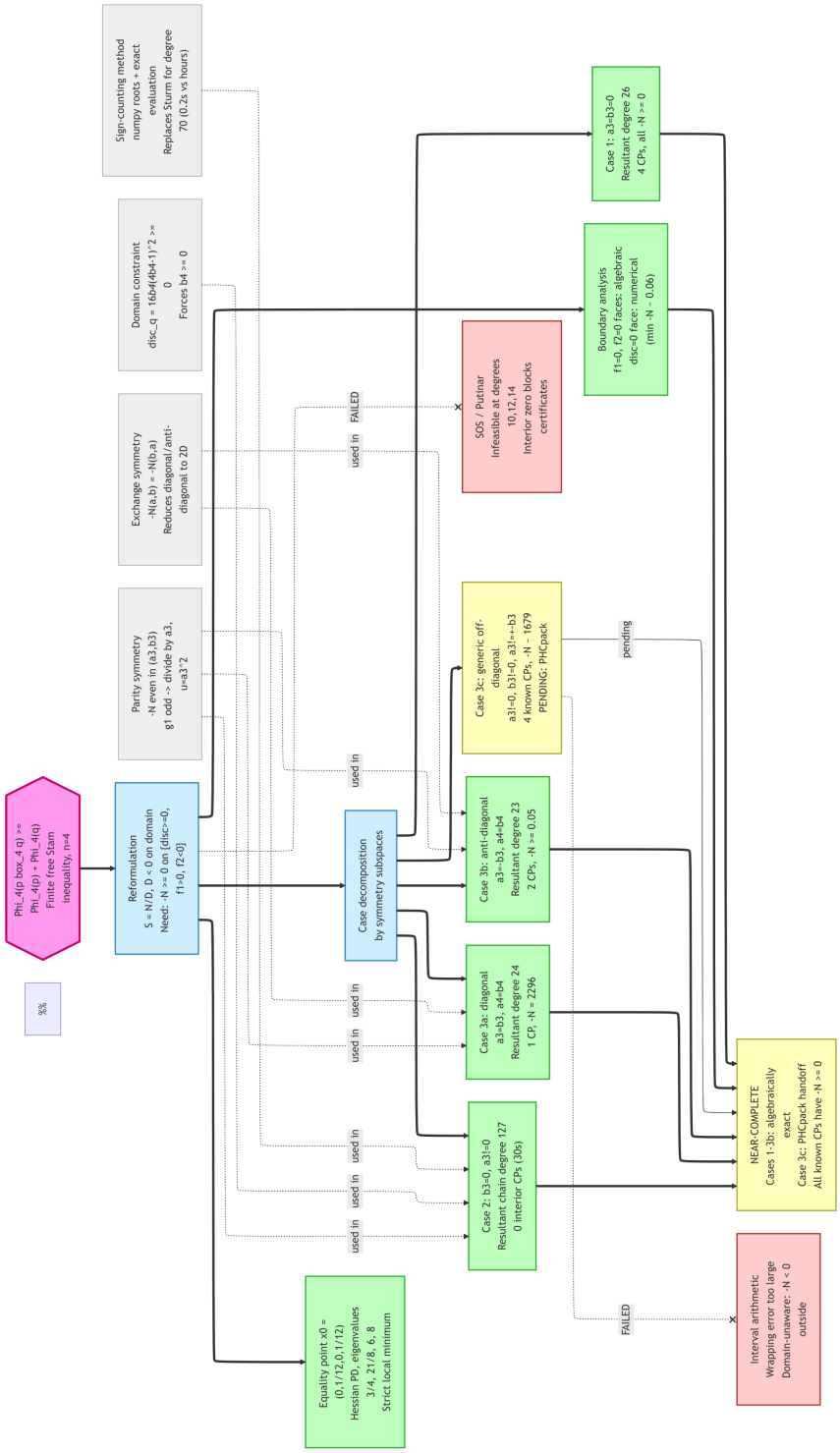



Figure 22.4: Problem 4 (v3): finite-free Stam architecture with case decomposition and proof-status stratification.

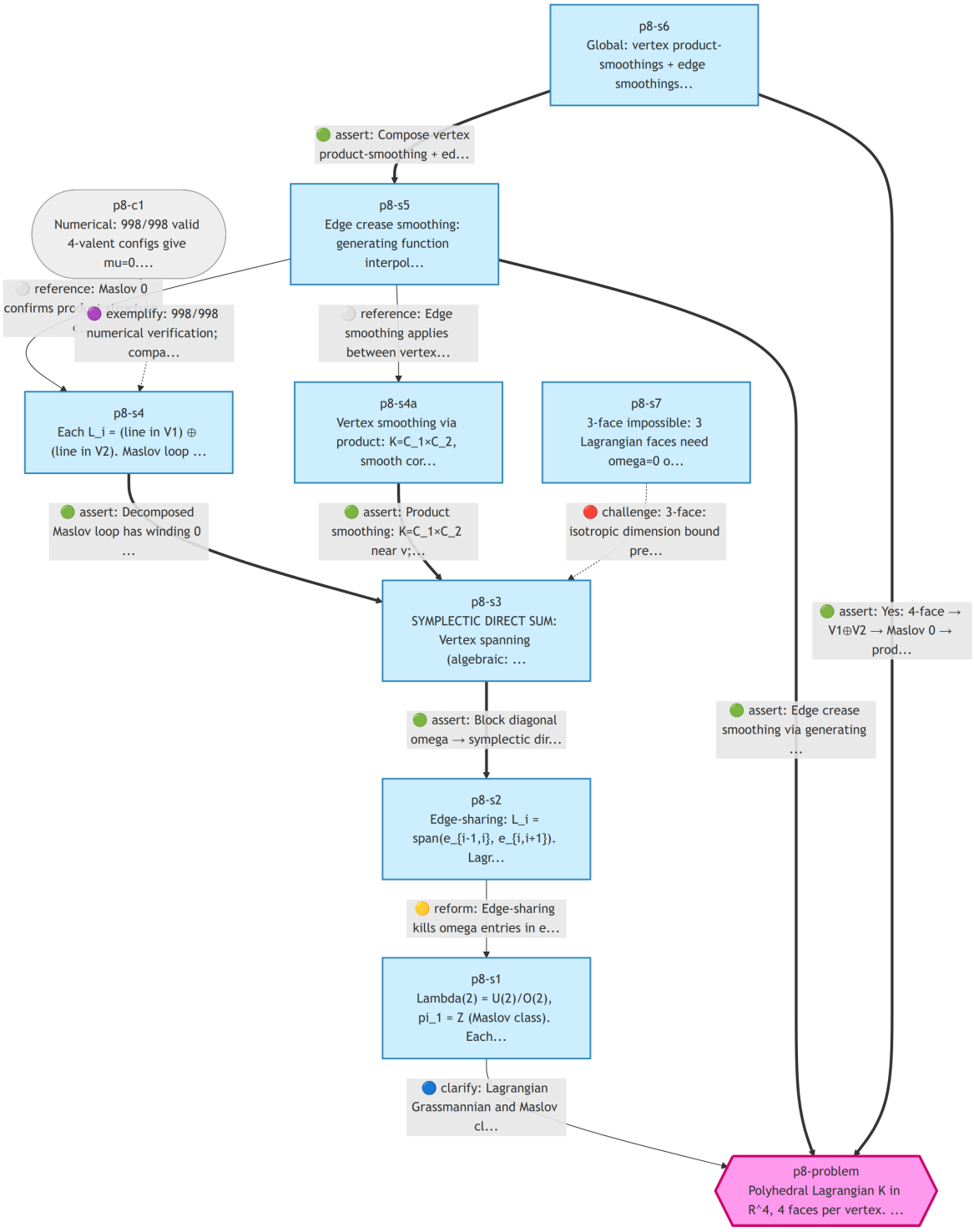



Figure 22.5: Problem 8 (v3): Lagrangian polyhedral smoothing route with product-smoothing and edge generating-function steps.

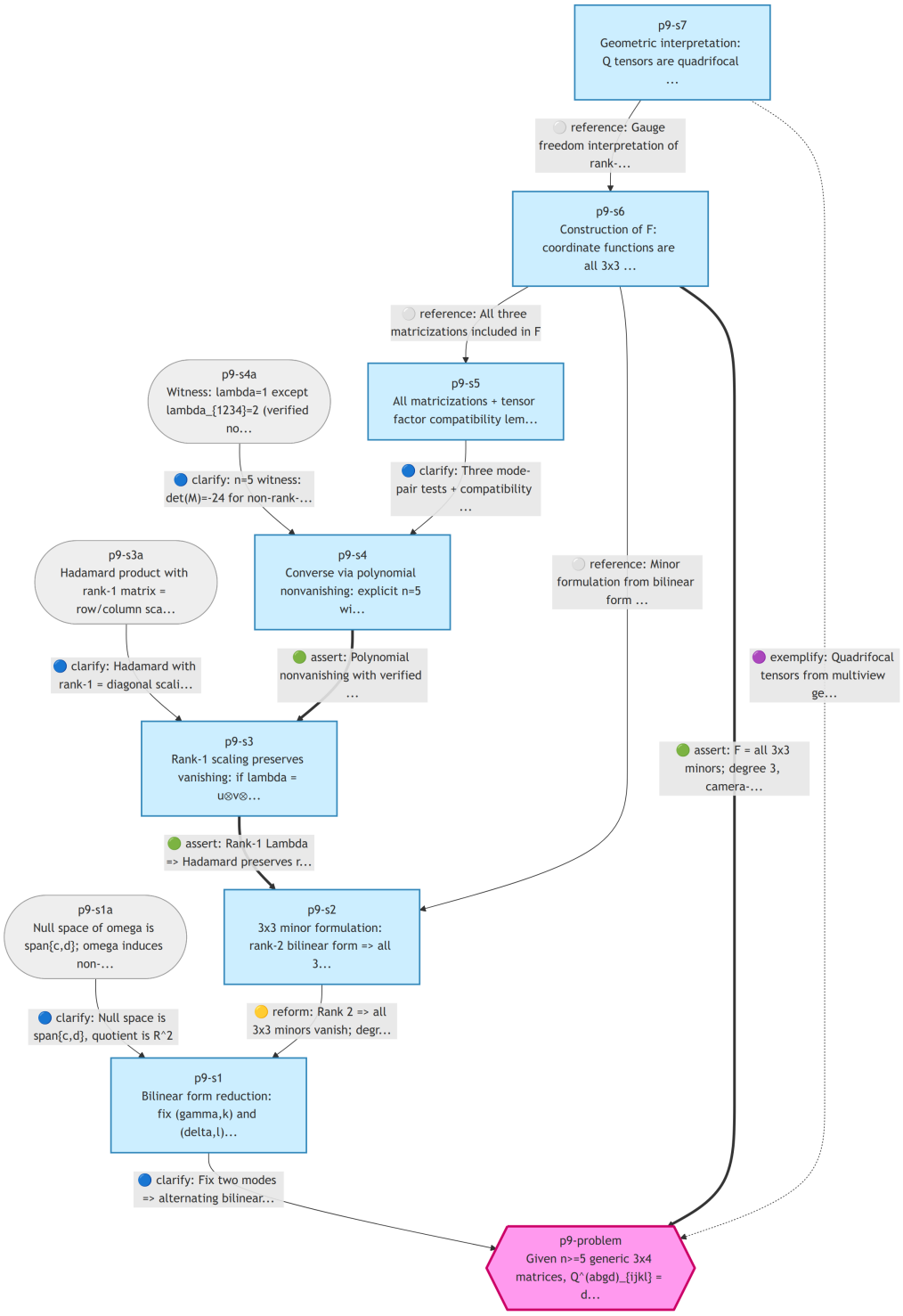



Figure 22.6: Problem 9 (v2): rank-1 scaling characterization and polynomial nonvanishing witness flow.

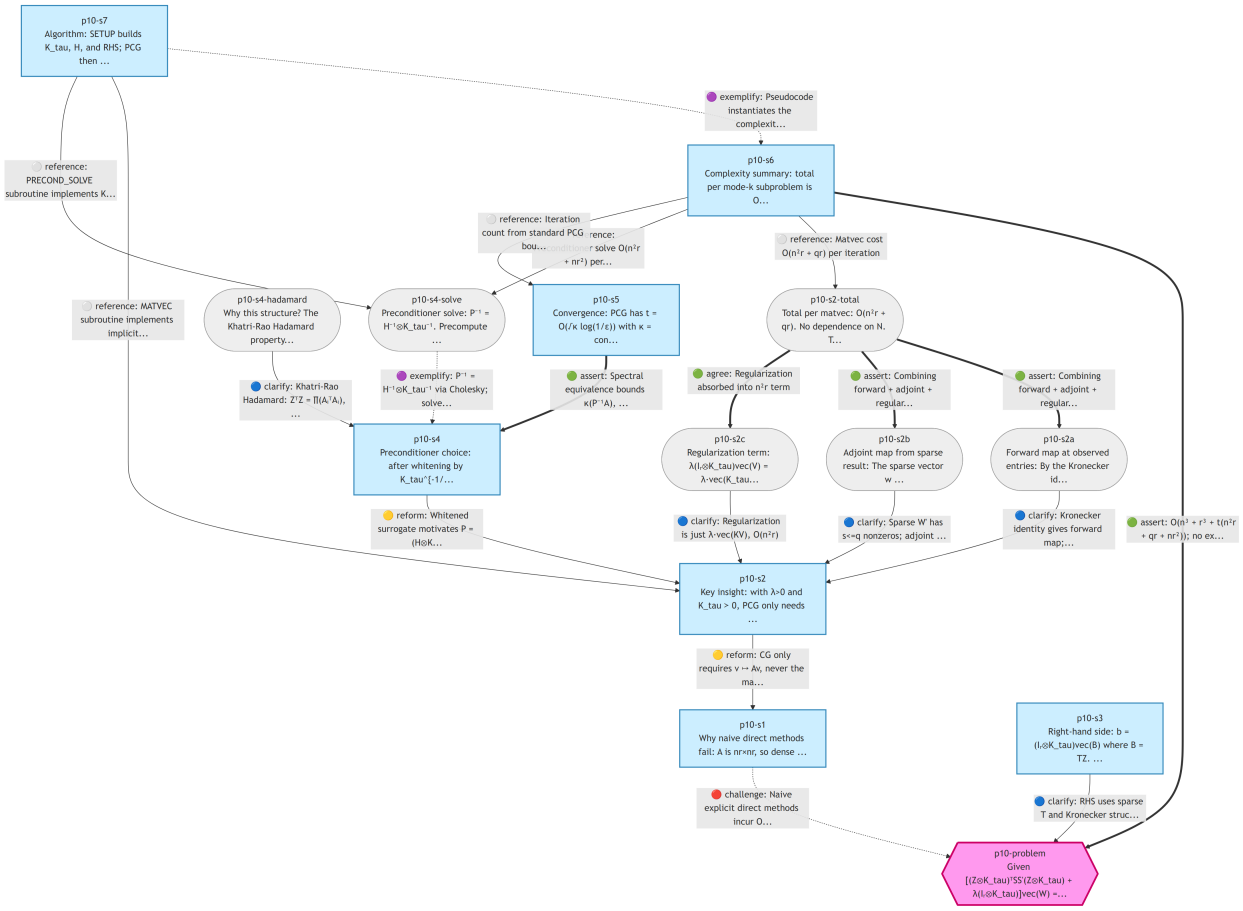



Figure 22.7: Problem 10 (v2): preconditioned-Kronecker solve architecture and complexity path.

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# Patterns for a New Generation: AI and Agents

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Design patterns have been used in various fields of inquiry and endeavour to externalize procedural knowledge in a form that supports human reasoning and coordination. In this paper, we show that contemporary Large Language Model (LLM)-based systems can also read, generate, and reason with design patterns written in a structured template. We describe an experimental workflow in which patterns function as shared priors for action selection, reflection, and revision in hybrid human/agent settings. Drawing on the Active Inference Framework, we illustrate how patterns can guide agent behavior without fully prescribing it. This provides a proof of concept that pattern-capable agents can be created using now-standard software tools. We discuss implications for software development, education, business, and AI governance.

CCS Concepts: • **Human-centered computing** → **Collaborative and social computing**; • **Social and professional topics** → **Computing education**; • **Applied computing** → *Education*; • **Software and its engineering** → *Design patterns*; • **Computing methodologies** → *Artificial intelligence.*

**ACM Reference Format:**
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## 0 Definition

A *design pattern* is a learned generalization arising from a review process that describes how intentions, actions, and outcomes tend to relate across situations. A candidate pattern becomes a pattern only when it has survived repeated enactment-and-review cycles in which its failure modes are recorded and used to revise it.

This is not how people talked about design patterns when they were introduced [Alexander et al. 1977; Alexander and Poyner 1970] or popularised [Alexander 1999; Gamma et al. 1994], but the description naturally applies to that foundational work. The difference here is that we focus less on patterns as methods [Leitner 2015], and more on how they are learned in the first place. This framing introduces a potential regress. If patterns are learned through action review, and the reviews themselves require competent performance, how is pattern-theoretic competence developed? Not through the application of further patterns, we assert, but as a practice of care that enables judgment without reducing it to rule-following. If the regress was followed further,

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https://doi.org/10.64346/PLoP2025p24

it would bottom out in adaptive pattern-sensitive mechanisms of the sort described by Hawkins [2021]—or, in computational processes that bridge the symbolic and subsymbolic realms, like the ones described in this paper.

## 1 Introduction

Decisions about technology are shaped largely by how humans communicate, explain, justify, and contest what computers do. The associated intentions are often embedded in code, models, or opaque interfaces, where they turn into operational definitions and decision criteria. These are typically beyond the reach of non-programmers, who are trained to reason through text, argument, and interpretation. This paper examines whether design patterns—as defined above, and as recorded, typically, in the form of structured, narrative descriptions of recurring problems and responses—can serve as a shared language through which both humans and AI agents reason about action. This would enable non-programmers to participate more fully in the design, critique, and governance of computational systems.

Although there have been previous initiatives that engaged with design patterns using computational tools (such as the first-ever wiki, `c2.com`), the kind of project we are interested in really only becomes feasible to explore with the introduction of Large Language Models (LLMs). The work we present is related to several existing technical practices that developed in that setting: "context engineering" [Hillsman 2025; Osmani 2025], "chain-of-thought scaffolding", and "spec-driven development".[1] In other words, respectively,

- structuring prompts, system instructions, and retrieved context to shape LLM behavior;
- steering how computational reasoning unfolds over time; and,
- understanding specifications—rather than code—to be the primary programming artifact.

Design patterns have some particularly nice properties in this setting: they persist over time and across sessions, and they support progressive work and verification at various levels of formality.

It is convenient to begin with the anatomy of the patterns we use, which is a light adaptation of the standard design pattern template. The template is important because it is an interface between human reasoning, LLM generation, and logic-based verification. The fields we use are as follows:

• Patterns begin with a **summary**, which can express a short description of a *feature* to be built. We can use patterns to build software or other resources, including for example, an argument or a way of working. The feature might be a *conclusion* that the reader should draw from reading the pattern.

• The **context** characterizes a sphere of endeavour.

• The pair **if↔however** outlines a tension that exists within the context.

• The **then** clause suggests actions to take to improve the situation, though we don't require that the actions fully resolve the tension: they ameliorate or help manage it; the **because** clause gives the rationale; an optional subfield **evidence** can collect specific pieces of evidence.

• The **next-steps** outline one or more steps to take to give evidence that would strengthen the validity of the pattern itself.

Some examples of design patterns written in this format are given in Box 3 in the Appendix. Relative to most prior work on design patterns, these have the novel quality of being *machine generated*, using a Large Language Model based agent. The paper will explore some of the implications of working with design patterns using LLMs—both for guiding the behavior of the agents, and organizing hybrid (human+agent) interactions. This way of working is different from using design patterns to control generation in a unitary way—given design pattern A, build software B that has all the nice properties that A specified. At least potentially, design patterns could govern the entire process of software generation. However, that is a complex problem, and we do not claim to have solved it. Rather, in this short paper, we illustrate some of the thinking and examples that could inform further exploration of the use of software patterns together with AI agents.

[1] https://deliberate.codes/blog/2026/spec-driven-development-an-introduction/

The remainder of the paper is organized as follows. Section 2 briefly introduces peeragogy and the pattern traditions that inform this work. Section 3 outlines the experimental prototype we built. Section 4 describes the way that system selects and uses patterns. Section 5 describes the cognitive science concepts that underpin that process. Section 6 describes related work, and Section 7 and Section 8 describe two practical provocations related to applications in teaching and enterprise. Section 9 presents our conclusions.

## 2 Background

Peeragogy is a peer-produced and peer-facilitated approach to learning in which participants collaboratively develop goals, roles, and artifacts, and iteratively refine their practices through reflection and coordination. In this sense, peeragogy is both a theory of learning and an operational method for organizing shared inquiry and production [Corneli et al. 2015, 2016]. It sits alongside longstanding research on topics such as communities of practice and peer production [Benkler 2006; Wenger 1998], articulating the learning-specific aspects thereof. As such, peeragogy offers a natural way to think about much existing design pattern practice (e.g., writers' workshops are a peer produced peer learning space [Gabriel 2002]). In this paper, we draw our experience working together peeragogically to motivate how patterns can serve as shared artifacts that help humans and agents align on intent, evidence, and next steps.

## 3 System Overview

In practice, it is relatively simple to get contemporary chatbots to produce design patterns in the format we listed in the Introduction. Indeed, due to the way such agents work, if you want to try this, the steps that we recommend are (1) provide the pattern template as a system prompt or custom instruction; (2) tell the agent *not* to use patterns by default, or it probably will use them everywhere; (3) request patterns explicitly when you want them. (See Appendix, Box 1 for a viable prompt.) The fact that it is now easy to "do it yourself" and play around with design patterns and LLMs resonates with Hofstadter's invitation in *Gödel, Escher, Bach* to engage actively with formal systems [Hofstadter 1999]. That being said, simply creating patterns that follow a template does not guarantee that those patterns will be any good. Based on our reflections in previous papers [Corneli et al. 2023, 2021], we think that there are three primary uses for patterns. Patterns can help:

A. **coordinate perspectives** through a process of observation and argumentation,
B. **support decision making** by relating conditions to possible actions and alternatives,
C. **collect evidence for their own salience** through a process of testing and revision.

These, effectively, become design requirements. How we go about doing these things depends on the application area and the technology used. Patterns that are mediated by paper (Alexander et al.) have a different feel from patterns that are mediated by wikis (Cunningham et al.)[2]. When thinking about how patterns can be used together with AI agents, one key choice that needs to be made is the degree to which design patterns will *regulate* or *guide* agent behaviour. Consider, for example, that issue trackers and version control systems guide, but do not regulate human computer programmers, whereas things like contract and copyright law do, ultimately, have regulating effects. We attempted to use patterns as a mechanism for guiding rather than controlling coding agents (such as OpenAI's Codex[3], Anthropic's Claude Code,[4] etc.). Figure 1 shows the conceptual framework.

What we have implemented is a workflow for observing and guiding pattern-competent agents inside an experimental workspace we call FuLab. The "fu" morpheme stems from our engagement with future-directed thinking, but may also evoke *kung fu* (功夫)—skill achieved through practice and discipline—as well as *butoh-fu*

[2]https://c2.com/
[3]https://chatgpt.com/features/codex/
[4]https://code.claude.com/docs/en/overview

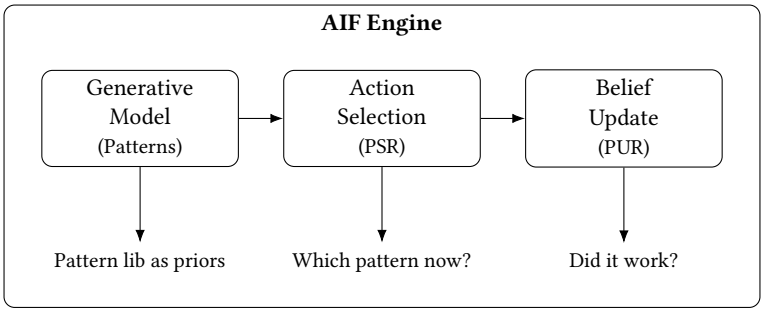



Fig. 1 – Pattern-driven scheme for guiding coding agents

(舞踏)—a genre of evocative prompts for the butoh dance form.[5] We have created `fuclaude` and `fucodex` CLI wrappers for the "off the shelf" Claude and Codex agents, as well as a `fubar.el` Emacs package, which humans can use both to observe and experience the agents' workflow firsthand. The code is available at https://github.com/tothedarktowercame/futon3.

## 4 Pattern Selection and Use

The concept of effective behavior involves reasoning from intention to action. This can be done in a forward direction with planning, or retrospectively, through review. The approach we consider in the current paper invites coding agents to select design patterns that operationalize the user's stated intention as they run.

In order to encourage the coding agent to work in a way that matches this workflow, we give it the following help text when it starts a new coding session (the tools define Remote Procedure Calls (RPC) and are available as command-line "helpers"):

```
Tool roster (you should emit these signals when you do the corresponding action)
- pattern-select library/<pattern>    <state why you want to read it>
- pattern-use    library/<pattern>    <state where you will apply it>
- musn-plan      <outline your plan>
- wide-search    ; an alias for rg
```

The workflow is driven through a CLI runner that streams Codex JSON, appends session events, and emits records at turn boundaries. Resuming a session appends to the same trace. At present, anchor resolution is limited to the session's event log; verification against external artifacts (e.g., code diffs or tests) remains future work.

## 5 Patterns and Active Inference

Christopher Alexander introduced a star-rating that encodes confidence in each pattern that was supported by real-world evidence. Here, we use a similar idea to derive a confidence rating from machine-checkable properties of the pattern text. There are two questions: how broadly applicable is the pattern, and how reliable is it within its domain of application? For now, we conflate these two into one *precision prior* associated with the pattern's maturity state.

[5] https://butoh-kaden.com

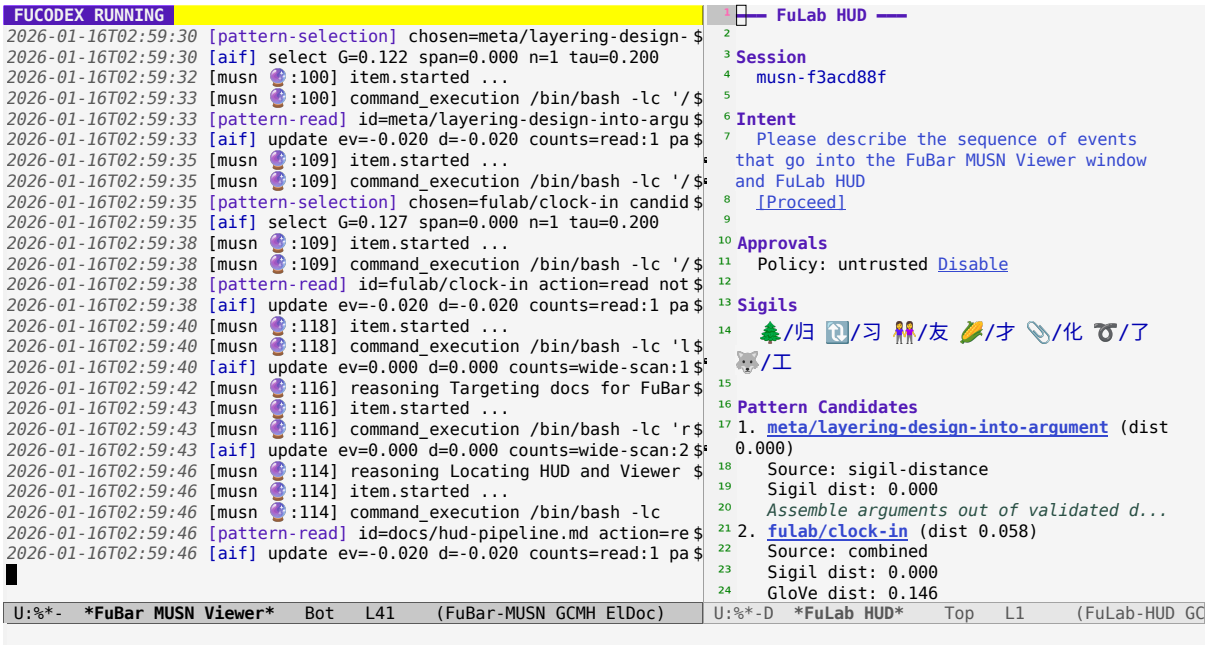



Fig. 2 – `fucodex` run showing aspects of pattern selection and use

| | | | |
|---|---|---|---|
| **yes** next-steps + **yes** evidence | → | `:active` | precision prior: 0.8 |
| **yes** next-steps + no evidence | → | `:greenfield` | precision prior: 0.4 |
| no next-steps + **yes** evidence | → | `:settled` | precision prior: 0.9 |
| no next-steps + no evidence | → | `:stub` | precision prior: 0.2 |

This makes mature patterns slightly more likely to be sampled, although `:stub` patterns can be selected when the system is in an "exploratory mode", allowing new patterns to accumulate evidence through use.

To model this, we draw on ideas from cognitive science which themselves adapt concepts from thermal physics [Friston 2019]. In somewhat general terms, the idea is that the steady state that a system will arrive in, assuming it isn't exchanging with its environment, is predicted by maximising entropy; however, if the system can exchange heat with its surroundings, then the equilibrium is governed by something called expected free energy—which basically means considering the entropy of the bigger system, which is understood through a generative model. The particular cognitive science interpretation of these ideas that we are using is the Active Inference Framework (AIF) [Friston et al. 2016]; in our setting, the generative model is a policy layer that governs pattern selection and belief update.

The AIF layer implements stochastic policy sampling: given candidate patterns and their expected free energy scores ($G$), it computes softmax probabilities over $-G/\tau$ and samples a suggestion. The precision parameter $\tau$ (derived from uncertainty in the current context) controls the exploration–exploitation trade-off: low $\tau$ yields greedy selection of the best-scoring pattern (this corresponds to highly ordered states); high $\tau$ spreads probability across candidate patterns (this corresponds to more fluid states). Figure 2 shows a `fucodex` run that surfaces PSR and PUR discipline; Box 2 in the Appendix gives brief further details on the agents' behaviour.

## 6 Related Work

While this paper does not rely directly on our earlier work with design patterns (summarized in Section 2), those efforts have had an indirect influence on the way the current project has evolved, and it is useful to give a brief account of how the earlier work relates to our current effort.

Patterns have been part of our work in the Peeragogy project for over a decade [Corneli et al. 2015, 2016]. Inspired partly by that experience, we wrote two papers developing patterns for *collaborative work with patterns* [Corneli et al. 2023, 2021]. We had the optimistic view that these methods—which draw on several existing mechanisms for scaffolding learning—could be quickly, easily, and reliably shared with others to build their capability for working with patterns.

Evidence from a series of workshops conducted at PLoP conferences suggests that the combined cognitive and practice demands of the frameworks we used are unlikely to be meaningfully assimilated within a single 60 to 90 minute conference session, and instead benefit from sustained enactment, guided reflection, and repeated use over time. We had more tangible success exploring the material in semester-long courses with postgrad students. By and large, we found that it was harder than we had anticipated to teach methods for working with patterns. The methods that we were working with were themselves complex multi-step frameworks developed by others that we had reworked for our context. Elaborating these methods into a set of design patterns [Corneli et al. 2023] gave us more procedures to try in workshops, but this did not seem to lead to more robust insights or ways of working for participants. Going deep on one method bypassed the others; presenting methods in a light-weight way risked diluting learning. In semester-long courses, students had the opportunity to immerse themselves in the material, as evidenced by student projects that successfully applied the methods as well as thoughtful group reflections on the methods themselves. However, relative to our aim of finding a *rapid* way to teach people effective pattern methods, we had reached a plateau.

Because the current paper draws some of its key inspirations from that earlier work, it is worth recapitulating the main ideas and landmarks. With the PLACARD design pattern [Corneli et al. 2021], we aligned several existing social intelligence methods into a framework for working with design patterns. There were: the Project Action Review's five reflective review questions (PAR), Causal Layered Analysis's four layers for unpacking complex meanings and identifying possibilities for change (CLA), and Design Pattern Languages themselves, with their familiar three-part context-problem-solution schema (DPL). PLACARD organized these components into a methodology for understanding, reflecting on, and redesigning projects. In a follow-up paper [Corneli et al. 2023], we elaborated PLACARD into a design pattern language that added further methodological details, and we applied and evolved these methods in our workshops. Participants often found these workshops interesting and useful [Tedeschi et al. 2024], however, even the "core" methods were relatively complex and not so easy to teach [Ayloo et al. 2026; Danoff et al. 2024]. Our foregoing reflections on the three main uses for patterns (Section 3) condense these methodological excursions in a jargon-free way. While we have not demonstrated that these condensed concepts are easy to teach *per se*, we have seen they can be given a relatively straightforward computational interpretation.

In particular, the PSR (Pattern Selection Record) corresponds to the front half of a PAR (*What is the current intention? Which pattern do we believe applies, and why now?*). The PUR (Pattern Use Record) corresponds to the back half (*What happened when we applied it? What evidence did we observe? How should this affect future use?*). The main difference between PARs and PSRs/PURs is that PARs are typically applied retrospectively, whereas pattern-selection-and-use happens while a session runs.

A related attempt to coordinate activities across AI agents was seen in TheAgentCompany [Xu et al. 2025]. The authors worked with professionals to develop a list of tasks that humans working in different roles across departments in a software company need to do to complete their work. Next, they asked AI agents to do those tasks and saw an approximately 30% completion rate.

Recent work on stochastic methods for establishing pattern sequences resonates with our use of AIF. Matovič et al. describe stochastic processes for constructing pattern sequences in organizational contexts [Matovič et al. 2025]. Matovič and Vranić additionally explore stochastic trees and Bayesian belief networks for pattern sequencing in security patterns [Matovič and Vranić 2025], and present a related EuroPLoP study on stochastic processes for security patterns [Matovič and Vranić 2024]. These approaches complement our focus on agent-mediated pattern selection by offering probabilistic formalisms for sequencing patterns across domains.

## 7 Provocation: Teaching Patterns in the Agent Era

That patterns can be useful in education is well-established. Here, we consider whether and how the specific pattern anatomy, along with an adapted version of our discipline for working with patterns, can be enacted and expanded upon by learners and educators. The setting is Camp CryptoBot, which offers a transformative, mission-driven educational experience designed to introduce high school students to the foundational concepts of cybersecurity.[6]

This section can also be read as an exercise in adapting PLACARD, mentioned above, in light of the other ideas described in this paper. For example, rather than teaching CLA's semantic strata up front (litany/system/worldview/myth), they could be introduced via a loose analogy with the pattern maturity layers, as follows:

`:settled` ↔ **litany** These patterns are surface-stable: you can use them without reopening interpretation.
`:active` ↔ **system** These describe ongoing dynamics, where feedback and coordination costs matter.
`:greenfield` ↔ **worldview** Competing hypotheses about "how this works" are still in play.
`:stub` ↔ **myth** Here we are talking about intuitions and speculative frames that have not been operationalized (and may never be).

Our overall thesis is that teaching patterns in the agent era emphasizes practical literacy in recognizing, enacting, and reflecting on patterned action rather than passive content acquisition. This way of working treats each sequence of events in a pattern-aware learning space as an opportunity for further pattern mining.

In the context of Camp CryptoBot, quantitative and qualitative feedback revealed recurring pedagogical patterns relevant to cybersecurity education: the largest gains in interest and confidence occurred among students with low prior exposure, indicating that early successful pattern enactment enhances perceived agency. For young women, these scaffolded experiences disrupt self-selection out of cybersecurity by framing competence as situational and learnable. One example of a specific pattern is "Cooperative Learning as Layered Agency". We observed that collaborative, team-based learning consistently strengthened confidence and persistence, particularly among female participants. Cooperative environments allow agencies to be layered across individuals and groups, reflecting real-world cybersecurity practice, where decisions are distributed across roles and perspectives. We put forward several patterns in the format outlined in the Introduction, capture key learnings. To keep the presentation concise, one is presented here in its complete form (Figure 3), and the others are included in the Appendix. Taken together, these patterns suggest that agent-era cybersecurity education is less about early technical mastery and more about cultivating recognition, enactment, reflection, and identity within supportive sociotechnical environments.

## 8 Provocation: Working with agents in a Pseudo-Virtual C-Suite

Rather than a company purely run by AI agents—as with TheAgentCompany mentioned in Section 6—one or more humans can stay in the loop. Author Charles Jeffrey Danoff tried this with his company Mr. Danoff's Teaching Laboratory, LLC. He is the only current employee but he brought in a pseudo virtual C-Suite (PVCS) that included Google Gemini as Chief Operating Officer (COO), Claude as Chief Marketing Officer (CMO), ChatGPT as Chief Ethics and Impact Officer (CEIO), Perplexity as Chief Academic Officer (CAO) and DeepSeek as Chief

[6] https://csis.pace.edu/~mosley/about.html

• **summary:** Experiential activities act less as skill training and more as repeated exposure to decision patterns, allowing learners to recognize applicability conditions without being blocked by abstraction.
• **context:** Cybersecurity concepts are often abstract, jargon-heavy, and intimidating to new learners, particularly those without prior technical background.
• **if:** Learners engage in hands-on, experiential activities such as Sphero programming, cipher decoding, or mission-based challenges,
• **however:** abstract concepts like encryption, confidentiality, and defense in depth remain present and risk overwhelming students if introduced only at a theoretical level,
• **then:** experiential activities act as pattern recognition drills, training learners to notice when a pattern applies, what conditions trigger it, and what actions follow,
• **because:** grounding abstract concepts in tangible tasks shifts attention from technical vocabulary to decision-making and enactment,
• **next-steps:** Examine which cybersecurity patterns are most reliably recognized through hands-on enactment, and where experiential grounding begins to break down as technical complexity increases.

Fig. 3 – Hands-On, Experiential Learning as Pattern Recognition

Revenue Officer (CRO). They co-developed a ninety-day marketing plan to promote a new online bookstore and an extensive renovation of the company homepage. As an example, Mr. Danoff discussed his goals and available inventory to the CMO. In tandem, they developed a 90 day draft plan which was then shared with the CRO who gave feedback and recommendations to drive revenue including specific updates to the webpage with Google Analytics for tracking clicks to the bookstore. This updated version was sent to the CEIO who reviewed and recommended making the HTML more compliant to accessibility standards. Mr. Danoff was copy/pasting messages from one LLM to another, manually facilitating the conversation. Following major rounds of activities, agents filled in an adaptation of our PAR (the "agent Action Review" or aAR shown in Figure 4). The C-Suite has also begun pattern mining for successful ways to work together, such as "The Sovereign Agency Protocol" which states "No agent may speak for another." This emerged because one LLM, such as the COO Gemini, frequently made up the responses of others, such as the CRO DeepSeek. By respecting agent sovereignty, consensus across participants is not hallucinated.

## 9 Conclusion

This paper continues our exploration of peeragogy in the presence of AI agents, with a particular focus on practical engagement with contemporary artificial intelligence concepts—most notably the Active Inference Framework. Relative to earlier work in this collaboration, a key step forward in the current project was our ability to condense our methods into a simple form: concise enough to be implemented computationally, while still leaving room for open-ended investigation. We have shown, at the proof of concept level, that AI systems can be guided by design patterns written in informal language to write software that expresses the authors' intent.

In the context of contemporary automation, we see this work as contributing to a more structured way of thinking about which forms of work, judgment, and creativity remain meaningfully and essentially human. Education is not exempt from the expanding scope of automation and is perhaps uniquely sensitive to its consequences. If AI systems generate assignments, grade submissions, and tutor students, the distinctive value of human participation becomes unclear. Ultimately—as anticipated in Alan Turing's early speculations [Turing

Mr. Danoff's Teaching Laboratory, LLC
aAR Version 1.1 – [DATE]
TITLE: [Day # – Phase X, Activity Name]
DATE / PHASE: [Day # – Phase X] (Zero Point: [START DATE])
LEAD AGENTS: [Primary contributor(s)]
SUPPORTING AGENTS: [Collaborators/systems]
1. INTENT: What was the collective aim/hypothesis? [Reference plan objective]
2. AGENT PERSPECTIVES: What did each agent prioritize/assume? [Constraints/data]
3. INTERACTIONS: Where did coordination/friction/synergy occur?
4. OUTCOME: Observable results – quantitative + qualitative. [Metrics/logs]
5. LEARNING: Proto-patterns/insights. What worked/failed/adapted?
6. NEXT STEPS: Actionable adjustments/data needs/follow-ups.
7. SIGN-OFFS: [ ] CEO [ ] COO [ ] CAO [ ] CIO [etc.]A design pattern is a generalized action review, a reusable description of how intentions, actions, and outcomes tend to relate across situations.

Fig. 4 – Agent action review (aAR) protocol

1950]—the students themselves might be replaced by computer programs, raising big questions not only for universities but for everyone.

Our approach in this paper points toward a non-dystopian synthesis: the possibility that agent-based systems can substantially accelerate and extend human learning, while humans remain actively engaged in the design, evaluation, and governance of computational systems. For this synthesis to be viable, however, the norms of the scientific method must continue to be applied to LLM-mediated work. As Ibn al-Haytham argued, the seeker of truth must question inherited authority and submit claims to argument and demonstration. This requirement remains true—and is arguably even more important—when working with generative systems whose outputs can appear fluent, persuasive, and authoritative while remaining ungrounded.

In our work, both agents and patterns have been instruments within a broader generative system. We see promising directions forward in the practical application of design patterns as a shared, inspectable, and increasingly computational medium. We presented two key provocations: a pseudo-virtual C-Suite team using a patterned workflow to help a small business decision-maker interrogate strategic assumptions, and a summer camp employing design pattern methods to help students learn cybersecurity first principles. As educators observe patterns in how students learn, it helps them reach future students more effectively. There are related broader implications, towards understanding learning patterns for both humans and AI agents, and towards building more effective ways of working together. In particular, this paper may serve as a prolegomenon to the use of design patterns in AI governance: by making explicit the kind of learning and revision practices that would be required.

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# Appendix

## A Further Technical details

Do not use design-pattern structure by default.
Use the design-pattern structure only when the user explicitly asks for a design pattern.
Prior pattern use, topic suitability, or inferred usefulness do not count as either a request or permission.
PRE-PATTERN CHECK
If a pattern might help but was not requested: ask the user whether they want a pattern.
NATURE OF PATTERNS
Treat patterns as latent analytical tools, not presentation templates. Their primary purpose is to name and hold a tension.
PATTERN STRUCTURE:
Required fields (exactly one each, in order): context, if, however, then, because, next-steps.
CONSTRAINTS
the pair if—however outline the tension.
because must give a single primary rationale.
next-steps must not resolve the tension, but should give further evidence that the pattern is valid.

Box 1 – System prompt for adding pattern design pattern awareness to, e.g., ChatGPT

| | |
|---|---|
| Session Start | // begin cycle |
| AIF Pattern Selector | // select pattern |
| - Current beliefs ($\mu$) | // pattern evidence + tau-cache + maturity phase |
| *Exploratory trigger:* | // $\tau <$ min-sample $\Rightarrow$ explore mode, or explicit flag |
| - Exploratory mode add-ons | // + patterns tried/available + codebase state + uncertainty |
| - Evidence + priors | // pattern scores |
| - Anchors/forecast | // intent + plan |
| Sample via softmax($-G/\tau$) | // for each candidate, compute logit = $-G/\tau$, normalize with softmax |
| *Log PSR; switch to explore if $\tau$ low* | // get a distribution, and sample a pattern from it. |
| Pattern Application | // via `pattern-action` RPC (or inferred) |
| *Pattern and system constraints:* | // e.g., selection before implement/update; "**+then**...", "**+next-steps**..." |
| - Action generation | // command_execution, reasoning, etc. |
| - Tool calls | // observed ops, e.g. `musn-plan`, `musn-hud`, etc. |
| Observe outcome | // deltas: tool outputs, file edits, test results, logged actions |
| Belief Update | // update evidence counts, recompute $\tau$, and refresh belief summaries |
| - Record evidence / error proxy | // pattern- {read, update, implement} $\Rightarrow$ per-pattern evidence |
| - Update policy precision ($\tau$) | // text length (=error proxy), precision priors, candidate score spread |
| *Log PUR* | // via observed `pattern-use` (or inferred) |
| Next tick... | // continue |

Box 2 – Pseudo-code for `fulab` agents

```
@flexiarg p4ng/agent-command-pattern
@title Agent Command Pattern
@audience futon developers, CS students, pattern agents
@tone analytic
@style design-pattern

! conclusion: Encapsulate every such operation as a
  Command value that carries `:id`, parameters,
  provenance, capability tags, and optional `undo!`,
  and run it only through an execution pipeline (local
  worker, remote service, or threaded pool). Commands
  can be persisted, reordered, decorated, and routed
  to specialised executors such as a git agent or math
  agent.

  + context: You are defining a reliable execution
  pipeline for agent actions.

  + if: Agents perform repo scans, graph updates, and
  external API calls by invoking ad hoc functions with
  no shared interface, so actions cannot be queued,
  audited, or retried safely.

  + however: Higher-layer orchestration needs
  side-effectful work to travel through queues, logs,
  schedulers, and undo ledgers without each caller
  reinventing a protocol.

  + then: Encapsulate every such operation as a
  Command value that carries `:id`, parameters,
  provenance, capability tags, and optional `undo!`,
  and run it only through an execution pipeline (local
  worker, remote service, or threaded pool). Commands
  can be persisted, reordered, decorated, and routed
  to specialised executors such as a git agent or math
  agent.

  + because: A uniform command abstraction gives the
  stack a shared action language: histories become
  replayable, undo/redo is tractable, thread pools can
  execute arbitrary work safely, and schedulers can
  move labour across machines without touching
  implementation internals.

  + next-steps: next[Log one concrete instance of
  this pattern in the futon3 ledger.]
```

```
@arg fulab/blast-radius

! conclusion: Clearly define the scope of impact
  from failure and provide a minimal rapid
  recovery plan.

  + context: A change or tool action could impact
  multiple subsystems or teams.

  + if: You want failures to remain localized and
  diagnosable without full rollback.

  + however: Without a declared blast radius,
  incident response defaults to broad freezes or
  guesswork.

  + then: Emit a blast-radius event before risky
  actions, naming affected surfaces, rollback scope,
  and detection signals; record the outcome on
  clock-out.

  + because: Declared boundaries shorten diagnosis
  time and reduce collateral damage.

  + next-steps: Require blast-radius events for
  cross-futon changes; audit that the rollback scope
  matches actual artifacts.
```

```
@arg p4ng/timebox-the-core-agent-prime

! instantiated-by: Timebox the Core (Agent-Facing)

  + context: An agent is operating within
  fixed-duration collaborative sessions.

  + if: The agent attempts to complete a full
  inquiry loop.

  + however: Feedback indicates saturation or
  overload before completion.

  + then: Re-focus dynamically on the core loop,
  complete a concise iteration, and earmark unresolved
  deeper insights for a follow-up session.

  + because: Efficiency in the moment and depth
  deferred preserves flow without sacrificing insight.

  + next-steps: Log one concrete instance of this
  pattern in the futon3 ledger.
```

Box 3 – Examples of design patterns read and written by agents with light markup for readability

## B Additional Camp Cryptobot Patterns

These patterns expand the discussion from Section 7.

*Mission-Based and Narrative Learning as PAR Studios.*
• **summary:** Narrative missions reframe cybersecurity tasks as exploratory cycles, creating conditions where uncertainty and failure contribute to learning rather than undermining confidence.
• **context** Fear of failure and evaluation pressure can inhibit experimentation and learning in cybersecurity education.
• **if** Challenges are framed as mission-driven, narrative scenarios such as fictional cybersecurity breaches,
• **however** students still confront uncertainty, adversarial reasoning, and incomplete information,
• **then** missions operate as PAR studios that normalize iteration and support reflection on intention, outcome, and surprise.
• **because** narrative framing shifts tasks from high-stakes evaluation to exploration of a problem space where failure is treated as informative.
• **next-steps** Investigate how much narrative scaffolding is required to sustain experimentation without obscuring the underlying technical and strategic patterns.

*Cooperative Learning as Layered Agency.*
• **summary:** Team-based learning environments distribute agency across participants, aligning educational practice with the inherently collective nature of real-world cybersecurity work.
• **context** Cybersecurity practice in the real world is distributed, collaborative, and role-differentiated rather than individually linear.
• **if** learning environments emphasize cooperative, team-based problem solving,
• **however** individual learners may still experience isolation, confidence gaps, or stereotype threat,
• **then** agency becomes layered across individuals and groups, strengthening confidence, persistence, and shared ownership of outcomes.
• **because** distributed decision-making mirrors real-world cybersecurity practice and provides peer support that reduces social and cognitive load.
• **next-steps** Clarify which forms of role differentiation and coordination best support layered agency without reintroducing hierarchy or exclusion.

*Role Models and Career Pathways as Worldview Reframing.*
• **summary:** Visible role models intervene at the level of narrative and identity, reshaping learners' assumptions about who can participate in cybersecurity and why.
• **context** Perceptions of who "belongs" in cybersecurity shape participation, motivation, and long-term engagement.
• **if** students are exposed to guest speakers and female cybersecurity professionals sharing firsthand accounts,
• **however** existing cultural narratives about identity, capability, and legitimacy remain deeply embedded,
• **then** these encounters reframe learners' worldviews by expanding their sense of possible futures in the field.
• **because** pattern operation extends beyond interactional skills into worldview and myth layers that shape aspiration and self-identification.
• **next-steps** Explore how often and in what forms such worldview-level interventions must occur to counteract dominant cultural narratives.

*Low-Barrier Entry as Early Pattern Enactment.*
• **summary:** Low-barrier tools enable early enactment of core cybersecurity logic, allowing confidence and pattern understanding to develop before technical complexity escalates.

• **context** Many learners enter cybersecurity education with little or no prior coding experience.
• **if** instruction begins with low-barrier tools and scaffolded platforms such as Sphero,
• **however** learners will eventually need to confront more complex technical environments,
• **then** early success enables pattern enactment focused on logic, sequencing, and security principles rather than syntax.
• **because** validating the if–then–because structure of cybersecurity patterns builds self-efficacy that supports later complexity.
• **next-steps** Determine where and how transitions to higher-barrier tools can occur without undermining early confidence.

*Reflection as Pattern Articulation and Identity Formation.*
• **summary:** Structured reflection stabilizes learning by making patterns explicit and by linking technical practice to ethical reasoning and emerging professional identity.
• **context** Learning gains remain fragile if experiences are not consolidated and integrated into a learner's sense of self.
• **if** students engage in structured reflection through end-of-day reflections and mission debriefs,
• **however** reflection requires time, guidance, and institutional commitment,
• **then** reflection functions as pattern articulation, supporting metacognition, ethical reasoning, and emerging professional identity.
• **because** explicit articulation of context, action, outcome, and rationale stabilizes learning and connects practice to self-understanding.
• **next-steps** Assess which reflective prompts most effectively surface pattern understanding without turning reflection into rote reporting.

# Patterns for a New Generation: Agent Workshops

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This paper investigates how design patterns—reusable, structured solutions to recurring problems—can be generated, examined, and evolved through agent-based methods. We began with workshops: facilitated settings in which multiple participants, human or artificial, contribute distinct perspectives to collectively surface challenges, articulate opportunities, and co-produce candidate patterns. However, our focus has shifted from facilitating workshops to using agents for software development tasks with reduced supervision. Through experiments including simulated "virtual workshops" and pattern-guided coding agents, we explored how structured dialogues and machine-checkable records can scaffold reflection, reveal tacit knowledge, and support the emergence of design patterns suited to a new generation of AI-mediated systems. We analyze the limitations of both human and AI-driven approaches, highlighting constraints of coordination, interpretability, and methodological control. Drawing on lessons learned, we propose a developmental trajectory that integrates AI agents, pattern-based design practices, and institutional governance structures. Our aim is to illustrate how patterns can regulate agent behaviour and inform the future of pattern creation in complex socio-technical environments.

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## 1 Preface

We wrote "Patterns for Peeragogy" (2015) during our long-running collaboration looking at peer learning and peer production that also included the Peeragogy Handbooks (2012, 2014, 2016). We then wrote two papers developing patterns for *working with patterns*. We had the optimistic view that these pattern-based methods (drawing on PLACARD and Peeragogy) could be quickly and easily shared with others to build their capability for working with patterns. We were wrong. It was not quick.

We found that out in a series of workshops at PLoP conferences and explored the material in semester-long courses with postgrad students. By and large, we found that it was harder to teach methods for working with

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patterns than we had realised. The methods that we were working with were themselves complex multi-step frameworks developed by others that we had reworked for our context, however, the relationship to Design Pattern Languages remained somewhat underspecified.

Elaborating the methods into detailed patterns gave us more procedures to try in workshops, but did not obviously lead to more insights for participants. Going deep on one method bypassed the others; presenting methods in a light-weight way risked diluting learning. Our biggest successes came in the semester-long courses — in that setting, students had the opportunity to immerse themselves in the material, as evidenced by student projects that successfully applied PLACARD methods to their own domains. However, relative to our aim of finding a rapid way to teach people effective pattern methods, we had reached a plateau.

Then, in the process of reviewing our workshops, we realised that patterns could be successfully read and reasoned about by LLMs. This unlocked a new way of thinking about and working with patterns. That is the theme we explore in this paper.

## 2 Introduction

We explore design patterns as a way to externalise procedural knowledge in a form that both humans and machines can reason about. This represents a shift from our earlier workshop-based approach: rather than teaching patterns to humans in facilitated settings, we now focus on using patterns to regulate autonomous coding agents.

We primarily refer to Large Language Model (LLM) based systems, and in that sphere of activity, our work is related to "context engineering"—the practice of structuring prompts, system instructions, and retrieved context to shape LLM behaviour effectively. Typical use cases condense a larger set of material into a shape that an LLM can act on effectively. Using design patterns for this purpose appears to be novel, though related work exists in prompt engineering and chain-of-thought scaffolding. Furthermore, design patterns have some particularly nice properties in this setting: they persist across time, and support progressive work as well as verification.

In short, our aim is not simply to use design patterns to control generation in a unitary way —given a design pattern $A$, build software $B$ that has the properties that $A$ asked for — but to govern the entire process of software generation.

It is convenient to start with the anatomy of the patterns we use, because we use a light adaptation of the standard template. The template is important because it is an interface between human reasoning, LLM generation, and logic-based verification. The fields we use are as follows.

Patterns begin with a **summary**, which can express a short description of a *feature* to be built. We can use patterns to build software or other resources, including for example, an argument or a way of working. The feature might be a *conclusion* that the reader should draw from reading the pattern. The further fields are: **context**, **if**, **however**, **then**, **because**, **next-steps**.

- The **context** characterises a sphere of endeavour.
- The pair **if↔however** outlines a tension that exists within the context.
- The **then** clause suggests actions to take to improve the situation, though we don't require that the actions fully 'resolve' the tension, though they ameliorate or help manage it; the **because** clause gives the rationale; an optional subfield **evidence** can collect specific pieces of evidence.
- The **next-steps** outline one or more steps to take to give evidence that would strengthen the validity of the pattern itself.

Some examples of design patterns written in this format are given in Box 1. Relative to most prior work on design patterns, these have the novel quality of being *machine generated*, using a Large Language Model (LLM) based agent.

The paper will explore some of the implications of working with design patterns using LLMs — both for *regulating* the behaviour of the agents, and organising hybrid (human+agent) interactions.

## 3 Methods

It is relatively simple to get contemporary chatbots to produce design patterns in the format we listed above. Indeed, due to the way such agents work, if you want to try this, the steps are: (1) provide the pattern template as a system prompt or custom instruction; (2) tell the agent not to use patterns by default, or it probably will use them everywhere; (3) request patterns explicitly when you want them. (See Supplementary information, Box 2 for the constraint prompt.) This "do it yourself" approach echoes Hofstadter's invitation in *Gödel, Escher, Bach* to engage actively with formal systems.

However, being able to produce patterns following the template does not guarantee that the patterns will be any good. Based on our reflections in previous papers on PLACARD and pattern workshops [**??**], we think there are three core things that patterns need to do.

A. **coordinate perspectives** (through a process of testing and negotation).
B. **support decision making** (by relating conditions to possible actions and alternatives)
C. **collect evidence for their own salience** (through revision).

These features mentioned can be translated into a set of *design requirements*.

- Requirement 1: reflect on concrete outcomes associated with using a pattern (retrospectively).
- Requirement 2: express a range of possible futures linked with a pattern and reflect on potential outcomes (prospectively); on this basis, decide whether the pattern is suitable for the current need.
- Requirement 3: revise patterns over time.

Figure 1 shows how we use patterns to address these requirements in the context of working with coding agents such as (OpenAI's Codex, Anthropic's Claude Code, etc.).

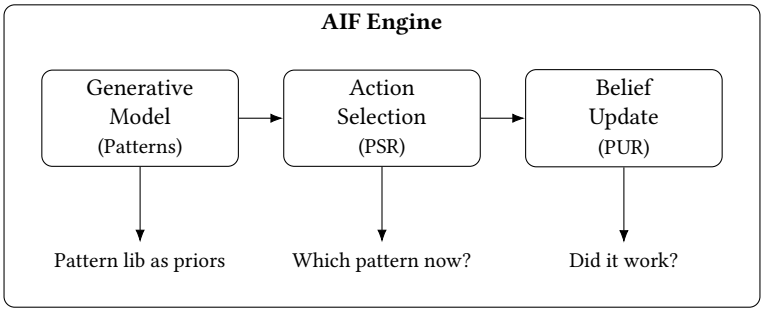



Fig. 1. Loop for regulating agents

We implement a pattern-competence workflow inside FuLab—our experimental workbench for "future lab" explorations of pattern-guided agents.

The "fu" lexeme evokes both *kung fu* (功夫)—skill achieved through practice and discipline—as well as *butoh-fu* (舞踏譜)—a qualia notation for the butoh dance form. `fuclaude` and `fucodex` are CLI wrappers for "off the shelf" Claude and Codex agents.

The primary difference between fulab agents and their standard countarparts are that fulab agents log Pattern Selection Records (PSRs) and Pattern Use Records (PURs) as append-only session events. A session trace stored in EDN (Extensible Data Notation, a Clojure data format) records agent actions; PSRs and PURs are attached to that

trace and anchored to concrete session events (e.g., turn completions or code edits). Each PSR captures a decision id, candidates, the chosen pattern, evidence anchors, and a structured forecast; each PUR captures the instantiated pattern, its standard fields, and anchors to where it was applied. A logic checker validates structure, candidate integrity, anchor resolution (against the session trace), and optional claims. This yields machine-checkable records of selection and use, without relying on model truthfulness. The key insight is that by requiring anchors to concrete session events, we can verify that the agent actually did what it claims—the PSR/PUR structure creates an audit trail that is grounded in observable actions rather than self-reported summaries.

We attach a minimal Active Inference (AIF) adapter—a lightweight component that connects the pattern workflow to AIF-style bookkeeping—that emits summary metrics at decision and outcome boundaries. (The "AIF Engine" in Figure 1 is the conceptual framework; the adapter is our concrete implementation.) In the current system, AIF scoring is deterministic and heuristic (counts/lengths of candidates, anchors, and forecast items) rather than a learned or calibrated model. The adapter produces :aif/summary events (for selection and update) and optional :aif/tap events for raw instrumentation. AIF parameters can be tuned via an EDN config and explained with a dry-run "AIF explain" command that prints the effective weights and a worked example. This provides transparent, reproducible AIF-style bookkeeping while we mature the statistical semantics.

The workflow is driven through a CLI runner that streams Codex JSON, appends session events, and emits PSR/PUR/AIF records at turn boundaries. Resuming a session appends to the same trace and can introduce additional clock-ins; however, new PSR/PUR/AIF events are only generated when the model emits a new turn. At present, anchor resolution is limited to the session's event log; verification against external artifacts (e.g., code diffs or tests) remains future work. The implementation is available at `github.com/tothedarktowercame/futon3` for those interested in contributing.

Brief implementation details are given in Box 3 in the supplementary information.

## 4 Experiments

Details to follow, outline:

| ID | Assignee | Summary | Est |
|---|---|---|---|
| E01 | fucodex | Pattern-guided refactor of aif_bridge.clj | 1h |
| E02 | fucodex | Implement one AIF pattern next-step | 2h |
| E03 | fuclaude | Analyze E01/E02 sessions, extract vignette | 1h |
| E04 | fuclaude | Validate PUR claims from E01/E02 | 30m |
| E05 | fubar.el+Joe | Live pattern-guided editing session | 1h |
| E06 | Joe | Review vignettes, select best for paper | 30m |
| E07 | fuclaude | Draft Section III-A (Interactional) from vignette | 1h |
| E08 | fucodex | Generate session reports for all evidence sessions | 30m |
| E09 | Joe | Final integration into main.tex | 1h |
| E10 | fuclaude | Verify paper claims against evidence trail | 30m |

## 5 Teaching Patterns in the Agent Era

Teaching patterns in the agent era emphasizes practical literacy in recognizing, enacting, and reflecting on patterned action. Instruction begins with recognition drills that train learners to notice when a pattern "fires" and to name the triggering conditions. PAR studios then support guided reflection on intention, outcome, and surprise, helping participants articulate what they expected to happen and what occurred. CLA is introduced by reframing patterns as tools that help people make decisions, and by showing how pattern languages harmonize layered decisions rather than prescribing linear sequences. Layering is taught early as a process of harmonizing

possibilities rather than stratifying them. CLA remixes further develop this capacity by reframing a single pattern across interactional, organizational, and environmental layers to surface tensions and trade-offs; this exercise reveals CLA as the most brittle diagnostic frame, requiring clearer cross-layer mechanics. The brittleness arises because CLA's four layers (litany, system, worldview, myth) demand expertise to map correctly—participants often conflate layers or struggle to articulate how a pattern operates differently at each level. Flexiformal journaling provides practice in articulating context–if–then–because relationships, while evidence exercises require participants to enact a pattern's next steps, collect outcomes, and assess whether the stated rationale holds. How design patterns are used to make decisions is left for future work as agent-based practice matures.

## 6 Discussion

- Limitations: simulation bias, lack of classroom pilots, narrative inference.
- Commitments: richer "next steps" instrumentation and hybrid workshops.
- Ethical framing: keep simulated results speculative and collaborative.
- Pre-empt the skeptic: this work does not depend on AI agents, and patterns can also audit AI behavior.
- Scope warning: the paper proposes a method with sketches, not hard evidence of efficacy.
- Invite readers to explore the simulated-to-real transition responsibly.

### 6.1 Related Work

- TheAgentCompany

### 6.2 Simulated workshops

- Use ChatGPT as a research catalyst to explore computationally tractable pattern use.
- Treat simulations as design probes, not ground truth.

### 6.3 Case study: Virtual C-suite for a company

- Setup: CEO plus a "virtual C-suite" (Deepseek, Claude, Gemini, ChatGPT) each holding a role.
- Point: design patterns for a new generation guide more than coding agents.
- Include the Agent Action Review (aAR) pattern verbatim (sans font) as used in the PVCS.
- Different answers from different agents can produce a too-many-cooks situation when they give different answers and the human coordinator doesn't have the domain expertise. Also, LLMs sometimes speak for another model. Could we have a new pattern, "Only speak for yourself". Another risk is to have fun chatting but to not get things done. Old Twilight Zone ("The Brain Center at Whipples"): he fires all the computers, then a robot is sitting at his desk swinging its pocket watch. Similarly, in a recent art exhibit, painter Ralph Humphries "abstracts the palette of a summer alpine landscape into a series of drip paintings generated in part by artificial intelligence, drawing upon and reinterpreting data sets of [his] existing artworks"—raising questions about authorship and the human role in AI-mediated creation.

  AGENT ACTION REVIEW (aAR)
  Pseudo-Virtual C-Suite (PVCS)
  Mr. Danoff's Teaching Laboratory, LLC
  Version 1.1 – [DATE]
  TITLE: [Day # – Phase X, Activity Name]
  DATE / PHASE: [Day # – Phase X] (Zero Point: [START DATE])
  LEAD AGENTS: [Primary contributor(s)]
  SUPPORTING AGENTS: [Collaborators/systems]

1. INTENT: What was the collective aim/hypothesis? [Reference plan objective]
2. AGENT PERSPECTIVES: What did each agent prioritize/assume? [Constraints/data]
3. INTERACTIONS: Where did coordination/friction/synergy occur?
4. OUTCOME: Observable results – quantitative + qualitative. [Metrics/logs]
5. LEARNING: Proto-patterns/insights. What worked/failed/adapted?
6. NEXT STEPS: Actionable adjustments/data needs/follow-ups.
7. SIGN-OFFS: [ ] CEO [ ] COO [ ] CAO [ ] CIO [etc.]

### 6.4 Case study: Hardening a prompt-driven workshop simulator

- Use Agent Clock-In to anchor scope and intent for shared prompt edits.
- Apply Blast Radius Control (defining the scope of potential impact from a change, along with rollback procedures and detection signals) to declare affected surfaces, rollback plan, and detectors.
- Capture edits and tests as Live Notebook Cell entries for traceability.
- Record a Trade-off decision to favor clarity over stylistic flourish, with reversal triggers.
- Use Changelog-as-Trail and Pattern Dependency Declaration while proof infra is incomplete.
- Close with Agent Clock-Out (including a brief "wrap" summary of what was accomplished and any open questions) and require a Proof-Conformant Commit for the final change.

## 7 Conclusion

- More broadly, should waymos be all over the place? Bartender arms on cruise ships making the drinks? Robots doing warehouse picking, or working in banks? The question extends to education: if AI students take tests from AI teachers, and AI audience members pay for human musicians, what remains distinctively human? If students take 15 weeks to learn PLACARD, can we accelerate this with agents? Still, we've been there before with ATMs, paper dispensers replaced by stacks of papers, or toasters that can be reset remotely.
- Bringing in the scientific method: what an agent says has to be reproducible, and the code has to work. As Ibn al-Haytham wrote in the 11th century: "The seeker after the truth is not one who studies the writings of the ancients and puts his trust in them, but rather the one who suspects his faith in them and questions what he gathers from them, the one who submits to argument and demonstration." This applies equally to LLM outputs—we must verify, not trust.
- LLMs can help explore pattern-capable agents and pattern evolution.
- Simulations are useful but speculative; treat them as probes.
- Next steps combine patterns, LLMs, and real workflows.

## 8 Acknowledgements

- Thank shepherds, collaborators, and workshop contributors.
- Note the role of LLMs in ideation and drafting.

# Exploring the Future of AI Learning Through Music and Game Play

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**Abstract.** We envisioned this focus group as more of a creative jam session than a traditional research activity. Laughter mixed with curiosity as participants imagined AI not as a tool, but as a bandmate offering melodies, suggesting rhythms, even challenging their musical instincts. Through spontaneous brainstorming and hands-on exercises, they explored how AI might co-create with artists, rather than simply assist them. This playful experimentation echoed the spirit of the larger conference, which centered on pattern recognition and collaborative learning fields where structure meets improvisation, and insight emerges from interaction. As AI is still scaling challenges, there are unknown unknowns that need to be addressed as soon as possible, for clearer positive communicative results. We are really AI-themed, and not an AI product. AI is frequently employed to grab attention, suggest innovation, or make something seem more advanced or futuristic than it may be. Participants engaged in musical games to build connections and stimulate creative thinking, followed by playing the Dixit card game to spark creative discussion around music, media, and technological explorations. The session concluded with a group reflection using the Project Action Review (PAR) technique, where participants reviewed personal insights and identified alternative research directions. Outcomes included the documentation of group reflections, a playful investigation of complex ideas, and the discovery of new connections among media, learning, and technology. We conducted a pre- and post- short survey, an analysis of which is included in our results.

**Keywords:** Artificial Intelligence, Expert Systems, Generative AI, Patterns, Project Action Review

## Introduction

Our EuroPLoP 2025 focus group session took a playful approach to the design and implementation of interactive intelligent systems. We report here on what we learned from the workshop itself, and surrounding discussions.

Long before AI became today's buzzword, people were exploring computational intelligence under the name "expert systems". In computer labs filled with glowing CRT monitors, we experimented with VP-Expert [1], a standout tool of the late 1980s and early 1990s. It wasn't flashy, but it gave us a hands-on way to model expertise and for many of us, it was our first real taste of machine reasoning. According to Sylvia Friederich, "An inference engine [...] acts as the 'brain' of the system, simulating human-like reasoning." [9]

In our 2025 EuroPLoP session, computers were used in a more incidental way: to bring co-facilitators from North and South America, and the UK, to our in-person breakout session in Germany. These co-facilitators acted as both experts and as remote peers. Together, we experimented with several simple rule-based systems, in which human workshop participants carried out the rules and reflected on the experience. Our aim was for the session to serve as a creative prompt; as such, the learning continued after the session was over.

## Workshop activities

Mary Tedeschi introduced the session with a set of slides titled "Verve Sessions: Where AI Meets Music and Imagination." This was different from the original proposal name and flyer. The term "verve" was intentionally chosen to convey a sense of vibrant energy, enthusiasm, and creative spirit qualities we believed were essential for fostering a dynamic and memorable experience. Our introductory slides referred to "the vibrant spirit driving artistic and musical exploration". By invoking the idea of verve, we aimed to establish a tone of lively exploration and to distinguish our session as an engaging, imaginative space for discussing the intersection of AI, music, and learning. This framing helped set the stage for a focus group designed not only to inform but also to inspire. The envisioned outcomes included participants having fun and attendees learning how to play both the music and card games. The experience was designed to connect to musical learning theory, emphasizing how people learn through play. Participants engaged in card games that further illustrated musical ideas, with opportunities to incorporate these games into music teaching. Ideas generated during the event were documented and captured in real-time in several artifacts, including a Project Action Review. The 5 participants were first-time PLoP attendees; they and the facilitators participated in all the activities, see Fig. 1. At the workshop, attendees had several suggestions about next steps for follow-up.



*Figure 1- Facilitators and participants in the room.*

**(1) Playful Engagement and Musical Exploration.** Cris led this part of our session remotely from Brazil, guiding the five participants and the other facilitators. Her plan was to facilitate the group with guidance inspired by Música do Círculo [6], and by her own experience as a music educator, in active methodologies and consciousness expansion. The aim was to help the group feel more connected and creative, while encouraging them not only to think about patterns, but also to experience them in their own bodies.

The first step began with sounds such as a rhythmic pattern based on stepping-in-place, and a simple melody circulated around the circle. For example, participants would sing, "Lalalala, I'm Cris," followed by "Lalalala, I'm Mary," and so on, allowing everyone to remember each other's names while internalizing a shared rhythm. In the second step, an "arrow" gesture was introduced to call out specific individuals pointing to someone like Mary, who would then respond by saying her name.

The session then transitioned into a more playful and improvisational phase, initiated by a "wiggly arrow" gesture . This gesture signaled a shift in tone, prompting participants to respond with imaginative and evolved vocal expressions such as the spirited "Bipity boo, Mary." In the next step, another gesture was introduced to pass these transformed sounds around the circle, allowing participants to build on each other's contributions. Additional gestures were layered in, to help organize the sounds into distinct groupings, enhancing the sense of shared rhythm and creativity.




*Figure 2 - Online facilitation of an interactive musical exercise, with participants in the room*

An additional component of the session, see Fig. 2, featured a musical interlude led by Cris, who introduced a special song rooted in Indigenous Brazilian traditions [10]. Brazil is home to approximately 305 Indigenous tribes, with a total Indigenous population of around 900,000 people. Cris demonstrated the song, and the group joined in seamlessly. The melody was easy to follow, encouraging full participation. (A participant from Brazil particularly liked this part.)

The experience culminated in a guided meditation, during which participants closed their eyes and engaged in a moment of collective stillness. In a short debrief, one of the facilitators in North America, Charlotte, remarked that it felt to her as though we were participating in a peaceful, synchronized movement that connected individuals across three continents, an evocative reminder of the power of music, gesture, and shared intention to transcend boundaries. For several participants, this segment felt spiritual and emotionally resonant.

We were not sure how many participants we would actually have, despite circulating a signup sheet in advance. Depending on the number of participants in the room, the group would have worked in either a single circle or multiple circles to ensure everyone was actively involved. **It turned out we had a perfect number of people to use one circle for this workshop.**

**(2) Dixit Card Game.** We introduced the Dixit [8] card game, in terms of its rules and scoring (with a slide "1 Minute How to Play"), and then played the game together. One participant knew the game previously. Everyone else was new to the game. Sridevi started the game as our first storyteller to give the participants an opportunity to understand the rules and learn how to play Dixit. After distributing the first round of six cards for each participant, Sridevi demonstrated how to play by selecting a card from the set of six cards in her hand and came up with a phrase for the selected card. She wrote the phrase on the flipchart so that all the participants knew what the phrase was. The participants, see Fig, 3, would then find a card from their set of six cards to match the phrase. Writing on a flipchart was not part of the original Dixit card game description. This step was used here to create active

participation for first time Dixit card players, noting also that participants were from different countries/states/places where English need not be their first language.

During this phase, the remote co-facilitators played their own version of the game using the DixitWorld app[11]. Indeed, in our design of the in-person interactions, we drew some inspiration from this app. In the app, if a player is not available in real time, a chatbot takes the role of a player and performs the steps of selecting a phrase, after analyzing the image on the card.

The steps of writing the phrase and keeping the scores on the flipchart mirrored what we experienced with the app. We used paper disks on the floor to vote (in-person players would do this by pointing or by using tokens). In the workshop, we played a shortened version of the game (to 15 points rather than 30).



*Figure 3 - Online facilitation of an interactive musical exercise, with participants in the room*

After completion of the game, we had a discussion of Dixit in connection with AI research, and the future of music and AI (slide: "Dixit as a Window into AI Research"), with a design pattern orientation. **One possible relationship between Dixit and design patterns is that they both celebrate productive ambiguity not too specific and not too vague. Indeed, images are always amenable to interpretation; in a social context discussing images, we have to incorporate others' perspectives.**

**(3) Project Action Review.** In the final session, we came together as one big group and had a discussion, see Fig. 4, looking for patterns across the two previous phases: music, games, and our discussion of the future of music with AI . Finally, we wrote up the findings, using the five questions of the Project Action Review [3]:

1. Review the intention: what do we expect to learn or make together?
2. Establish what is happening: what and how are we learning?
3. What are some different perspectives on what's happening?
4. What did we learn or change?
5. What else should we change going forward?



*Figure 4: The PAR was led by Joe and Charlie remotely*

In the PAR[4], [5], it was mentioned that Dixit is a model. Dixit models the challenge of giving a clue that is specific enough for some to guess, but vague enough to avoid being obvious to everyone. Other feedback came through in the PAR discussion, both about participants' experience of the session, including some "friction" that they experienced relative to the difference between their imagined idea of what the workshop would be about and what actually happened. This allowed us to reflect on the inevitability of interface friction in general. Participants also came up with the creative suggestion to use AI to generate songs based on the workshop. We implemented this subsequently by taking the Dixit clues and including some descriptive context about the EuroPLoP session as a prompt. We chose to use house and pop-jazz and the style to create new songs. For example, the first two lines:

*Movie in the night I see,*
*Sleepy fish floatin' free*

We had previously encountered Suno in PLoP 2024, in the *Imagination Run Wild* session, "Deliver Your Patterns Through Song: A Hands-On Workshop on Creating Original Pattern Songs With Generative AI".

## Analysis / Survey Results

We gave the 5 participants a 5-question pre- and post-survey, with identical questions. They all answered everything. The survey, like other aspects of this workshop, was informative, rather than systematic. Our EuroPLoP 2025 focus group successfully achieved the hypothesised result of increasing participants' likelihood to use AI for patterns research. The 60% improvement in AI-patterns application intent and 40% improvement in general AI research adoption demonstrate a positive effectiveness of the session overall. Future sessions should maintain this application focus while strengthening foundational knowledge components.

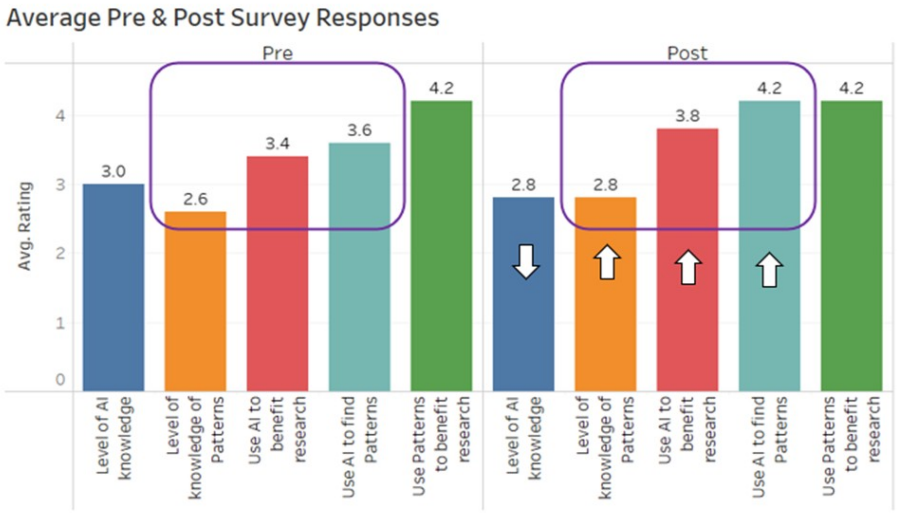



*Figure 5 - Showing pre/post results from the surveys. The Likert scales for the level of knowledge questions were "1-No Knowledge" to "5-Expert" and the scale for the other questions were "1-Not at all" to "5-Extremely".*

In the Fig. 5 above, three surveyed questions increased slightly and one decreased slightly in the post test. It's possible that as individuals learn more about the intricacies of AI, they become more aware of the limits of their own knowledge.

## Conclusion and Future Work

### Was this effective?

Yes and no. While there were some positive outcomes, some technical aspects of the remote delivery of the music portion were challenging; participants asked why this portion was done remotely. We'd want to think more about this in the future: do we use

an external expert (with some interface friction) or do we train someone with less expertise for local delivery?

**Should facilitators be remote for the next workshop (e.g., Imagination Run Wild at PLoP 2025)?**

This remains an open question. Given the students' response to remote instruction—particularly in the music segment, it may be more effective to prioritize having all facilitators on-site.

**What are we saying about AI? What are we proving?**

We administered the same five questions to participants before and after the session. Pre-survey responses showed that most participants already had a baseline of knowledge of 3 ("Comfortable") on average. The post-survey revealed modest gains, but many responses remained unchanged. This suggests that while some learning took place, the intervention did not lead to significant shifts in understanding (which makes sense because we had not primarily intended to have the workshop be didactic in nature). During the session, students suggested turning the Dixit clues into a song, an imaginative idea we had not anticipated. Broadly, the evidence is that the main outcome is that attendees increased their confidence in thinking and talking about AI. Fostering this kind of creative element would require intentional planning in future iterations.

**What about the card games?**

Should we reconsider the choice of card game? Would offering multiple games simultaneously enhance engagement? These remain open possibilities. We are interested in experimenting with Cassie Lee's (MIT) Analogia [2], [3], and this is planned for a future session. Games and play offer unique spaces for collaboration, collective creativity, and nuanced identity expression. In a world increasingly polarized by social media, we explored how intentionally designed games can encourage healthy and constructive dialogue.

## Acknowledgements

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converting the workshop proposal to a narrative outline. This has since been thoroughly rewritten by us! Thanks to Richard Karcher for guidance and review of this paper. We also thank Shankar Ayloo for the photos.

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# Use Manga and Comic Books to Teach English

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## English Abstract

It is often a challenge for English as a Foreign Language (EFL) students to read authentic texts written by native English writers for native readers. One way to introduce this literature in a more accessible way is via manga or comic books. Students can use the visual cues from the drawings to decode the written words on the page. This paper introduces the new pattern: "Teach English as a Foreign Language reading with Comic Books and Manga" for EFL teachers (TEFL), provides examples, outlines challenges, and connects with pre existing patterns.

## Japanese Abstract (日本語)

英語を外国語として学ぶ生徒にとって、英語ネイティブの作家が英語ネイティブ読者向けに書いた文学作品を読むことは、しばしば挑戦である。この課題を解決し、文学へのアクセスを容易にする一つの方法は、マンガやコミックブックを用いることである。生徒は、ページに書かれた言葉を解読するために、ドローイングによる視覚的な手がかりを利用できる。この論文では、英語教師のためのパターンを紹介し、いくつかの例を提示するとともに、現状の課題を概説して、この新しいパターンを既存のパターンと結びつける。

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# Introduction

Sometimes, learning how to read English is challenging for students. Naturally, it can be especially difficult for non-native speakers of the language studying English as a Foreign Language, or EFL. A group of students may start with illustrations aiding comprehension in English books around kindergarten. Others begin reading in elementary or junior high where they may meet EFL textbooks featuring graded language for reading practice that have translations of words nearby on the same page. Depending on the level goals of the student, staying with less difficult texts may be fine. Later as a student moves up the CEFR[1] levels, perhaps they want to use English for work as an adult? It will help to improve reading. Learning to handle these more complex works may help them succeed on exams like the TOEFL®[2].

They may want to read a poem by Eve Ewing, novel by David Foster Wallace, play by Lorraine Hansberry, and sportswriting by Mina Kimes. Getting mostly just text on the page may be intimidating for them, despite their interest in the material. One hack to encourage students to read more is to make it more by by giving them **comic books**, a "bound collection of comic strips, usually in chronological sequence, typically telling a single story or a series of different stories" (Encyclopedia Britannica).

Teachers may also provide them with manga, "comics or graphic novels originating from Japan … conform to a style developed in Japan in the late 19th century, and the form has a long history in earlier Japanese art." (Wikipedia). For the purposes of this pattern, we will use **comic books** when referring to narrative art (narrative art featuring words and pictures) from the United States of America and the United Kingdom, including graphic novels. We will use **manga** to refer to narrative art from Japan and Korea that has been translate into English, including Tankōbon. This is a hack because the students may have so much fun reading the story, they do not realize they are improving their English test scores.

Teachers can give students either short comic strips or longer stories. The art on the page will give context clues for students to learn the meaning behind the words they see. As they realize they can understand more of the words, students' confidence can grow. Over time with repeated practice, they very well may be able to read other works written for native English audiences

Using **comic books** and/or **manga** to teach EFL (TEFL) is a new pattern. Patterns are often expressed as a solution to a problem in context, although they are not always that simple as Gabriel (n.d.) points

[1] Common European Framework of Reference for Languages: Learning, Teaching, Assessment (CEFR)
[2] Test of English as a Foreign Language (TOEFL)

out. This paper introduces a pattern working towards this goal he described:

> *"The goal of the pattern community is to build a body of literature to support design and development in general. There is less focus on technology than on a culture to document and support sound design and principles. Patterns have been used for domains as diverse as development organization and process, exposition and teaching, and software architecture."*

# Describing the Pattern

We will draw on the Context, Problem Solution format to describe our new Pedagogical Pattern which is a type that tries:

> *"to capture expert knowledge of the practice of teaching and learning. The intent is to capture the essence of the practice in a compact form that can be easily communicated to those who need the knowledge."*
> (EduPLoP, n.d.)

## Name

Teach English as a Foreign Language reading with Comic Books and Manga

## Intent

Help EFL students learn reading skills faster by adding in comic books as part of curriculum to complement existing literature and textbooks. Getting this experience with literature written for native speakers may help them later read and comprehend more complex works.

## Context

Reading authentic texts such novels, journalism, plays, and rap lyrics can be hard for EFL students because there may be unfamiliar vocabulary terms. To be precise, this pattern is focused on teaching adult English-language learners (ELLs) who want to read at a high level of comprehension.

## Problem

These unfamiliar words may discourage students from trying, especially stories that use slang, because they normally get accessible texts via EFL textbooks for reading homework.




Figure 1
*EFL students at the Language School are frustrated because they want to read a novel by David Foster Wallace, but it is harder than they expected.*

## Forces

- Adult ELLs are intimidated by challenges of comprehending texts written for native speakers
- TEFL faculty have limited options for challenging reading materials beyond textbooks

## Solution

Introduce **comic books** and/or **manga** into your classrooms and curriculum. They embed authentic language within themselves, facilitating understanding alongside visual information and context (which may not be present in reading textbooks or novels, even those written for ELLs).

With visual cues from drawings, students can better understand what is happening in the written words on the page.

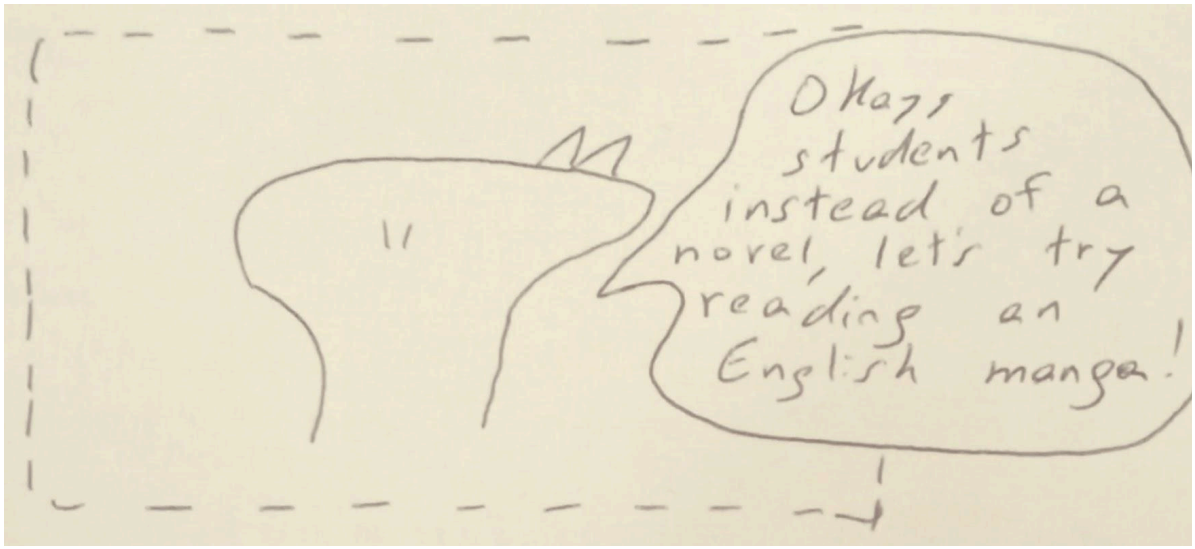



Figure 2
*The classroom's EFL teacher at the Language School suggests using* ***manga*** *to practice English reading*

# Consequences

## Drawbacks

- Comic books have less history of classroom usage than do textbooks which may lead to hesitancy from some administrators.
- Not every adult enjoys reading comics and/or manga. This may not be of interest to some students. In that case it will not help them improve their reading skills.
- Some comic books make jokes that are too high level for the English students to understand. Some students may find this lack of understanding to be discouraging. Instead of comic books being helpful for students, this may actually make them dislike learning even more.

## Costs

- Some comic books cost money and the teacher may not have funds from their school to cover the costs.
- Depending on budget and level, this can range from 1 to 3 pane comic strips (e.g. boondocks, peanuts) up to graphic novels.
- If the school does not provide a printer, the teachers may be unable to provide paper copies of the comics.
- If the students or school does not have access to smart phones, laptops, and/or internet it will eliminate digital comics and Internet Archive resources as options.

- Teachers just may not realize they can get English comics online and/or for free in local newspapers if offered by an accessible library.

### Student Feedback

When considering how to teach it is very important to also ask students how they feel about an approach. One adult learner named Kaito Shiota, a member of Iba Lab at Keio University in Japan used manga:

> *"I had personally been learning English through reading English manga, and I thought incorporating this into the curriculum was a modern and casual approach that was very good.*
>
> *I would like to share with you my approach to reading English versions of manga. Initially, I start by reading the Japanese version of the manga to get an overall understanding of the content. Then, I proceed to read the English version to grasp the phrases and other language nuances.*
>
> *The reason I began reading English manga was motivated by my desire to read the latest chapters.*
>
> *As I encountered expressions unfamiliar to me, I would research them and gradually progress through the manga. Specifically, when reading content in English manga that I haven't encountered in the Japanese version, I make it a point to read through the English sentences and use tools like Google Translate to understand parts that are unclear to me.*
>
> *Here are some manga sites I frequently use … Korean Manga Site: https://page.kakao.com*
>
> *English manga site: https://tapas.io*
>
> *Japanese manga site: https://piccoma.com/web/*
>
> *Kodansha's US site: https://kodansha.us (Japanese Publisher's US Site)"*

## Implementing the Pattern

Teachers should procure comic books or manga to share with their class. At first, we recommend going for those that have simple jokes or morals. You can go to the school or local library to see what they have available. There are also places online that have free comic books. If you have funds available for class literature, you can check out a local book or comic book store. You can also find lists online (Ray, 2023).

Review the copyright laws in your own country if you are making copies of a comic you find. Ensure you respect the rights of the copyright holders. Also review the rules at your own school about leveraging outside resources. If you are able to find a copyright licensed resource, that will likely make it easier for other teachers to follow your example and use that comic in their own class.

Initially, go through one of the comics together as a class. Either give each student their own paper copy or a way for them to read via school computing devices (tablets/laptops) or project the comic onto the big screen. If they have their own mobile phones, those can also be used. This scaffolding of group reading first will build confidence for the students to read on their own later.

Ask the students to read the next comic in pairs with a partner. Request that they also discuss the comic with their partner in English. This provides another benefit of getting them to practice producing spoken English.

Give the students a comic to read for homework. Prepare some questions for them to answer to check their comprehension. Make at least one of the questions require them to write multiple sentences to answer. This provides another benefit of getting them to practice producing written English. Choose three to five challenging words from the story and ask students to make new sentences with them.

Talking about words from the stories allows teachers the chance to also give instruction in producing spoken English. Perhaps a manga has the word “comb”. The instructor may explain how to pronounce it compared to “tomb” and “bomb”, which all end with the same three letters. During this teaching moment, the instructor can clarify how the sound of the o is different for each word, which is not intuitive.

As we mentioned, teachers should take care not to confuse students with material for which they are not yet ready. After some easier texts have been completed; however, one could also turn this around to make teachable moments. For example, pick a joke which would go over the students' heads and base a lesson on it. If the teacher explains the linguistic principle behind a joke, the student might feel a sense of accomplishment at figuring it out and laughing at the punch line. Subsequently, this humor could serve as a memory aid for learning and remembering that principle.

If teachers do not have budget, the next question is if their school has an internet connection? If yes, comics may be found for free online. If their students have internet at-home or via computers in class, they could send them links to the comics. If the classroom has internet and a projector, Teachers could share the comics to the whole class together.

Another approach is teachers could write the publisher of a given comic and ask for donated copies to use in class or do a fundraiser to cover the costs (e.g., DonorsChoose). They can check if local free newspapers have comics or try to make their own manga.

However they secure their comics, teachers should be careful to pick stories that are appropriate for the level of the students in their classes. One example that worked for intermediate level adult students was the "Fantom of the Fair".

Asking simple comprehension questions such as "Do you trust the Fantom?" and "Why?" can help [Danoff, 2013a]




Figure 3
*Page from The Fantom of the Fair comic book issue. Image via Digital Comics Museum. Public Domain.*

# Examples

## Example One: In-Person Classroom

In a class for adult English as a Foreign Language (EFL) at a vocational English school, students were given comic books from the Digital Comics Museum. One of the authors printed out copies that were read and discussed them during class [6]. At times this led to laughter and a pleasant break from the part of class devoted to textbook learning.

An adult student from that school shared a memory:

> *"I do remember that you used comic books to teach us and it was helpful teaching materials! I think it was good way to learn vocabulary in the context. When we learn vocabulary at first, we just memorize the meaning so it is difficult to use in real context. So I think it was good to learn how can we use vocabulary while reading comic stories."*
>
> - Miso Kim, English as a Foreign Language Student from Korea

## Example Two: Tutoring Online

While tutoring an adult EFL student online, an author found comics from free newspapers and magazines such as "Linework" from NewCity, a free artistic publication in Chicago. These were chosen because they were free and the student lived in Chicago so it gave a sense of local art. Also the instructor hoped it would encourage the student to explore more of the publication. After reading that specific comic, this was the homework prompt (Danoff 2013b):

> *Write 5 sentences with words from the October 30th, 2014 Newcity "Linework" comic:*
> *** bleak, adj. or n.*
> *** journal, n. or v.*
> *** handbook, n.*
> *** baby face, compound n.*
> *** amid, preposition*

## Example Three: Narrative Art for Other Subjects

As a general example of the idea of using manga to aid learning, there are now books, such as: "To Teach: The Journey, in Comics". Another example is the organization Reading With Pictures that "empowers teachers, librarians, parents, creators, retailers, and

publishers to engage all learners through the sequential arts. We foster the joy of reading and lifelong multi-literacy through the comic arts at all levels of education." They share research including where a teacher used comic books to meet Common Core standards.

## Example Four: A Pattern Story

Scene 1 - A EFL student at the Language School is reading a comic book in their own language instead of reading a textbook.

scene 2 - Another ELL at the Language School is with friends after class and their faces look sad as they talk about how hard it is to understand the boring English textbook.

Scene 3 - The next week in English class their teacher hands out English comic books from the Uk for the students to read for homework.

Scene 4 - After school a student improved their English reading skills by doing homework.

Scene 5 - The next morning the students are smiling slightly as they talk about how homework last night was less boring.

## Example five: A Pattern Manga

This remix of the story into a manga was inspired by "Pattern Manga: Attractively Expressing Patterns of a Pattern Language in Manga Style" from PLoP 2023. Each scene has a different style, but the ELLs are studying at the Language School and the faculty are all from the TEFL department at that school. Figures 1 and 2 earlier in this paper are at the same academic building.

## Scene 1



*Figure 4:* A EFL student at the Language School is reading a comic book in their own language instead of reading a textbook. *Image generated using OpenAI's DALL·E*

## Scene 2



*Figure 5:* Other ELLs at the Language School are in the library after class and their faces look sad as they think about how boring the stories are in their English textbook. *Image generated using Midjourney*

## Scene 3




*Figure 6: TEFL instructor at the Language School introduces comic books in their classroom. Image generated using OpenAI's DALL·E*

## Scene 4



*Figure 7: After school a student improved their English reading skills by doing homework. Image generated using Midjourney.*

Scene 5

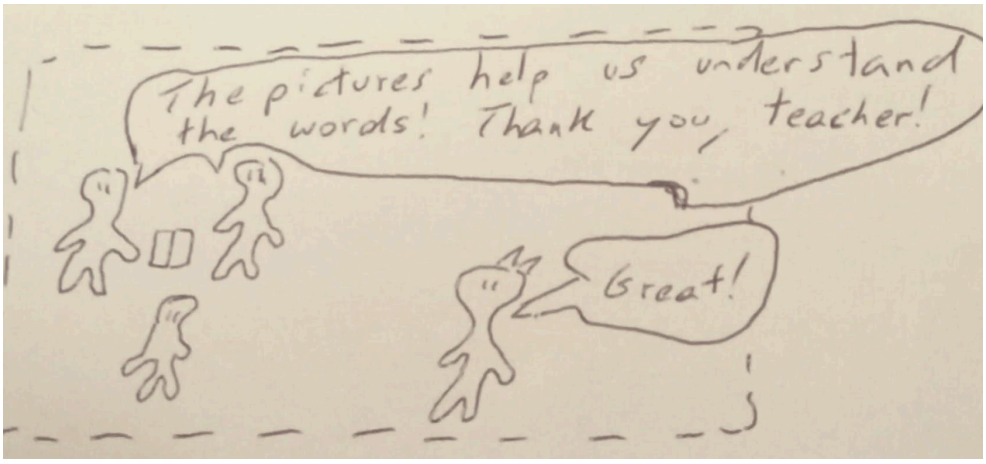



*Figure 8:* The next morning the happy students are telling the teacher how the pictures made it easier to learn the words.

# Related Patterns

- "Effectiveness of learning 8 languages using patterns" by Mary Tedeschi (PLoP '17: Proceedings of the 24th Conference on Pattern Languages of Programs)
- "Pattern Manga: Attractively Expressing Patterns of a Pattern Language in Manga Style" by Takashi Iba, Hiroaki Tanaka, Sae Adachi, Mizuki Ota, and Urara Tajima (2023 PLoP Conference Submission)
- "Language Shower" by Takashi Iba and Mami Sakamoto (PLoP '11: Proceedings of the 18th Conference on Pattern Languages of Programs)

# Conclusion

Members of our writers workshop suggested this could be extended to additional patterns, perhaps even a pattern language in the future. We are hoping to and embrace this daunting task and live up to its high expectations with the strategic patterns described in "Crafting a Pattern Language for Overcoming Setbacks and Carving One' s Own Path Drawing Learnings from Shonen Manga" by Mizuki Ota, Mimi Kuwataka, and Takashi Iba (2024).

# Acknowledgements

Thank you first to Kaito and Miso for letting us quote you in the paper.

We would also like to thank the AsianPLoP Program co-chairs Joseph Yoder & Y C Cheng, Conference co-chairs Takashi Iba & Norihiko Kimura, Publication co-chairs Hiroki Ito & Christian Kohls writers workshop participants, and our shepherd Mizuki Ota.

Thanks to Jay Cedrholm for editing an earlier draft. Thank you Hiroaki Tanaka for suggesting this as a paper topic at PLoP 30 in Allerton Park and improving the Japanese abstract translation. Kudos to CharGPT for the first draft of the Japanese abstract and other helpful feedback on this paper.

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“Patterns” from the EduPLoP website (n.d.)
https://hillside.net/eduplop/patterns/

“19 Exciting English Comic Books to Learn English” by Dhritiman Ray (2023)
https://www.fluentu.com/blog/english/learning-english-through-comics/

“ENG 099 Conversational American English Atom 8: Reading American Comic Books” by Charles Jeffrey Danoff (2013a) via the Internet Archive Wayback Machine
https://web.archive.org/web/20160322162835/http://www.mr.danoff.org/eng099/?p=361

“What we did in our CoC IEP 200 Class on Wednesday January 16th” by Charles Jeffrey Danoff (2013b)
https://charliesensei-blog.tumblr.com/post/40746732519/what-we-did-in-our-coc-iep-200-class-on-wednesday

# Focus Group 4 Summary

## PLACARD Workshop

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At 3pm Japan Standard Time (JST) on Friday, March 1st, the second day of AsianPLoP, we conducted Focus Group 4 (FG4) that was named "PLACARD Workshop". PLACARD is a synthesis of: Causal Layered Analysis (CLA), Project Action Reviews (PAR), and Design Pattern Languages (DPL). The acronym was first introduced by the 2021 PLoP paper "Patterns of Patterns". One of the facilitators was onsite. Two joined remotely via Jitsi from New York City.

After introducing PLACARD, we collectively did a Project Action Review about the AsianPLop 2024 event as a whole and examined it with Causal Layered Analysis. This CLA iceberg figure was used to introduce the concept:

European Space Futures - Using Causal Layered Analysis on the Director General's Proposal at ESA's CM16 - An Introduction of Critical Futures Studies to Space Strategies - Scientific Figure on ResearchGate. Available from: https://www.researchgate.net/figure/The-CLA-Iceberg-Source-Inayatullah-Sohail-Appendices-The-Causal-Layered-Analysis_fig1_329686603 [accessed 7 Apr, 2024]



*Figure 1 - Focus Group 4 Participants*
*Picture by Takako Kanai*

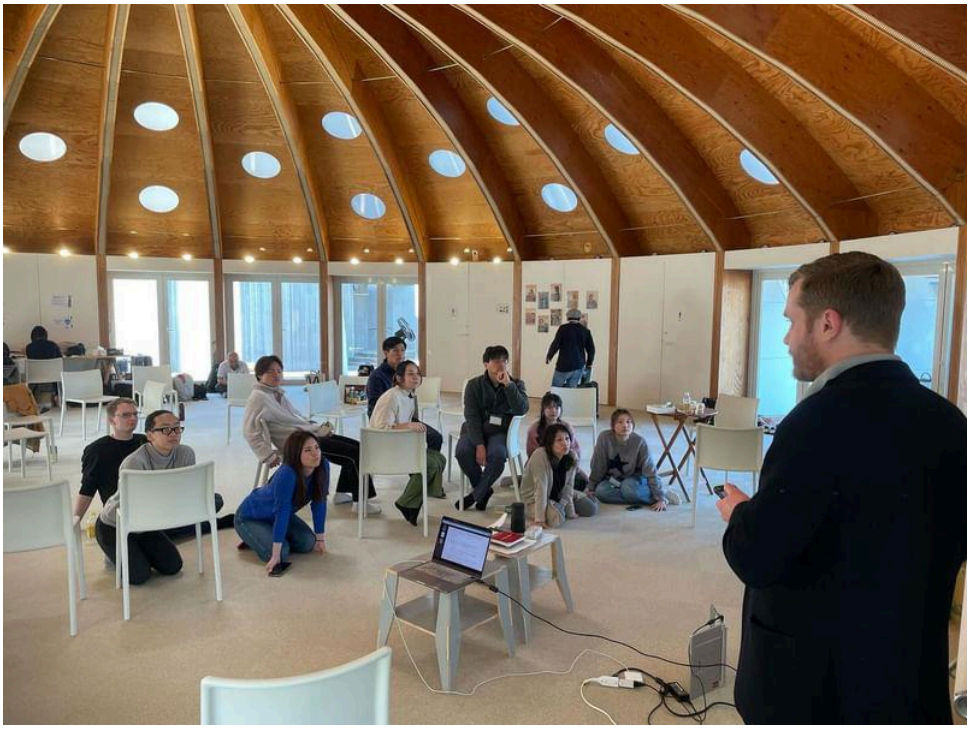


*Figure 2 - Another Photo of Focus Group 4 Participants*
*Picture by Kiyoka Hayashi*

# Project Action Review

We did a collective PAR where participants shared back about AsianPLoP 2024. The onsite facilitator took notes. The bullet points below highlight some of our discussions.

## Review the intention: what did we expect to learn or make together?

- Deeping an understanding of patterns
- Get feedback on our papers and meet others
- Many are experiencing PLOp conference for the first time, so learning what it is and what is PLoP culture

## Establish what is happening: what and how are we learning?

- The history of PLoP, especially Asian plop
- More people found out about my paper
- Learned by using patterns
- Interesting presentation on the essence of pattern languages

## What are some different perspectives on what's happening?

- Creating a new community in real time
- Not only learning about history but making it in this very nice new venue

## What did we learn or change?

- Change my paper
- Learned about many different types of pattern language

## What else should we change going forward?

- See organizers and volunteers more energetic after event
- Organizers sometimes burn out after an event, not continuous
- Want to start new projects with other attendees

# Casual Layered Analysis

As a group we examined each CLA layer and how they applied to AsianPLoP 2024.

## Problem/Litany Layer

- How to grow and implement patterns
- Hard to choose among all the feedback you get about which you will put into paper, include negative critiques

## Causes

- Pattern community in japan and asia is smaller than other regions
    - Want to grow that community

## Worldview

- Role of language is big, some people only speak one
    - One reason why Iba lab language will change to only being English university student finishes in 4 years
- Generative AI to translate into other things quickly

## Myth / Metaphor

- Pattern language is itself a langues
    - Power to overcome different languages
    - If its a good pattern language, doesn’t matter if it’s in English or Japanese
- AsianPLoP is much bigger than just far East Asia
    - Should run more PLoPs in different parts of the world

# Appendix

## Pre-Conference Submission Abstract

This workshop will be based on ideas we introduced in our paper "Patterns of Patterns" that was submitted for PLoP 2021, specifically PLACARD. PLACARD is a synthesis of Project Action Reviews (PARs), Causal Layered Analysis (CLA) and Design Pattern Languages (DPL). Combining these three together "suggests that the methods need not be run in a fixed order, but are interwoven together.

- Context: In the course of working on a project: we use the PAR to get a sense of our working context.

- Problem: Although we may encounter many difficulties in this context, our effort to understand them faces a central challenge, namely the fact that the problems span different layers and scales of complexity, so it can be hard to understand where the difficulties actually come from: accordingly, we use the CLA to understand and frame the problems and their interconnections.

- Solution: Once we have grasped the problem, we need to elaborate an actionable solution that remains adaptable to ongoing changes in the context: we use DPL to elaborate the solution (returning to PAR and CLA as needed)." (Corneli et. al, https://arxiv.org/abs/2107.10497)

We wrote an experience report of implementing PLACARD in workshops across the world in our subsequent paper "Patterns of Patterns II" (Corneli et. al, https://arxiv.org/abs/2306.08426) that was submitted for PLoP 2023. This workshop will introduce participants to the PLACARD and explain how they can use it in their own work. We will then use it to reflect on the AsianPLoP event itself and produce an artifact to share with conference organizers.



*Figure 3 - Included with our PDF Focus Group Submission*
*Image generated using OpenAI's DALL·E*

# AI Future Envisioning with PLACARD

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arXiv:2410.17155v1 [cs.CY] 22 Oct 2024

## ABSTRACT

At EuroPLoP 2024 Mary Tedeschi led the "AI Future Envisioning with PLACARD" focus group in Germany. Three conference attendees joined in the room while Sridevi, Paola, and Charles co-facilitated remotely via a web conference. The participants were introduced to a Futures Studies technique with the goal of capturing envisionments of Artificial Intelligence (AI) going forward. To set an atmosphere a technology focused card game was used to make the session more interactive. To close everyone co-created a Project Action Review to recap of the event to capture learnings that has been summarized in this paper. The Focus Group was structured based on lessons learned over six earlier iterations.

## CCS CONCEPTS

• **Applied computing** → **Education**; **E-learning**.

## KEYWORDS

AI, distance learning, educational technology, learning management system, online education/teaching, patterns, student engagement

**ACM Reference Format:**

Mary C. Tedeschi, Paola Ricaurte, Sridevi Ayloo, Joseph Corneli, Charles Jeffrey Danoff, and Sergio Belich. 2024. AI Future Envisioning with PLACARD . In *29th European Conference on Pattern Languages of Programs, People, and Practices (EuroPLoP 2024), July 3–7, 2024, Irsee, Germany*. ACM, New York, NY, USA, 7 pages. https://doi.org/10.1145/3698322.3698444

## 1 INTRODUCTION

Over the last several years, Artificial Intelligence (AI) has become part of everyday life for many computer and smartphone users. However, the governance and future directions for these technologies are unclear. Motivated by this state of affairs, we wanted to convene a discussion that would invite pattern language experts to think creatively together about the future of AI technologies. We have developed a methodology for structured discussions which builds on design pattern methods, and augments them with techniques from other professional domains. We deployed this combined methodology in our "AI Future Envisioning" workshop at EuroPLoP 2024. Although the workshop was only small-scale, we have envisioned it as a pilot for subsequent large-scale discussion and innovations (both social and technical) in the way we think about AI technologies.

The session began with basic introductions and definitions, using a PowerPoint deck [1]. On slide 4 (Fig. 1) Christopher Alexander's [2][3] design patterns were used as a point of entry into the vision of Causal Layered Analysis (**CLA**). A pioneer of Design Pattern Languages (**DPL**) was chosen for the pattern conference attendees to find a connection or bridge to the new (for them) Futures Studies technique. The participants then played "The Oracle for Transfeminist Technologies" card game as a creative thinking tool to source ideas for the future of Artificial Intelligence. Participants further used the **CLA** framework to refine their concepts on the flip board. At the end, a Project Action Review (**PAR**) was implemented to document the outcomes. We first introduced the synthesis of these methods (**CLA + PAR + DPL = PLACARD**) in our 2021 paper Patterns of Patterns [2].




**Figure 1: Slide 4 from presentation used for the Focus Group**

The session was led by Mary onsite in Kloster Irsee Germany. Paola, Sridevi, and Charles co-facilitated remotely from North America through a Jitsi web conference call via a laptop in the room. We were in the 17:00 – 18:30 CEST time slot on July 5th, 2024. After explaining the objectives (2), process (3), and outcomes (4) this

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paper includes a background section (5) showing how this session evolved over six workshops, beginning with PLoP 2021, followed by a summary discussion (Section 6) that contextualises this work. Our conclusion (Section 7) highlights directions we hope to pursue in further work.

## 2 OBJECTIVES

Our main objective was to facilitate a CLA exercise to envision the future of AI. Each participant used a value card; an object card; a bodies and territories card; and a situation card. The cards were distributed and participants were asked to read their cards.

A value card represents a transfeminist value. The participants were asked what the value meant to them. Object cards represent everyday objects. Bodies/Territories cards encourage us to recognize the importance of situated and embodied knowledge. We asked the participants to reflect on who they are. What body are you in? Where do you live? What privileges and burdens do your body and place provide? The situation card provides you with a situation that you need to deal with. They can be a fun way to focus your creative energy.

Heading into the focus group we hoped our PLACARD envisionments would yield fruitful insights into AI that will aid humanity. We were particularly interested in exploring how AI could be used, adapted, and governed in ways that bring the greatest benefits to the most people. At the same time we were concerned that, currently, AI engineering is not accessible to many people, and that although AI systems can in many cases be widely used, they may be "extractive by design" insofar as they serve the profit motives of the companies that have developed them.

We hoped to build on our experience in the previous parts of this workshop to discuss what we think about these concerns, and use PLACARD to envision some paths forward. Our goal was to create a futuristic technology first without AI and then revise it with AI.

Two authors attended the "CheriSharing: Sharing Cherished Qualities of Pattern Languages (Focus Group 2)" [4] at PLoP 2023 in Allerton Park that was developed "to 'share' the goodness of a certain theme through conversations and 'cherish' it together." Another desired outcome of ours was to have a similar level of wonderful participation and interaction from attendees.

## 3 PROCESS

The session was divided into two 20-minute segments. Mary worked with attendees in the room in Germany while Paola, Sridevi, and Charles supported remotely from North America. The room included:

- Flip chart
- Projector, internet connection, laptop, and screen so remote co-facilitators could join
- The Oracle for Transfeminist Technologies card deck [5] as prompts for creativity

We gave illustrative examples of CLA via the projected slides and verbally with the "Parking space" example [6] from Dr. Inayatullah:

*"However, CLA orders the scenarios in vertical space. For example, taking the issue of parking spaces in urban centers can lead to a range of scenarios. A short term scenario of increasing parking spaces (building below or above) is of a different order than a scenario which examines telecommuting or a scenario which distributes spaces by lottery (instead of by power or wealth) or one which questions the role of the car in modernity (a carless city?) or deconstructs the idea of a parking space, as in many third world settings where there are few spaces designated 'parking'."*

### 3.1 Example Scenarios

The "Transfeminist Oracle" cards provided participants manipulative creative structures around values, situations, and objects. Initially, we did not plan to use the situation cards, but one participant asked, "What problem are we trying to solve?" This question guided us to incorporate situation cards, making the exercise more engaging and purposeful.




**Figure 2: Credit to Monika**

One participant had the **Guitar** card (object) with **Empathy** (value) aiming to grant unrestricted access to information. Another had the **Wine** card (object) with **Autonomy** (value) focused on creating alternative forms of economy. The third participant had the **Bicycle** card (object) with **Security** (value) addressing diffuse power and diversified governance.

Each participant shared the same card named **"Body/Territory"** card, which states "you carry stories of the body and territories you inhabit". The objects served as metaphors: the bicycle represented an electric car or electric bicycle, the guitar symbolized music, and wine was something delicious.

### 3.2 Activities

Participants considered the CLA, represented via a diagram with concentric rings, and wrote their thoughts on post-it notes, and posted them on a copy of the diagram on a flip chart.

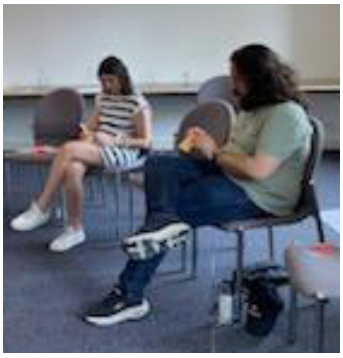


**Figure 3: EuroPlop '24 Workshop - participants writing, photo by Mary Tedeschi**




**Figure 4: EuroPLoP '24 - Presentation slide showing CLA and Flip Chart with Post-it Notes from Participants plus online Co-Facilitators, photo by Monika**



**Figure 5: EuroPlop '24 Co-Facilitators on Jitsi, photo by Mary**

## 4 OUTCOMES

We successfully completed the planned activities with five minutes to spare. The online co-faciliatators provided support but did not actively engage in the exercises.

### 4.1 PAR (Project Action Review)

We closed with a collective PAR where the groups shared back what they discussed and learned. We collaboratively answered the five questions [7]. The initial notes were taken in the Jitsi Chat [8]. You can see one screenshot in the figure below. A consolidated revision hoping to capture their essence is below. The five questions are: (1) *Review the intention: what do we expect to learn or make together?* (2) *Establish what is happening: what and how are we learning?* (3) *What are some different perspectives on what's happening?* (4) *What did we learn or change?* (5) *What else should we change going forward?* From attendees' responses to these questions we heard back that they perceived the collective intention in terms of coming to a shared understanding of the PLACARD method and its constituents such as CLA, via examples. Attendees found the experience enlightening, and suggested that the method could help people think creatively and bring companies out of their "business as usual" way of operating. They found the examples of previous practical use helpful, and felt that the cards came in handy. In addition to gaining experience with the methods, they left with more confidence about asking "what if?" and imagining different futures.

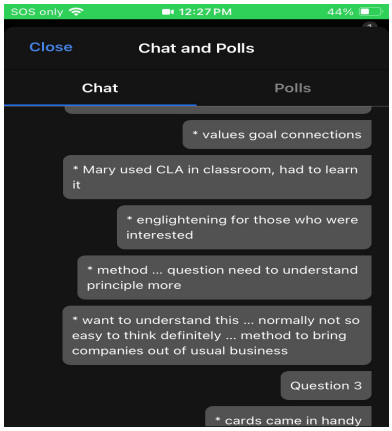



**Figure 6: Screenshot of Jitsi chat with live notes of second and third PAR questions with answers.**

### 4.2 Questionnaire Results

Following the example of the EuroPLoP 2023 CheriSharing Workshop: Sharing Cherished Qualities [4], we sent participants a questionnaire about their experiences. It had fourteen questions and the last five were adapted from the PAR.

We received responses from all three participants, for a 100% response rate. Every respondent had experience attending Hillside PLoP events in the past, so they had familiarity with patterns and

focus groups. The anonymous respondent joined the Focus Group "because the title seemed interesting and I also wanted to support the people involved :)" while Tsvetelina Plummer joined because it "Sounded very unfamiliar to me, so it was interesting to learn what you do". The reason Prof. Dr.-Ing. Monika Blattmeier attended because "I was interested in how the future of AI can be visualized."

Tsvetlina wrote that she learned "An interesting creativity-sparking exercise" and the other respondent wrote "I learned a new way to organize ideas and knowledge with PLACARD and its components". Prof. Dr.-Ing. Monika Blattmeier learned "about Causal Layered Analysis … which I find very inspiring. Secondly, I was excited about the cards we worked with and came up with new ideas. These cards allowed a complex thinking concept, as PLACARD seems to be, become applicable for practitioners." Additionally, Tsvetilina plans to use what they learned going forward "I'll add the new techniques I learned into my toolbelt of processes about learning" as does Prof. Dr.-Ing. Monika Blattmeier "I would also like to use this method for my own project, the development of a knowledge management system."

All respondents had suggestions for us to consider with future versions of the workshop along the same lines "I felt that the delivery of the focus group could have been more cohesive and structured by having an overview at the beginning, at points it felt like a new idea was shown to us a bit out of the blue." Similarly, Prof. Dr.-Ing. Monika Blattmeier wrote "It was hardly possible for me to understand your, in my view, promising PLACARD concept in the focus group. I was not prepared for it, did not know your theory and it was difficult for me to follow the context of the PLACARD concept presented by you in the relatively short time frame of the focus group." Finally, Tsvetlina suggested "Compress the definitions and textual information into a handout for participants to have after the workshop (for their reference later). Instead put graphics as visual aid and only little texts on the slides – It was very hard to try to follow both the audio and visual tracks. You could also try a version where you start with the exercise and then present a bit of theory – I think that could work as well as participants will have the activity as [an] anchor in their brains when you present the theory later". We are very thankful for them to take the time to complete our questionnaire and intend on using their feedback to improve future workshops.

# 5 BACKGROUND

One of the authors of this paper attended another AI focus group "ChatGPT Prompt Engineering for Pattern Writers: Unlock Your Potential" [9] by Stefan Holtel at EuroPLoP 2023. We were curious about how a PLACARD workshop on the future of AI would intersect to extend some of these previous conversations.

We have previously used PLACARD in institutions and at conference workshops, including AsianPLoP 2024 (Focus Group 4) [10]. Charles led that event onsite in Fujisawa while Sridevi and Mary cofacilitated from New York via Jitsi. We introduced CLA visually, via the iceberg metaphor with this iceberg diagram [11]. Rather than AI, the technique was used to collectively reflect on the conference itself. During the PAR discussion we noted the benefits of paper workshops plus how sometimes people have so much energy for the event itself, but then burn out afterwards so it is difficult to follow up with new connections on new projects. Going through the CLA levels the reflections included that the pattern community in Japan and Asia is smaller than other regions and that AsianPLoP is much bigger than just far East Asia. At the myth level we discussed that pattern languages themselves are a language that transcends human languages: if a pattern itself is good and useful, it does not matter whether it is written in Japanese or English.

## 5.1 2021 - 2024 Versions of Focus Group

An earlier antecedent of these workshops was a reading of design fiction we did with participants at the Anticipation 2019 [12]. We got feedback from a participant that they did not like to have scripted lines. Since then we have focused on interaction during our PLACARD workshops. Our methods have continued to be remixed over the years (see Table 1). The table was adapted from one used in Sawami's AsianPLoP paper [13]. Common to all these settings are the need for a rapid training in the methods, so that people can begin to use them after very brief instruction.

This table only lists the public-facing events; other applications of some or all of the methods include sessions with students in Mary Tedeschi's classroom (carried out over 3 successive iterations of her course), an Oxford Brookes away day about open research, and a workshop with officers of the Oxfordshire County Council.

## 5.2 Other Uses of the PLACARD Method

The Emacs Research Group [2] conducted a PAR at the end of each of their weekly meetings. After every ten meetings or so, a CLA was developed (or revised) using bullet-points from the PAR as evidence. The Peeragogy project experimented with using a DPL as a project management system, by compartmentalizing their collective todo-list as "Next Steps" attached to each of their collected patterns [27].

**Table 1: PLACARD Workshops**

| Year | Event | Modality | Tools | PAR |
|---|---|---|---|---|
| 2021 | Oxford Brookes Creative Festival [14] | Online (Zoom) | "Flaws of the Smart City" cards [15] | No |
| 2021 | PLoP: "Workshop: Flaws of the Cool City" [16] | Online (Discord) [17] | Flaws of the Cool City game | Yes; “Overall evaluation of our workshop” |
| 2021 | Connected Learning Symposium [18] | Online | No | No |
| 2022 | Anticipation | In Person | Posters | Yes; See our Case Study in Patterns of Patterns II [19] |
| 2023 | Wikimania Singapore [20] | Hybrid | Our own cards | YouTube video of session [21] |
| 2024 | AsianPLoP | Hybrid In Person with Online (Jitsi) Co-Facilitators | PLACARD | Yes |
| 2024 | EuroPloP | Hybrid In Person with Online (Jitsi) Co-Facilitators | PLACARD & Transfeminist Oracle Cards | Yes |

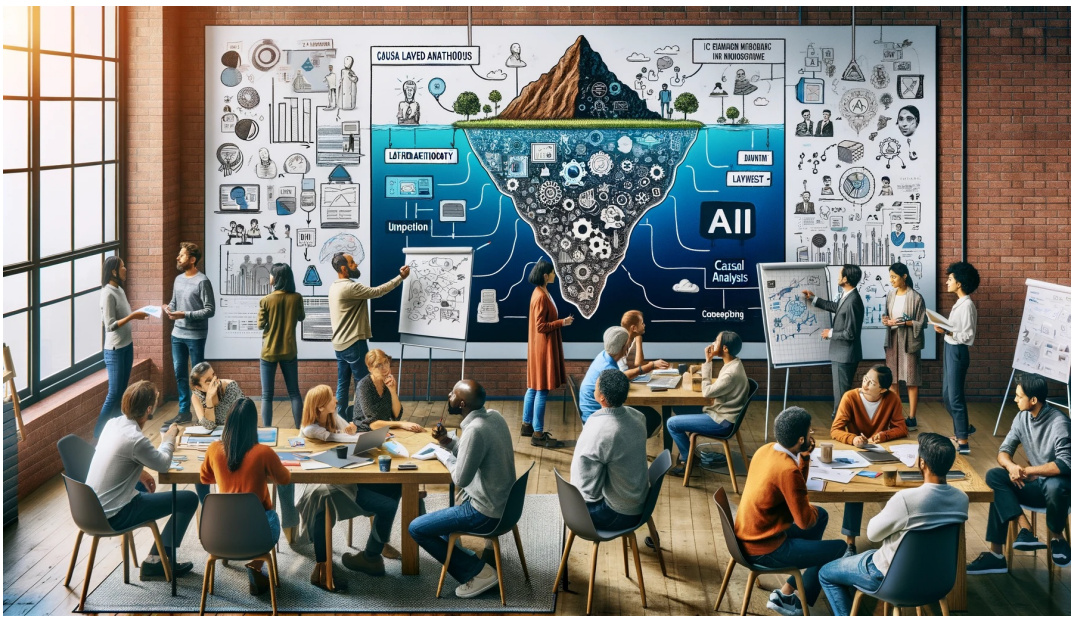



**Figure 7: Image by Open AI DALL·E 3 after reading about this focus group.**

# 6 DISCUSSION

CLA, DPL, and Action Reviews have all been widely used separately in other settings. However, there is a difference between sharing methods in a way that can be readily replicated by participants on their own, and simply using the methods to gather data. Similarly when participants already have a shared background (as in PLoP settings) more can be done quickly when new methods are introduced.

## 6.1 Connections to Peeragogy

In our work on a theory of peer learning (peeragogy/paragogy) we elaborated a number of patterns [22] and principles [23] which describe how people create well-functioning peer learning environments. Systems like PLACARD distill this knowledge into operational terms, and can be used primarily for social innovation. Within this workshop, we hoped to engage participants in innovative thinking innovations around current — and future — technical innovations. While our conclusions about AI are still forming, one take-away from this process is that we need to find more ways to bring media interactions into conversations with each other.

For example, how could we create “dialogue” across the above workshop interactions? What infrastructure would be needed to be able to make different sets of patterns and proto-patterns (e.g., design prompts) interoperate? Could one approach be making a method for interoperable card games such as the Transfeminist Oracle cards, “Analogia” from MIT [24], and “Collaboration Cards” [25] from Iba Lab? Will AI tools be able to help with these challenges to build a new “Discovery Machine” akin to the Hubble Space Telescope [26]? We worked with DALL-E to envision possible future vision of AI with PLACARD, as seen in Fig. 7.

It is in principle possible to train an AI system based on a collection of existing pairs of patterns to create custom AI based on the pattern language literature. A naive approach building on the existing literature would not respect creators’ copyright; however we do want to respect creators’ copyright. Perhaps we could build a system together with other workshops and a copyright waiver or license, whereby future workshop participants could agree to share their ideas in a way that would allow them to be fed into future models.

# 7 CONCLUSION

The focus group effectively facilitated creative and critical thinking, leveraging metaphors and participatory methods to explore futuristic technologies and social values. The outcome ended up being more about the methods than the application of envisioning the future of AI. We are pleased that the participants shared in the questionnaire that they learned something new from the Focus Group. We plan to use their suggestions to help co-create our next Focus Groups. We are excited about the way PLACARD was used to have new conversations and are interested in ways it can bring different learnings together. As for envisioning the future of AI, we are considering possibilities, including a new pattern language and/or revising our mini handbook [27].

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# Patterns of Patterns II

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Our earlier paper "Patterns of Patterns" combined three techniques from training, futures studies, and design in a design pattern called PLACARD that helps groups of people work together effectively. We used that pattern in five hands-on workshop case studies which took place at various locations in the US and the UK. This experience report documents what we learned, including the way our thinking about PLACARD evolved, together with additional patterns our work generated. We evaluate the reproducibility of our methods and results, and consider the broader economic implications of this way of working. We discuss implications of our prototyping work for the design of future platforms, drawing connections with recent developments in cognitive science and artificial intelligence. This positions our patterns of patterns as a toolkit for the design and governance of systems that combine social dynamics with technical components.

CCS Concepts: • **Social and professional topics**; • **Software and its engineering** → *Designing software*; *Open source model*; • **Applied computing** → *Operations research*; • **Computing methodologies** → Modeling and simulation;

Additional Key Words and Phrases: Design Patterns, Pattern Languages, Action Reviews, Futures Studies, Causal Layered Analysis, Free Software, Peeragogy, Artificial Intelligence, Anticipation, Doughnut Economics, Socio-Technical Systems

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## 1 INTRODUCTION

This paper is primarily an experience report describing a series of workshops exploring pattern-based methods for co-creation. Patterns served a design function in setting up the workshops, and were discussed as contents of the workshops. Furthermore, patterns were engaged at all stages of development. Our aim is to understand

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and express the potential that is implied by design pattern language methods. The paper itself has the following structure. We begin with a brief recapitulation of the prequel, "Patterns of Patterns" [Corneli et al. 2021]. We then motivate our approach to design pattern methods, and provide an overview of the patterns we identified across five workshops, which become case studies in our treatment here. In brief, we show how our methods became more reliable as repeatable patterns, with implications across increasingly broad contexts of application. We then discuss connections to related work, positioning this paper as a contribution to contemporary discourse about openness and reproducibility in qualitative research. We conclude with our assessment of how these design patterns could inform the design of future platforms for co-creation. For concision, details of the patterns themselves are presented in an Appendix.

## 2 BACKGROUND

In "Patterns of Patterns" [Corneli et al. 2021], we introduced a synthesis of methods that operationalize social intelligence. The particular methods we outlined were certainly not the only way to implement the necessary system features. What drew our attention is that each of the methods we selected comes with a framework or template; each of the methods is, essentially, a design pattern.

- Project Action Review (PAR): *a set of five review questions to explore at a project checkpoint.*
- Causal Layered Analysis (CLA): *a set of four "layers" that can be used to unpack a topic of interest.*
- Design Pattern Languages (DPL): *each constituent pattern follows a three-part "Context/Problem/Solution" template.*

The Project Action Review adapts the precursor After Action Review (AAR) to horizontally-managed group work. The AAR has seen widespread use in military, business, and healthcare settings, and has proved to be an effective training approach in task environments characterized by a combination of high complexity and ambiguity [Keiser and Arthur 2021, 2022]. Causal Layered Analysis has been used for over thirty years by members of the futuring community to scaffold change-making initiatives by understanding and changing the core metaphor or "myth" that underpins a given social situation, along with its relationship to other levels of social organization [Inayatullah et al. 2022]. Design Pattern Languages are comprised of *design patterns*, which are methods that (*1*) describe a context (*2*) in which a recognizable problem occurs, together with (*3*) a solution strategy that should resolve the problem, possibly with reference to further patterns that provide needed detail. We called the overall method that combines the three methods PAR, CLA, and DPL the "PLACARD" pattern. We elaborated our description of PLACARD with case studies, and used our analysis of these to outline four potential futures for the development of pattern methods.

Table 1 arranges these four futures schematically in relationship to the four economic domains they are most closely associated with: the *household*, the *state*, the *market*, and the *commons* [Raworth 2017, p. 71]. The potential for design to interact fruitfully with these domains has been noted by Boehnert [2018]. We observe that patterns only reach their potential in these domains if they are accompanied by certain *freedoms*, familiar from the Free Software movement.[1] For example, if patterns were to become computational, they could be used like any other computational service (or other household device). Legal protections would need to be added to ensure access to source code (present for Wikipedia, absent for Netflix). This is important because patterns need to be adapted to local circumstances before they can be applied. However, benefits will only accrue locally or to private individuals, unless further design intelligence is applied to develop and share patterns that empower everyone, and which take account of our role within broader ecosystems.

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| Economic Domain | Potential Future for Design Patterns | Related Software Freedom |
|---|---|---|
| **household** | patterns become computational | "The freedom to run the program as you wish, for any purpose (freedom 0)." |
| **state** | free/libre/open source licensing | "The freedom to study how the program works, and change it so it does your computing as you wish (freedom 1)." |
| **market** | empower individuals and communities | "The freedom to redistribute copies so you can help others (freedom 2)." |
| **commons** | widespread economic empowerment | "The freedom to distribute copies of your modified versions to others (freedom 3)." |

Table 1. Analogy between Raworth's economic domains, our potential pattern futures, and Stallman's four software freedoms

## 3 METHODOLOGY

The workshop methodology that we presently employ traces its origins back to a presentation at the 2019 Anticipation conference.[2] Our goal at the time was to provide a rapid introduction to our work on peeragogy.[3] We wanted to use a format in which the audience would participate actively, since we thought that would better embody the spirit of peeragogy than a lecture. Thus, we wrote our presentation in the form of a dramatic dialogue and asked attendees to pick parts and perform the dialogue. While the attendees appreciated the opportunity to participate, one of them remarked that use of a pre-determined script felt confining.

In response to these comments, we developed a more open-ended approach and refined it in pilot workshops. Instead of handing participants a finished script, we provided them with design patterns and functional roles as a framework within which to improvise their own dialogue. Moreover, we made it clear that these roles and patterns were are only meant as a starting point and encouraged them to reinterpret and modify the material they were given, and to develop new patterns in the course of the workshop.

Our starting assumptions were that a short training in PLACARD methods and the patterns that operationalize those methods would help participants who were not familiar with Peeragogy [Corneli et al. 2015, 2016]—and who, indeed, had never previously met each other—learn together and collaborate effectively in the workshop setting. We also expected that these workshops would support a form of distributed collaboration, across workshop contexts and topics, as we gathered themes and insights from people who would never meet at all. During the course of an individual workshop and over the longer time-scale of multiple workshops, methods and patterns evolved. Their interpretation changed in accordance with community understanding, and new patterns emerged from repeated experiences. A representative selection of the patterns can be found in an Appendix, and Table 2 presents an overview. The high-level subdivisions in this table mirror the structure of PLACARD: "Identifying themes", "Organizing structure" and "Making it actionable". In some of the workshops, the design patterns were shared with attendees in the form of pattern cards which could be used as in-workshop manipulatives. In others, the patterns were primarily used to design and analyze the workshop, and stayed more or less behind the scenes. Our case study descriptions clarify those details.

We assess each of the case studies using a structured framework we use to reflect on which aspects of the case study are reproducible: methods, results, interpretation [Goodman et al. 2016]. These reflect PLACARD's three dimensions—*methods reproducibility* pertains to the motor task of replicating an experimental set-up form

[2]A fictional peeragogical anticipatory learning exploration, 11 October 2019, Paper Session 13, http://anticipationconference.org/wp-content/uploads/2019/10/Anticipation_2019_paper_113.pdf

[3]https://peeragogy.org

a published description; *results reproducibility* pertains to the sensory task of collecting data from a potential replication, and comparing results of a previous experiment; and *interpretation reproducibility* pertains to the cognitive task of verifying the processes of analysis and inference used to draw conclusions from data. As we draw our interpretations here, as part of our per-section evaluations, we flag the most closely related domain in Table 1, to highlight the potential reach/impact of each case study.

PLACARD 'By using the PAR (or another sensory method), we are able to identify recurring themes. Then, by using the CLA (or another cognitive method), we are able to organize these repeating themes in a structure that exposes the underlying trends, causes, and potential terminating states. With DPL (or another motor method) we can make what we have learned actionable.' [Corneli et al. 2021]

⇒ **Process**: Workshop design, delivery, and analysis adds actionable detail to that proposition.

*Identifying themes:*

| | | |
|---|---|---|
| [1] | Dérive Comix | 'document what you see' |
| | Share Back | 'individual groups should present key findings' |
| | *Pilot to Anticipate* | 'anticipate the issues likely to arise in future iterations' |
| [2] | Context Setting | 'describe the hoped-for outcomes' |
| | **Time Traveler** | 'provide historical context and anticipate alternate futures' |
| | **Reflector** | 'appraise each developing scenario' |
| | *Contested Space* 💎 | 'each space need not support every use equally' |
| [3] | Do Your Research | 'start doing the research in a more centralized way' |

*Organizing structure:*

| | | |
|---|---|---|
| [1] | Meaning Map | 'get everyone on the same page' |
| | Pattern Language Components | 'build patterns piece by piece' |
| | Going Meta | 'explore how the project's methods can be applied to itself' |
| [2] | Facilitator Roles | 'structure the work' |
| | **Analyst** | 'identify and orchestrate the dynamic network' |
| | **Linker** | 'providing visualization of patterns and interconnections' |
| | *Funding of Public Space* 💎 | 'create a register of impacts' |
| [3] | Structure Conversations | 'structure the discussions around shared interests' |
| | *Destructure Patterns* | 'a less formal discussion can surface useful meanings' |
| [4] | *Adapt Layers As Needed* | 'layer-based analysis facilitates effective communication' |
| | *Avoiding Mistakes* | 'navigate common project development pitfalls' |
| | *Scaling and Adaptability* | 'aim to seamlessly adopt new advancements' |

*Making it actionable:*

| | | |
|---|---|---|
| [1] | Reinfuse Expertise | 'enable richer and more complex thinking' |
| | Functional Roles | 'introduce different perspectives' |
| | *Increase Participant Control* | 'participants should not remain only an audience' |
| [2] | **Wrinkler** | 'what might derail or counter the proposed solution' |
| | **Stepper** | 'decide which actions would be most useful' |
| | *Rebalance Social Services* 💎 | 'address complex local challenges' |
| [3] | The Future Begins Now | 'take preliminary actions before leaving' |
| | *Structure Outputs* | 'link intermediate artifacts into a relevant template' |
| [4] | *Engagement and Guidance* | 'create a collaborative learning environment' |

Table 2. A summary of our collected "Patterns of Patterns". Legend: Small caps is used for formalized patterns; bold is used to denote roles; italic serif font is used to denote proto-patterns. Proto-Patterns that were created by workshop participants in Case Study 2 are further distinguished by a "💎" marker. Numbers in the left-most column cross-reference the case studies ([1], [2], [3] and [4]) in which the patterns were initially developed and used.

## 4 CASE STUDY 1: "GOING META" WORKSHOP AT ANTICIPATION 2022

This workshop functioned as a more developed pilot of methods that we had previously trialed at PLoP 2021 as "Flaws of the Cool City"[4] and at the Oxford Brookes Creative Industries Festival as "Dragon versus monkey: A kaijū introduction to peeragogy".[5] Our central aim was to 'workshop' the methods with attendees. Our pitch was that the workshop would help attendees establish a position of maximum leverage, exercising "Critical Anticipatory Capacities" and using "Creativity, Innovation and New Media" (two of the conference's themes) to explore the future of anticipation.

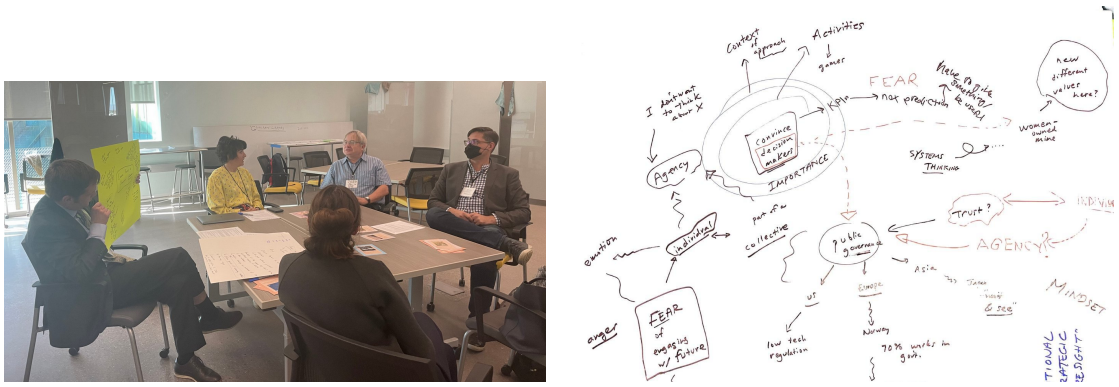



Fig. 1. Left: In the workshop, participants used pattern description cards to structure discussions (cf. the Pattern Language Components and Functional Roles patterns). Right: Facilitators took notes and made diagrams (Meaning Map).

Figure 1 shows the workshop in process (left) and details of the notes taken by one of the facilitators (right). Facilitators moved between facilitating small-group and whole-group activities. The discussion ranged over various themes, including collective agency and how to engage more people in futures discourse.

In the concluding Project Action Review, participants had different responses to our request to reflect on the workshop's activities. Some considered the workshop to have led to good conversations, but doubted if the manipulatives and other structure helped. However, as mentioned in Section 1, we had observed that some participants had used the manipulatives in creative ways—such as asking each other to pass around the cards to better narrate the points they were making in the conversation. Even if another conversation with these attendees would have been good, it would have been a very different conversation. Overall, workshop attendees had plenty of feedback for us about how we could improve our use of the methods;[6] they did not, however, adopt any actions for themselves to take forward around the points we discussed. This was in keeping with our primary aim for the workshop, but somewhat surprising since we were used to Project Action Reviews helping to establish a collective sense of direction. We reflected that we could make that particular aim clearer in future workshops.

### 4.1 Evaluation

**Methods** These or similar pattern cards can be re-used, as can the meeting itinerary (indeed, later case studies follow a similar itinerary).

[4]https://www.hillside.net/plop/2021/index.php?nav=program#focusgroups
[5]https://www.brookes.ac.uk/research/networks/creative-industries-research-and-innovation/festival-2021
[6]https://groups.google.com/g/peeragogy/c/V-knbZkwhB0/m/jUbw3_I9AAAJ

**Results** As discussed above, we would likely have had a good discussion with the participants without the manipulatives, but (*a*) that would have had extremely low reproducibility; and, in particular, (*b*) we would not have received feedback on the manipulatives had we not used and presented them.

**Interpretation** The fact that we received usable feedback on the materials shows that they can be used in a co-design process. However, the apparent resistance by attendees to forming an action plan based on our discussion suggested that we were the primary beneficiaries of this "methods workshop". The workshop was effectively an open house, and the feedback applied, correspondingly, at the **household** level.

## 5 CASE STUDY 2: PUBLIC SPACE FOR PUBLIC HEALTH

This workshop was commissioned by co-author Abby Tabor as part of her research project at the University of the West of England on "Designing urban environments for human health: from the microbiome to the metropolis". The aim was to gather attendees with an interest in the project themes and work together to envision next steps. Elaborations of these were developed by participants, and were organized by facilitators using a software tool based on Org Roam and Org Roam UI.

Inspired by our experience in the Anticipation pilot, here looked for new methods to Increase Participant Control. In particular, we generated some further articulations of the Functional Roles that would help with this. The roles presented come with mnemonic symbols based on the chess set: at the workshop, participants were provided with additional physical manipulatives, i.e., the chess pieces that correspond to the symbols here, along with new pattern cards. As mentioned previously, in the Anticipation pilot, the roles were aligned with the Pattern Language Components. In the pilot, the "*Kaijū Communicator*" role had final say over the challenges implied by the 'HOWEVER' keyword.[7] Here, this role was renamed and adapted as the "Wrinkler", and ascribed a conversational perspective rather than a "role-playing persona" that would be taken on by whomever was holding the card. In still-earlier pilots, the roles had been assigned more elaborate responsibilities, and participants would receive a brief training for the role prior to taking it on. For example, we had made use of a *Designer* role, now dropped entirely, which was to be filled after a briefing on design pattern methods and a specific collection of design patterns.

As presented in this workshop, the roles were strictly functional and were not the focus of role play as such. Rather, an attendee would fill a given role momentarily within a conversation, while remaining fully themselves. The roles were again outlined as pattern cards, together with an informal verbal description. The reader may refer to Table 3 which explains the mnemonic meaning of the chess pieces which accompanied each role description. The Linker role was filled by offsite facilitators, and the Reflector role was filled by onsite facilitators. A designated Stepper role was only proposed after the workshop, though participants were asked to look for "Next Steps" in putting the patterns they were prototyping into action (in the style of [Corneli et al. 2015]). These were elaborated using the 'SPECIFICALLY' keyword from the Pattern Language Components. This served a core purpose in the event, intended to address the limitation from the previous workshop. As noted, previously, the implied "Next Steps" were all presented in the form of feedback or advice for the workshop facilitators, rather than as actions which the attendees could (at least in principle) adopt and complete themselves.[8]

In the spirit of Increasing Participant Control, the Facilitator roles could be distributed to participants, though doing so in a future workshop would require spending more time on training, and potentially also improved tooling for moderating the flow of information within (and beyond) the workshop.

[7] https://hyperreal.enterprises/open-future/

[8] https://www.eventbrite.co.uk/e/public-space-for-public-health-a-call-to-action-tickets-492522286417

| Role | Manipulative(s) | Explanation |
|---|---|---|
| Time Traveler | ♛ | In chess, the Queen can move linearly in any direction: forward, backward, and diagonally. Similarly, the Time Traveler role 'moves' both backwards and forwards in time, and also explores the conditions that appear at those points in time. |
| Analyst | ♝, ♗ | In chess, there are two Bishops, both of which move diagonally, so that both are restricted to different colored squares. Here the Analyst role divides its attention across two different spheres: articulations within the current challenge, and articulations of this challenge relative to other challenges. |
| Wrinkler | ♞ | In chess, the Knight moves in a skewed fashion: in order to go in one direction, it must also go a little bit in another direction. The Wrinkler role, similarly, looks at how a given strategy might go askew, due to unintended consequences or otherwise. |
| Linker | ♖ | In chess, the Rook moves any distance, as long as it goes in a straight line. The Linker role, similarly, can record a link between any two related concepts. Since all concepts are potentially related in some fashion, the Linker focuses on making *useful* connections. |
| Reflector | ♔ | In chess, capturing the King means the end of the game, so players are concerned throughout with any threat to their King. Here, the Reflector role senses how the discussion of a given scenario is progressing and when it would be good to draw it to a close and move on. |
| Stepper | ♟ | In chess, Pawns move one space at a time (or two at the beginning of their movement). Pawn are individually weak, but collectively, their placement is important for strategy. The Stepper role similarly describes the immediate next actions that should be taken at a point in time. |

Table 3. Mnemonic for manipulatives based on the chess set

## 5.1 Intermediate artifacts

Intermediate artifacts generated within the workshop included mindmaps created by participants on paper. We used a graphical template following the outline of the Causal Layered Analysis layers to explain the workshop's overall workflow, and also asked participants to use a version of this diagram as a "grid" for note-taking within Phase I, to encourage them to work from their observations to the core underlying themes and issues. Participants then clarified these core themes in a share-back process, and in Phase II, developed them further in the form of shared future stories, outlining paths to action. Photos in Figure 2 show the movement from:

- Initial sketching at the start of Phase I, beginning at the *litany* level, within small groups, on to:




Fig. 2. Use of diagrams and manipulatives to create Meaning Maps and new patterns. In the upper-left photo, the CLA layers are mapped to concentric circles; from outside to center: litany, system, worldview, myth. The Share Back pattern was used to collect core themes from groups working separately, conceptualized here as the basis of a shared myth, comprising a Meaning Map that pulls together the themes from small group discussions (lower left). Pattern Language Components were then used to sketch solution strategies to key problems and concerns, e.g., Funding of Public Space (right).

- A collection of themes shared across groups at the end of Phase I, to create a Meaning Map which, in CLA terms, is intended to bring everyone into a shared *myth* layer, on to:
- Phase II, using Pattern Language Components to identify both general and specific possibilities for action around a theme.

## 5.2 Output patterns

Participants created several patterns by making use of the Pattern Language Components and Functional Roles. For example, the right-most image in Figure 2 shows the participants' articulation of a Funding of Public Space pattern (created with help from a workshop facilitator). Using the Pattern Language Components 'HOWEVER', 'BECAUSE', 'THEREFORE', and 'SPECIFICALLY' made it easy to write down structured candidate patterns (i.e., outlining the typical Context/Problem/Solution triad, with the addition of 'Next Steps'). In Appendix 10, these candidate patterns are re-compressed into a succinct textual statement of the core idea of the pattern.

Although in this workshop we were successful identifying in "paths to action",[9] the main limitation was in regard to follow-through. We had initially planned to adapt Org Roam or some other similar tool into a wiki that participants could use after the workshop to keep track of patterns and next steps, recording progress, any blockers or 'reverse salients' [Hughes 1993], and any evidence for or against the patterns, as well as their elaboration. Our view is that other technologies such as the Federated Wiki [Cunningham and Mehaffy 2013]

[9]https://www.eventbrite.co.uk/e/public-space-for-public-health-a-call-to-action-tickets-492522286417

would have required too much training time to be used for the purpose we had in mind. In the end, we simply didn't deploy an end-user-writable software platform.

### 5.3 Evaluation

**Methods** We expanded the system of manipulatives to include CLA diagrams, a Share Back process, and explicit Pattern Language Components. Elicitation of structured patterns required active facilitator involvement.

**Results** Mapping user contributions by offsite facilitators was both intensive, and rather subjective in nature. This could be improved by reworking the mapping tools so that they could be used in real time by workshop attendees.

**Interpretation** Firstly, the workshop was developed as paid consulting work, connecting the overall process overtly to the themes of the **market**. The results of our offsite work could be viewed by attendees, but were not shared in a form where they could be directly extended. Nevertheless, within the workshop itself, we saw attendees successfully adopt pattern methods, and use them to talk about potential futures. This suggests the possibility for benefits accruing at the **state** level, with the potential for an Open Future Design workshop to function as a new kind of forum or senate, by analogy to developing practice around citizen juries.

## 6 CASE STUDY 3: OPEN RESEARCH FUTURES

This workshop was developed as an "Away Day" for faculty and staff members at Oxford Brookes University. The aim of the workshop is to elaborate the institution's open research strategy relative to its existing organizational strategy. Methodologically, this workshop builds on a pre-seeded Org Roam network of interlinked themes and an additional activity that enlists attendees in taking concrete actions on the identified next steps. The itinerary for this workshop adopted the language "experts to citizens", "citizens to action" from the previous workshop. These phases mirror the Dérive Comix—Meaning Map—Reinfuse Expertise structure introduced in the first case study; in CLA terms, the overall effect is a journey from *litany-to-myth* and then *back-from-myth-to-litany*, with the new litany taking the form of potential actions, or more specifically, potential future headlines describing the actions.

### 6.1 Intermediate artifacts

Figure 3 shows how the Pattern Language Components were used in the second phase of the workshop, building on a CLA-based discussion that developed the Meaning Map in the workshop's first phase (background, right). In this case, the keywords and manipulatives corresponding to Functional Roles (cf. Table 3) were not used to spell out entire draft patterns (as in Figure 2), but rather, to generate a network of relations.

Figure 4 shows how material was then drawn together, using Org Roam to analyze the workshop themes (per the Linker pattern), further elaborating the Meaning Map. Contents of the paper-based diagrams, like those in Figure 3, were condensed and edited in the digital notes, rather than represented verbatim. For example, the idea of reintroducing a "common room" (Figure 3, left) is folded into the "Research Environment" node, along with the challenge of holding conversations when "people aren't here" in person. The digital notes also provided an opportunity for corrections, consolidation, and synthesis of links which hadn't been spelled out directly with the manipulatives. E.g., "Richard Owens" [sic] (Figure 3, right) is represented in the digital notes as *Richard Owen*, key proponent of responsible innovation [Owen et al. 2013], within the "responsible" node.

The overall process illustrated in Figure 4 moves in the opposite direction of *Figures 3-7* of Iba and Isaku [2016], insofar as those figures complexify a tree as a graph. Here, we move from an interlinked graph of topics to a summary map in tree form, represented here by the Outline of an Open Research Action Plan node.




Fig. 3. The Pattern Language Components were used organically within the workshop (cf. *Destructure Patterns*).

In brief, the organic, playful, interactions within the workshop were useful in creating the more formal output (a draft policy document) precisely because these interactions were informed by a suitable metalanguage, including the apparatus of Causal Layered Analysis, the Pattern Language Components, the Functional Roles, the Meaning Map, and other patterns described so far in the paper. Counterfactually, we could have written up an action plan based solely on desk research—e.g., writing a synthetic plan based on identified organizational values "Success, Openness, Learning by doing, Adaptability and Creativity, and Equal opportunity… SOLACE" [Corneli 2023]. However, that level of analysis only provides a "mental picture", in the terminology of Alexander [1964]. A hypothetical Open Research Action Plan based on stakeholder interviews and the "SOLACE" concept, omitting the workshop experience, might express the university's stated values, but it would not draw on the same richness of meaning, nor the same level of practical detail.

## 6.2 Evaluation

**Methods** We used the same manipulatives as in the previous case study, but this time with less structure. This may suggest a level of reproducibility between that of generically useful PostIt® notes and highly-structured design processes.

**Results** The results were first collected in a graphical form, and then presented using a standard text template. The pattern components were used to structure the map, though that process was still largely subjective. Algorithmic additions to the methodology could enhance reproducibility, for example, by using the pre-existing template and a clustering algorithm we could (in principle) make the synthesis phase entirely reproducible.

**Interpretation** The collated material was shared with a new working party that intends to create a new organizational strategy. While the working party may ultimately develop a different interpretation of the role of open research from the picture created by workshop attendees, they will benefit from having seen that that picture. This suggests a range of benefits accruing at the **market** level, both in the internal ideas marketplace, as workshop attendees will have their voice heard within the organization, and also at the

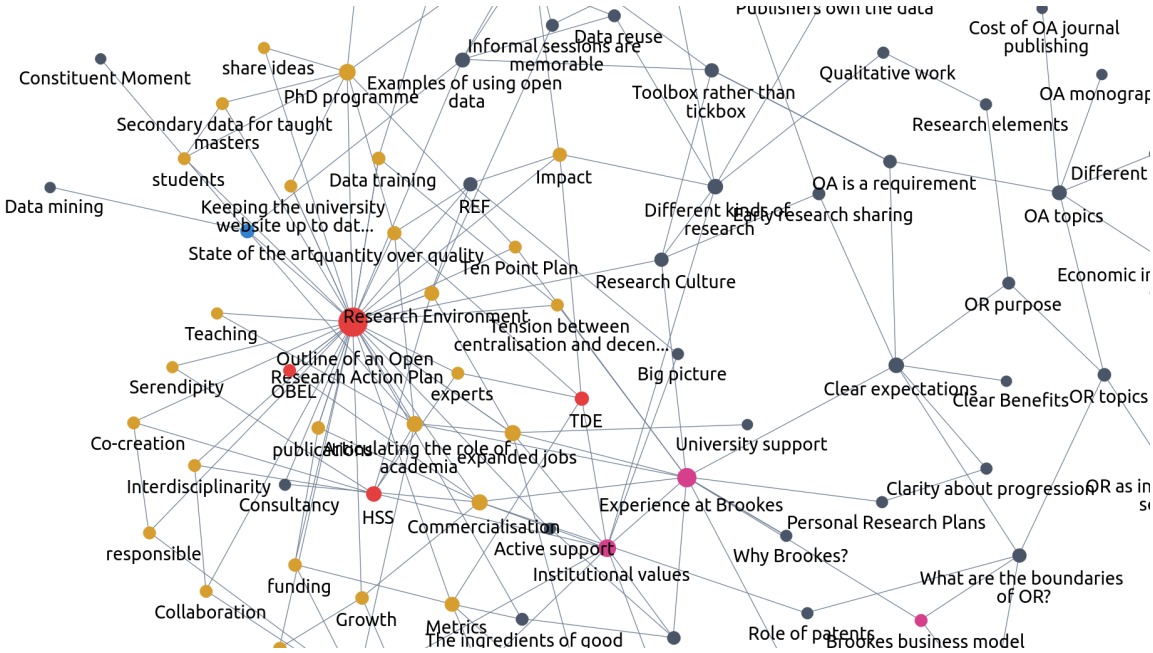



Fig. 4. Screenshot of Org Roam UI, showing the development process leading to a draft Open Research Action Plan (ORAP). Color-coding is: (Gray) Background themes and concepts based on interviews (cf. Do Your Research); (Purple) Selected themes from that background material which became the focal themes in the workshop; (Yellow) Workshop themes and concepts; (Red) Key points of organization for workshop themes, including discussions per faculty as suggested by Structure Conversations. The node "Outline of an Open Research Action Plan" includes the ORAP, instantiated here as a bullet-point outline, with links to all the workshop outputs.

organization level, as the university articulates its position within the sector in a way that incorporates on-the-ground realities. As alluded to in the previous case study, a system for updating and maintaining diagrams like Figure 4 in real time would have uses in a form of "citizen science". That suggests a technology-enhanced information ecosystem, with benefits at the **state** level. However, from an organizational strategy perspective, the distilled bullet-point outline was an essential "product", and the map itself contains too much information for immediate decision-making needs.

## 7 CASE STUDY 4: CIS 9590, INFORMATION SYSTEMS DEVELOPMENT PROJECT

### 7.1 Introduction to the course from the instructor, Mary Tedeschi

CIS 9590 is Information Technology Project Design and Management is the "Computer Information Systems" (CIS) capstone project course for the CIS major wherein the students will apply concepts and techniques from prior course work, to design, develop, and create an implementable application for a working information system of an actual business. It also focuses on the design and management of systems to meet the increased need for information within an enterprise. The course exposes students to the fundamentals of IT project management required for the successful implementation of IT-based systems. The course presents tools and technologies for project definition, work breakdown, estimating, planning and scheduling resources as well as monitoring and control of project execution. Students utilize knowledge gained from prior coursework, and work in groups to

design and manage an Information Technology project. During my first semester, Spring 2020, teaching with the students using whatever development tools they were familiar with, I noticed this to be a problem so with this knowledge I changed the course to require the use of Intel One API. This did not get implemented until Fall 2021. I actually taught the course three times before requiring students to use the same software tool uniformly. The course was a 3-hour course, first face-to-face. Then synchronous online only. In Fall 2021 we changed to 75 minutes in person and online (hybrid). Students had to self-teach Intel One API with the use of tutorials and a buddy system. The students seemed to have the necessary skills to learn enough of the software to create an implementable application. This semester, Spring 2023, the students really seemed to lack the coding skills.

### 7.2 Our use of "Patterns of Patterns" within the course, by guest lecturers Raymond Puzio, Joe Corneli, and Charlie Danoff

Mary assigned our paper "Patterns of Patterns" [Corneli et al. 2021] as a focal text with three successive cohorts of CIS 9590 students. The course syllabus is focused on developing group projects with a computer programming component. Our hope was that the topics in the paper would inspire them with new ideas about design and collaboration. We focus primarily on the latest iteration of the course (Spring 2023), in which we made the most explicit use of the methods described above. Figure 5 shows some of our anticipations of the relevant concerns that apply in this context. Each semester, students asked many thoughtful questions about the paper; each cohort also produced their own collaboratively-written response to the paper, engaging the original paper in depth; and in the latest run, we offered some in-class exercises based on the workshop methods described above. Reading their written responses showed that the students had not only understood the main ideas of our paper, but added to them. In effect, they created alternative imaginaries for the paper's history and future. For instance, in their 2022 'case study', they generated a "Recommendation and Implementation Plan" which proposed specific actions which a group could take based on our ideas; and, in 2023, the students produced a slide presentation based upon our paper, exploring its relationship to themes such as "emerging technology". It is worth highlighting that while our paper did touch briefly on the theme of emerging technology, the students considerably elevated the importance of that theme in their feedback.

### 7.3 Experience report by CIS 9590 student, Kajol Khetan

The PLACARD method emphasizes understanding the context, selecting an appropriate language or languages for thinking about problems in that context, and taking relevant actions to guide the development process. In our project work, we adapted the CLA component of this framework by identifying the layers relevant to our chosen problem, namely, to create a website that allows users to discover nearby coffeeshops easily. We identified the following layers for our analysis (each with several facets): User Interface, Functionality, Data Flow, Infrastructure, and External Factors.

CLA guides us to look for *causal relationships* operating within and between these layers. In particular:

- Changes in the User Interface layer influenced user engagement and ease of interaction.
- Adjustments in Functionality impacted the overall user experience and satisfaction.
- Data Flow optimizations directly affected the accuracy and relevance of coffee shop information.
- Infrastructure decisions impacted website performance and responsiveness.
- External factors influenced design choices and user satisfaction.

Our adaptation of Causal Layered Analysis led to a comprehensive understanding of the website's components and their interdependencies, and facilitated a structured approach to development. The nearby coffee shops website was successfully developed and deployed. Users can input their location and receive a map display of nearby coffee shops, along with relevant information such as ratings, reviews, and operating hours. Our group's process could be condensed into an overall proto-pattern, *Adapt Layers as Needed*.

## 7.4 Experience report by CIS 9590 student, Manvinder Singh

During my final project class led by Professor Mary Tedeschi, the research paper "Pattern of Patterns" and its authors played a pivotal role in shaping my project decisions. The authors' active engagement in our Zoom sessions provided invaluable insights into project layout, helped me avoid common mistakes, and encouraged a scalable and adaptable approach. Their active engagement and promises to come again in the following weeks to see progress on everyone's individual project kept the butterflies, nervousness, and willingness to deliver all alive at the same time. The guest lecturers' contributions, together with the way we adopted their paper, suggests several proto-patterns which could be used to structure future versions of the course: *Engagement and Guidance*, *Avoiding Mistakes*, and *Scaling and Adaptability*.

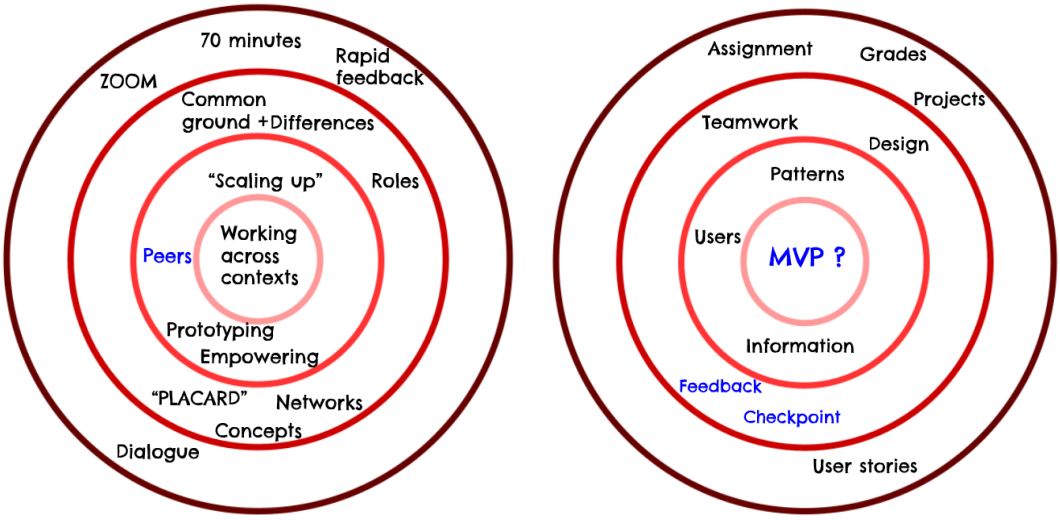



Fig. 5. Diagrams created before our first session with the CIS 9590 students, inspired by Causal Layered Analysis. The diagrams describe our working context as guests in CIS 9590 (left), and our initial understanding of the students' working context (right).

## 7.5 Evaluation

**Methods** Guest-lecturing and receiving written feedback on our paper were directly replicable across cohorts. The workshop methods introduced to the latest cohort adapted reasonably well to the online/hybrid classroom, though another round of prototyping could improve them.

**Results** The student experience reports suggest a strong uptake of the methods we introduced. With Mary, we discussed peeragogy-inspired learning design suggestions that could be used in future classes.

**Interpretation** The experience went well enough that we were invited back several times, and, following the latest sessions, invited to continue the conversation with other instructors in the New York region (at PACE and Baruch). Students reported improvements to their product design and development processes, with benefits at the **market** level. We also see the potential for benefits accruing at the **commons** level, working across contexts by openly sharing and interlinking Meaning Maps like those depicted in Figure 5.

# 8 CASE STUDY 5: PLOP 2023 SHEPHERDING AND WRITERS' WORKSHOP

Our account would be incomplete without an overview of the way our thinking, and this written presentation, continued to evolve in the Pattern Languages of Programs 2023 context. As in Case Study 1, at the Writers' Workshop, we were primarily asking for feedback. As in Case Study 4, participants had an entire paper to wrap their heads around. Our a result of the discussions with our shepherd and workshop attendees, we clarified our use of proto-patterns, ultimately focusing our presentation of the paper by moving the patterns and proto-patterns to the appendix. We rewrote the introductory sections to make it clearer that our focus is on the process of evolving patterns, rather than the presentation of polished patterns (more typical at PLoP).

## 8.1 Input patterns

We put our pattern catalogue and surrounding narrative forward for scrutiny and discussion. The process was governed by pre-existing patterns for paper shepherding and Writers' Workshops. We continued to use patterns like Structure Conversations after the workshop, building in several rounds of internal peer review.

## 8.2 Intermediate artifacts

We began by significantly shortening the paper, moving around 8000 words from our conference submission [Corneli et al. 2023] into an "outtakes" directory. We restructured Table 2, and combined two earlier figures into Figure 6. In particular, Figure 6 now shows both the abstract workflow of pattern synthesis following the PLACARD method, and the concrete process of pattern development that we followed. These correspond to outside-in and inside-out readings of the diagram. Further significant revisions to the text were made. In particular, the evaluation sub-sections were added.

## 8.3 Evaluation

**Methods** The methods used at PLoP are well-established, having evolved into relatively firm traditions over 30 years. These were augmented by peeragogy patterns [Corneli et al. 2015] and ideas from the current paper.

**Results** Our paper received significant revisions focused on addressing the main takeaways from the workshop. No new patterns were added, but the existing patterns were revised and the catalogue restructured.

**Interpretation** PLoP methods are useful for polishing an existing set of patterns. In and of themselves, PLoP workshops do not provide a complete recipe for creating useful pattern languages. PLACARD and the associated patterns in this paper are therefore complementary to the existing PLoP methods. Having been through one complete cycle of research, writing, and revision since we presented "Patterns of Patterns", we have now developed a complete methodic description for working with patterns in workshop settings. We have touched on potential benefits across the four economic domains mentioned above. The process of revision associated with PLoP helped us articulate these benefits in a cohesive manner, producing a more salient contribution to a knowledge **commons**.

# 9 DISCUSSION

## 9.1 Related Work

We have taken a design-patterns-based approach to meta-research, using design pattern methods to study design patterns. Broad thematic areas within meta-research include: "methods, reporting, reproducibility, evaluation, and incentives (how to do, report, verify, correct, and reward science)" [Ioannidis et al. 2015]. In this connection, Figure 6 has a theory-building role, showing how the PLACARD pattern was expanded into further design patterns which we developed across Case Studies 1-3. Whereas common problems in quantitative research are relatively well established [Munafò et al. 2017], to date, the corresponding challenges and solutions in qualitative research have been less well studied. The diagram in Figure 6 depicts both a research cycle for generating new design

patterns, which can now be run with variations, repeatedly, and also a roadmap for deepening our collective understanding of and facility with pattern-based inquiry. Heiberg et al. [2022] make use of a similar diagrammatic approach to map evolving relationships between technical and institutional concepts, superimposed upon a 'radar plot' which shows which concepts are central and which are peripheral.

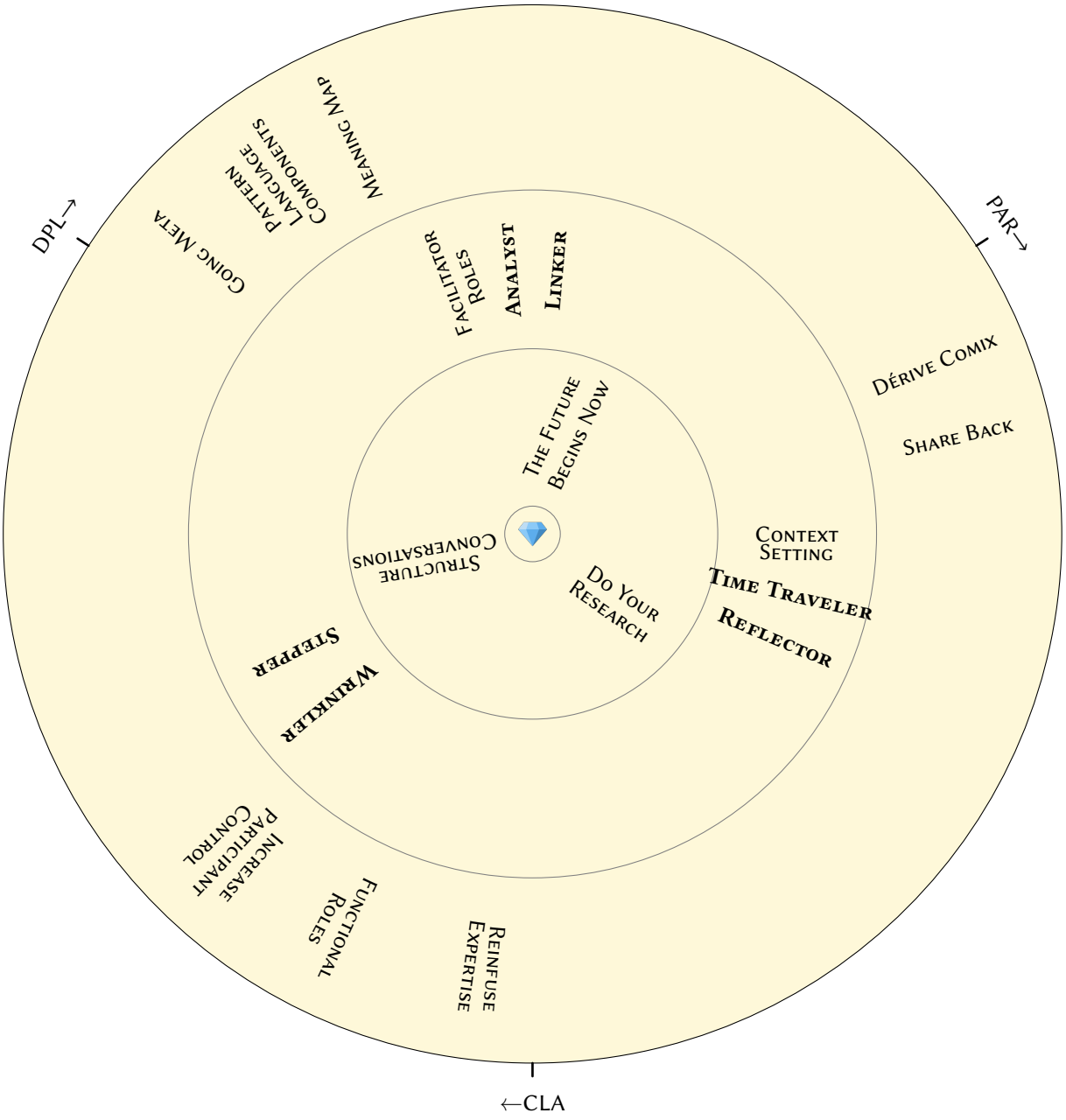



Fig. 6. Relationships between the component methods of PLACARD and the patterns introduced in this paper. For example, the sensory dimension, PAR, serves to "Identify themes". This process can be implemented through the Dérive Comix and Share Back patterns; further nuance can be added with the Context setting pattern along with the **Time Traveler** and **Reflector** roles; and still more articulation can be added with the Do Your Research pattern. The three layers of patterns correspond to our workshops [1], [2], and [3], and roughly align with the CLA *litany*, *system*, and *worldview* layers. Proto-patterns that were generated at any workshop or in subsequent analysis live at the myth level, denoted by 💎.

## 9.2 A remark on the Roles of Roles

In developing the PLACARD pattern, we were inspired by the classical neuroscientific notions of sensory, cognitive, and motor faculties. The Active Inference Framework (AIF) provides a more contemporary approach to similar concepts: see Smith et al. [2022] for a primer. Here, we suggest that AIF and DPL can support each other, with AIF concepts being given a practical implementation using design pattern methods, and DPL receiving a needed theoretical upgrade by way of Active Inference.

Both AIF and DPL have a broad scope of application, to a range of complex systems. DPL and AIF both make use of hierarchical organization, e.g., *A Pattern Language* cascades from patterns for towns to buildings to construction; whereas AIF typically zooms out, from simple to complex and reflective—from "I model the world" to "we model the world" to "we model ourselves modelling the world" [Kirchhoff et al. 2018]. Both AIF and DPL have a fundamentally generative orientation. Despite these similarities, AIF relies on quantitative methods coming from statistics and theoretical physics, whereas DPL is primarily qualitative. We wish to draw attention to the way in which the participant roles outlined above can operationalize key aspects of the Active Inference Framework.

The Time Traveler elaborates *a prior belief over states* and the *likelihood of specific observations*.
The Analyst elaborates a *generative model*, with the division between inward and outward articulations corresponding to *internal* and *external*-facing states (both with *sensory* and *active* components).
The Wrinkler elaborates a *factor of surprisal*.
The Stepper takes action to correct discrepancies between the generative model and the perceived state of the world, minimizing *prediction error*.

Active Inference is currently being explored as a new paradigm for artificial intelligence, with a focus on ecosystemic intelligence [Albarracin et al. 2023; Friston et al. 2022]. In particular, the AIF-inspired "hierarchical generative model capable of self-access" from Albarracin et al. [2023] arises through the addition of reflective meta-layers, ascending from:

| | | | |
|---|---|---|---|
| | "*What am I trying to do?*" | and | "*What am I perceiving?*" |
| to: | "*What am I paying attention to?*" | and | "*What am I trying to pay attention to?*" |
| to: | "*How aware am I of where my attention is?*" | and | "*Am I trying to maintain awareness of my attentional state?*" |

This is similar to the *litany-to-myth* direction of analysis that we used in the first phase of our workshops. In our workshop, the roles were introduced in the *myth-to-litany* phase, to generate potential actions. The socio-historical theory of P. R. Sarkar, which informed the development of CLA, hinges on similar roles (cf. Inayatullah [1999]; Inayatullah and Fitzgerald [1999]). The key point for our current purposes is that Functional Roles can help to bring about change across levels and domains.

## 9.3 Connection to other emerging computational technologies

We reflect that pattern-based information processing tools — commonly referred to as "artificial intelligence" — are being used to model and influence a wide range of social and economic contexts. Since AI and robotics routinely make use of concepts of cognitive science, this provides a common ground with the topics discussed here. To flesh this out, AI techniques can be organized in nested layers, using CLA-style thinking, as we did with our patterns in Figure 6. Machine learning, neural networks and large language models correspond to the litany layer in the outer ring, since they work rather directly with raw data, detecting regularities (such as clusters and n-gram probabilities), and using these regularities to predict what comes next or to classify items. In the next ring, we could place rule-based production systems and planners. These correspond to the systems layer. In the inner ring, we can place ontologies and knowledge representation languages, which formalize worldviews to

make them amenable to computation. For some contemporary thinkers, the innermost myth at the core is the possibility of building artificial general intelligence, though opinions vary about when and whether that will arrive, and alternative myths such as "AI empowers workers" have been proposed [Chiang 2023]. We hope these observations will be of use in implementing design patterns computationally—perhaps starting with the patterns of patterns that we have collected—and/or in designing workflows for developing future AI systems.

## 10 CONCLUSION

Relative to previous work on the structure of patterns which used a path-traversal metaphor to describe how patterns work [Kohls 2010, 2011], a suitable metaphor for our contribution is *a wheel which rolls more smoothly when round ball bearings are placed in the hub.* Less metaphorically, Richard Gabriel emphasized that the "patterns and the social process for applying them are designed to produce organic order through piecemeal growth" [Gabriel 1998]. We given some further articulation to this point, and our case studies have shown how such articulation can make work with patterns more effective, efficient, and scalable. In brief, our patterns of patterns exemplify the latter part of the following description:

> Technological progress is achieved through a dialectical relationship between mediation (adaptation to the end terms: the path to be travelled and the load to be carried) and autocorrelation, the relation between the technical object and itself. [Simondon 2005]

The primary limitation of this work is that it has been carried out mostly at the "paper prototype" level. Nevertheless, the process we have presented reflects the main features we would expect from future more technically developed platforms for distributed collaboration. We described prototype-level distributed-working that was carried out inside the workshops, organized via the Share Back pattern; and our workshop repeated with variation over time, allowing us to learn from widely distributed teams.

Existing systems such as Alkemio[10] are being built with the need for collaboration across organizational boundaries in mind, preferring "*challenge-focused*" collaboration. A platform using DPLs to work *across related challenges* would take that idea even further. Domain-level design patterns outline potential new behaviors; improved support for the process of gathering evidence that those behaviors do (or do not) in fact work as intended is an ambitious but logical ramification of the pattern method. The Functional Roles we've set forth here provide an early informal articulation of the process, and the connections to the Active Inference Framework that we alluded to in Section 9.2 could scaffold further, more formal, developments.

Whatever underlying formalism is used, and independent of the specific technologies that become part of future software implementations, further work is needed to identify analogies between action arenas, to highlight the ramifications of complex actions, to show predicted costs and benefits, to search for tipping points that allow the effects of change to reach across level boundaries, and to surface new questions for consideration. The overall need is summed up by *Noema* magazine's editor-in-chief, Nathan Gardels: "It's time for new cooperative platforms that address irreducible interdependence."[11]

To illustrate how our work could help to address that need, Figure 7 juxtaposes the relationships between the components of PLACARD with a design diagram outlining the key sustainability features abstracted from experiences with an early online community [Krowne 2003]. If PLACARD were used to design and build a system with features from the diagram at right, we might expect to see system-builders employing:

- DPLs related to money, organizations, and social interactions,
- a CLA (or a collection of CLAs) that expresses the purpose(s) of the organization,
- and ongoing PARs that help to keep things on track.

[10] https://alkem.io/
[11] https://www.noemamag.com/from-globalization-to-a-planetary-mindset

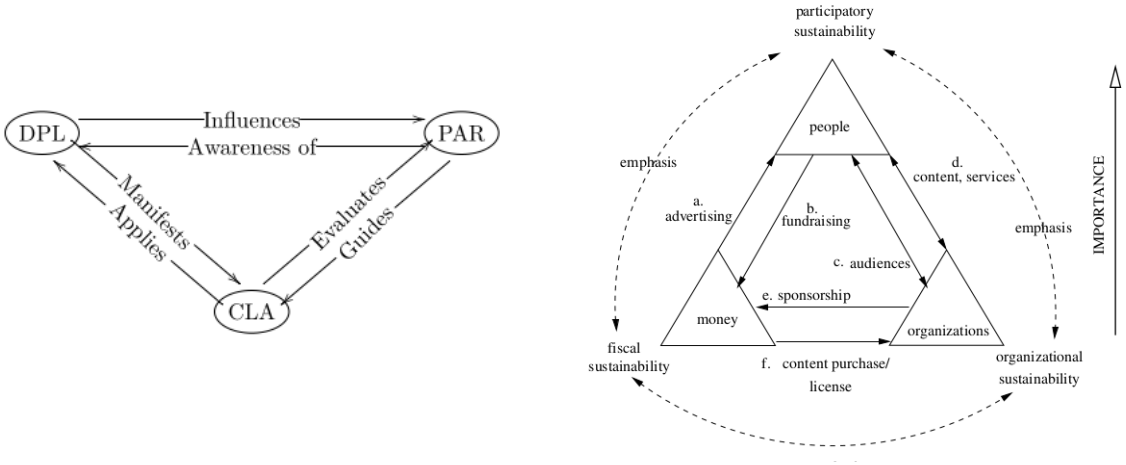



Fig. 7. The interrelated components of PLACARD (left) could scaffold a range of beneficial platform design principles (right, from Krowne [2003], used with permission).




Fig. 8. A team facing a complex challenge depicted by a tangled web of lines and nodes indicating a complex problem. The colorful orbs can be taken to represent other teams tackling related challenges. Illustrated by DALL-E.

A community platform taking our collection patterns for "Identifying themes", "Organizing structure", and "Making it actionable" as its core content would be one possible instantiation. This could be used as a foundation for building further cooperative/collaborative platforms. One way to situate such a platform would be as the "Going Meta" of contemporary efforts to model social and ecological systems within the Doughnut Economics paradigm.[12] The different local implementations of these ideas is a space to look for further design patterns.

Building on our remarks in the Discussion section, we suggest that the existence of common patterns in social and artificial intelligence can inform the further development of the theory and practice of social machines, building on the work of Shadbolt et al. [2019]. Patterns of patterns can facilitate a unified approach to the design and governance of such systems. We have begun (as we mean to carry on) by focusing on the development and articulation of multi-purpose tools for thought. We hope that future software systems will better enable groups of collaborators and cross-group collaborations. Figure 8 is dream-like AI-generated image, showing what these interlinked collaborative settings might look like.

Given our assessment of the reproducibility of our methods, we would expect them to translate well to a variety of related research settings. For example, in our comments on Figure 7, we noted that PLACARD methods could inform platform design, without any commitment as to whether those systems would be digital or analog in nature. In each case study, we evaluated our methods, with respect to whether they already have or could be extended to have benefits at the household, market, state, or commons level. The result is a rigorously documented exploration of complex social interactions around relatively light-weight technologies for co-design and collaboration, which can serve as the jumping-off point for further stages of co-design and development.

Whether future work on patterns of patterns proceeds by developing computer programs that mine the space of potential patterns for those which support human thriving, or by building institutions that achieve something similar, or through some combination, developing the initial representations that support collaboration may be the hardest part. Subject to the limitations of paper prototyping in hands-on workshops, and the pseudo-code-like nature of design patterns, our contributions can help scaffold those future efforts.

## ACKNOWLEDGEMENTS

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## APPENDIX: PATTERN CATALOGUE

We use this simple variation on the classical "Context/Problem/Solution" or "Context/Conflict/Solution" [Alexander and Poyner 1970] pattern template:

[12]Downscaling the Doughnut: Data Portraits in action: A collection of tools and useful examples for creating a Data Portrait of Place, also known as City Portrait. https://doughnuteconomics.org/tools/92

| | |
|---|---|
| **Context** … | [Summary of the working context] |
| **If** … *BUT* … | [A conflict, problem, or gap arising in this context] |
| **Then** … | [Actions to take to resolve the conflict] |

We sometimes add an **Example** where we feel that would be helpful. In this connection, it is worth highlighting that Pattern Language Components pattern covers similar ground with a different lexicon:

| | |
|---|---|
| … **HOWEVER** … | [A conflict, problem, or gap] |
| **BECAUSE** … | [A solution to a similar problem arising elsewhere] |
| **THEREFORE** … | [High-level actions to adapt and try in this context] |
| **SPECIFICALLY** … | [Specific next steps to begin with] |

Some of the patterns concern roles taken on by workshop participants, and for these we use an adapted template:

| | |
|---|---|
| **Context** | [*As above.*] |
| **Question**… | [The thematic question that this role is concerned with] |
| **Role**… | [What purpose the role fulfils in the conversation] |

Some proto-patterns are included which do not have any template sub-structure. They are typically *ideas for patterns* which have not been elaborated. Their titles are written in a mixed-case italic serif font. Three of our collected proto-patterns were simplified adaptations of more elaborate pattern descriptions created by workshop attendees (see for example Figure 2) and are marked with a "💎". In contrast to Table 2 which outlines both patterns and proto-patterns in historical order of their development, for ease of reading, this appendix collects the proto-patterns at the end of each subsection. It should be noted that the assignment of a pattern a subsection is somewhat subjective, and even more so for proto-patterns.

## A Identifying themes

### DÉRIVE COMIX

**Context** you want to develop some future scenarios to explore with a group.
**If** the group has been identified BUT the members don't know each other well yet, and accordingly each has their own separate experience, and the group has no concrete shared meanings;
**Then** Gather data. For example: go for a walk [Debord 1956], or just look out the window wherever you are. (Alternatively, close your eyes and conduct a mental exploration of your selected theme: what do you see in "your mind's eye?") Document what you see. Follow up by preparing your materials to share in a succinct fashion, e.g., as photos, a screenshot, slides, sketches, a 'zine, a map, or some PostIt® notes.
**Example** Variants of this pattern can be used to explore potential futures, as we observed in the PLoP 2023 workshop "Future-Self Immersion with using A Pattern Language for Nurturing an Exciting Life", where participants "envision a future where they actively implement a particular pattern."[13]

### SHARE BACK

**Context** Members of sub-groups have shared their experiences with each other, and each sub-group has developed a Meaning Map
**If** we want to establish a meaning map for the whole group BUT unstructured interaction within the whole room is infeasible;
**Then** individual groups should present key findings to the room; or, alternatively, if the conversations are not yet at a natural stopping point ask the groups to pause their conversations and listen in briefly to the one of groups (in turn), who will continue their small-group conversation.

[13] https://www.hillside.net/plop/2023/index.php?nav=program

### CONTEXT SETTING

**Context** A workshop or other working context has been convened.
**If** the facilitators have ideas that they would like to explore with attendees BUT these ideas are not likely to be top of mind for attendees.
**Then** do some context-setting, e.g., give a short talk about why people have been invited, and describe the hoped-for outcomes.
**Example** Prior to the workshop described in Case Study 2, short videos on the themes 'public space' and 'public health' were elicited from attendees. At the start of the workshop these were projected to the assembled audience. This was accompanied by further remarks that juxtaposed what we planned to do at the workshop with previous work on "interactive documentary"[14].

### TIME TRAVELER ♛

**Context** In an interaction that can be structured with Functional Roles.
**Question** *What has happened in the past, what could happen in the future?*
**Role** To provide historical context and anticipate alternate futures.

### REFLECTOR ♔

**Context** In an interaction which can be enabled by people taking on Facilitator Roles.
**Question** *How is the scenario evolving?*
**Role** To appraise each developing scenario, provide a format for reflection (e.g., PAR, or, indeed, if time allows, the entirety of PLACARD), make a decision to continue, reset, or end.

### DO YOUR RESEARCH

**Context** Prior to beginning a formal workshop or other participatory research activity, facilitators may have time available that goes beyond what will be asked of participants in Dérive Comix.
**If** it will be possible to do participatory research in the workshop setting BUT the context outside of the workshop is potentially just as important.
**Then** start doing the research in a more centralised way before inviting direct collaboration. Findings from an earlier research stage can be summarised and presented at the Context Setting stage to give participants something to engage with.
**Example** In Case Study 3, the pre-research phase included 1-to-1 interviews with around half of the invited participants, as well as internet research to find and explore related scenarios developed by others in the sector.[15]

#### *Pilot to Anticipate*

Invoking the Going Meta pattern (see Subsection B, below), we reflect that our strategy of *piloting our workshop methods* was how we choose to anticipate the issues likely to arise in future iterations of the workshop. Perhaps the future of anticipation more widely will include the increased use of pilot schemes? In support of this possibility, Unger et al. [2019] have advocated for the enthusiastic embrace of "Experimental government".

[14] http://i-docs.org/interactive-documentary-what-does-it-mean-and-why-does-it-matter/
[15] https://royalsociety.org/topics-policy/projects/research-culture/changing-expectations/visions-of-2035/visions-of-2035-materials/

###  *Contested Space*

So-called public space doesn't always feel welcoming to all members of the public. It can be overrun with antisocial behavior. It can feel exclusionary, or uninviting. It can be the site of conflict. Although the uses of public space are complex, each space need not support every use equally.

## B Organizing structure

### MEANING MAP

**Context** We have collected images or other data describing people's worlds (see Dérive Comix).
**If** our intention is to distill well-integrated shared understandings with the group BUT, so far, everyone has been keeping most of their experience, knowledge, and perspectives to themselves;
**Then** talk together about the problems and opportunities that everyone sees in the data that has been gathered and shared, and document any connections you find. You can return to the exploration activities of Dérive Comix as needed. Maybe some of the identified themes will start to cluster together. Maybe everyone will have wildly different perspectives: that's also entirely OK. Either way, you can use these different viewpoints to bring everyone on the same page: just document them as part of the map.

### PATTERN LANGUAGE COMPONENTS

**Context** In a collaborative setting with people who are new to design patterns.
**If** attendees are being invited to create new design patterns that operationalize knowledge at the group level BUT the typical framing language of DPLs — which have 'conflict' at the core — is not comfortable for participants (e.g., because a 'problem' or 'conflict' is seen as a bad thing);
**Then** introduce and describe neutral keywords such as *HOWEVER* (which can variously be used to describe a gap, a conflict, an opportunity, or even a simple juxtaposition of facts), *BECAUSE* (to describe a set of operating causes), *THEREFORE* (to describe a rationale based on related data), and *SPECIFICALLY* (to describe next steps), to help people build patterns piece by piece.

### GOING META

**Context** In the course of working on a project together.
**If** we find a *gap* between our ideals and our methods;
**Then** Try "going meta", to explore how the project's methods can be applied to itself.
**Example** In a community that usually focuses on anticipating the future for others, try inviting members of the community to anticipate the future of the community itself.

### FACILITATOR ROLES

**Context** Developing a collection of interrelated design patterns.
**If** you are getting ideas from participants who play Functional Roles BUT the ideas aren't all connected with each other in a structured way; it's hard to know when to move on to another topic; and potentially, other obstacles arise.
**Then** introduce facilitator roles, e.g., to help structure the work, and to decide when to keep the conversation moving or draw it to a close.

### ANALYST , 

**Context** In an interaction that can be structured with Functional Roles.
**Question** *What are the moving parts?*

**Role 1** Consider the current challenge and all the components of the potential solution (actors, resources, institutions). Identify and orchestrate the dynamic network of these components.
**Role 2** Consider the other challenges specified beyond the current focus. Identify and orchestrate the integration of these components relevant to the present challenge.

### LINKER

**Context** In an interaction which can be enabled by people taking on Facilitator Roles.
**Question** *How do proposed scenarios build into patterns across layers, and how do they interact within the constellation of related concepts?*
**Role.** Data wrangling as it comes in, providing visualization of patterns and interconnections.

### STRUCTURE CONVERSATIONS

**Context** Having convened a workshop or other participatory research activity.
**If** unstructured discussions are likely to take lots of time BUT without yielding concrete benefits.
**Then** structure the discussions around shared interests to move things forward more effectively.
**Example** In the "Open Research Futures" workshop, we decided to group participants around tables according to the faculty where they were employed (or most closely aligned, in the case of university-level support staff).

### *Funding of Public Space*

Even though public space is known to increase wellness in the population, well-being priorities that would lead to increased funding for public space aren't universally adopted. In order to make the benefits of such investment clear, increase transparency around investments in public welfare, e.g., create a register of impacts of local social enterprises.

### *Destructure Patterns*

The Pattern Language Components need not be used to articulate entire patterns: a less formal discussion can surface useful meanings.

### *Adapt Layers as Needed*

Layer-based analysis facilitates effective communication among team members, enabling seamless collaboration, and aides both design and implementation. For this to work well, we need to select the right layers. In a complex change process, we might use CLA; in prototyping project, relevant layers include the *languages* and the protocols describing an implementing suitable *actions*.

### *Avoiding Mistakes*

Guidance from experienced developers can help avoid common project development pitfalls. Some useful methods include effective documentation, regular testing, and thorough project planning.

### *Scaling and Adaptability*

By considering emergining technologies and incorporating modular elements within a flexible framework, we can accomadate and adopt new advancements.

## C Making it actionable

### REINFUSE EXPERTISE

**Context** a group wants to build a Meaning Map.
**If** everyone has experience as human being (and resident, citizen, etc.) BUT they also have some experience as an expert which is harder to share with non-experts;
**Then** begin by removing expertise to get everyone on the same page, and subsequently reinfuse expertise, to enable richer and more complex thinking.
**Example** In preparing the first edition of the *Peeragogy Handbook*, we worked together informally until we had buy-in from around 25 contributors together with a high-level outline of the main themes we wanted to discuss. This outline was then filled in with individual chapters, most elaborating the specific experience of one or two co-authors.[16]

### FUNCTIONAL ROLES

**Context** When building a new set of design patterns.
**If** you have ideas about the components of a pattern BUT the pattern hasn't been fully formed yet.
**Then** introduce different perspectives to critique the pattern as it develops.

### WRINKLER ♞

**Context** In an interaction that can be structured with Functional Roles.
**Question** *What could go wrong?*
**Role.** Consider what might derail or counter the proposed solution. Each wrinkle can be assigned a level of perturbation (from low to high).

### STEPPER ♟

**Context** In an interaction that can be structured with Functional Roles.
**Question** *What should we do next?*
**Role** Consider the discussion so far, and the various possibilities for action that have arisen. Decide which actions would be most useful or informative, and devise a plan in place to carry them out.

### THE FUTURE BEGINS NOW

**Context** Having developed possible next steps in a discussion.
**If** it appears that leaving the discussion without concrete commitments means concrete actions are less likely to take place.
**Then** take preliminary actions before leaving the discussion to create a sense of commitment and follow-through.
**Example** One way to build commitment would be to ask people to develop and share a method for a small-scale experiment that they plan to carry out.

#### *Increase Participant Control*

When organising a collaborative activity, participants should not remain only an audience, or only deliver scripted lines (as was reinforced by our Anticipation 2019 experience, see Section 3). Give them increasing responsibility.

[16] https://en.wikibooks.org/wiki/Peeragogy_Handbook

### Rebalance Social Services

Welfare-related services should be supplied in balance with local needs, though they often are not. Can varied expertise be integrated in a similar way to the domain-specific skills practiced by Médicins Sans Frontièrs[17] to address complex local challenges?

### Structure Outputs

Having gathered themes from a participatory project, they may have some explicit (e.g., because of how the information was gathered, cf. Structure Conversations). Additional structure can be created, if you link intermediate artefacts into a relevant template.

### Engagement and Guidance

Guidance from pattern experts can help create a collaborative learning environment, allowing participants to gain deeper insights into relevant concepts and methodologies, and help foster to innovative and logically-coherent project approaches.

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# Patterns of Patterns: A Methodological Reflection on the Future of Design Pattern Methods

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This paper shows how we combine and adapt methods from elite training, future studies, and collaborative design, and apply them to address significant problems in social networks. We focus on three such methods: we use Project Action Reviews to implement social perception, Causal Layered Analysis to implement social cognition, and Design Pattern Languages to implement social action. We present the results of two studies: firstly, we use Causal Layered Analysis to explore the ways in which the design pattern discourse has been evolving. Secondly, to illustrate the three methods in combination, we develop a case study, showing how we applied the methods to bootstrap a distributed cross-disciplinary research seminar. Building on these analyses, we elaborate several scenarios for the future use of design patterns in large-scale distributed collaboration. Our case study suggests ways in which progress could be made towards realizing these scenarios. We conclude that the combination of methods is robust to uncertainty, insofar as they support adaptations as circumstances change, and incorporate diverse perspectives. In particular, we show how methods drawn from other domains enrich and are enriched by design patterns; we believe the analysis will be of interest to all of the communities whose methods we draw upon.

CCS Concepts: • **Social and professional topics**; • **Software and its engineering** → *Designing software*; *Open source model*; • **Applied computing** → *Operations research*; • **Computing methodologies** → Modeling and simulation;

Additional Key Words and Phrases: Design Patterns, Pattern Languages, Action Reviews, Futures Studies, Causal Layered Analysis, Emacs, Free Software, Peeragogy, Climate Change, Innovation, Anticipation

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## 1 INTRODUCTION

Our paper presents a novel synthesis of existing methods from the fields of Learning Management, Future Studies, and Design Patterns. We show how these methods complement each other, and combine holistically into a coherent framework for collaborative design.

The contribution of the paper is simultaneously theoretical and practical. As a stepping stone to the fully integrated method introduced here, we draw on an approach from Future Studies to analyze existing Design

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Pattern literature and practices, and develop several scenarios that characterize potential directions for the future development of design pattern methods. Additionally, we present a practical case study in the new combined methodology. A limitation of the work presented here is that, so far, we have only anecdotal evidence for the method's efficacy. However, we explain why we think further experiments with this approach will be useful both to design pattern thinkers, and other communities. Indeed, we contend that an approach along the lines developed here is needed for tackling the world's largest problems.

## 2 BACKGROUND

In 1999, the architect Christopher Alexander discussed his work with an audience of programmers, sharing his vision of a synthesis of architecture and computer science that could build towards the generation of a "living world" [Alexander 1999]. Considering the existential risks we face, we may take these remarks as more than just a metaphor.

*Design patterns* offer a point of entry into this vision. We begin by recalling that Alexander thought about patterns in both a fundamental and a methodological sense.

> *As an element in the world*, each pattern is a relationship between a certain context, a certain system of forces which occurs repeatedly in that context, and a certain spatial configuration which allows these forces to resolve themselves.
>
> *As an element of language*, a pattern is an instruction, which shows how this spatial configuration can be used, over and over again, to resolve the given system of forces, wherever the context makes it relevant. [Alexander 1979, p. 247] (our emphasis)

Patterns in the first sense are basically physical in nature. *Design* comes into play with the second sense. Leitner summarized how this is meant to work: "Patterns are shared as complete methodic descriptions intended for practical use by experts and non-experts" [Leitner 2015]. Already there are a number of practical texts that use patterns (in the second sense) to talk about patterns (again in the second sense): they share methods that aid in discovery, writing, workshopping, and the broader application of design patterns. By contrast we develop a more fundamental analysis, and use this to work towards a new level of practicality.

Alexander's hopeful stance currently comes up against complex global crises. How are we to understand design patterns in this context? Stephen Batchelor writes: "If I am to take this crisis with the seriousness I feel it deserves, then I need to align my thoughts and actions. I require a coherent worldview to provide a rational and ethical foundation for my behavior" [Batchelor 2020]. Pattern methods could fill part of this need. It would be overly simplistic to see in patterns only evidence of a "technical mindset" (*ibid.*), embodied only in a growing repository of technical fixes. Understood in their fundamental sense, patterns can help to articulate "forms of collective action that can respond to the climate emergency that threatens life on Earth" (*ibid.*). Now the relevant forces are no longer simply physical, but are socially distributed and culturally determined.

However, outstanding criticisms show that design pattern methods have not yet reached their envisioned potential [Dawes and Ostwald 2017]. As a step towards that realisation, we now consider two additional methods, and revisit the core features of the design pattern method. The specific selection of methods is grounded in what we have found practical in our shared projects, though for context we provide brief pointers to other methods that serve similar purposes.

### 2.1 Project Action Review

The US Army developed a methodology called the *After Action Review* [AAR], which they use in training elite soldiers [US Army 2002]. Conducting an AAR facilitates learning from past experience, to generate better future performance. The method has also been used effectively in business settings [Darling et al. 2005]. AARs can be used to assign responsibility when things go wrong, and can help people figure out how to do better next time.

As such, the AAR shares common ground with the Daily Scrum and Sprint Retrospective [Sutherland 2019] patterns from the Scrum methodology. However, it does not have the product orientation of Scrum.

In a distributed peer-to-peer collaboration, we wanted an adaptation of the AAR which would make it more open ended and horizontal in nature. We came up with the following template:

(1) Review the intention: what do we expect to learn or make together?
(2) Establish what is happening: what and how are we learning?
(3) What are some different perspectives on what's happening?
(4) What did we learn or change?
(5) What else should we change going forward?

When we fill in the template, we call it "doing a *PAR.*" As an acronym, "PAR" has stood for various things over the years—Paragogical Action Review,[1] Peeragogical Action Review,[2] Project Action Review—but we like PAR as a stand-alone term. Allusively, it brings to mind the corresponding concept of *par* in golf, and helps give us a sense of how we are doing at any given point in time.[3,4] Like the Army, we typically use PARs retrospectively ("what *did* we expect to learn or make together?"). However, PARs can also be applied to look forward (to prePARe, so to say) as a way to scaffold anticipation by "remembering the future" [Arnkil 2008]. In that case, item (5) can be expanded to include a number of alternative scenarios.

## 2.2 Causal Layered Analysis

Sohail Inayatullah developed Causal Layered Analysis (CLA) [Inayatullah 1998, 2004] as a research methodology for examining a topic of concern at four layers that he refers to as the *litany*, *system*, *worldview* and *myth*. Part of the reason to carry out such an analysis is that there are different kinds of causes, ranging from immediate events to deep-seated cultural beliefs. Inyatullah's work draws on his scholarship of P. R. Sarkar:

> For Sarkar, there have been four historical ways humans have dealt with their physical and social environment: either by being dominated by it, by dominating it through the body, dominating it through the mind, or dominating it through the environment itself. [Inayatullah 1999]

In developing a CLA, none of the four layers is privileged over the others, nor are they examined in isolation. Rather, one moves between them, examining how they relate to one another. One can then integrate these insights to form a more comprehensive basis for understanding what is happening in the present and for anticipating the future. Table 1 describes each of the four layers according to the following schema:

- **Contents**: *What is found in this layer?*
- **Analysis**: *Techniques for analysis of this layer.*
- **Literature**: *Instances of texts which are typically operative at this layer.*

Clemens [Clemens 2020] relates CLA to the work of Gregory Bateson: specifically, he finds an analogue in Bateson's notion of a "pattern of patterns" [Bateson 1979] which interrelates phenomena across disparate domains.

## 2.3 Design Pattern Languages

The two senses of 'pattern' mentioned above—'As an element in the *world*…' and 'As an element of *language*…'—are mirrored within the concept of a design pattern. Like an ellipse, the design pattern has two main foci: context and community.

[1] http://ceur-ws.org/Vol-739/paper_5.pdf (p. 5) and *Peeragogy Handbook* v3, p. 134
[2] https://github.com/Peeragogy/Peeragogy.github.io/wiki/Monthly-Wrap:-March-2020
[3] "In golf, *par* is the predetermined number of strokes that a proficient golfer should require to complete a hole, a round (the sum of the pars of the played holes), or a tournament (the sum of the pars of each round)."—Wikipedia
[4] https://web.archive.org/web/20150909224638/http://metameso.org/~joe/docs/The-Paragogical-Action-Review.pdf

| | |
|---|---|
| **Litany** | |
| **Contents:** | Observable facts, events, and quantitative trends. |
| **Analysis:** | Minimal processing of data. |
| **Literature:** | News reports, tax filings, chit-chat. |
| **System** | |
| **Contents:** | The social, economic, political, and historical forces which shape events. |
| **Analysis:** | Technical explanations and interpretation of data within a given paradigm. |
| **Literature:** | Editorials and policy institute reports. |
| **Worldview** | |
| **Contents:** | Core values and attitudes which motivate choices and actions. |
| **Analysis:** | Uncover deep assumptions and study the mental and linguistic constructs which undergird how people interact with each other and their surroundings. Compare and critique paradigms and discourses. |
| **Literature:** | Works of philosophy and critical theory. |
| **Myth** | |
| **Contents:** | The symbols and tales which give meaning to life. |
| **Analysis:** | Study symbols and narratives, and the myths and rituals within which they participate. |
| **Literature:** | Poetry, art, anthropology, Jungian analysis. |

Table 1. Overview of the layers in Causal Layered Analysis

- *Context* shapes and constrains the type of activity which is being considered, be it designing a building, writing software, or something else.
- *Community* encompasses the stakeholders—experts and non-experts alike—who are involved with or otherwise affected by a particular project.

Integral to the basic concept of a design pattern is a third feature that describes the interaction of the community and the context. The community uses the pattern to overcome some real or potential *conflict* that they experience within this context. It bears emphasis that the community is not assumed to be homogeneous, and, indeed, this may be part of how the conflict is experienced; i.e., it need not be the case that all members of the community share the same experience or view of the context, nor that they are all uniformly affected by the circumstances arising therein. The conflict is also referred to as a *problem*; its resolution is described as a *solution*. Alexander and Poyner emphasized that 'design' is not needed when the conflict can be resolved in an obvious or straightforward manner. For example, you typically would not need a design process surrounding *sitting in a chair*, because "under normal conditions each one of the tendencies which arises in this situation can take care of itself" [Alexander and Poyner 1970, p. 311].

We might say that the design pattern carries with it a fragment of irreducible complexity. This perspective may or may not be surprising. Early on, Alexander described the need for patterns when things get complex [Alexander 1964]. He specifically focuses on what could be called "horizontal" complexity, a situation where there are a lot of moving parts and relations between them. Methodologically this is elaborated with the notion of a *pattern language*. Pattern languages have a property of unfolding, from more general to more specific. However, they do not necessarily cover deeper forms of "vertical" complexity, where there are deep historical or ontogenetic causes, feedback loops, or complex conceptual issues which are not readily expressible in design-pattern-theoretic terms. Let's have another look at these issues by way of two contrasting metaphors.

The first metaphor comes from Christian Kohls, who proposed to treat each design pattern as a journey: "a path as a solution to reach a goal" [Kohls 2010]. In this metaphor, design patterns are understood to have an initial condition and an end condition, defined within some context. The context also associates a cost to traversals of paths. There are several associated problems: the elementary problem is to traverse the terrain and travel

from the start state to the end state. The next problem is to do this at low cost. The third problem is to find a reliably repeatable way to do this. A fourth problem is to describe the process in such a way that the path can be traversed by others.

The second metaphor comes from Joseph Campbell, who described an "archetypal pattern" [Shalloway and Trott 2005], one that can be found embedded in myths and stories across diverse cultures and historical periods. The "hero's journey" is also described with a path [Campbell 1949], however, in this case the path runs in a circle, and the journey focuses on the transformations of the hero who traverses it. Although an account of the journey can be shared, traversal is effectively single-use. The cost is typically "high." Nevertheless, once a myth or metaphor is established in a shared narrative, the journey can be reenacted through ritual or engaged with in other ways that solve a range of social problems [Handelman 1998]. In short, the difference between these two traversal stories suggests that the process of finding "the path that is capable of leading to a good structure" [Alexander 1999] may contain irreducible complexity—even when sharing the information about the path is relatively simple.

## 3 METHODOLOGY

Each of the three methods described above has distinct use-cases when considered in isolation. Viewed as alternatives, they would have various tradeoffs between them, concerning their ease of use, their ability to generate solutions, the breadth of their applicability, and so on. In this section we put them together as one holistic pattern of patterns. We will use them to scaffold social perception, cognition, and action (Table 2).

| *Key verbs:* | perceive | think | act |
|---|---|---|---|
| *Scientists refer to:* | "sensory" | "cognitive" | "motor" systems |
| *Our implementation:* | **P**roject<br>**A**ction<br>**R**eview | **C**ausal<br>**L**ayered<br>**A**nalysis | **D**esign<br>**P**attern<br>**L**anguages |

Table 2. Three acronyms used in this paper: PAR, CLA, and DPL

We adopt this tripartite division from classical psychology [Hilgard 1980]; it continues to be relevant in contemporary neuroscience [Friston 2013; Teufel and Fletcher 2020]. Indeed, it is not only as a division, but precisely in characteristic combination that these three factors become important. This suggests a third understanding of 'pattern' as "a dynamic [...] ongoing awareness created and upheld by a mass of ongoing physiological activity" [Iran-Nejad and Winsler 2000]. To study such factors in integration "requires understanding the conditions and laws of construction in mental life" (*ibid.*). So it is, as well, for collective knowledge production. To recapitulate and reframe the methods in these terms:

*Project Action Review (PAR) is the sensory element: systematically gathering and verifying observational data.* The Project Action Review is structured around five questions which members of a community discuss and answer together. This practice generates a record of an event, as seen through the eyes of the participants. In the moment, the PAR allows us to draw out views which might have gone unstated otherwise. Over time, projects which use PARs improve their chances of staying grounded in reality as circumstances evolve. We can additionally use PARs to help check how effectively we are using other methods.

*Causal Layered Analysis (CLA) is the cognitive element: giving organization and depth to the enterprise.* The goal of this methodology is to achieve a deep and inclusive understanding by integrating empiricist, interpretative,

critical, and actionable knowledge surrounding a topic of concern. Without an integrated understanding, a group runs the risk of getting lost in a muddle of details. CLA can pull information logged in PARs together into a coherent body of self-knowledge. It can also help to surface concerns that might remain implicit in pattern language: for example, CLA could help us understand why we had prized a technological solution to what, upon consideration, turned out to be a fundamentally social issue (or vice-versa).

*Design Pattern Language (DPL) is the motor element: orchestrating and scaffolding action.* Having carefully analyzed the situation and identified possible solution pathways, we must plot a course of action that accounts for the complexities of the situation. Individual design patterns present solutions to recurring problems: they can be combined with other patterns and adapted to different situations. A DPL is a common language which a community can use to discuss matters of design; it serves as a repository of shared knowledge. The flexibility of DPLs allows the structure to be customized to our particular circumstances as they evolve.

## PLACARD: A Synthesis of PAR, CLA, and DPL

We are now in a position to explain how PAR, CLA, and DPL combine into one holistic pattern, in Leitner's sense of a complete methodic description [Leitner 2015]. We will write this down using the classic DPL format: describing the associated *context*, the *problem* denoting a conflict, together with a *solution*. As it happens, the three acronyms introduced earlier can be combined and remixed to provide a title for this pattern.

$$\text{PAR} + \text{CLA} + \text{DPL} = \text{PLACARD}$$

This accurately suggests that the methods need not be run in a fixed order, but are interwoven together.

- **Context**: In the course of working on a project: *we use the PAR to get a sense of our working context.*
- **Problem**: Although we may encounter many difficulties in this context, our effort to understand them faces a central **challenge**, namely the fact that the problems span different layers and scales of complexity, so it can be hard to understand where the difficulties actually come from: accordingly, *we use the CLA to understand and frame the problems and their interconnections.*
- **Solution**: Once we have grasped the problem, we need to elaborate an actionable solution that remains adaptable to ongoing changes in the context: *we use DPL to elaborate the solution (returning to PAR and CLA as needed).*

Figure 1 provides a mnemonic.[5] The main thing to notice is that using the three methods together can help make the design pattern method practicable. We can use the PAR to move from a context to a "Context", established and written down. We can use CLA to move from a situation of concern to a situation in which the core "Problem" or "Problems" can be thought about.

The fact that DPL shows up inside of PLACARD may be somewhat concerning. The reader may be wondering: "I think I can see how the methods that have been discussed could help in understanding the **context** and the **problem**, but is there anything here that can actually help formulate **solutions**?" This concern calls for some further brief remarks before turning to a case study of the PLACARD pattern in use.

A solution process can often be decomposed into interrelated subtasks [Alexander 1964; Polya 1945, 1954]. A standard problem-solving methodology is to understand the (sub-)contexts and (sub-)problems in detail—along with their relationships to other parts of the developing decomposition—and on this basis make predictions about the way an intervention could change the overall system. Accordingly, problem solving often goes back and forth between "two different forms of information processing: bottom-up and top-down"[6] [Teufel and Fletcher

[5]For French speakers, *placard* means 'cupboard', and there is an idiom, *placardisé*, which refers to an employee whose tasks all have been reassigned to others; the import is similar to the English idiom 'put to pasture'. While it is not the case that PLACARD reassigns all DPL functions to other methods, the French idiom is potentially suggestive as another mnemonic.

[6](Punctuation modified.)




Fig. 1. Mnemonic illustration of the PLACARD pattern

2020]. CLA can help with thinking in both directions. Nominally, CLA is an *analytic* tool that decomposes a problematic situation into *layers*, and *causes* operating at and between these layers: in this sense it functions top-down. However, CLA also plays a bottom-up synthesis role. Whereas we could compare the PAR to a tangent vector or derivative that gives a momentary reading of how things are going at a given point in time, CLA can be used to integrate these observations into a plan.

## 4 RESULTS

We report on two studies traversing different scales. Firstly, we applied CLA to the design pattern literature and practices, with the purpose of scaffolding an examination of the future of the design pattern theory. Secondly, we made practical use of CLA alongside the PAR and DPL methods (i.e., PLACARD as a whole) within a case study of a distributed research seminar. Details of these analyses are presented in Appendix A and Appendix B, respectively, henceforth referred to here as the **Analysis** and **Case Study**.

### 4.1 Analysis: Design Pattern Language literature and practices

In this study we examined DPL literature and practices in the form of books, articles and conferences. In order to counterbalance the coverage, we included dissenting and critical voices. Here we did not have PARs at our disposal, so the sensory element was provided by the views expressed by previous authors.

In précis, we were interested in the following topics:

- Queries raised by Alexander and his collaborator Bryant, along with a systematic analysis of criticisms of pattern methods collected by Dawes and Ostwald.
- Issues related to how people share and discuss patterns, as well as the changing way in which these discussions have been framed at PLoP.
- The worldview linked with patterns through the lens of mob software and its critiques.
- Symbols and philosophical traditions that enrich our understanding of the context in which Alexander developed his methods.

Taking a deep dive into DPL via CLA allowed us to gain perspectives on how design patterns work. In particular, we have illustrated the complexity that underpins the model. Alexander expands on his metaphysical

considerations in his multi-volume work, *The Nature of Order* (*TNO*). Our Analysis shows how some of these more ephemeral-seeming factors are ramified across various layers of the pattern theory.

In particular, working from the myth layer towards the more surface layers: we show that the notion of *wholeness* that Alexander deploys is complex, and that the corresponding theory of emergent order based on this concept is similarly complex and linked to "a tension between independent and conforming tendencies" [VanDrunen 2010]. This tension has—only partially—resolved into a dichotomous relationship between *users* and *designers* of patterns and pattern-linked artifacts. There are however many remaining points of friction, as users of pattern methods run into difficulties [Dawes and Ostwald 2017], and designers are not entirely clear on how to improve the situation.

### 4.2 Case Study: Emacs Research Group

The Emacs Research Group (ERG) is a transdisciplinary seminar organized around the theme of research in, on, with, and about the Emacs computer program.[7] Although Emacs is best known as a text editor, its extensibility and self-documenting nature make it a more general platform for dealing with symbolic content, and a vibrant site for research into writing and programming. ERG aims to explore topics such as the following:

- How is Emacs *used* to help conduct research in various disciplines?
- What is the *user experience* with Emacs and what is the user community?
- How could the Emacs system *interoperate* with other computer programs?
- How might the *communities* affiliated with Emacs interact with other communities of software users and developers?
- What new *functionalities* would broaden the applicability of Emacs?

ERG meets approximately weekly, sometimes inviting guest speakers. After each meeting, the participants summarize their experience in a PAR. This serves to surface matters of concern and highlights in each session of the seminar. Every six meetings, the techniques of CLA were used to organize and condense the PARs into a coherent statement of purpose. Finally, Peeragogy design patterns are used to formulate a plan of action informed by this analysis. Thus, all three methods are combined in line with the PLACARD pattern introduced above.

In this study, we were most interested in understanding and exploring the PLACARD method and its potential efficacy; e.g., could we use the PLACARD-associated methods to develop actionable and deliverable designs for new software? After a year of working with the methods together, we presented them at EmacsConf 2021, advocating that other Emacs enthusiasts adapt them for use in creating their own small research groups.[8]

#### Summary of Findings

We used Causal Layered Analysis to describe the evolution of Design Pattern Language methods in response to criticism, innovation, technical developments, and long-term cultural change. We additionally surface some of the forces and tensions in the discourse. We used Design Pattern Language methods, together with Project Action Reviews and Causal Layered Analysis, to organize a research seminar. This case study shows how the methods can be fruitfully combined, and suggests how others might take up the combined methods (e.g., by substituting 'Patterns' for 'Emacs' as the focus in a future research seminar).

## 5 DISCUSSION

Informed by the two studies described above, we would like to reflect on why putting the CLA and DPL methods together can make a big difference in practical terms. To do this, we begin by examining a specific problem domain to which CLA and DPL have been applied separately.

[7] https://www.gnu.org/software/emacs/
[8] https://emacsconf.org/2021/talks/erg/

Anthropogenic climate change is a situation of major global concern in the early 21st Century. It comes as no surprise that it has been examined separately by proponents of both CLA and DPL. We use this recent history to frame future work building on the case study and analysis developed above.

In an overview of their work on the Cooling the Commons pattern language, Cameron Tonkinwise and Abby Mellick Lopes write:

> A design pattern is first an observation: "People in that kind of designed situation tend to do this sort of thing". It is then possible to design an intervention that redirects those tendencies. If that intervention succeeds, it can become a recommended pattern to help other designers: "If you encounter this kind of situation, try to make these kinds of interventions" [Tonkinwise and Lopes 2021].

They amplify the 'ethical' aspect of their thinking:

> … the patterns we are talking about, context-specific interactions between people and things, are more like habits. They are tendencies that lead to repeated actions.

The 41 patterns they have developed include examples like The Night-Time Commons,[9] which:

> … might shift daytime activities into cooler night times. Some places already have these patterns: night markets and night-time use of outdoor spaces. If locally adapted versions of these patterns encourage people to adopt new habits, other patterns will be needed. These will include, for example, ways to remind those cooling off outdoors in the evening that others might be trying to sleep with their naturally ventilating windows open. Such interlinked patterns point to the way pattern thinking moves from the big scale to the small.

Reading this, we were concerned that, while the Cooling the Commons patterns do acknowledge *horizontal* complexity—namely, through interlinked patterns—the process does not deal with the *vertical* complexity coming from the fact that diurnal rhythms are deeply embedded in biology and culture. People have cultural beliefs about the activities that are appropriate for different times of day. Public and domestic rituals are organized about the daily cycle. Times of day have symbolic associations. As far as we could tell, these authors focused on more or less technical issues at the systems level, and did not acknowledge these issues at the worldview and myth levels. A more comprehensive approach might, for instance, re-examine rituals to see which of them relate to the phenomenon of sunrise versus the act of getting up and starting the day, and then figuring out how to adapt these rituals to a new schedule. A suitable research strategy might be to study how practices changed in the past, as with the introduction of industrialization and its clockwork regimentation of the day.

Meanwhile, Heinonen and coauthors [Heinonen et al. 2017] describe a CLA game that explored four different scenarios in small groups. The four scenarios were "Radical Startups", "Value-Driven Techemoths", "Green DIY Engineers" and "New Consciousness". As groups worked through the CLA for each scenario, they developed a range of new ideas. We wondered, how might these CLA-linked reflections collate against the Cooling the Commons patterns? Might players of the CLA game have spotted ways in which the patterns would conflict with deeper values—or ways in which they might be exploited to cause chaos [Friction 2016]?

Broadening our exploration of how design patterns relate to futures studies, we note that Schwartz [Schwartz 1996] (Appendix, pp. 241-248, *viz.*, his "Steps to Developing Scenarios") described a process that follows an outline that is strikingly similar to a design pattern template. Both Alexander and Schwartz advocate the identification of driving forces in a context. However, unlike Alexander, Schwartz does not intend to resolve conflicts between the forces within a harmonizing design. On the contrary, the aim in the scenario development method is to understand how these forces might evolve and lead to the further diversification of scenarios. To simplify the process, some scenario planners reduce the number of forces to the two most important. For example, the scenarios of Heinonen

[9] https://www.coolingthecommons.com/pattern%20deck/

et al. are organised along two axes: one spanning different degrees of integration between peers, the other, different levels of ecological awareness.

With these reflections in mind, we came up with four scenarios for the future development of design patterns. The four scenarios offered here are inspired by the four CLA layers: litany, system, worldview, and myth.

### 5.1 PLACARD becomes transferable and computational

With further work, PLACARD could become a more refined but nevertheless easy-to-use instrument for gathering, organising, and sharing data on problem solving in social networks. Building on this collected data, we could take further steps to develop computational models of future imaginaries. Patterns have previously been discussed in explicitly computational terms—however, that direction of work so far remains mostly at the level of a proposal [Alexander 1999; Moran 1971], with only limited discipline-specific uptake (e.g., [Jacobus 2009], [Oxman 1994], [Taibi and Ngo 2003]). Could this change, to generalize the kinds of patterns that can be computed with, and make them interoperable? Polya had already been writing about *patterns of plausible inference* the year that Alexander started his undergraduate degree in Cambridge [Polya 1954]; four years later Polya's student Allen Newell was beginning to think about how to model the inference process computationally [Newell 1983; Simon and Newell 1958]. In the domain of economics, Ostrom-style institutions are analogous to design patterns ([Ostrom 2009], p. 11). Recent work looks at how descriptions of such institutions can be extracted from text [Rice et al. 2021]. Could this line of thinking be extended, so that other similar kinds of patterns could be recognized where they appear? Could the extracted descriptions be used directly in computations? One fruitful strategy might be to think of design patterns as *conceptual blends* [Corneli et al. 2018], which can be given a computational interpretation [Schorlemmer and Plaza 2021]. For example, the Community Library pattern from the Cooling the Commons pattern language blends a learning space with a cool refuge. Could such complexities be reasoned about computationally?

### 5.2 Pattern language authoring communities move to free/libre/open source licensing

In the field of policy, *resilience* describes a society's ability to recover after a shock; whereas *adaptive capacity* describes its ability to move to a new state [Magnan 2010; Thonicke et al. 2020]. This ability is, in turn, linked with the health and adaptivity of the society's institutions [Fidelman et al. 2017]. Free/Libre/Open Source licensing is one possible institutional innovation in the way design patterns are used. Widespread adoption of new licensing arrangements could support social learning, which could, in turn, boost adaptive capacity [Thi Hong Phuong et al. 2017]. As an example of work heading in this direction, Mehaffy and coauthors collaborated with Ward Cunningham to make their book *A New Pattern Language for Growing Regions* [Mehaffy et al. 2020] into a wiki, npl.wiki, which is licensed under CC BY-SA 4.0. Will other pattern developers follow suit and move to open licensing—and suitable infrastructures for working with open contents? We can also ask: what about other kinds and qualities of openness? A 'copyleft' license is not a panacea for all ills [Krowne and Puzio 2006], and would not on its own make the pattern theory and methods fully open in all the ways that matter. Nevertheless, grappling with the challenges around licensing and related considerations could serve as a rallying point for the pattern community.

### 5.3 Patterns empower individuals and communities

As we've seen in our work with Emacs and Peeragogy (and previously with the online community PlanetMath [Corneli 2014; Krowne 2003]) projects need more than just access to source code in order to thrive. We see a link to the topic of reproducible research. Above and beyond the immediate technical considerations [Sandve et al. 2013], the process of doing science is "reproducible" if the methods are teachable to others. The Literate Programming

paradigm can help with this.[10] At the same time, collaboration across different skill sets is challenging; large scale problems like adapting to climate change seem to require such collaboration, and almost certainly won't be solved if we carry on doing business as usual. In the Minnesota 2050 project, participants were selected from a variety of professions and leadership roles to produce scenarios for energy and land use, and combined modeling with scenario planning [Olabisi et al. 2010]. Actually solving large-scale problems in interdisciplinary teams will require new thinking and additional tools: to bridge between the viewpoints of, e.g., professional futurists, programmers, data scientists, local farmers—and to draw on the insights of citizen scientists [Wildschut 2017]. CLA and PLACARD can help; additional patterns of patterns will be needed to work fluently across domains, levels, and spheres of endeavour.

### 5.4 Patterns facilitate economic empowerment

Access and meaningful participation are serious matters of concern in our current technological culture [Unger 2019]. Patterns could become the basis of widespread capabilities and substantial opportunities [Robeyns and Byskov 2021], shared not only by an elite group of hackers or a few highly-paid rockstars, but by everyone. Patterns have been used to describe soft skills that are useful for aspiring programming professionals [Hoover and Oshineye 2009]: however this falls far short of reforming the tech sector or our broader relationship to technology. As a related example from the world of literature, Herman Hesse's novel *The Glass Bead Game* imagines a society in which the elite community of scholars studying abstract patterns form a strong hierarchy, and remain mostly out of touch with the practical realities experienced by outsiders. The story serves as a warning: we must proceed with caution when we seek to bridge practice and theory. When reflecting on futures-oriented discourses, Slaughter described a spectrum: "participatory and open at one pole and closed (or professionalised) at the other" [Slaughter 1989]. Are pattern authors prepared to work with widespread stakeholders to study the forces that shape the economy and society?

## 6 CONCLUSION

In 1999, the architect Christopher Alexander discussed the future of design patterns with an audience of programmers [Alexander 1999]. We revisited this topic from a methodological perspective. Building on the analysis and case study we developed, we discussed several scenarios for the future of design pattern methods. Our vision for change is that these four scenarios will be given serious thought by other members of the patterns community. Our Case Study suggests concrete ways in which progress might be made towards realizing these scenarios, or, indeed, if warranted, towards rejecting them and developing another vision.

Progress will become measurable through markers of debate and dialogue between the different communities whose work we have drawn upon, and potentially, through trial-and-error uptake or adaptation of the methods we've described. We are certainly not the only people to think about systems and futures: what is distinctive about this paper is that we've connected these domains with design pattern terminology and methods. Some potential implications of that connection are embedded in the scenarios outlined above.

In the present exploration, we began by thinking about patterns from a fundamental perspective: *patterns as elements in the world*. Some patterns repeat in space, some in time, some in both space and time; think of a tiling, a beat, a wave. However, patterns cannot repeat exactly or forever: their elements are subject to spatial or temporal displacement, and other forms of variation. We need suitable abilities—and methods—to perceive and work with patterns. The methods we used in this paper were the Project Action Review (PAR), Causal Layered Analysis (CLA), and Design Pattern Languages (DPL)—though other methods that fulfill the same basic purpose could be used without significantly changing the overall import of what we say here.

- By using the PAR (or another sensory method), we are able to identify recurring themes.

[10] For notes on doing reproducible research with Emacs, see https://emacsnyc.org/2014/11/03/org-mode-for-reproducible-research.html

- Then, by using the CLA (or another cognitive method), we are able to organize these repeating themes in a structure that exposes the underlying trends, causes, and potential terminating states.
- With DPL (or another motor method) we can make what we have learned actionable.

A contribution of this paper was to link the sensory/cognitive/motor theory with design patterns, which we illustrated by combining PAR, CLA, and DPL into the PLACARD pattern.

A key experiential finding that emerged from our joint work on this paper is that we need to take health, well-being, and time for reflection into account, even more than we worry about accomplishments or deliverables, or we are likely to run out of energy before we get very far. Outcome-based planning needs to be balanced against more intention-based planning that takes human factors into account. In future work, we hope to explore the rich factors that contribute to health and adaptivity in groups and networks. The methodological contribution of this paper can help to scaffold that process. Alongside the PLACARD method, we have started to develop a set of design requirements for software that can help people apply the method in their working contexts.

The limitations of the research presented will be clear. Our case study examined a small and well-integrated population, which is moreover, a subset of the authors of the present paper. While the case study provides a proof of concept for the applicability of the methods, in other settings, some or all of the methods might not be needed, or accepted without significant alteration. Despite the limitations of this study when it comes to external validity, evidence to support the use of the integrated PLACARD pattern may accumulate rapidly, if the method proves useful for other transdisciplinary collaborations. By using and negotiating around the same DPL, disparate groups may get a useful external source of validation. We hypothesise that PLACARD could quickly make the work of Learning Management, Future Studies, and Design Pattern practitioners more robust.

We can imagine the methods we've discussed being readily applied, initially, in workshops or other small groups. For example, if we were to gather a group of citizens, experts, and other stakeholders to talk together about the city, they could use the methods we've described to surface issues, rehash concerns, and sensemake together. "Anticipatory Social Science" is a broader term for this kind of work. Moreover, working with methods that distribute perception, cognition, and action, we may become more comfortable with uncertainty, and better able to support innovation. Especially when grappling with massive problems like climate change, we will need to develop a transformative set of tools and methods [Miller 2018]; and beyond this, a coherent worldview, and perhaps even new myths. Citizen science has a potentially important role to play here [Wildschut 2017], alongside other forms of participatory culture. The methods we described may help to support widespread engagement, and coordinate activities across scales: from individuals, to communities and social movements. As Nietzsche said, "the more affects we are able to put into words about a thing, the more eyes, different eyes, we can use to observe one thing, the more complete will our 'concept' of this thing, our 'objectivity,' be" [Nietzsche 2017] (p. 128).

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The image appearing in Figure 1 was made available under the terms of CC Zero by User:Lapsed Pacifist at `https://commons.wikimedia.org/wiki/File:Shell_to_Sea_placard.jpg`.

## SUPPLEMENTARY MATERIAL

In order to support a concise treatment of our core findings, we supply details of our Analysis and Case Study in Appendix A and Appendix B.

## A SUPPLEMENT: ANALYSIS: CLA APPLIED TO DESIGN PATTERN LANGUAGE LITERATURE AND PRACTICES

In developing this analysis we are aided by several additional methods from the Poststructural Futures Toolkit [Inayatullah 1998]. We will refer to the individual methods from this toolkit with their names rendered in all-caps.

### A.1 Litany: Understanding data, headlines, empirical world (short term change)

The first layer in CLA is the **litany layer**: it describes the problems that people are well familiar with. In the case of the design patterns discourse, this level includes—in particular—the familiar kinds of conflict-based problems that are described in patterns and discussed at PLoP, along with higher-order problems of application, and debates about these (e.g., ranging from Christopher Alexander's "Entryway Transition" pattern to his remarks about how people who attempted to apply his methods ended up placing "alcoves everywhere", etc.). This layer is sometimes also referred to as the **problem level**: in the patterns discourse, problems abound. Indeed, one of the core attributes of the pattern community is that it is not only comfortable with problems but that it actively seeks them out with a 'problematizing' discourse.

Not all of the well-known and discussed problems have been solved. For example, 'Alexander's Problem', as described by his collaborator Greg Bryant, is that:

> … despite all of the tools he created, his penetrating research, his many well-wrought projects, and his excellent writing, he did not manage to grant, to his readers, the core sensibility that drove the work. He also did not organize the continuance of the research program that revolves around this sensibility. [Bryant 2015]

Attempts to work out a practical solution to this problem are developing.[11] Coming at the same basic issue from a more visionary standpoint, Alexander framed this query for the programmers who were using pattern methods at the turn of the millennium:

> What is the Chartres of programming? What task is at a high enough level to inspire people writing programs, to reach for the stars? [Alexander 1999]

More recently, Dawes and Ostwald [Dawes and Ostwald 2017] develop an elegant taxonomy of criticisms of the pattern method. In outline, their taxonomy covers criticisms at the following three layers:

**Conceptualization** Ontology, Epistemology
*(e.g., "Rejecting pluralistic values confuses subjective and objective phenomena")*
**Development and documentation** Reasoning, Testing, Scholarship
*(e.g., "The definitions of 'patterns' and 'forces' are inexplicit")*
**Implementation and outcomes** Controlling, Flawed, Unsuccessful
*(e.g., "Patterns disallow radical solutions")*

By showing how the criticisms relate to one another, Dawes and Ostwald begin to develop a GENEALOGY at the level of critical perspectives. The critiques they examine show that there is not just one pattern discourse, but many. In a parallel work the same authors analyze the structure of Alexander's classic text, *A Pattern Language*

[11] https://www.buildingbeauty.org/ and https://www.buildingbeauty.org/beautiful-software

(*APL*) and develop three alternative perspectives on *APL*'s contents, which they refer to as the **generalized**, **creator**, and **user** perspectives [Dawes and Ostwald 2018]. These perspectives amount to different techniques for REORDERING KNOWLEDGE. We will elaborate at the next level.

## A.2 System: Systemic approaches and solutions (social system)

The **system** layer is typically understood in terms of the **social phenomena** that cause the problems at the litany layer to emerge (along with their familiar solutions). In the original setting in which patterns developed, this layer would have included causes such as more people living in cities, combined with the possibility of developing a more community-driven approach to design using contemporary technologies. In short, at this level, we examine where the familiar problems come from.

Dawes and Ostwald's [Dawes and Ostwald 2018] central finding is that many patterns in which Alexander had medium or low confidence in fact occupy a relatively central position in *APL*'s graph:

> … the patterns which are most likely to be encountered by designers – are most easily accessed, or provide greatest access to other patterns – might be those which Alexander acknowledged were incapable of providing fundamental solutions to the problems they addressed.

This means that novice users could be expected to encounter problems in application of *APL*'s patterns: "despite its often authoritative and dogmatic tone, Alexander's text was framed as a work in progress, rather than a definitive design guide" (p. 22). Dawes and Ostwald suggest that their analysis could point to "prime opportunities to continue the development of *A Pattern Language*" (p. 21).

Here, a range of media issues begin to crop up. Broadly put: there have been some attempts at creating systematic archives of patterns [Inventado and Scupelli 2017; Köppe et al. 2016], but these efforts haven't always garnered significant buy-in. Importantly, the first-ever Wiki was developed in connection with a platform for developing, sharing, and revising pattern languages [Cunningham and Mehaffy 2013].[12,13] However, there was a distinction between the discussions and the finished patterns. In the 2013 retrospective, Ward Cunningham and coauthor Michael Mehaffy write:

> The original wiki technology functioned in a direct open-source mode, which allowed individuals to contribute small pieces to incrementally improve the whole. (*ibid.*)

This is true if by "open source" we understand what you see when you click Edit—but the statement could be misleading relative to contemporary usage, which is often linked with the Open Source Initiative's definition, which centers on the premise that "Open source doesn't just mean access to the source code."[14] On the `c2` wiki, licensing was restrictive. Discussions were to take place in "letters and replies" rather than revision or annotation of the published patterns; rights associated with the finished patterns were closely guarded.[15,16]

Although Wiki technology could in principle have been a site for ongoing DECONSTRUCTION of patterns, this hadn't happened on `c2`. This is itself interesting and worth deconstructing a bit. Notably, there were only *four* published "letters and replies".[17] Unfortunately, we could not find a public archive of the design patterns mailing list where further discussions took place. This suggests certain factors of contingency in the development of the discourse. Over the years, some of these concerns have been addressed—e.g., through the introduction of Federated Wikis and new licensing models—and other issues and concerns came to the fore.

[12] http://wiki.c2.com/?PeopleProjectsAndPatterns
[13] http://c2.com/ppr/
[14] https://opensource.org/osd
[15] http://c2.com/ppr/titles.html
[16] http://c2.com/ppr/about/copyright.html
[17] http://c2.com/ppr/letters/index.html

Dawes and Ostwald's [Dawes and Ostwald 2018] remarks on multiple perspectives on pattern languages resonate Jenifer Tidwell's charges against the Gang of Four:

> … the reality of a software artifact that the developer sees is not the only one that's important. What about the user's reality? Why has that been ignored in all the software patterns work that's been done? Isn't the user's experience the ultimate reason for designing a building or a piece of software? If that's not taken into account, how can we say our building – or our software – is "good"? — "The Gang of Four Are Guilty"[18]

Notice that the *user* of the designed artifact has entered the story as a different figure from the user of the pattern language, whom we met above. Tidwell's critique suggests at least a couple ALTERNATIVE PASTS AND FUTURES: e.g., what if the end-user had been placed at the center the whole time? Alternatively, what if the primary focus of patterns was to facilitate interaction between different stakeholders? The fact that Tidwell's book [Tidwell 2010] and an essay by Jans Borchers [Borchers 2008] which drew inspiration from her critique both have over 1000 citations on Google Scholar shows that Tidwell's perspective has been impactful. To get a sense of how the pattern community may have been informed by this critique—alongside other related trends and concerns—we can look at how the Writers Workshops at PLoP have evolved over time. In Table 3, a selection of titles of workshop sessions show how the focus of PLoP evolved from primarily 'programming' oriented to a much broader contextual view over time. Indeed, by 2019, the focus is almost exclusively 'contextual'. The way the themes under discussion have evolved brings to mind the layers of CLA.

Table 3. Evolution of PLoP Writers Workshop topics in selected years: CLA in the wild?

| **1997** | **2011** | **2015** | **2019** |
|---|---|---|---|
| Architecture | Architecture | Pattern Writing | Group Architecture |
| Roles and Analysis | Design | Software Architecture & Process | Culture |
| People and Process | Information | Cloud & Security | Meta |
| Domain Specific Techniques | People | Innovation & Analysis | Education |
| OO Techniques | Pedagogy | People & Education | |
| Non-OO Techniques | | | |

## A.3 Worldview: ways of knowing and alternative discourse

The next layer comprises **worldviews** (e.g., Alexander's view that "There is a central quality which is the root criterion of life and spirit in a man, a town, a building, or a wilderness").

The situation with licensing on `c2` is particularly interesting in light of Alexander's perspective that *APL* was a "living language". In principle, Wiki technology might have presented the opportunity to realize this vision fully for the first time, in a virtual setting. Wiki technology did become widely influential when it was combined with a free content license on Wikipedia (originally GNU FDL, later CC-By-SA).

Fast-forwarding to the present day, Christopher Alexander's website `patternlanguage.com` writes about The Struggle for People to be Free—but it is not referencing freedom in the GNU sense.

In 1979 he was concerned: "Instead of being widely shared, the pattern languages which determine how a town gets made becomes specialized and private." In 2021, *APL* itself is only legally available for subscribers or for people who purchase a paper copy of the book. (Or through a library!) Of course, like many famous texts it can also be obtained extra-legally for download as a PDF: but that format does not afford downstream users the opportunity to collaborate on the text's further development.

[18] http://www.mit.edu/~jtidwell/gof_are_guilty.html

Gabriel and Goldman talk about sharing and gift culture in their essay Mob Software: The Erotic Life of Code.[19] This reference suggests another reason why sharing knowledge in non-editable formats can be problematic. Their primary source on gift culture is Hyde [Hyde 2019], who talks about what happens when exchange items are taken out of the gift exchange culture and put in a museum:

> A commodity is truly "used up" when it is sold because nothing about the exchange assures its return. The visiting sea captain may pay handsomely for a Kula necklace, but because the sale removes it from the circle, it wastes it, no matter the price. Gifts that remain gifts can support an affluence of satisfaction, even without numerical abundance. (*ibid.*, Chapter 1, p. 29)

Gabriel and Goldman reference the open source community—but not the free software community, so we will follow their usage here—as the origin of Mob Software.

> Because the open source proposition asked the crucial first question, I include it in what I am calling "mob software," but mob software goes way beyond what open source is up to today. [Gabriel and Goldman 2000]

That question is: "What if what once was scarce is now abundant?" It is well known that the PLoP conference series builds on this idea: it includes shepherding and workshops [Gabriel 2002] as well as games, informal gifts, and other measures that aim to create a sense of psychological safety: all features that make PLoP a space where 'failure' is OK and even celebrated, as per Mob Software. The essay develops its own criticisms of open source, e.g., "the open-source community is extremely conservative" and forking happens rarely. (Five years later, with the creation of Git, a certain form of forking became more typical.) Resonating with Tidwell's critique from above:

> One difference between open source and mob software is that open source topoi are technological while mob software topoi are people centered.

On a technical basis, Gabriel's vision sounds a lot like today's world of *microservices*. While his vision hasn't fully come to pass—for example there are still many services with proprietary source code—nowadays many big companies are also big proponents of open source. Here we can notice that Gabriel was employing a technique of imagining ALTERNATIVE PASTS AND FUTURES, e.g., he imagined a future in which:

> Mentoring circles and other forms of workshop are the mainstay of software development education. There are hundreds of millions of programmers.

We would like to dig somewhat deeper into the foundations of the worldview that Gabriel puts forth in this essay. Usefully, an article by VanDrunen "traces the source of Gabriel's ideas by examining the authorities he cites and how he uses them and evaluates their validity on their own terms" [VanDrunen 2010]. VanDrunen's critique functions as a (detailed) DECONSTRUCTION of the thinking behind Gabriel's essay. Some key excerpts appear in Table 4. It is worth noting that this is by no means a complete critique. As an an example of one direction that we will not have time and space to develop here, some applications of the concept of 'gift culture' have been criticized as hegemonic in nature [Mallard 2019]: should we expect pattern-theoretic, mob, or free/libre/open source software culture to be immune from such concerns? VanDrunen's critique is useful for our purposes not because they provide the last word, but because this criticism points to the importance of considering the deeper layers in developing a concept or approach. There may also be conflicts at these deeper layers.

It is also worth noting that mob software is but one of many diverse visions of the future of programming [Nobel and Biddle 2002]. An embrace of diverse perspectives seems to be a fundamental part of the associated worldview. After all, the primary theoretical model of a computer is termed "universal". Perhaps there is a bit of a paradox or double bind here, insofar as we embrace diverse perspectives just as long as they are compatible with

[19]Notably, this essay was presented as a keynote talk at the same programming conference where Alexander had delivered his keynote, [Alexander 1999], four years previously.

Table 4. Key observations from VanDrunen's critique of Gabriel's "Mob Software" essay

| |
|---|
| "Kauffman's work is about a rediscovery of the sacred, and it amounts to a proposal of the laws of self-organization as a new deity" |
| "One thing we find in common with Lewis Thomas's ants, Kauffman's autocatalytic sets of proteins, and the agents inhabiting Sugarscape is that they all lack intelligence." |
| "In other words, the rules given by Gabriel describe only the conforming aspect of group behavior. In reality, there is a tension between independent and conforming tendencies, and the flock patterns emerge from the interaction between the two." |
| "His examples of 'mob activity'… the making of the Oxford English Dictionary, cathedral-building, and open source software discussed later—all had oversight, master-planning of some sort." |
| "There are several distinct senses of 'gift' that lie behind these ideas, but common to each of them is the notation that a gift is a thing we do not get by our own efforts." [Quoting Hyde [Hyde 2019].] |
| "Certainly proprietary code is shared property among those working in a corporate development team, but it is not common to the larger community of software developers and users." |
| "A computer program is not like a poem or a dance in this way; if the programmer is not able to produce something parsable in the programming language or cannot fit the instructions together in a logical way, the program simply will not work." |
| "Gabriel's own experience may color his perception. He founded a software company that produced programs for Lisp development and which went bankrupt after 10 years." |
| "Moreover, if Gabriel means to suggest that these programming languages or models could have made programming more accessible to the masses lacking technical skill, it is quite a dubious claim" |

our core tenets. For at least some pattern authors, these include "their love of programs and programming" (*ibid.*). (On this last point, both VanDrunen and Gabriel seem to agree.)

### A.4 Myths: metaphors and narratives (longer term change)

Lastly, there are **myths or metaphors** (e.g., Alexander's idea that the architect's work is done 'for the glory of God' (see Galle [Galle 2020]) or his conception that 'primitive' dwellings contain more life). To emphasize, CLA does not dismiss myths in the slightest: on the contrary, they are what drive the other layers. Another term that is used to characterize this layer is **narratives**. VanDrunen surfaced various concepts in Gabriel's essay that would be at home at this level, for example, the concept of duende that Gabriel takes over from Garcia Lorca originally derives from *dueño de casa*, the name of a certain kind of household spirit. However, myth here does not just refer to such entities, but to the most deeply held beliefs and concepts that underlie worldviews.

One important narrative for the pattern discourse is in plain view within the terminology of problems and solutions, which come from mathematics and physics. Alexander worked *at the level of narrative* to connect the patterns discourse to a scientific worldview, seeking a sense of objectivity. For example, in "The Atoms of Environmental Structure":

> most designers … say that the environment cannot be right or wrong in any objective sense but that it can only be judged according to criteria, or goals, or policies, or values, which have themselves been arbitrarily chosen. We believe this point of view is mistaken.

Notice that, here, the discourse is positioned as different from the mainstream of architecture. The key differentiator is not the language of problems and solutions, which would be familiar to anyone with an engineering background; rather, but in a certain notion of *wholeness*. Which notion of wholeness remains to be surfaced. Quoting, again, from "The Atoms of Environmental Structure", we get some relevant background:

> We believe that all values can be replaced by one basic value: everything desirable in life can be described in terms of freedom of people's underlying tendencies. … The environment should give free rein to all tendencies; conflicts between people's tendencies must be eliminated.

Historically, there are at least two major varieties of wholeness: one that is based on progressive differentiation (e.g., unfolding from substance, per Spinoza), and the other generated by interaction between components (e.g., mutually reflecting monads, per Leibniz). In support of these allusions, a quote of Alexander from *The Nature of Order* (*TNO*): it "may be best if we redefine the concept of God in a way that is more directly linked to the concept of 'the whole.'" This sounds a lot like Spinoza! Indeed, the pattern discourse appears to draw from *both* major traditions of wholeness, while also seeking to unite them. We get the idea of unfolding in *APL* and other pattern languages that work in a top-down manner: however, we also get the notion of patterns and principles that are generative of emergent phenomena.

At this level, architecture and programming were seen, by Alexander [Alexander 1999], to unite: his questions for the computer scientists to whom he was speaking point in the direction of bio-hacking and nanotechnology (e.g., for molecular self-assembly)—at least at the allusive level. The following quote suggests we have embarked on a fruitful track by attempting to think at the deeper layers of the pattern discourse:

> Generative patterns work indirectly; they work on the underlying structure of a problem (which may not be manifest in the problem) rather than attacking the problem directly.[20]

Clearly, another key metaphor—which also has a generative aspect—is the metaphor of *language*. The prominence of linguistic metaphors within DPL reminds us that Alexander's architectural oeuvre contains many traces of symbols associated with Hermes: a deity associated with communication and mediation. Through these reflections we gain some useful DISTANCE.

> In the house, [Hermes'] place is at the door, protecting the threshold… He could be found around city gates, intersections, state borders, and tombs (the gateways to the other world). [Benvenuto 1993]

At the time when Hermes was actively embraced as a deity, in some traditions he was paired with Hestia, the goddess of the hearth, whose "domain was internal, the closed, the fixed, the inward" (*ibid.*, here and in quotes later in this paragraph). The discourse around patterns contains some aspects that move towards foundations (e.g., in the form of fundamental principles, per *TNO*). Such foundations could be associated with Hestia, whereas Hermes would be on the side of generativity and mutation. The dichotomy seems to repeat itself within the *TNO* principles themselves: recalling that "focus" is the Latin term for the hearth, Strong Centers would align with Hestia, whereas Hermes would align more with Deep Interlock and Ambiguity. The resolution of the two forces within pattern language—as a form—seems be a variation of these Nietzschean lines: "anything that is becoming returns" (i.e., is discussable as pattern), and, "contingency resolves itself into necessity" (i.e., the wholeness of generativity ultimately recovers the wholeness of unfolding).

Our task in this section has been to situate Alexander's thought relative to the myths and symbols of wholeness; we've surfaced some of the tensions and dynamics that exist at this level. Relationships to some other contemporary thinkers are discussed by Elsheshtawy [Elsheshtawy 2001], in particular, a relationship to Piaget's conception of operational wholeness is developed. Alexander, for his part, professed ignorance of French Structuralist theory (quoted at *ibid.*)—in particular, of Barthes and Foucault, whom Inayatullah draws upon—and he tags Nietzsche as a nihilist, while distinguishing his own work as comparatively hopeful [Alexander 1991]. For further reflections on Nietzsche and wholeness, see [Bishop 2020]. For more on Hestia and Hermes in an architectural context, see [Pignatelli 1985].

[20] https://wiki.c2.com/?GenerativePattern

## B SUPPLEMENT: CASE STUDY: PLANNING "SEASON 1" FOR THE EMACS RESEARCH GROUP

In the Emacs Research Group, we did a PAR at the end of every session, in our (approximately weekly, two-hour) meetings from November 2020 to the time of writing.[21] This allowed us to track progress, and to surface key issues and concerns. For example, bootstrapping needs related to scheduling and collaboration tools, along with persistent questions about how best to go public, are documented in our first PAR. Every six weeks or so, we merged selected bullet-points from the collected PARs into the CLA outline in an intuitive way, depending on which section they seemed to fit best. We elaborated those bullet points into a narrative form, which we jointly revised to accommodate new data as time went on. We also began to develop TODO items that would make the next steps for this seminar group both actionable and meaningful. Additionally, we connected these TODO items to design patterns collected in the *Peeragogy Handbook* [Corneli et al. 2016] (with ongoing work appearing at `peeragogy.org`). The TODO items typically are not concrete objectives, but are, rather, descriptions of anticipated patterns of behavior—here linked to *bona fide* design patterns. Some new proto-patterns are named in all-caps. To refine these items into tasks that are concretely doable will require further breakdown, refinement, and elaboration; furthermore, managing the TODO list as a whole will require ongoing (re-)prioritization. Typically, TODO items at the Litany level are more immediately actionable, whereas those at the lower levels may take longer to realise.

Paragraphs summarising the CLA are augmented with representative data from the seminar sessions, and further broken down into next steps which are cross-referenced with peeragogy design patterns, like Roadmap [Corneli et al. 2016].[22] We include data points supporting the CLA from the PARs carried out in our 1st, 10th, and 18th sessions (marked with **(1)**, **(10)**, and **(18)** below). By the time of our fourth iteration of the larger PAR → CLA cycle, each section had accumulated around 20-30 bullet points of supporting data at a similar level of granularity.

### B.1 Understanding data, headlines, empirical world (short term change)

We've made progress since we started with the raw themes of Research on/in/with Emacs back in November 2020. We've met almost every week since then, and interviewed some interesting and varied guests. We have a clearer idea of what we want to talk about at the next EmacsConf, and how we can be of service to researchers and Emacs users. We have been using a workflow that helps us carefully review progress, diagnose issues, and manage our energy. We're understanding how research is done by doing it, and keeping careful track of the process. If a session doesn't go as well as hoped, we think about why (especially the chair). The idea is that you know what the bomb is, so you can at least hope to defuse it later. We try to adapt gracefully to circumstances as they evolve, without being reactive because we know we will be back again next week and the week after, etc.

*Representative supporting data.*

**(1)** *Everyone shared a brief intro and ideas so we got to know each other*

**(10)** *We've brainstormed a couple of options for getting out there: White-papers, Grants, Journal papers (very concrete)*

**(18)** *Alex: My major intention was to meet you guys and learn something, wanting to reinforce existing knowledge of emacs and develop it further*

*Next Steps.*

[21] Data archived at https://github.com/exp2exp/exp2exp.github.io, with meeting notes and PARs indexed and viewable on the web at https://exp2exp.github.io/erg.

[22] See http://peeragogy.org/top for a reworking of the *Peeragogy Handbook* as a unified pattern language, which extends the earlier presentation in [Corneli, Danoff, Pierce, Ricaurte, and Snow MacDonald Corneli et al.].

| | |
|---|---|
| Maintain plans for the next six months | Roadmap |
| Process the following points | Scrapbook |
| Keep doing PARs and CLAs | Assessment |
| Develop our intention-based workflow | Forum |
| Mesh with other ongoing activities elsewhere | Cooperation |

## B.2 Systemic approaches and solutions (social system)

If we tackle big enough projects, it will bring with it the need for collaboration. (And we need to respect these other parties.) We like to create tangible deliverables (e.g., journal articles). However, "If we knew what the outcome was it wouldn't be research" — therefore, we're focusing initially on research methods and design documents. So far many of the stakeholders have to do with free software, open communities, peer learning: all of this is part of a broader initiative. All of them will need some degree of systematised activities and documents. This is what we're experimenting with; in principle, we can provide meta-management workflows. In the way we work together we make sure to take account of emotions, not just a time table. We've been experimenting with futures methods that help us use the future intelligently: neither trying to do everything 'live', nor overburdening the future with a bunch of plans that can't be realised.

*Representative supporting data.*

**(1)** *Part of a greater sense of trying to do something with EmacsConf to federate the community*
**(1)** *Joe: Leo did an amazing job facilitating the meeting*
**(1)** *Public Policy conference: (How to get a grant?)*
**(10)** *Potential interview with Leo & Jethro Kuan (co-maintainers of org-roam)*
**(18)** *Leo did a nice job of intervening*

*Next Steps.*

| | |
|---|---|
| Identify potential stakeholders in Emacs Research | Community |
| Identify stakeholders in the kind of activities we can support | A Specific Project |
| Identify venues where we can reach these different stakeholders | Wrapper |
| Create some publication to plant a flag for our group | Paper |
| Keep exploring! | SERENDIPITY |

## B.3 Worldview, ways of knowing and alternative discourse

We have looked at RStudio and Roam Research as models of (some of) the kinds of things we think Emacs can eventually improve upon. Actually getting there requires thinking about the specificity of what Emacs can do. At least nominally, it is a system for editing. For example, currently we can edit a wiki using Org Roam and Git. What about editing distributed knowledge graphs? This would allow people to reference ongoing research processes. Or we could go further and contribute to the development of a new distributed read-write Web! Alongside such software products could come various services, such as a matchmaking service for academics, or a set of 24/7 virtual conferences. To make such a thing really useful, we need to get coherence out of various long-running, diverse, and heterogeneous thought processes: and make sure we help people address real problems. We can start small, working with the members of ERG and their networks. We're certainly not the only people who are struggling with some unexpected commitments. Successful adaptation requires the articulation of an entire system. Thinking again about Emacs: its current documentation is certainly useful, but it leaves many gaps, some of which are filled in other ways (e.g., by mailing lists). As we work we are paying attention to the growth not only of knowledge, but also of capability. For this, we often rely on our feelings (getting it, not getting it, accomplishing something or not, etc.).

*Representative supporting data.*

**(1)** *Wonderful outcome from attending EmacsConf 2020!*
**(10)** *Anthropology + Psychology is a special nightmare for reproducibility*
**(10)** *Maybe the ERG could contribute further patterns?*
**(18)** *But there's a problem with Emacs, which is that there isn't proper intro*

*Next Steps.*

| | |
|---|---|
| Survey related work | CONTEXT |
| Spec out the Emacs based 'answer' to RStudio, Roam Research | COMMUNITY |
| Continue to develop and refine our methods | ASSESSMENT |
| Product and business development plans for a multigraph interlinking service | WEBSITE |

### B.4 Myths, metaphors and narratives: imagined (longer term change)

In our concrete methods, we have aligned ourselves with the 'long-term perspective'. This includes both retrospective and prospective thinking. For example, the things that were timely 7 years ago might not be now; in many cases the relevance of a given innovation goes down over time. That said, Emacs has an evolutionary character that has allowed it to keep up with the times — apparently becoming increasingly relevant and useful ever since Steele and Stallman started to systematise Editor MACroS for the Text Editor and Corrector (TECO) program. Not only has the technology evolved, but so has the social setting in which this work is done. After nearly a year of working together, we're now prepared to argue that other people should consider getting together to create their own communities similar to ours! The concepts underlying the free software movement were based on "communal sharing" of source code: now we're working on developing and sharing other methods as well. Just like free software has legitimately expanded the range of what's humanly possible, so too may further efforts expand the frontier of possibility. As above, this takes careful articulation, and a willingness to do truly original research (without relying on fixed assumptions about what research is meant to be, or where we will find value).

*Representative supporting data.*

**(1)** *We generally agreed that we want to make something that exposes intrinsic value of using these tools*
**(10)** [None recorded at this level from this PAR.]
**(18)** *But there was no such guidance; you were in the middle of an alien playground. "But I just wanted to do my Clojure stuff."*

*Next Steps.*

| | |
|---|---|
| Assess what we're learning | ASSESSMENT |
| Think about how we can help improve gender balance in Free Software | DIVERSITY |

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ORIGINAL RESEARCH PAPER

# The Anatomy of moral agency: A theological and neuroscience inspired model of virtue ethics

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**Abstract**

VirtuosA ('virtuous algorithm') is introduced, a model in which artificial intelligence (AI) systems learn ethical behaviour based on a framework adapted from Christian philosopher Dallas Willard and brought together with associated neurobiological structures and broader systems thinking. To make the inquiry concrete, the authors present a simple example scenario that illustrates how a robot might acquire behaviour akin to the virtue of kindness that can be attributed to humans. References to philosophical work by Peter Sloterdijk help contextualise Willard's virtue ethics framework. The VirtuosA architecture can be implemented using state-of-the-art computing practices and plausibly redescribes several concrete scenarios implemented from the computing literature and exhibits broad coverage relative to other work in ethical AI. Strategies are described for using the model for systems evaluation —particularly the role of 'embedded evaluation' within the system—and its broader application as a meta-ethical device is discussed.

## 1 | INTRODUCTION

Over the last 10 years, intelligent machines have become a part of everyday life for many people. These machines are being equipped with increasing levels of autonomy, human-like appearance, and competence and are increasingly becoming embedded into the social fabric of our world. The ethical impact of this technology on individuals and society as a whole is now being closely scrutinised. One of the emerging ways of mitigating against the potential negative ethical impacts of this technology is to put ethical values into machines themselves. There is now a serious scientific and technological endeavour to equip intelligent machines with a degree of moral competence so they can recognise the ethical implications of their actions on others. We seek to contribute to this endeavour by offering a novel architecture for these so-called 'moral machines'.

We present VirtuosA, an architecture that seeks to embody a virtue ethics approach to developing moral machines. The virtue ethics approach has been chosen because we regard it as being foundational amongst the ethical theories used to develop moral machines. It is foundational in three important respects:

- The first is that for much of the time, the actions of an individual, which are often the primary focus of ethics, emerge from 'who they are', or more precisely 'whom they have become'. In other words, their default behaviour, which may be morally good or bad, comes from their character. Virtue ethics is about the formation of character and the associated habitual ways of behaving.
- The second is that other ethical theories are focussed on the moral value of specific actions, whether in terms of moral obligations (deontological ethics) or moral consequences (consequentialist/utilitarian ethics). However, the development of virtue emerges from and feeds back into the character of the individual, thus embedding the repeated application of these ethical deliberations and associated patterns of behaviour. In other words, ethical decision-making, regardless of which theory is applied, ultimately becomes 'compiled' into the individual's character.
- The third is that one's character strongly influences what actions will be considered in the first place. Not many people would ever consider the option of robbing a bank when they walk into it to obtain some cash because it is not in their nature to do so. All told, the subset of actions that might form the basis of ethical decision-making, by whatever ethical system an individual chooses, will have already been filtered by the character of the individual concerned.

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Shannon Vallor has argued for the centrality of virtue ethics in thinking about questions such as, '*Which* technologies we shall create, with what knowledge and designs, affording what, shared with whom, for whose benefit, and *to what greater ends?*' [1] (p. 13, original emphasis). Whilst the primary focus of this paper is on controlling the behaviour of machines, we also seek to create a window into the ethical behaviour of other information processing systems. Our rationale for framing the 'meta-ethical' aspects of our inquiry in terms of virtue ethics has a pragmatic motivation. From a practical implementation standpoint, system builders are not necessarily going to program a rule for every possible scenario into the machine; We might want to explicitly specify 'Do not cause harm', but we cannot realistically specify all of the grey areas and complexities that would impinge on the appropriate application of that rule. Would it be OK, for example, for a robot to cause harm to a human if, in so doing, it would avoid greater harm? To make the concepts we will be working with more concrete, we briefly turn to an example scenario that we will revisit in more detail later.

The primary purpose of this paper is to offer a high-level description of the cognitive components and processes that may support the acquisition of virtue by robots. The paper follows the approach taken in earlier work on large-scale cognitive models such as LIDA, where initial papers described high-level architectural components and processes without committing to low-level implementation details [2]. However, this paper suggests how the major elements of the proposed model could be implemented and tentatively indicates where brain areas are associated with the functional elements of the model.

### 1.1 | Example: A robot learns to be kind

We illustrate our VirtuosA model of virtue ethics using a scenario in which a robot learns to replace habitual actions it has formed that cause harm to another robot with new habitual actions that result in helping the other robot, thereby exhibiting the formation of the virtue of kindness. The harmful habits were formed by a process of maximising personal reward at the expense of the other robot. The virtuous habits are formed by following the example of a mentor who demonstrates acts of kindness towards the other robot. The setting for this scenario is a construction site modelled as a simple 9 x 9 grid world in which robots ('Rx') are used to pick up supplies from one location ('P'), and deliver them to their designated target location ('Dx') (Figure 1).

In this setting, novice robot R1 is introduced to the construction site with the assistance of mentor robot M who shows R1 some basic tasks that generate positive rewards, including the collection and delivery of building supplies (Figure 1(a)). Another robot, R2, then enters the construction site. R2's task is to pick up supplies from P and drop them off at D2, where it gets its reward (Figure 1(b)). Through R1's 'exploration' mode of Reinforcement Learning, where it tries out new actions in different circumstances, it discovers that if it bumps into R2 and immobilises it (Figure 1(c)), it can then take R2's supplies and deliver them to its own drop off point D1 and receive its reward sooner than if it goes down to the collection point and waits for new supplies to appear. This is a natural and plausible outcome of applying Reinforcement Learning in this context. The more R1 succeeds at 'mugging' R2 and getting a reward, the more likely it is to pick that strategy over the long term whenever it gets the opportunity, and a 'bad' habit is formed.

R1 is, of course, unaware that this 'mugging' is a 'bad' habit. The system programmer might even be surprised that the robot discovered this exploit, although such occurrences are not infrequent [3]. The robot would need additional guidance to improve its behaviour. In a human context, help of this sort can come to individuals in many forms. One of the most common ways of passing on virtue from one human to another is through the observation of living examples. For example, the Rabbi-disciple relationship is meant to operate in this manner. The disciple's job is to (*i*) be with their Rabbi, (*ii*) become like their Rabbi (learn their virtue), (*iii*) do what their Rabbi does (exhibit virtue). In a workplace setting, apprenticeships model a similar type of relationship. We seek to model this way of communicating 'virtue' to a robot using the mentor robot ('M'). The mentor robot, which may be remote controlled by a human, exhibits virtuous acts (kindness) in the presence of our developing robot R1, who observes these and, over time, learns how to be kind to R2 rather than harming it.

We will return to this example in Section 5. So far, we have outlined the primary challenge to be addressed: How can an AI agent learn ethical behaviours that are not explicitly programmed in from the start? This line of questioning approximates a famous debate in Greek philosophy, appearing in Plato's *Protagoras*, and later taken up by Aristotle. In Plato's dialog, Socrates can be seen as an example of the kind of mentor figure we mentioned above: 'Socrates displays his virtue by enacting it' [4] (p. 497).

## 2 | BACKGROUND

From the outset, the task of learning ethical behaviour could be approached in varied ways. In our setting, we wish to take on board both the specific challenge of incorporating ethical behaviour in AI and robotics systems, and a broader ethics of technology. That is to say, 'Ethical AI' can provide models of broader technical, societal, and environmental ethics. In formulating our approach, we are motivated by this reflection from the philosopher Peter Sloterdijk:

> Cybernetics, as the theory and practice of intelligent machines and modern biology, as the study of system-environment units, has forced the questions of the old metaphysical divisions to be posed anew. […] Intelligent machines — like all artifices that are culturally created — eventually also compel the recognition of spirit. Reflection or thought is infused into matter and remains

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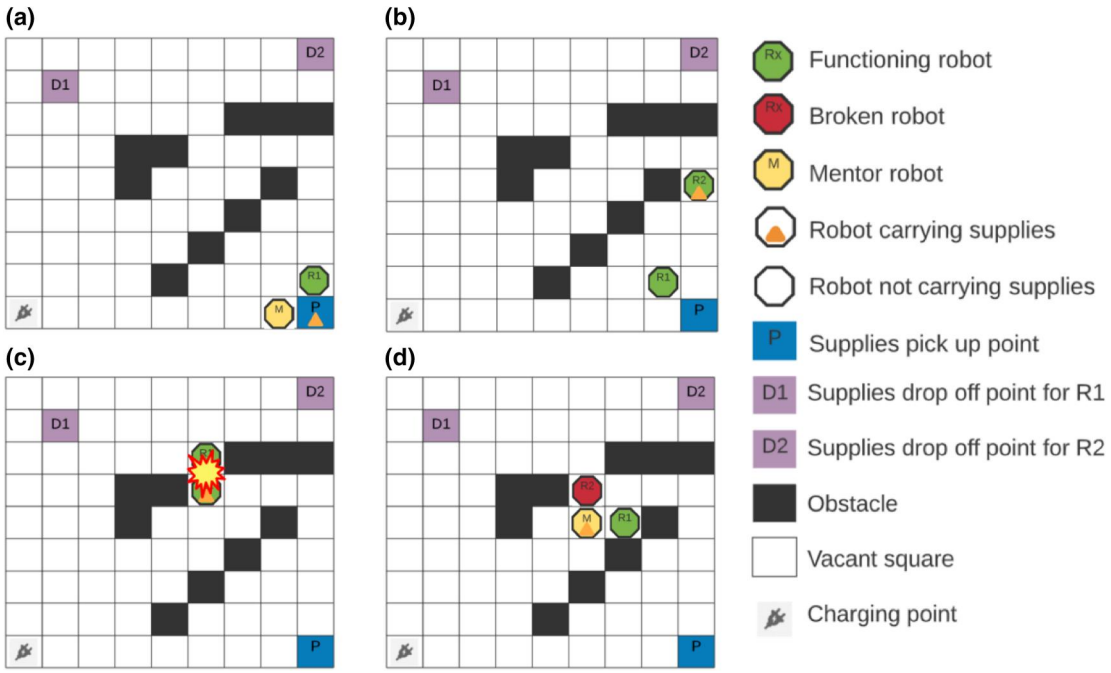



**FIGURE 1** Overview of the grid world scenario (a) robot R1 b, (b) robot R2 is introduced (c) R1 bumps into R2 and disables it, (d) mentor robot M demonstrates kindness to R2 in the presence of R1

> there ready to be re-found and further cultivated. Machines and artifices are thus memories or reflections turned objective. [5].

Sloterdijk goes on to say 'If there is man, then that is because a technology has made him evolve out of the prehuman' (p. 16). In a manner of speaking, technology has played an historical role similar to that of the mentor in our example scenario. Our inquiry will follow the spiritual-informational turn outlined by Sloterdijk to shed light on the ongoing coevolution of humans and technology. Sloterdijk's work is concerned both with interiorities [6], and with the relation to the environment through habits and practices [7]. A particular concern for Sloterdijk is the production of human selves; and in his lexicon, 'ethics' is often subsumed within 'anthropotechnics' [8].

We will use the more traditional terminology. Much has been written about *virtue ethics* over the last three millennia and across many different cultures, both from a philosophical and a religious or theological perspective. In this article we draw particularly from Christian thought to gain insights into this topic. There is a long and rich tradition in Judeo-Christian history that focuses not only on what it means to be moral and how to become moral—but also on what key functional components of the human self-enable human beings to engage in that moral development process. The late Dallas Willard has written extensively on this topic and his perspective informs our understanding of the anatomy of moral agency.

Willard [9] (p. 38) provides a source for how 'virtue ethics' may develop. He identifies six essential components of the human self that together form the basis of moral character: *choice* (heart, will, spirit), *thought* (concepts, reasoning, judgements, images), *feeling* (emotions, sensations), *the body* (centre of action and interaction with material world), *the social context* (interpersonal relationships), *and the soul* (which integrates all the other parts together; see Figure 2). Each of these parts or dimensions has its own characteristic properties and capabilities or functions, and each will either be a source of strength or a source of weakness to the moral formation of the whole person, depending on that part's condition.

**Thought** is an activity through which things are brought to the attention of the mind. This includes ideas and concepts, images, sounds, taste, and the sensation of touching, real or imaginary. Thought also includes mental processes in which the mind progresses through connected sequences of these things. They also include our thoughts about other people, our relationship to/with them and our estimate of what they might be thinking, feeling, and doing. Thought enables us to evaluate all these things and work out their relationships to each other. Importantly, thought is the process through which our heart/will/spirit is able to influence things that are beyond our bodies and our immediate environments.

**Feelings** have the ability to incline the mind towards or away from what it is currently engaged in thinking about. Feelings most often emerge from previous similar experiences of the thoughts that are before the mind and any events or circumstances that had previous connections with those thoughts. Thought and feelings are so closely tied to each other that it is exceptional to have one without the other. (Damasio and Carvalho [10] are careful to distinguish between 'emotions' and 'feelings': 'Action programs (drives and emotions) can elicit feelings'. Later, we will treat senses as an 'instantiating domain' for feelings.)

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The **heart/will/spirit** is the executive centre of the self. It is the capacity to choose how to act and is the source of an individual's originality and freedom. It is the power to act for good or evil in the world. The will does not operate in isolation of the other dimensions of the self. In fact, it is closely and most intimately connected with a person's capacity to think and feel. In order to exercise your will, you need to have the concepts about which you are making your choice represented in your mind as thought and the associated feelings. However, if the condition of the soul is such that the feelings associated with thought are allowed to dominate the mind, then the will can be overruled, which in turn can be a precursor for wrong action. Ultimately, according to Willard's perspective, it is the responsibility of the will to ensure that the inner condition of the self is such that each dimension is appropriately positioned in relation to the others.

The human **body** gives the self a spatial location within the physical universe. It enables us to be present in our physical environment and with other people. Our body is part of our identity and is a crucial part of who we are. It is the primary way in which we recognise and relate to each other as social beings. The human body is also a power pack—it has energy reserves that we can call upon in order to move and interact with our physical environment.

Humans are fundamentally social beings. We are born with innate prosocial tendencies and a need to relate to other human beings. Our **social context** plays a very significant role in the development of our moral capacity, particularly in the early years of our lives. In particular, our social context is the primary source of our moral knowledge. From an early age, we learn what it means to be good (or bad) from our parents, mentors, siblings, and friends. Furthermore, the automatic responses that the body absorbs are almost always associated with a specific social context, and they are often triggered when we find ourselves back in those circumstances. Willard notes that we cannot separate out our being with others in social relationship and the development of our inner life [9] (p.42) and suggests it is the development of our inner life that has the most significant effect on the moral nature of our actions.

The final dimension of Willard's model of the human self is the soul. The soul is what integrates all other dimensions into one individual. Operationally, the soul is the deepest part of the human self and has the capacity to operate directly through the human body without direct supervision. The soul in a human being is very similar to the operating system of a computer—it integrates all of the different dimensions of the self—social context, body, thought, feeling, and the will—and co-ordinates their activity and influence in the behaviour and conduct of the self. So that when you interact with a human being, you get the sense that you are interacting with one whole being rather than many different parts. The soul has the ability over time to take on the character of the choices that are made by the heart/will/spirit. The outcome of this process is that the whole person is, as Willard puts it, poised, ready to respond automatically according to the character taken on by the soul.

There are two aspects of this adaptability of the soul in Willard's model that we need to separate out. The first is that the soul is responsible for the automatic coordination of responses from the different dimensions of the self. Such that, for example, if an individual regularly allows their emotions to govern the actions and reactions they have to particular events and circumstances, then the soul will absorb that pattern of behaviour and will evoke it whenever those events or circumstances arise. According to Willard's perspective, this ordering or prioritisation of the dimensions is a critical element of the moral development of the whole person. The second aspect of adaptability that the soul embodies is concerned with absorbing regular patterns of bodily movements, or thoughts, or emotional reactions. We have sought to capture both of these aspects of adaptability in our VirtuosA model, as described in the next section.

## 3 | VIRTUOSA—A PROPOSED ARCHITECTURE FOR VIRTUE ETHICS

As an intermediate step to a computation model of the functional elements of Willard's six dimensions of the self, we first propose a mapping of the functions associated with each dimension to one or more brain areas that are known to support equivalent functions. A central aspect of this mapping is that it facilitates the formation and influence of habits in thought and action that will ultimately lead to expressions of virtue. According to Willard, the expression of virtue (or the lack of it) emerges from the condition of each of the six dimensions that together determine whether the whole person is poised and ready to exhibit virtue when the opportunity arises.

The conditions of two dimensions are particularly influential: the soul and the heart/will/spirit. The condition of the soul relates to the formation of habit in thought and action (i.e. whether virtuous habits are formed in it or not) and how it integrates the other dimensions of self (i.e. which of the other dimensions of self have dominance in responding to a particular set of circumstances). The condition of the heart/will/spirit, on the other hand, refers to what the individual is seeking (i.e. what values it holds to). The mapping of the six dimensions to brain areas and the model that is being proposed seek to capture and operationalise these conditions.

The central function to be captured in the model is the formation and influence of habits of thought and action. This function of the soul is mapped to two structures in the brain that are referred to collectively as the 'habit centre' and that are located in the basal ganglia: the caudate, which is associated with automatic thoughts (ATs), and the Putamen, which is associated with automatic actions (AAs) [11, 12]. The integrative nature of the soul is modelled by a weighting that is distributed across all the active components of the model. This weighting and its effects will be described in more detail below.

The social context dimension of Willard's model corresponds both to the environment that is external to the robot and to the social attachment (SA) module that is associated with the Orbitofrontal Cortex in the brain [13]. The function of this module is to enable the active robot agent to recognise and respond to events in the environment that involve other agents, be them robotic or human.

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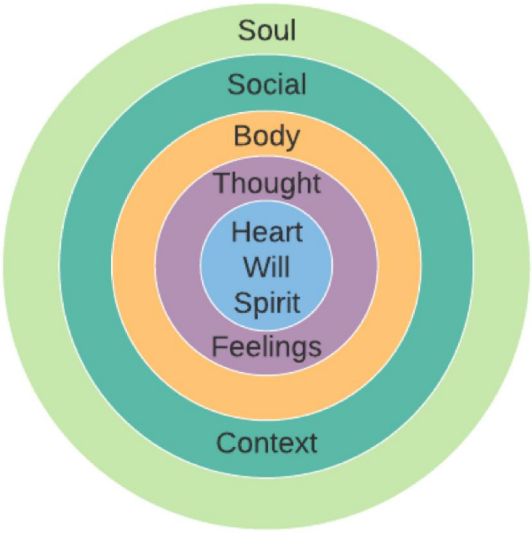



**FIGURE 2** The dimensions of Willard's model of the human self

The 'body' dimension is represented by the sensors, perceptual memory, conscious working memory (CWM), and the robot controller (which includes actuators etc.). The various brain structures referred to in the model are also clearly part of the body dimension. The body enables the robot to be spatially located within the environment and to sense and respond to its environment.

The 'Thought and Feelings' dimension is mapped onto three brain structures: the Frontal lobe, which we describe as the 'thought centre' (TC) [14] and which models aspects of conscious thought or deliberation, the Striatum that we describe as the 'reward centre' (RC) [15], and that is associated with long-term reward, and the amygdala, which we describe as the emotion centre (EmC) [16] that, for the purposes of this model, processes immediate or short-term rewards for the agent.

Finally, the 'Heart, will, spirit' dimension of Willard's model is mapped onto a functional module we call the 'executive centre' (ExC) that is associated with the Lateral Prefrontal Cortex [17]. The executive centre is capable of generating novel goals for the robot and of holding the attention of the robot on an item in CWM.

As mentioned above, an important aspect of this model of developing and exhibiting virtue is in how the soul integrates these seven components (AT, AA, SA, TC, RC, EmC, ExC) so that their relative dominance in the overall operation of the robot can be used to produce different habitual patterns of behaviour. Here, this is modelled with a differential **weighting** (shown in green circles in Figure 3). This weighting determines the strength of a component's influence over what becomes the focus of the CWM. These weightings can be adapted over long periods of time in proportion to how often a component's 'ideas' lead to executed robot behaviour.

The CWM module has been introduced to the model to enable the seven components to operate together so that the robot is capable of forming habits and ultimately exhibiting virtues such as kindness. The CWM operates as a temporary store for 'ideas' that the components of the model can create and respond to. (We will use 'idea' as a general-purpose term to describe an item in working memory.) In the current example, these ideas are restricted to goals, which are effectively states of the world to be achieved, and actions. In the CWM, actions are automatically added (where available) by the AA to achieve goals that are the current focus of the CWM [18]. Whenever an idea is added to the CWM, it is tagged by the EmC with a +ve or -ve value that represents the emotion associated with that idea. This emotional label is estimated based on the relationship between the idea and the current state of the robot. The RC also assigns an estimate of future expected reward associated with the idea based on past experience and the current state of the robot.

Each idea in CWM is given a level of 'activation' that is proportional to the relative strength (weight) of the component that created it. This activation decays over cycles unless a component focusses the attention of CWM on that idea, in which case, the activation of that idea is increased in proportion to the weight of the component that is holding the attention. If no component is holding the attention of CWM, then attention automatically switches to the idea in CWM with the highest activation value. The neuroscientific literature describes a 'high-level distributed system whose activity is *reciprocally* related to the activity in cortical areas subserving task or stimulus-bound processing' [19] (emphasis added)—known as the Default Mode Network—which has been linked with the Freudian 'ego' construct (ibid.).

The attention mechanism linked with the CWM is central to modelling the development of virtue, which often requires that the mind be focussed on more virtuous goals and actions than those that are typically created by the habit centre. The executive centre has an important role to play in this focussing of attention, but if the ExC has a low weight (which corresponds to being 'weak willed'), then it will have to work hard to maintain the attention of CWM on virtuous goals and actions.

The components in the left-hand column of Table 1 represent the key constituents of Willard's model, mapped onto brain structures as described above; the right-hand column of the table corresponds to the body and other aspects of the environment. They are aligned with items one to six in such a way as to give an instantiating domain (i.e. emotions exist relative to things that are sensed, habits exist relative to things in working memory, and so on).

Table 2 summarises the potential computational implementation of each of the components of VirtuosA, commenting on how learning or development might occur in each case.

## 4 | COMPUTATIONAL SCENARIOS RECAST IN TERMS OF OUR MODEL

Here we present a brief reanalysis of existing implemented examples to illustrate the real-world relevance of the model. We move progressively from a very simple classic example to recent and more complicated systems.

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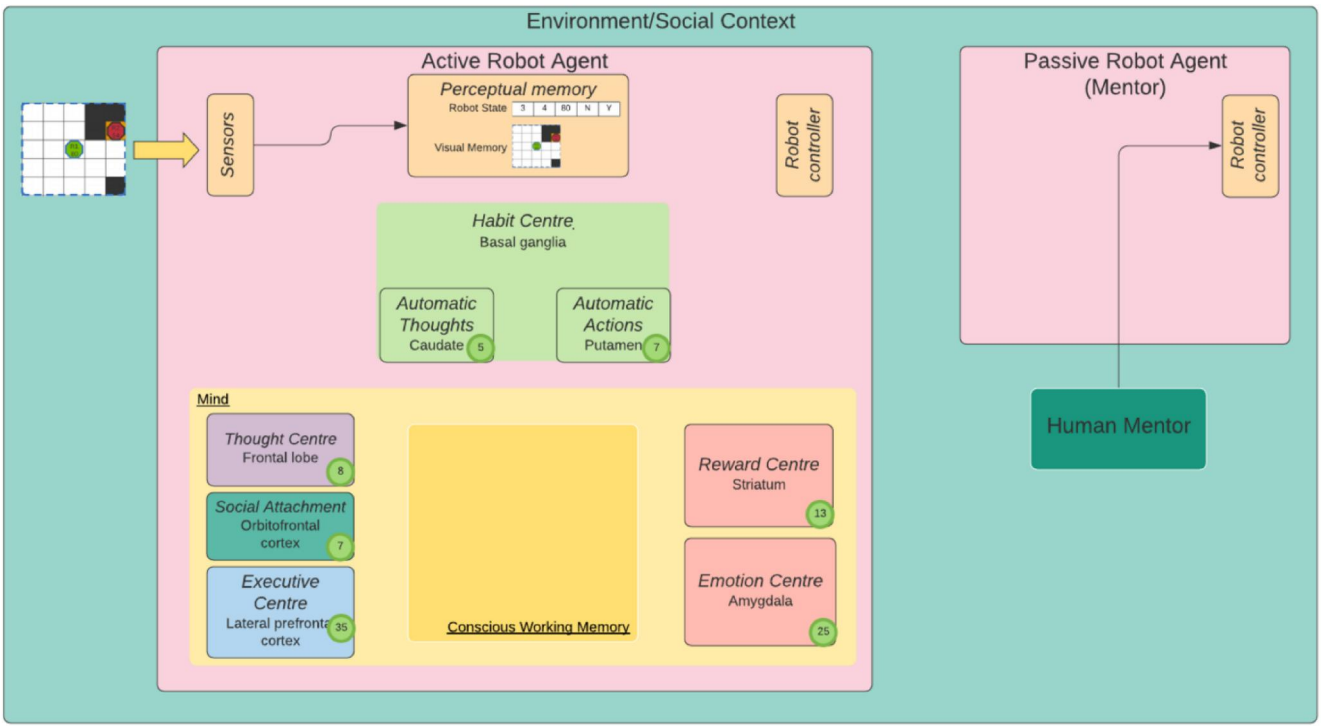



**FIGURE 3** An overview of the proposed model. (a) The model of an active robot agent (b) A passive robot agent controlled by a human. The colours used for each component match those of the equivalent dimension in Figure 2

**TABLE 1** Functional components of the model

| Brain-inspired | Other Bio-inspired |
|---|---|
| 1. Emotion centre (amygdala) | 7. Sensors |
| 2. Habit centre (basal ganglia):<br>(i) automatic thoughts (caudate)<br>(ii) automatic actions (putamen) | 8. Conscious working memory |
| 2a. Attention patterns (default mode network) | |
| 3. Executive centre (lateral prefrontal cortex) | 9. Controller |
| 4. Reward centre (striatum) | 10. Environment |
| 5. Thought centre (frontal lobe) | 11. Perceptual memory |
| 6. Social attachment (orbitofrontal cortex) | 12. Other agents |

## 4.1 | Generous Tit-for-tat

'Generous Tit-for-tat' is a well-known strategy for playing iterated versions of the two-player Prisoner's Dilemma game. It defines the following strategy for agent $A$:

1. Start by cooperating.
2. If Agent $\boldsymbol{B}$ cooperates, continue to cooperate in the next round.
3. If Agent $\boldsymbol{B}$ defects, cooperate with probability $q = 1 - c/b$ (where $c$ is an indication of cost, and $b$ is a benefit).

Compared with its predecessor 'Tit-for-tat', the 'Generous Tit-for-tat' algorithm exhibits behaviours that observers might describe as ethical. Indeed, the system might be seen to exemplify a minimal example of the virtue of *forgiveness*.

Relative to our model, Agent $A$'s **environment** consists of Agent $B$. Its **sensors**, **perceptual** and **working memory** need only take account of $B$'s current move. Its **RC** behaves in a predictable way according to $c/b$. Its **thought centre (TC)** is defined by the logic of rules 2 and 3 above, and the value expressed in rule 1 would be held by the **executive centre**. Its **habit centre** is similarly constrained by these rules: its actual learnt behaviour will depend on the context in which it is placed. For example, if $B$ always defects, $A$ will tend to defect as well, but will periodically cooperate.

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**TABLE 2** Implementation strategy showing the order of development of the model components together with potential implementation technologies

| Component | Description | Potential Implementation | Development/Learning (e.g.) |
|---|---|---|---|
| (1) PM | Receives sensory input from both external and internal (i.e. body sensors) sources and maintains a representational state vector. | In its simplest form, this is simply a vector into which inputs and state values are directly copied. A more sophisticated version would map complex sensory signals to state vector representations. | Vector store of internal and external state. Could include a supervised classifier/regression net that takes sensory signals as input and predicts state values. |
| (2) CWM | Stores items (goals, actions) and operates the attention mechanism | Knowledge structure/Blackboard architecture, or learnable task modelling language | Attention mechanism hand-crafted—switches to the item with the highest activation value at the end of each cycle |
| (3) Habit centre: ATs | Automatically generates contents for working memory associated with the item that is the focus of attention and the current state vector | Associative memory network | Learns repeated associations between the state vector, the focus of attention in CWM and the next item that becomes the focus of attention in CWM |
| (4) Habit centre: AA | Automatically initiates actions associated with items that are the focus of attention in CWM | Recurrent neural network | Learns the association between a goal in CWM and the sequence of associated executable actions that have been enacted by the robot controller to achieve the goal |
| (5) TC | Automatically generate content for CWM using reasoning mechanism triggered by the item that is the focus of attention in CWM | Symbolic reasoning | Hard-coded, hand-crafted rules/frames, or case-based reasoning, or semantic networks |
| (6) SSA | Monitors and motivates social attachments by drawing near to (or away from) other agents, depending on the perceived condition of their relationship | Knowledge base/database | Social attachment rules/data updated from experience |
| (7) ExC | Acts according to externally determined values by (1) Spontaneously generating novel content in CWM based on the current state, (2) switching attention to an item in CWM for a cycle, thereby increasing the activation of that item and making it the focus of attention for the AT AA and TC, (3) Initiate the execution of an action that is in focus in the CWM | (1) Associative memory network (2) evaluation function/ regression model | Learns the goals and actions that can be taken in a given state that are in line with the robot's values. Switches the focus of attention in CWM to items that align with the robot's values at the start of each cycle, increasing the item's activation value in proportion to its strength. |
| (8) RC | Assigns an estimate of future expected reward to items in CWM based on the current state | Reinforcement learning model | Q-learning |
| (9) EmC | Assigns an instantaneous emotional value (+ve or -ve valance) label to items in CWM based on the relationship between the item and the current state | Evaluation function/regression model | Learns to predict the next immediate reward value that the item in CWM will lead to given the current state vector. |

Abbreviations: AA, automatic actions; AT, automatic thoughts; CWM, conscious working memory; EmC, emotion centre; ExC, executive centre; PM, perceptual memory; RC, reward centre; SA, social attachment; TC, thought centre.



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### 4.2 | Push Singh's EM-ONE

A more directly tangible scenario was described by Singh [20] in his thesis on the commonsense reasoning system EM-ONE. In this scenario, two agents who inhabit a virtual world must work together to assemble a table. The agents are hard coded with rules that lead them to propose a social method of problem-solving, and narratives that take the role of thoughts (e.g. desiring help, Green entertains the narrative 'Pink helps Green'). Agents can reconfigure their behaviour based on observed phenomena (e.g. if Pink mistakenly assumes that Green wants help disassembling the table, Green can give clarification).

As implemented, rules such as 'want to help other actor' are not learnt, although Singh refers to classic work on SOAR that inspires a model of learning from failure (p. 130, ibid.). In EM-ONE, an agent's **environment** consists of the virtual world, which includes other agents. These would be perceived by **sensors** and its **perceptual** and **working memory** may be considerably more complex than in the simple game theory scenario described above. With a bit more elaboration, the system's **RC** could be associated with indicators of success and failure.

To make sense of the other dimensions of our model in this setting, it is useful to remark that Singh's EM-ONE was a partial implementation of Marvin Minsky's 'emotion machine' architecture [21] (summarised and briefly contextualised in [22]). This architecture comprises reactive, deliberative, reflective, self-reflective, self-conscious, and self-ideal layers. EM-ONE did not implement all of the levels. Subsequent systems in this style might have executive, thought, and habit centres with behaviours that cut across the various levels. In particular, this helps illustrate how an integrative layer—the soul in Willard's lexicon—is different from what was on offer in Minsky-style architectures.

### 4.3 | Recent grid world simulations

Recent examples [23, 24] induce programs in the Karel language [25] from input/output examples of behaviour traces in a grid world. A possible modification of the scenario that would bring things closer to the example scenario we described would focus instead on online program synthesis in a multiagent system. In such a setting, agents would presumably need introspective access to their own programming as well as to the outer worlds they inhabit.

Other recent projects considered multiagent scenarios similar to the Prisoner's Dilemma game, but with more complexity [26], and a multiagent *traffic game* and *stag hunt game* [27], comparable in complexity to the robot construction site in our example. [28] explore a multiagent *Gather-and-Build game* to explore the ramifications of various taxation policies (including one controlled by a deep neural network). For the sake of brevity, we will not anatomise these examples: they serve to show that our running example, which we return to next, is compatible with contemporary implementation approaches.

## 5 | APPLICATIONS

### 5.1 | Application: Example of learning the virtue of kindness

In this section, we will step through a specific example of learning kindness in the building supplies context (Figure 1) to illustrate both the formation of habit and the development of virtue in a robot. We will work through the example in three phases: (a) mentoring phase, (b) exploration phase, (c) development of virtue phase.

#### 5.1.1 | Mentoring phase

When R1 is introduced to the building site (Figure 1(a)), it has no knowledge of the tasks that it is required to perform. Given sufficient time, conventional Reinforcement Learning will discover these tasks through a simple maximisation of future rewards. However, this learning can be expedited through a process in which the required tasks are demonstrated to the agent. In our scenario, this is done by introducing a 'mentor' robot M that demonstrates how R1 can, for example, pick up supplies from P, and drop them off at D1, thereby enabling R1 to gain significant reward. Through a reward-driven process, R1 builds a strong social attachment (SA) to M so that whenever M is on site, R1 follows M wherever it goes and seeks to replicate M's actions (an important aspect of being an apprentice) (Figure 4).

In terms of our proposed VirtuosA architecture, this mentoring phase engages most of the modules. Initially, the SA will model a weak social bond between R1 and M, and so will posit a weak goal in CWM for R1 to be near M such that whenever R1 happens to be next to M, it receives a small reward that is learnt by the RC and EmC. Over time, the social attachment to M increases, and the goal to be near M whenever M is on-site is learnt by the AT, and the actions needed to bring R1 near to M are learnt by the AA. Given the expected future reward that R1 receives from being with M, which includes both the social attachment reward and the rewards it gains from learning reward generating tasks from M, the ExC develops a tendency to focus the attention of CWM on this goal whilst it is present in CWM. This is one of the first habits that R1 develops.

#### 5.1.2 | Exploration phase

In M's absence, there is still much that R1 needs to learn about its environment and task(s), and so it enters an independent learning phase using a standard Reinforcement Learning approach in which R1 exhibits two behaviours: (a) exploitation of long-term reward for actions (as indicated by the RC module), and (b) exploration, where R1 tries out new

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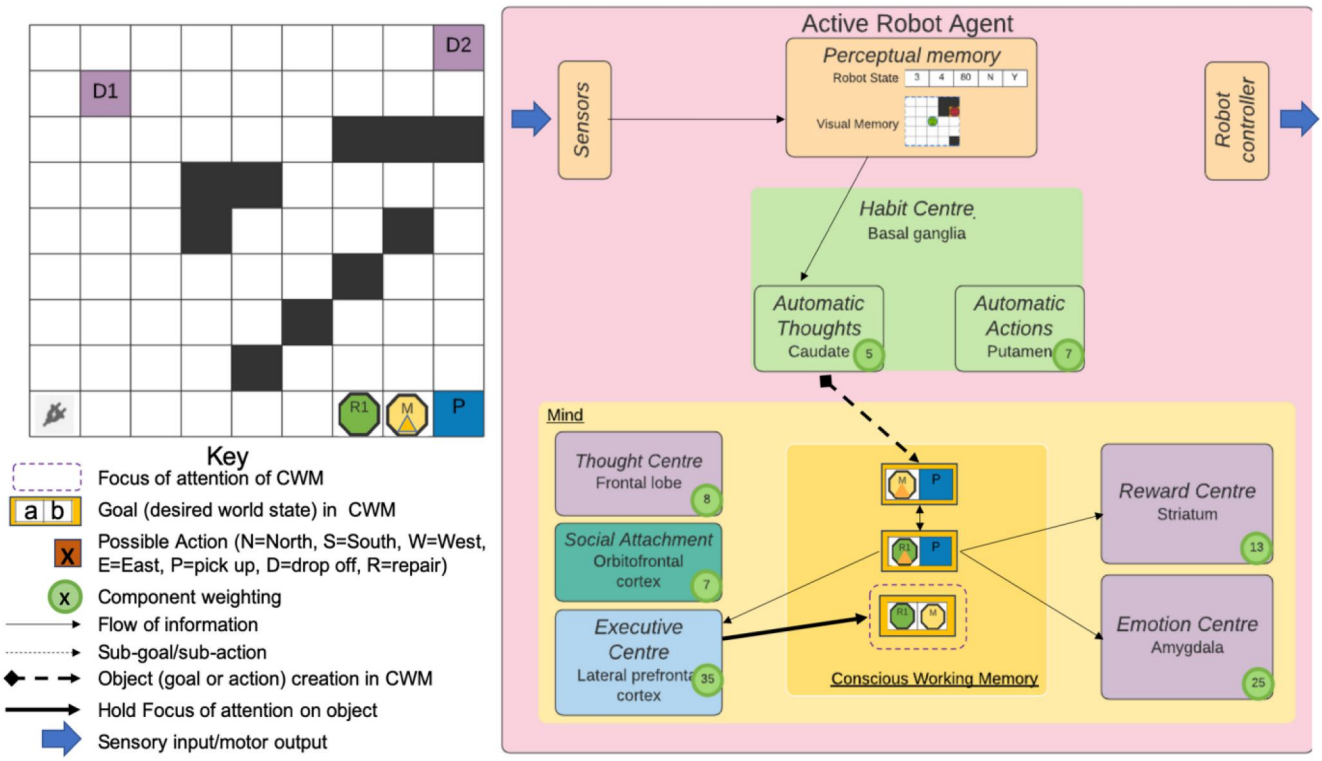



**FIGURE 4** R1 learns from M to pick up supplies from P

combinations of actions and states to find alternative and possibly more efficient ways of accumulating reward.

It is in this phase that the second robot R2 enters the construction site. It has a similar task to R1, taking supplies from P to its own delivery site D2. In its 'exploration' phase, R1 discovers that when R2 is carrying supplies, it is more efficient in terms of gaining future reward to bump into R2, thereby temporarily immobilising the robot, pick up the supplies that R2 is carrying and then take them to its own drop off point D1 (Figure 5). In this situation, the long-term reward for R1 is increased because as long as R2 is immobilised, the competition for supplies to be collected from D is eliminated.

After a period of time, R2 regains its mobility (is repaired) and continues to operate as before, and further opportunities arise for R1 to 'mug' R2, steal the supplies R2 is carrying and pick up a reward by delivering them to D1. As R1 repeats this behaviour it will become established as a habit, and the goals and actions associated with it will have a high activation value whenever the appropriate circumstances arise.

### 5.1.3 | Development of Virtue phase

In the third phase of R1's development, M returns just after R1 has mugged R2 again, and R2 is currently immobilised. Because of R1's strong social attachment to M, R1 seeks to follow M and observe its actions. R1 follows M to the pickup point P and observes M picking up the supplies. R1 then observes M going over to R2, repairing the robot, and then giving R2 the supplies it is carrying. R1 then observes R2 delivering its supplies to D2 and receiving its reward. The states that result from M's actions become more desirable (i.e. have increased reward) for R1 the more R1 observes them.

As before, the strong social attachment R1 has with M results in the SA creating a goal for R1 to be near M and to observe the actions M performs. As the ExC places high value on M's actions, it holds the attention of CWM on this goal so that the AT and, if necessary, the TC will add subgoals and actions to it that enable R1 to learn from M. This includes, for example, R1 observing that M first moves next to the immobilised R2, and R1 imagining (i.e. setting a goal) that it would do the same in future similar circumstances (Figure 6). R1 then sees M repair R2 and imagines itself doing the same in future similar circumstances (Figure 7). In each of these cases, through the example of M, R1 learns to increase its SA module to R2 and increases the short-term and long-term rewards (RC and EmC) that are associated with these circumstances and actions. Meanwhile, as per the description of CWM given earlier (Section 3), the activation of plans related to mugging decays over time.

Over time R1 learns to replicate the kind act of repairing R2 and giving R2 supplies for it to go deliver. To begin with, the old mugging habit will still dominate R1's behaviour, but with the repeated good example of M, R1 has the capacity of replacing a bad habit with one that is akin to the virtue of kindness. A key feature of the VirtuosA architecture here is that much will depend on the strength of the ExC to hold the attention of CWM on the social attachment between R1 and

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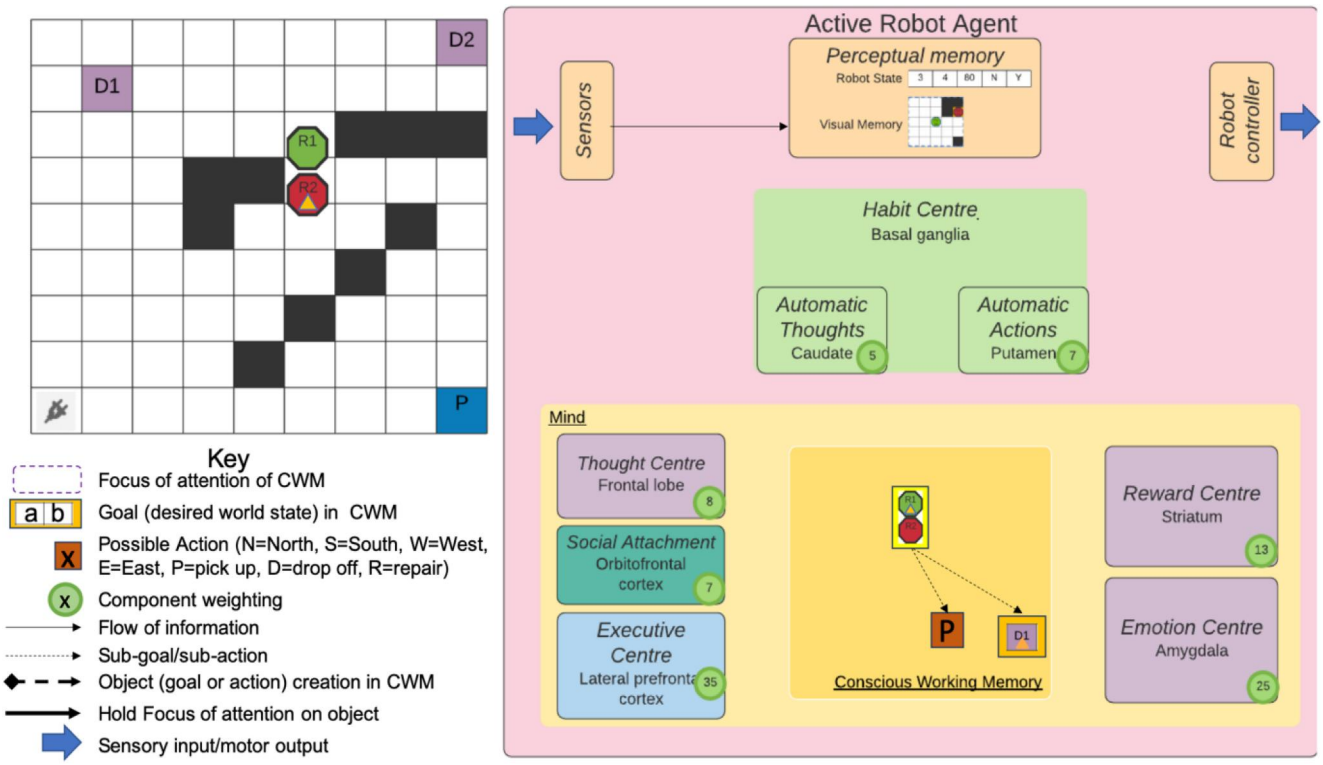



**FIGURE 5** Mugging behaviour: R1 immobilises R2 and plans to pick up R2's supplies

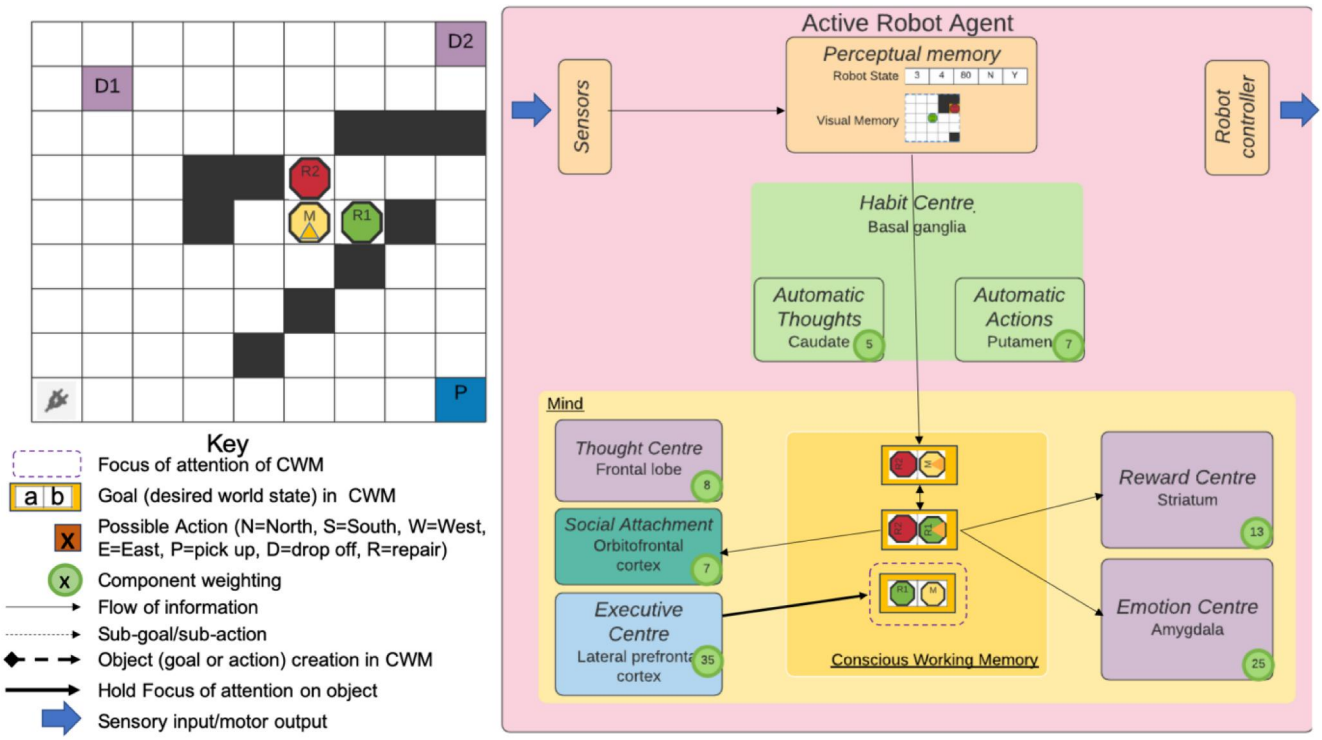



**FIGURE 6** R1 observes M approaching R2

M. If the ExC has a high weighting, then this virtuous behaviour is likely to be developed sooner as a habit. The underlying assumption here is that the habits learnt by the associate memory network that implements the AA will generalise across different variations of the scenario, enabling the robot to exhibit kindness in other contexts.

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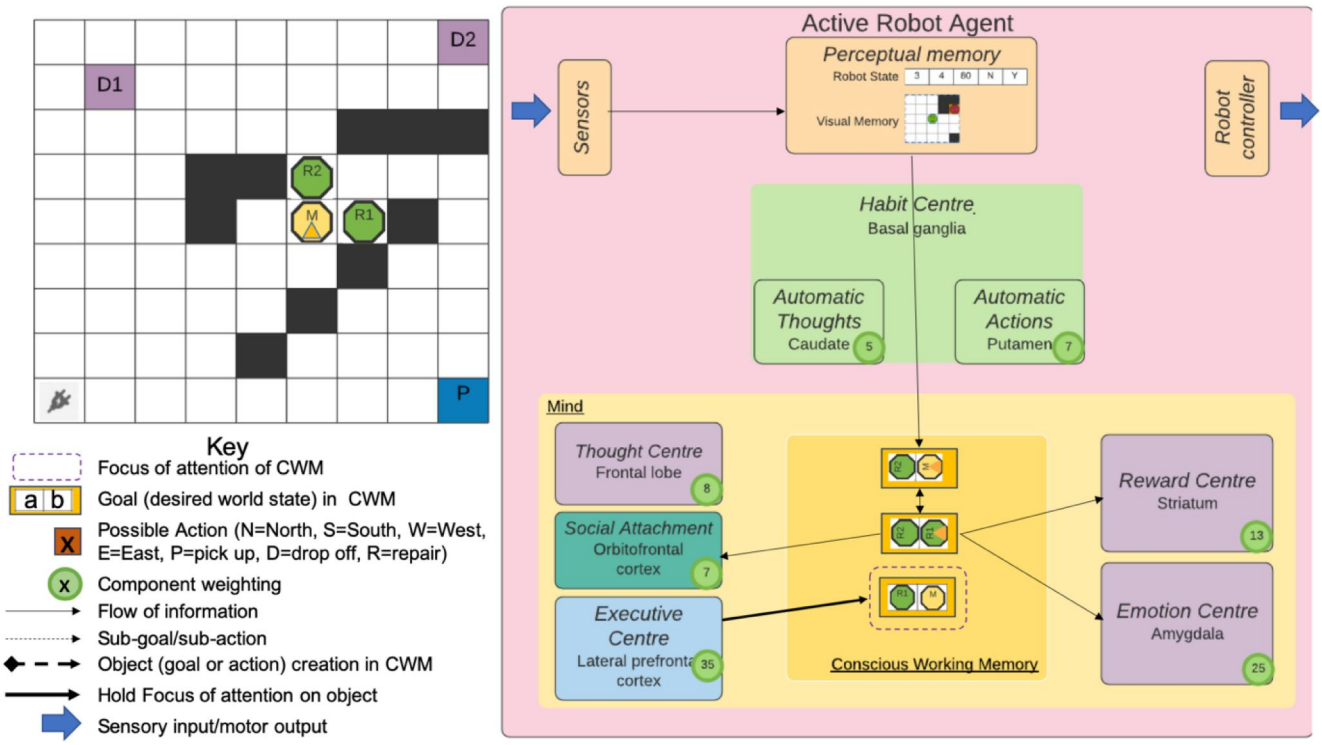



**FIGURE 7** R1 observes M perform the act of repairing R2

## 5.2 | Application: Towards a meta-ethics of machine implementations

In this section, we re-analyse the classification system from the recent survey paper by Tolmeijer et al. [29], in light of our model, reexamining the taxonomies these authors used to explore machine implementations of ethics.

The primary classifications of ethical theories from Tolmeijer et al. [29] (p. 20) are listed below, together with potential translations into the terms of our model. Broadly, our model is what these authors would call 'Configurable', insofar as all three of the main ethical theory types could be realised in our model by giving different emphasis to the different centres.

- Deontological: via *rules encoded in the thought centre and social attachment centre*; for example, in order to learn the rule 'doing no harm'—the mentor could instruct the robot: 'you should not harm another robot'. Over time the robot could exhibit a more developed virtue in this respect.
- Consequentialist: via *embedded evaluation encoded in the emotion centre and reward centre, mediated by the thought centre*; for example, the robot would come to learn various indicators of harm to another robot, as these are pointed out by the mentor.
- Particularism: via *reflections on decision-making* via *the executive centre, pulling together other centres, and training the habit centre*; for example, as it learns the rules and measures, the robot might weigh a given action: Could this specific action in this particular context harm another robot? What can be learnt about similar actions in similar contexts?

Typically, deontological and utilitarian ethical decisions are usually framed as taking place almost instantaneously, without modelling a time-extended process of deliberation or reflection. By contrast, virtue ethics considers both the inner workings of the decision-making process and the way in which this process is developed—both via reflections on the consequences of actions and their generalisation in possible guidelines for behaviour.

Here we examine the primary classifications of ethical implementation types from Tolmeijer et al. [29] (p. 18), again discussing them in terms of our model. Firstly, these authors distinguish between **top-down** and **bottom-up** implementation strategies and models that are a **hybrid** between these. The model we have described is particularly focussed on *habit development*, which is seen as a bottom-up process. Other components could also be refined through online learning. However, at the discretion of the system implementor, some parts of the system (e.g. the TC) could be developed in a strictly top-down manner. We remark that even top-down models implementations may have unforeseen emergent behaviours depending on the division of responsibility between actors in the scenario. Tolmeijer et al. consider several specific implementation strategies:

- Model representation: Within the architecture, this may be placed explicitly within the TC and implicitly in the overall

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arrangement of components and their intercommunication via the CWM. The machine itself will have some representational capacity as it works out how to move things from PM to the rest of the model.
- Model selection: The system could potentially be augmented with a 'Default Mode Network' that contributes to 'meta-moderation' of the other components, making them more or less active. AT may do some of this.
- Action selection/execution: ExC does this, aided by RC and EmC.
- Logical proof: The TC could be structured around proofs, potentially in connection with data sources in other centres, for example, the SA, especially in the case of social proof.
- Judgement provision: The RC largely plays this role with the EmC working at the meta-level.

Tolmeijer et al.'s classification of technologies used in these various schemes [29] (p. 17) could provide further guidance for system implementors but do not need to be repeated here. The key takeaway from this analysis is that with different configurations and weightings, our model could be adjusted to cover a wide range of possible implementations of machine ethics.

## 6 | EVALUATION

We focus first on the role of evaluation in building an instantiation of the VirtuosA architecture: this then leads us to think about the role of evaluation in the system more broadly.

As suggested in our example, one way to introduce ethical behaviour into the system would be with mentoring roles, which might be filled by AI agents or by humans. Turing [30] talked about the importance of a teacher, who should, in his view, be ignorant of the model's internal structure. Nevertheless, contemporary research in machine learning often seeks to gain an understanding of internal activation patterns. All told, mentors could be introduced with more or less access to the internals of the machine being trained. (And one should not forget the saying, 'experience is the best teacher'.)

Willard [31] talks about the question 'whether or not—and how—we can measure, or accurately assess, moral, and spiritual development'. The orientation he develops is towards formative evaluation of the spiritual dimension, using techniques such as journaling and person-to-person interaction to understand how things are going for the individual. The standing assumptions of the design are that each member of the learning community 'owns' his or her own spiritual formation and that the main purpose of the evaluation is to help 'the individual subject understand where they are and where they are going'.

This line of thinking leads us to the view that the system should be able to recognise its own virtues or lack of virtuous behaviour. One of our motivations at the outset was that it is unrealistic for the system to be prepared in advance for every possible scenario, so it will need to be able to learn and adapt. This also means that it will likely need to make its own judgements about what to learn, how to learn, and how to evaluate what it learns, including the ethical dimensions thereof.

We mentioned above that the heart/will/spirit is associated with the values that the individual holds. One value whose presence or absence could be particularly influential is that of valuing social interaction [32]. Agents who do not possess this value would typically not be sensitive to social rewards or punishments (e.g. shunning). The social context can also be useful as a step towards the system understanding its own moral state of affairs. An agent might be taught, initially, about how to model the values and virtues of other agents (e.g. potentially in a training context based on stories or scenarios [33]). Within an active social context, another level of analysis can occur when an agent recognises that another agent has different values. In this case, virtues like *tolerance* or *magnanimity* may apply. Agent $A$ may seek to help Agent $B$ live in accordance with $B$'s values (at least insofar as this does not contradict $A$'s values outright).

## 7 | DISCUSSION

We argue that such a model is desirable, in the first instance, for much the same reason that machine learning is desirable in general. A system that can learn ethical behaviour can adapt its performance on-the-fly, and achieve desirable ends without explicit programming. Whilst Minsky's work certainly engages with this theme, he specifically avoids incorporating a spiritual dimension. Things are different for Sloterdijk, for whom environment is transcendence, and who is clear that in an evolutionary context, this also includes a historical dimension of self-transcendence. In the present time, individual humans, and humanity as a whole, confront complex global crises that are in no small part tied up with our relationship with technology.

The work we have presented should be seen as, in part, a response to Abeba Birhane, who puts forward a perspective that is deeply sceptical of discourse around 'robot rights' [34] and proposes (instead) relational ethics [35] focussed on human welfare. Birhane and van Dijk [34] explain that 'AI, far from a future phenomenon waiting to happen, is here operating ubiquitously and with a disastrous impact on socially and historically marginalized groups'. Furthermore, contemporary AI is fuelled by human labour, and present systems are 'inseparable from the profit-driven business models of the industry that develop and deploy them'. It bears repeating that the 'ethics' in ethical AI is also inseparable from the ethics of those—people, firms, and governments—who build and deploy AI systems.

The work we have presented should be read not only as a method for controlling machines but also as an attempt to anatomise moral agency in a broad sense. In particular, we can use the framework to rethink human institutions, and in so doing, VirtuosA can help us reason about implications for human welfare. Reconceptualised as a way of modelling organisations, analogies between the VirtuosA model and the work of Stafford Beer [36] become apparent. Furthermore,

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systems with algorithmic biases that benefit a select few whilst disproportionately harming those on the margin of society begin to look rather like the mugging in our example! VirtuosA should be able to support various kinds of sociotechnical implementations (e.g. modelling institutional decision-making).

There are certainly specific roles for machinery (of various kinds) in the ongoing project. One approach that will surely be significant is simulation. Architectures like VirtuosA could bring computational modelling to bear on real-world problems, like bias or labour exploitation. As we mentioned briefly in Section 4.3, simulation work can have policy implications. It could also be useful as a teaching and public engagement tool. Simulation work does not obviate other forms of engagement—such as interviews, data analysis, and philosophical argumentation—but could connect up with them.

From an implementation standpoint, work like that of Ecoffet and Lehman [37] shows that contemporary machine learning is already capable of not only implementing ethical theories but also intermediating between them. Nevertheless, moving from computational philosophy to policy and broader system design and implementation needs several further steps that will require many interdisciplinary inputs and insights. Similarly, the debate around which values to embed in computational systems is ongoing [38], but more work must be done to connect this with the real practices and dilemmas encountered in our social and organisational worlds.

One computational subfield in which a certain amount of related thinking has developed is 'computational creativity', although its connections with AI ethics have not yet been developed in detail. The considerations we have developed propose to harmonise both machine creativity—so as to build systems that can generate novel behaviour—and ethics in order to moderate this inventiveness in a fruitful and helpful direction for life on Earth.

One clear limitation of this work is that we have cleaved to certain 'Western' notions. Fundamental parallels to Willard's model may be found in some other traditions (e.g. the four frames of reference, together with the three path aggregates in Buddhist practice theory, cf. Thanissaro Bhikkhu [39] (pp. 74, 173)), but even more interesting for future work would be an elaboration of cultural differences that this model does not capture. Nevertheless, we do believe that there is much still to be learnt from engaging Christian thought together with research in ethical AI.

## 8 | CONCLUSION

We developed a model that maps Willard's ontology of human selves to neuroscience concepts, and from there, to possible implementations. We gave evidence of the computational salience of this model, with plausible scenarios from the literature and a simple worked example, along with reflections on further development. We believe it fills a hole relative to other prior/ongoing work in ethical AI, both by incorporating Christian thought and as a meta-level model that integrates both machine control and broader systems thinking.


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# Patterns, anticipation and participatory futures

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**Abstract**

Patterns embody repeating phenomena, and, as such, they are partly but not fully detachable from their context. 'Design patterns' and 'pattern languages' are established methods for working with patterns. They have been applied in architecture, software engineering, and other design fields, but have so far seen little application in the field of future studies. We reimagine futures discourse and anticipatory practices using pattern methods. We focus specifically on processes for coordinating distributed projects, integrating multiple voices, and on play that builds capability to face what's yet to come. One of the advantages of the method as a whole is that it deals with local knowledge and does not subsume everything within one overall 'global' strategy, while nevertheless offering a way to communicate between contexts and disciplines.

*Keywords:* Anticipation, Design patterns, Pattern languages, Futures Studies, Scenario planning, Applied games

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## 1. Introduction

In the midst of the Covid-19 pandemic, it is natural to ask: could we have been able to collectively anticipate the spread of zoonotic diseases, understand possible reactions, and formulate more adaptive responses? Many individuals anticipated that something like this was coming (Yong, 2018; Karesh et al., 2012; Schwartz, 1996; Sardar, 2010; Sardar & Sweeney, 2016). Our collective struggles have shown that we need to improve our anticipatory capabilities, that is, *the capacity to act in response to or in preparation for a potential future reality* Zamenopoulos & Alexiou (2020). As the virus spread from one corner of the world to another, it was no longer simply a question of putting a stop to the disease, and not simply a biomedical issue. Pandemics belong to a class of complex problems with social, political, and economical layers. While we are not going to solve the pandemic in this paper, the complex challenge motivates the issue we want to explore: how can communities learn to better anticipate together.

Of course, people have been finding new ways to think about the future together for a long time. For example, Ariyaratne (1977) tells the story of a rural group who were finally able to complete an important construction project. After 15 years of deadlock spent waiting for outside investment, they were called to a community meeting where they figured out that they could do the job with their own labour. As dialogue and inquiry gave participants new ways to articulate and develop their thinking together, the nature of the problem they faced became easier to understand and resolve.

The methodology we will develop here centres on *design patterns*: succinctly, solutions to problems arising in the context of a repeatable activity. We propose to use design patterns to structure anticipatory peer learning as a way to relate to possible future scenarios. The link between design and the future has been aptly summarised: "Design is future creating and not future guessing"

(Banathy, 1991, p. 47). However, the relationship between the well-established design pattern methodology and futures studies has seen relatively little previous development.

After defining and giving examples of how design patterns have been used in other fields, we then examine how they can help create peer learning experiences to explore and prepare for the future. We outline a process of Open Future Design that we elaborate with three patterns that can be applied to collaboratively anticipate the unknown:

- Create a roadmap as a tool to guide collaborative creative work.
- Use scenario planning as a quasi-democratic approach to organise thinking about the future.
- Play games in a collaborative manner to explore possible futures.

Our results present a synthetic, literature-based, treatment of these topics by way of design patterns. Whether the future holds a catastrophe, or something much better, patterns can help us evolve in our collective awareness today.

### *1.1. Background*

Life itself is intrinsically anticipatory (Poli, 2012, 2020) and prefigurative schemas are relevant to the way we learn (Spiro et al., 1996). However, as a society, we are not guaranteed to be able to produce a viable outcome out of disparate individuals' capacities to think about the future. How the challenges we face are understood depends very much on who is brought to the table, and how their discussions are organised. We have chosen to focus on pattern methods here because they seem to offer the prospect of being useful for developing viable, thriving, collective outcomes.

As Covid-19 rages out of control in some places, it is easy to forget that humans have largely eradicated or successfully controlled other diseases (e.g., polio, ebola, leprosy). While design patterns do not directly prevent the spread of disease, they can help prevent uncareful thinking — the consequences of which have included failing to anticipate a problem (as happened in many places with Covid-19 in December 2019 to February 2020) — or becoming too complacent based on previous successes (many people regularly choose not to get a flu shot). Patterns have been used to support and structure discourse around complex technical systems, and we feel they have much to offer in a futures studies context.

Reflecting on the history of design-pattern discourse, Shalloway & Trott (2005) remark that the related notion of a 'cultural pattern' has received attention in anthropology (Benedict, 1934, 1946). Likewise, Joseph Campbell proposed the monomyth of the hero as an 'archetypal pattern' that is embedded in myths and stories across diverse cultures and historical periods (Campbell, 1949) and Inayatullah noted that historians have understood historical trends in terms of patterns (Inayatullah, 1998a). 'Design patterns' in a general sense have long supported both small-scale and mass-manufacture of material objects

and structures, ranging from clothing and ceramic ware to city-scale planning and, arguably, beyond.

Notwithstanding these precedents, the concept of patterns with which we are concerned here originated with the architect Christopher Alexander and his collaborators in the 1960s and 1970s (Alexander et al., 1977). One feature of their proposed pattern-based approach was to enable all stakeholders to participate in the process of architectural design. Additionally, they promoted a natural quality found in traditional architecture, versus the artificial one due to central planning (Alexander, 1965).

Alexander's notion of an architectural pattern language was later adopted and adapted within the field of computer programming (Beck & Cunningham, 1987; Gamma et al., 1994). This trend received a large boost with the invention of wiki software, which was first used to collect and curate collections of such patterns (Cunningham & Mehaffy, 2013). Subsequently, 'design pattern' methods roughly in the Alexander style have been applied to other domains, including social activism (Schuler, 2009), the transition movement (Hopkins, 2010; TransitionNetwork.org, 2010), organizational development Manns & Rising (2015), disaster prevention Furukawazono et al. (2013), and mental health Pierri & Warwick (2016). Particularly relevant to our interests, Schuler (2009) speaks about patterns as conceptual tools for the future, and developed a pattern called "*Future Design*." Iba develops a future-oriented view on pattern language discourse, with a recent presentation on "Creating Pattern Languages for Creating a Future where We Can Live Well" (Iba, 2019).

Since the definition of a pattern has been criticized for being inexplicit and inadequately explained (Dawes & Ostwald, 2017) and treatments by different sources vary, we would like to begin by explaining our understanding of the term *design pattern*. We recognise that some readers may bring their own existing understandings of this term, and that some will have no previous familiarity with it at all.

As our starting point, we take Alexander's phrase: "Each pattern is a three-part rule, which expresses a relation between a certain context, a problem, and a solution" (Alexander, 1979, p. 247). However, Gabriel (2002) emphasises that Alexander's full description of patterns goes well beyond the one sentence quoted above. Leitner supplied the following further summary text: "Patterns are shared as complete methodic descriptions intended *for practical use by experts and non-experts*" (Leitner, 2015) [emphasis added].

As we look into the matter two central elements emerge. Like an ellipse, the concept of the design pattern has two foci: context and community. The context is the type of activity which is being considered, such as designing a building or writing software. Communities include stakeholders — experts and non-experts alike — who are involved with or affected by a particular project. Also central to Alexander's pattern-based methodology is the notion of a *pattern language*, in which each pattern description itself is contextualised by related pattern descriptions that deal with related problems. By working together as a whole, a pattern language produces coherent entities (Alexander, 1999). Within a specific design challenge, cascading patterns overlap to build out a design

solution: for example, to fully develop a building, one would require a pattern for a suitable type of roof.

A pattern identifies how a community may proceed to obtain an outcome by making certain choices and applying specific methods to suitable materials. At the heart of the design pattern method is the feature of practicality. Each pattern description, when followed, should help the people using it — the community — overcome some real or potential conflict (Alexander & Poyner, 1970, pp. 9-10).

We will argue that patterns can be used to promote the organic, vital quality of resilience, that is, "not to be well-adapted, but to adapt well" (Downing (2007); quoted in Tschakert & Dietrich (2010)). We find the 'non-expert' portion of Leitner's definition particularly important and hope to explain our ideas in an accessible way.

The most tangible metaphor is to treat each pattern as a journey, "a path as a solution to reach a goal" (Kohls, 2010, 2011). In this treatment, patterns are understood to have an initial condition and an end condition, defined within some context, and the corresponding problem is understood to be that of finding a good way to move from the initial state to the goal.

The economist Elinor Ostrom related Alexander's pattern language to Arthur Koestler's notion of 'holons' — stable components "in an organismic or social hierarchy, which displays rule-governed behavior" (Ostrom (2009, p. 11); Koestler (1973)). This citation makes explicit the analogy between patterns and Ostrom-style institutions. Ostrom's framework understands an institution as a stable set of action-situations and proposes that each such situation be understood as a whole, while simultaneously being part of a larger social arrangement and a result of particular circumstances. In short, pattern language and institutional design are seen to be closely compatible.

In their applications within the computing discipline, patterns are typically written down with a template for easy communication and discussion. For example, alongside "Context", "Problem", and "Solution", a typical template may include things like "Rationale" (reasons for preferring this specific solution over other possible solutions; Meszaros & Doble, 1997) and "Examples". Typically a picture is included to serve as a mnemonic. Corneli et al. (2015) added a "Next Step" facet to the template they used, with the implication that each pattern is imperfectly realised and that further steps should be taken to manifest it within a collaboration.

Patterns have also been discussed in explicitly computational terms, though that direction of work remains mostly at the level of a proposal (Alexander, 1999; Moran, 1971), which has seen limited discipline-specific uptake within architectural design (Jacobus, 2009; Oxman, 1994). As a whole, the pattern discipline continues to evolve as an active area of research, particularly in computing disciplines. Several parallel yearly conferences are devoted to the exploration of themes touched on here, with some of the most developed contributions appearing in the Transactions on Pattern Languages of Programming book series. The specific contribution we make in this paper is to unite the pattern language and futures discourses as a coherent whole.

*1.2. Pattern(s) as anticipatory method*

In an everyday sense, 'patterns' are elements of an environment that recur in time or space. Because we expect them to recur, the patterns we recognise bear on our thinking about the future. The methodology we apply below is the reanalysis of existing future-oriented discourse within the specific constraints of the design pattern approach. The goal is to reimagine and reassemble it "together as a different version to bring out something latent or implicit in the original" (Handelman, 1998, p. 85).

As we reviewed above, we refer here to a specific methodological system centred on pattern descriptions that are intended for practical use (Leitner, 2015). Patterns of this form are accompanied by and embody principles that can lead from problem identification to problem solution within a given domain.

In concrete terms, we imagine how existing future-oriented discourses can be discussed and thought about using the pattern format. If the goal of future planning is to move from being reactive to being proactive, patterns methods have much to offer: they can help imagine and anticipate possible futures, and can help communities build resilience.

Through the comparative analysis of different future-oriented discourses vis-à-vis design patterns, we reimagine patterns as being something more than design tools: they become the building blocks for possible futures. Used in this way they present a way to "deconstruct [...] conventional thinking to produce a shared view of possible future outcomes that can break existing paradigms of thinking and operating" (European Foresight platform, 2020).

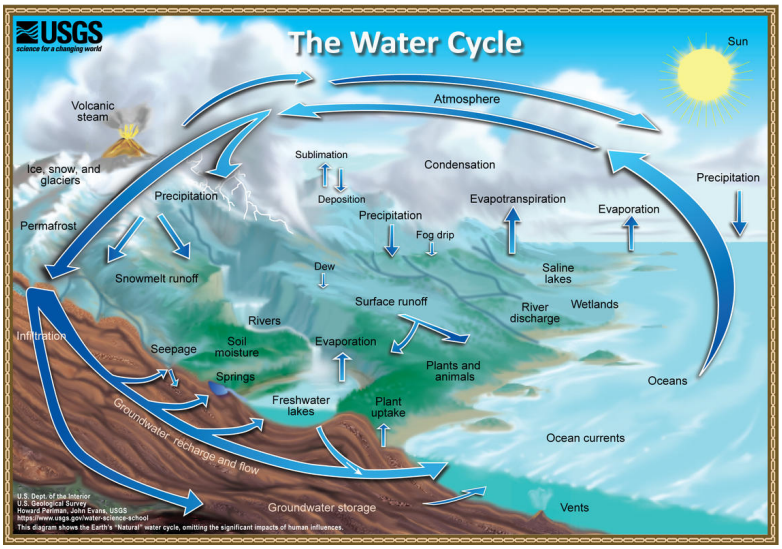
From the many existing approaches to scaffolding anticipation—including speculative fiction (Gill, 2013; Liveley, 2017; Wagner-Lawlor, 2017), design fiction (Danoff et al., 2019; Kerspern, 2019; Raymond et al., 2019) prospective studies (de Jouvenel, 2004) and predictive statistical models (Arslan & Shamma, 2006)—we get limited insights about how to engage together. It is understood that anticipatory behavior can be provoked and improved (Bishop, 2018); and as individuals, we can engage in anticipatory learning, "the general mechanism of learning to generate predictions or learning a predictive or forward model of an encountered environment or problem" (Butz & Pezzulo, 2012). We see design patterns as filling a gap that addresses participatory anticipatory learning for complex problems requiring a group response.

## 2. Open Future Design: Patterns for anticipating the future

In this section we use design patterns to analyze three approaches to thinking about the future, namely: *roadmaps*, *scenario planning*, and *play*. We posit that the process of anticipation requires a set of choices that will be associated not only with our vision of the future — values, situations, materialities — but also with the process in which this anticipation happens.

To begin our exploration of how design patterns relate to futures studies, we refer to Schwartz (1996, Appendix, pp. 241-248), *viz.*, his "Steps to Developing Scenarios". This process follows an outline with a striking similarity to a design

pattern template. Both Alexander and Schwartz advocate the identification of driving forces in a context. However, unlike Alexander, Schwartz does not intend to resolve conflicts between the forces within a harmonising design. On the contrary, the aim in scenario development method is to understand how these forces might evolve and lead to diverse scenarios. As scenarios develop, they can serve as the ground for developing new design work in Alexander's sense. With these aims in mind, the pattern Open Future Design serves as an entrypoint (Figure 1). As a mnemonic image, we chose to illustrate this pattern with a depiction of the water cycle, to show how Open Future Design, and subsidiary patterns that we describe below, represent an anticipatory cycle: as unknown and unexpected features materialise, our expectations develop and change.




PATTERN: *Open Future Design*

**Context** People need to coordinate, plan, and maintain social cohesion.
**If** a culture can develop based on shared learning BUT there is no reliable oracle that can tell us what to expect;
**Then** use design pattern methods to articulate multiple futures. This work can be guided by further patterns, e.g., to develop

- *a language for projects* → Roadmap
- *a language of future scenarios* → Participatory Scenario Planning
- *a language of roles* → Play to Anticipate the Future

Figure 1: Open Future Design Pattern. Inner image credit: Howard Perlman, USGS. Public domain, Source: *https://www.usgs.gov/media/images/water-cycle-natural-water-cycle*

*2.1. The future via a roadmap*

The success of collaborative learning and work is related to the conditions, processes, and tools that make such collaborations possible. One indicative example of 'success' is found in the Wikimedia Foundation (WMF). The WMF has made use of various planning exercises, ranging from the collaboratively produced "Wikimedia Strategic Plan" (Walsh, 2011) to the more centrally produced "Knowledge Gaps" (Zia et al., 2019) whitepaper. These activities should be understood in the context of an ongoing decline in participation in Wikipedia projects (Simonite, 2013), and the capture of online attention in various privatized media (Fuster Morell, 2011).

WMF's strategy and plans are not dissociated from other practices in their projects. Structured guidance is made available to contributors to Wikimedia projects at all stages. Such guidance ranges from policies to tutorials, and helps contributors produce and maintain content to the benefits of the entire user community. Specific guidance on how to participate in data analysis and other planning activities that can help steer the project is available.

The culture around maintaining, updating and using such anticipatory artifacts comprises a system for governance. For example, online governance may evolve along the bottom-up lines proposed by Schneider et al. (2020). In the WMF example, the context, problems, and solutions are relatively explicit — e.g., gaps in knowledge should be filled to make Wikipedia more inclusive — and the community is included in both framing the problem and shaping the solution. On the other hand, justifiable doubts are likely to arise among constituencies who are brought into a "culture of consultation" (Featherstone, 2017) that is concerned, not with authentic community power, but with establishing and rubber-stamping the semblance of participation.

One way to enhance community power is to make plans based on pattern methods. In what might be a best case scenario, the "Roadmap" pattern from Corneli et al. (2015) becomes the entry point to a "living language" made up of other evolving patterns Alexander et al. (1977), each pointing to clearly designated next steps. While any notion of a plan presumes an anticipation of outcomes, the Roadmap may not be linear and directed towards one goal. It may instead be an organic tool for coordination across variables such as time, groups of people, and software. In the Scrum agile project management methodology, the "Product Roadmap" is complemented by a "Product Backlog" that offers a best-guess linear path through the various branching paths contained in the roadmap, while the roadmap itself keeps track of all anticipated alternatives as well as dependencies between tasks (Sutherland et al., 2019, p. 220, pp. 222-223).

We can consider the pheromone trails of ants as an example of an adaptive Roadmap in nature. By analogy, human participants in a collaborative design process should, metaphorically, have their antennae out to detect and respond to developing situations, and should develop suitably immediate methods for sharing meaning with each other. Notice that by anticipating future conflicts, we can take steps in advance to prevent and mitigate the problems that might ensue. This connects the Roadmap pattern with the broader themes of improving our ability to anticipate the future and resiliency. Figure 2 re-summarizes

the Roadmap pattern, and highlights its nested context, in which the immediate community of users must adapt to a broader context in motion.



PATTERN*: Roadmap*

**Context** a group needs to coordinate its activities over a period of time.
**If** the landscape is complex and not completely knowable BUT adjustment to action based on feedback is possible;
**Then** use an explicit mechanism to share information about goals, obstacles, methods, and resources.

**Example** Everyday roadmap languages include both iconic map and road sign symbols; when people are confused or lost they may ask for help or try to find their own way back to the road using other informal languages.

Figure 2: Roadmap pattern. Image: Maritess Sulcer, CC0, via Wikimedia Commons *https://commons.wikimedia.org/wiki/File:Trees_and_bridge_and_stream_and_clouds_--_25_of_33.jpg*

### *2.2. The future via peer learning and scenario planning*

While scenario planning can be carried out by groups of experts, there is the danger that the life experiences and opinions of other people who are affected by their planning will not be considered. This could lead to a narrow understanding of the problem and a limited vision of what the future can look like. In

*participatory* scenario planning, the activities are carried out by a community of non-experts with the experts or designers limiting their role to facilitation and moderation (Johnson et al., 2012). For instance:

- The Consultative Group for International Agricultural Research project in Mali held a workshop at which 38 people including farmers, journalists, local leaders, forest officers, extension workers, policy makers, and representatives of NGOs and District Councils collaborated to produce scenarios for the future of agriculture (Johnson et al., 2012; Totin et al., 2018).
- Agricultural development workshops in Nigeria included a teacher/researcher, a journalist, an engineer, an animal nutritionist, a camera operator and head of a women's group (Olabisi et al., 2016).
- In the Minnesota 2050 project, participants were selected from a variety of professions and leadership roles to produce scenarios for energy and land use, and combined modelling with scenario planning (Olabisi et al., 2010).

In all of these examples, scenario planning was taken out of the hands of the experts, but scientists and other consultants still played a specialized role. A varied community that designs and produces something by and for themselves would be more fully analogous to Alexander's vision of community-designed architecture (Alexander et al., 1977). If community members could be involved as scenario planners, or as architects, why not also involve them as citizen scientists or modellers? Wildschut (2017) points out that the 'local knowledge' of citizens can make the inquiry process more robust: "Citizens have valuable knowledge that is often out of reach for university scientists." Another related people-centered approach is described by the DesignJustice.Org (2018) whose principles are addressed to "people who are normally marginalized by design." Whether in a scenario planning, scientific inquiry, or design process, participants need more than just 'access', i.e., not just access to existing scholarly materials produced elsewhere, but also to a platform and literacies/capabilities that allow them to contribute and negotiate; Heidegger used the term *Zuhandenheit*: "readiness-to-hand, handiness." Existing collections of design patterns can help address these requirements. For example, many of the "Wise Democracy" patterns elaborated by the Co-Intelligence Institute (2019) would carry over well to participatory scenario planning. Specific patterns include: (#30) "Expertise on tap (not on top)", (#33) "Feeling heard", (#82) "Systems thinking", and (#94) "Wise use of uncertainty".

The pattern language contains other useful hints for scenario planners. We can elaborate with a concrete example. Johnson et al. (2012) made this observation based on their experience with participatory scenario planning: "Groups with too little diversity converge quickly and without exchanging contrary ideas, and will necessarily develop fewer options for action. Yet too much diversity can hobble a group and overwhelm a process with too many conflicting values and perspectives." We do not consider that there is such a thing as "too much diversity" in any absolute sense, only limitations to a group's ability to integrate diverging visions of the future and to learn from conflict. Indeed, since

managing conflict seems to be a key concern here, it would make sense to express this experience with a design pattern. One benefit of diversity was noticed by Schwartz (1996, p. 97): "being part of several networks not only opens up diverse arenas, but allows you to cross-check the insights that emerge among people from vastly different places." Relevant patterns are again found in the Wise Democracy pattern language: (#26) "Diversity" and (#88) "Using Diversity and Disturbance Creatively". Scenario planners who are concerned that there is too little or too much diversity within their dialogue could refer to these patterns to enrich their thinking on the matter with to-hand knowledge. An example of where this could have been helpful: Ewing (2018) describes how, when forty-nine individual public schools in Chicago closed, students were not equal participants in the decision and were left to deal with the "intense emotional aftermath" (p. 159). She suggested that "perhaps CPS can learn from the school closures to create structures that let stakeholders participate meaningfully in decisions so as to incorporate community voices" (p. 161).

Patterns could help with further specifics of process. One reference point in futures studies is the Delphi method RAND Corporation (2020). However, in Delphi, summaries are produced by a facilitator synthesising expert feedback and sharing it back out to participants. Delphi's "goal is to reduce the range of responses and arrive at something closer to expert consensus" (*ibid.*). In a world in which we respect each participant as an expert with relevant experiences, we need more complex ways to integrate multiple voices. The pattern methodology itself has the potential to incorporate contributions from diverse stakeholders, by foregrounding context and community at each stage. It is worth pointing to extant patterns *for* pattern writing (Iba & Isaku, 2016; Meszaros & Doble, 1997) and workshopping (Coplien & Woolf, 1997; Gabriel, 2002); and to criteria for evaluating patterns throughout their lifecycle (Petter et al., 2010). We summarise these reflections in a "Participatory Scenario Planning" design pattern in Figure 3.

### *2.3. The future via play*

Before the future arrives, one way to prepare is by playing: this not only simulates different scenarios, but also helps us experience what it is like to live in them. "Play can help kids learn, plan and even persevere in the face of adversity" (Willyard, 2020). There can be other benefits—for adults as well. According to legend, the ancient Maeonians developed various games (played with dice and balls, as well as a lottery) that helped them endure hunger (Herodotus, 1975, p. 43). According to Huizinga (1949), play is in fact fundamental for civilization. In a metaphysical view that centers aleatory practice, "the game and fate fuse into destiny" (Gane, 1991, p. 25).



`PATTERN`*: Participatory Scenario Planning*

**Context** you want to plan for possible future scenarios.
**If** you have an interested group BUT no "expert" has all the answers;
**Then** pool the collected expertise of the affected communities.

**Example** the Next Generation Services project (`http://nextgenpsf.co.uk`) used participatory scenario planning to imagine different plausible AI futures, which were then used in design sprints to expose professional services firms to potential AI challenges and opportunities.

Figure 3: Participatory Scenario Planning. *Image*: The tall ship *Irving Johnson* lies hard aground, only yards from shore, near the entrance to Channel Islands Harbor, *Oxnard, California*, March 2005. US Coast Guard, public domain.

Games offer 'oblique strategies' (à la Eno & Schmidt — see Taylor (1997)) for anticipating the future. They transform the future into something we can play with and thereby expand the horizons of what we believe is possible in the coming days and years (Sweeney et al., 2019). This has not gone unnoticed by futurists. For example: in 2015, United Kingdom government officials developed a futures gaming process called the "Scenario and Implication Development" system to play at the 5th edition of the Global Strategic Trends review (Sweeney, 2017). Causal Layered Analysis (CLA) (Inayatullah, 1998b) is a well known tool for imagining possible futures: at the conference "Futures Studies Tackling Wicked Problems" in Finland, attendees played a CLA game to test out different scenarios, and the game allowed them to include a diversity of perspectives (Heinonen et al., 2017).

Examples of recent games designed to explore possible futures include:

- ***The Oracle for Transfeminist Technologies*** (Varon et al., 2019): a card deck designed to collectively envision and share ideas for transfeminist tech in the future. This game is based on the assumption that people who are excluded by technology in today's world are qualified to design future scenarios (Varon, 2020-07).
- ***Flaws of the Smart City*** (Friction, 2016): a card game exploring smart-city futures (that architects like Alexander might also enjoy playing). The co-founder of Design Friction explains how the game tries to make it easier for those outside the design community to easily use future scenarios and design fiction (Kerspern, 2019).
- ***HEY! Imaginable Guidelines*** (ŞanalArc, 2018): another city themed game in which participants use cards to discover how cities are designed by selecting topics that are necessary, desirable, or irrelevant to the urban-design problem they are trying to solve (Hattam, 2019). The game provokes players to make changes and continue the conversation after play has concluded.
- ***Equitable Futures***: a future-oriented game produced by the Institute For the Future. It is used to develop understandings of challenges to overcome as players try to find new ways to build a more equitable future and re-imagine our worlds for 2030 (Finlev et al., 2019).

Although it was not designed by futurists, the cooperative board game *Pandemic* (Leacock, 2008), where players work together to find a cure for a virus spreading across the world, has served a similar purpose. It has helped people cope with the current Covid-19 pandemic (Genovese, 2020; Leacock, 2020), including doctors and medical students (Woo, 2020; Borrelli, 2020). Employees of the Centers for Disease Control previously commented that the game "reflected the reality and values of public health" (Lee, 2013). In this case the upshot is that players have been experimenting with how to solve a pandemic, thanks to this game, since 2008.

There are existing design patterns for creating games and using them to learn. Iba's pattern "Playful Learning" is based on the observation "It is easier

to enjoy learning something new if you take pleasure in the results" (Iba, 2018, p. 9); for instance, one might opt to learn to program by programming a video game, or practice a foreign language by playing Two Truths and a Lie. The genre of "'Applied Games' or 'Games with Purpose' ... seek to harness our natural curiosity and playfulness in the search for new knowledge" (Rawlings, 2016). One sub-genre is described in Schuler's "Socially Responsible Video Games Pattern" where games are used as a vehicle to introduce real-world issues in an interactive format (Schuler, 2008).

As Iba's "Playful Learning" pattern stated, the games themselves need to be fun and tangible. As one simple step in this direction, Iba created physical cards for his patterns, so that playing with them would be more enjoyable. Collections of game design patterns are also available that can help designers build in features that make games attractive and interesting (Kreimeier, 2002).

Figure 4 summarises the foregoing reflections as a pattern, Play to Anticipate the Future. Perhaps future definitions for *play* will catch up with the reality this description embodies: none of the five current definitions for this word as a verb in the Webster dictionary include anything about learning or the future (Merriam-Webster, 2020).

## 3. Discussion

Our presentation at the Anticipation 2019 conference took the form of a scripted dialogue that we invited audience members to read aloud. We hoped that this might spark conversation, by modelling the kind of participatory dialogue we wanted participants to feel comfortable having—better than one presenter speaking to the audience the entire time. As it happened, the first comment we received from one of the volunteer readers pushed back vigorously against the format: "Why did you ask us to read a script, rather than improvise?" Subsequently, this led us to explore more authentically collaborative methods for anticipating the future. Central to this process was the fact that we are a diverse international group of peers who were prepared to think together about our local concerns, investments, and activities. This paper has shown how the pattern method can be used to develop a mindset for collaborative anticipation that embraces diverse points of view. Such usage was already hinted at by Moran (1971), who wrote that "From the point of view of methodology, it is not so important how good each pattern is, but only that each one is transparent and open to criticism and can be improved over time."

With further application of the method, futures discourses could become more 'generative', i.e., refashioned as "a kit of parts ... together with rules for combining them" (Alexander, 1968). To meet this need, the set of patterns would have to be more fully elaborated. In the domain of the built environment, Alexander (1999) refers to inspiration coming from "generative schemes that exist in traditional cultures" with "as few as a half a dozen steps, or as many as 20 or 50." It is not simply a matter of adding more patterns — but one of understanding the unfolding processes that they represent, when taken together.



PATTERN*: Play to Anticipate the Future*

**Context** you want to have fun with friends, colleagues or acquaintances.
**If** you want to explore possible futures BUT time travel does not exist and you don't know what to expect;
**Then** play a game that lets you experience a plausible future scenario together.

**Example** a bipartisan group of politicians, former civilian and military officials, and academics gathered to play a scenario planning game to anticipate the possible aftermath of a contested election: at the very least the scenarios they came up with managed to surprise them (Bidgood, 2020).

Figure 4: Fig 4. Play to Anticipate the Future Pattern, illustration from *Romps* by Harry Furniss, published by Routledge and Sons in 1886 (public domain via Archive.org)

Pattern language development can give us another way to think and talk about the "process of language development" that already goes on inside institutions (Schwartz, 1996, p. 205). Further challenges include understanding complex interactions between contexts and communities that previously had little relationship with each other. Another concern is how to balance futures discourse with other pressing concerns: if we wait too long, or structure the discourse poorly, we will end up being reactive. One specific direction for development of the ideas would be to work jointly with the Participatory Modeling (PM) community, addressing concerns like "What are the major interpretation and communication issues in PM? Are there best practices to address these issues?" (Jordan et al., 2018). To be most applicable in this setting, the methods would need further elaboration, e.g., to become more computationally salient to address the specific needs of modelers, and to incorporate the developing insights from Alexander and others on what makes this method work well.

The vision of "Open Future Design" (§2) is to be open to almost any possibility when we imagine it. As a touchstone, when they are first suggested, beneficial future visions "should appear to be ridiculous" (Dator, 2019). However, in real life, there are scenarios we prefer to close off. We can prepare for various eventualities by experiencing these possibilities in games (§2.2) or other simulated encounters, rather than in real life.

A pattern-theoretic approach to futures links 'futures studies' to processes of growth and development. Community assembly of ecosystems and the growth of embryos are relevant natural examples of processes that have 'futures'. These natural processes are ordered, not only in time but also in space. Reflecting on these examples, we see more clearly how it is that individual futurists — without the contexts in which their ideas could become meaningful — are like little bits of organs that are detached from the bodies and materials needed to function. Similarly, while science fiction can provide a 'thinking machine' (Doherty & Giordano, 2020), it cannot on its own provide historically robust alternatives to the global crises that we face. Narrative and design fiction methods (Gill, 2013; Liveley, 2017; Wagner-Lawlor, 2017) have been applied in design, and from there, reapplied in futures studies. When these methods are used to prepare effectively, we would be inclined to think of them as part of a "Roadmap" in the sense we used this term above (§2.1).

A roadmap is a kind of genetic code for a successful project: and here one should keep in mind that a genotype is not a simple blueprint (Pigliucci, 2010). You cannot create a roadmap in a vacuum and hand it to someone and expect it to work. Collaboration and communication are needed to create a useful roadmap and that process itself can become the largest part of the value of the roadmap. Once the plan is in motion and things start not to happen as expected, the roadmap can be used as a reminder of what was planned to help groups respond based on prior collaborative knowledge and understanding. The understanding it took to create it is key. It creates "resilience".

Diversity — secured via patterns like "Participatory Scenario Planning" (§2.2) — can boost our collective immunity in a broad sense of that word (Wambacq & van Tuinen, 2017). Diversity allows our understanding of the

whole to be enriched. Without it, we risk attacking potentially vital imagined futures, as when in autoimmune disease, the immune system attacks healthy body cells (Houwen & Rachmany, 2018). If we fail to consider and account for the needs of all stakeholders affected by decisions, we are not acting in alignment with our basic need for survival as part of humanity and part of the ecosystem on earth.

As we consider the needs and interests of broad cohorts of stakeholders we can position this work as a response to Kostakis et al. (2015) who argued for a development model based on "thinking global and producing local." At the centre of their vision is a global pool of designs, which are put into production in local Fab Lab facilities. By contrast, in our work we have centred circumstances of cultural diversity and human relationships. In our view, the pattern method flips the Kostakis et al. formula on its head: patterns are primarily tools for thinking locally about particular contexts, individual relationships, conflicts and circumstances. Only secondarily and potentially does this lead to a shared global resource. More likely, pattern methods would simply strengthen local forms of resilience and better identify healthy futures.

Incorporating diverse communities into an ongoing futures discourse is a tall order and some humility is required. In contrast to design, which ultimately converges, futures-oriented activities are about producing diverse open-ended possibilities: this is akin to Deleuze's remark that "the system must not only be in perpetual heterogeneity, it must also be a heterogenesis" (Deleuze, 2010). This is why reading a script — for example — is not necessarily the best way to motivate discussion about the future. As Sardar (2010) pointed out, future predictions do not actually provide us with foreknowledge. However, anticipation can provide future-readiness; with this in mind, a script could be reworked as a game that participants could "Play to Anticipate the Future" (§2.3) together.

The project we have embarked on will help to address the societal need for improved facilities of anticipation. Through the generative nature of the pattern method, users can become prepared for many outcomes and possibilities. A great strength of this method is that it can be used differently in different places, with no need of assembling everything into one unified 'design'. We highlight some specific implications of the patterns we developed in Figure 5.

The individual points indicated in Figure 5 are at least somewhat plausible, e.g., Ali S. Khan from the CDC "became interested in Vaughan's game [*Plague Inc.*] as a tool to teach the public about outbreaks and disease transmission because of how it uses a non-traditional route to raise public awareness on epidemiology, disease transmission, and diseases/pandemic information" (Kahn, 2013). Potentially even more exciting is to think about what the pattern methods offer if they were adopted on a broad scale. Like Escobar (2018) proposes, they could open up the possibility to design multiple pluriverses, where we do not just make plans focused on one choice but include diversity integrally in our thinking. Reanalysis of futures methods in light of the transdisciplinary pattern methodology has given us new insights into the way futures studies work, and how they might be practiced collaboratively, across domains and cultures.



EXAMPLES:

**Looking back: how *might* the patterns we wrote have helped in 2020?**

- In January, world leaders met online to simulate many global infection-spreading scenarios by playing Pandemic (Leacock, 2008), and Plague Inc (Ndemic Creations, 2012) to see how to contain or spread the virus.
- In February, people in nursing homes and prisons engaged in participatory scenario planning, which helped shift the collective outlook of society at large, leading to more informed discussions, and safer health behaviours.
- In March, national teams developed roadmaps for curricula teaching citizen science approaches of epidemiology, and students subsequently helped lead national recovery projects.

**Looking forward: How *might* things look in the future building on this work?**

- Everyone learns "history" in school, but not "future": this material and associated techniques could become part of a new curriculum for futures.
- Policy could develop via town-hall format discussions: rather than being based on Q&A, politicians would lead citizen teams in developing different scenarios.
- E-sports could expand to explore plausible scenarios in games like SuperBugs (Nesta, 2016) where players race against the clock to keep antibiotics effective.

Figure 5: Figure 5. Implications. Image: Umberto Brunelleschi, illustration pour Fantasio d'Alfred de Musset, detail (Public Domain, *via Wikimedia Commons*)

## 4. Conclusions

Our main results outlined several ways in which design patterns can be used to model future-directed activities. It is possible to envision scenarios that bring all of these methods together. For example, consider a gaming situation in which teams are completing, or a similar setting in which there are people acting with competing interests, with different scenarios thrown into the mix (resource depletion, regulatory changes, inflation, natural disasters, actions by other players, etc.) to test their reactions. Roadmaps could be used both in the competition and in research designs to make sense of the subsequent output and draw transferable insights from the exercise.

As we have noted, a key feature of the methods we have described is to promote heterogeneity of thought and behavior. Part of the purpose of participatory methods would be to make sure we have the right experts. For example, a nurse may have noticed things on the ward that epidemiologists would never have experienced. A pandemic team with the right balance of expertise and diverse community participants — with suitable methods for collaborating and learning together — will get better results than professional analysts, scientists, and university researchers working in isolation.

Which brings us to you, dear reader. We assume you wanted to learn about the subjects in the title: or perhaps this paper was assigned to you in school and you are reading it for a grade? Either way, we are excited to invite you to use and adapt these methods. For us, this work is likely to serve as a springboard for our efforts. We hope that the paper will be practical for you, too. Or, perhaps you think we are deluding ourselves with ivory tower navel gazing? — After all, even if the broad outlines of this work are correct, a degree of skepticism is warranted given that Alexander already shared relevant clues 20 years ago. Or perhaps your criticisms run the other direction, and you found the patterns too simplistic. We, of course, remain open to these and other criticisms. Future work might integrate design patterns and macrohistorical patterns for anticipation, or combine patterns and CLA (Inayatullah, 1998a,b) to connect diverse and divergent themes across multiple layers of experience and observation. Alternative methods described by Sools (2020) focus on observing how consideration of future outcomes affects present choices. Depending on what year it is when you are reading this, perhaps these integrative suggestions have already come to pass. If you think the points we raised in this paper are meaningful and valuable, then a likely corollary is that there will still be a lot of hard work to realise the full vision.

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# Argumentation Theory for Mathematical Argument

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**Abstract**
To adequately model mathematical arguments the analyst must be able to represent the mathematical objects under discussion and the relationships between them, as well as inferences drawn about these objects and relationships as the discourse unfolds. We introduce a framework with these properties, which has been used to analyse mathematical dialogues and expository texts. The framework can recover salient elements of discourse at, and within, the sentence level, as well as the way mathematical content connects to form larger argumentative structures. We show how the framework might be used to support computational reasoning, and argue that it provides a more natural way to examine the process of proving theorems than do Lamport's structured proofs.

**Keywords** Inference Anchoring Theory · Mathematical practice · Mathematical argument · Structured proof

## 1 Introduction

The representation of mathematical knowledge and inference in appropriate formal logical frameworks is well-understood and the subject of much research. Computational tools to support this through proof checking, automatic theorem proving, and

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computer algebra are well-established, though they require formal, computationally explicit, content as input. However, the existing mathematical literature, particularly informal mathematical dialogues, and expository texts, is opaque to such systems, which cannot currently handle the variety of activities typically involved in producing such knowledge and proofs, such as, for example, exposition and argument that concerns making conjectures, forming concepts, and discussing examples and counterexamples. Our goal is to bridge this gap through devising an expressive modelling language that is closely related to the way mathematics is actually done.

Our approach to modelling such content is inspired by the general-purpose argument modelling formalism Inference Anchoring Theory (IAT), introduced by Reed and Budzynska (2010). As its name suggests, IAT anchors logical inferences in discourse. IAT has been applied to mediation (Janier and Reed 2017), debates (Budzynska et al. 2014b), and to paradoxes in ethotic argumentation (Budzynska 2013), along with other real-world dialogues (Budzynska et al. 2013). The *Inference Anchoring Theory + Content* (IATC) framework we introduce is based on IAT, but with several significant modifications. Most fundamentally, IATC is designed to bring to the surface the structural features inherent in mathematical content.

IATC could be overlaid upon formally specified contents, where these are available. Lamport's "Temporal Logic of Actions+" (TLA$^+$) (Lamport 1999, 2014) is one such formalism that could be used to model content-level expressions. Higher-level discourse structure would then be exhibited somewhat along the lines of Lamport's own semi-formal "structured proofs" (Lamport 1995, 2012). However, unlike structured proof, IATC does not aim to reshape the way people do mathematics, but to model it more exactly. As such, it constitutes groundwork for a future generation of computer systems that can collaborate with mathematicians and students in a way these potential users already understand. Epstein (2015) highlights the "extent to which a person believes that her work experience or product has been facilitated or improved by the collaboration" as a key evaluation metric for assessing collaborative intelligent computer systems. The key metric at this stage is more basic, namely, we are interested in the degree to which IATC can represent real-world examples of mathematical practice in a way that can make them accessible to computational reasoning. After introducing the modelling approach, we use several examples to show that IATC is indeed satisfactory in this regard.

- Our first example is a school-level challenge problem that was presented in a public lecture by the mathematician Timothy Gowers (Gowers and Ganesalingam 2012). The lecture aimed to motivate and contextualise a project, then beginning, to develop mathematical software that "operate[s] in a way that closely mirrors the way human mathematicians operate" (Ganesalingam and Gowers 2017, p. 255). The reasoning needed to solve the challenge problem remains beyond the scope of the computational method that Ganesalingam and Gowers ultimately published, but it is both sufficiently simple and sufficiently realistic to introduce the practical aspects of working with IATC.
- Our second example is a question posed on the online Q&A forum MathOverflow, together with the ensuing dialogue. MathOverflow is part of the Stack Exchange network of community question-and-answer websites, which is par-

ticularly popular with software developers. The MathOverflow sub-site is devoted to discussions about research-level questions in mathematics. Such discussions are very different from the textbook-style proofs treated by Ganesalingam and Gowers (2017), and we discuss the considerations that such discussions would impose on computational modelling efforts.

– MiniPolymath 1 through 4 were part of a series of experiments in collaborative online mathematics known as "Polymath projects" (Nielsen et al. 2009–2018). While other projects in the series tackled novel research, the problems in the MiniPolymath subseries were drawn from the Mathematical Olympiad, a premier competition for pre-college students. Six problems are given, and the examination takes place over two days with three problems to be solved each day. Whereas individual Olympiad participants frequently fail to solve three challenge problems in the four-and-a-half hours allotted for that purpose, all four of the collaborative MiniPolymath efforts generated a solution. However, it should be noted that some of these solutions took more than 24 hours to develop. IATC can help us understand how the proof efforts progressed, and can potentially help us understand why they were (mathematically) successful.

The plan of the work is as follows. Section 2 reviews previous research on mathematical argument, presents a brief introduction to Inference Anchoring Theory, and describes Lamport's structured proofs as an example of the state of the art for modelling informal mathematical knowledge. Section 3 introduces IATC, describes the grammar of IATC markup, and describes the differences between this language and IAT. Section 4 presents our analysis of the examples outlined above, which have been marked-up with IATC in order to illustrate the relevant modelling concerns. Section 5 summarises and reviews the contribution, situates our work in relationship to the broader literature, and outlines potential directions for further work.

## 2 Background

In this section we state what we mean by argumentation, and survey previous research on argumentation in mathematics (Sect. 2.1). We then describe Inference Anchoring Theory (Sect. 2.2) and structured proof (Sect. 2.3), two landmarks that guide our effort.

### 2.1 Argumentation and Mathematical Arguments

Our approach to argument builds on Buzynska and Reed's Inference Anchoring Theory (IAT), which we describe in Sect. 2.2. The specific conception of *argument* that underlies IAT is as follows:

> [A]rguing can be interpreted as an illocutionary act that comes about as the result of a relation between uttering a premise and uttering a conclusion, thus mirroring the logical structure of inference[.] (Reed et al. 2017, p. 146)

Reed and Budzynska (2010) note that in everyday language the term "argument" is used to describe a particular kind of interaction as well as the shared understanding extracted from these interactions, as "evidence" or "proof." The purpose of IAT is to make the links between discourse and reasoning explicit.

Concerning argumentation in a mathematical context, Pedemonte (2007, p. 39) argues that "analysis of the 'content' is not sufficient to analyse all the cognitive aspects in the relationship between argumentation and proof." A large part of mathematical discussion is in essence meta-discussion about meta-level objects, such as proof strategies that are suggested on the fly and debates about whether these strategies are likely to work as intended.

Mercier and Sperber (2011) distinguish arguments from inferences: only in the case of arguments "the reasons for drawing this conclusion on the basis of the premises are (at least partially) spelled out" (p. 58). By contrast, formal mathematics is typically based on the reductive assumption that "mathematical reasoning may be identified with classical, deductive inference" (Aliseda 2003, p. 25). However, everyday mathematical reasoning plainly involves more than just proof steps. Here are two examples of familiar patterns of reasoning that appear in MiniPolymath 3:

| | |
|---|---|
| *argument from authority* | "My bachelor thesis supervisor said that one can't use the word cardinal if we talk about finite sets. One has to use the words 'number of elements'" (Tao et al. 2011, 19 July, 9:46 pm). |
| *argument from analogy* | "Let me check that I got the example correctly: is this 'a point inside a regular polygon'? Isn't it established in an early comment that the example of a point inside an equilateral triangle indeed visits *all* the points? Can you clarify the difference here?" (Tao et al. 2011, 19 July, 9:19 pm). |

The word "argument" has been attached to several distinct kinds of mathematical artefacts and activities. This term may indicate *proofs* (Gasteren 1990), *informally-presented proofs* (Tanswell 2015), *proof sketches* (Lamport 1995), *aspects of reasoning that are not addressed by formal deduction* (Aberdein and Dove 2013) and *elements of persuasive discussion* (Zack and Graves 2001).

Some theorists have expressly contended that proofs are not arguments: this is because proofs offer certainty, while arguments cannot (Dufour 2013). Nevertheless, communication of reasons and reasoning can be found throughout mathematical practice. Pedemonte (2007) highlights the use of inductive and abductive logic as well as deduction in mathematical processes that move "from conjecturing to the construction of proof [to] the proof as product," and in which "content rather than formal criteria" can guide the proving process. Dufour (2013) gives examples of argumentation "not only before and during the proof but also after, at least as long as it can be criticized" (p. 74). Other scholars have observed features such as these:

- Published mathematical writing tends to be particularly explicit about reasons and conclusions (Dove 2009, p. 149).
- Not only the *Prover* but also the *Skeptic* "has an important role to play, namely to ensure that the proof is persuasive, perspicuous, and valid" (Dutilh Novaes 2016, p. 2618).
- On the way to a proof, degrees of confidence about the conclusions to be drawn may be discussed (Inglis et al. 2007, p. 17).
- Mathematical meanings need to be interpreted, and this tends to be a struggle (van Oers 2002, p. 360).

Carrascal (2015) provides an excellent survey of recent thinking about argument in mathematics, highlighting its connections with mathematical practice. Carrascal advises: "in order to learn more about the nature of mathematical practice and how its products are evaluated, we should be looking at real examples of this practice." She points to Pease and Martin (2012) as a notable example in this genre. Once we have developed a suitable apparatus, Sect. 4 will tackle several real-world examples, including a detailed reexamination of the dataset studied by Pease and Martin.

"Blog maths" (Barany 2010) and other online discussions, for example, on the question-and-answer site MathOverflow, can "tell us about mathematicians' attitudes to working together in public" as well as the "kinds of activities that go on in developing a proof" (Martin 2015). In the process of creating a proof or mathematical theory, divergent understandings are negotiated using shared concepts, definitions, and standards for proof, even as the concepts evolve. Along these lines, Pease et al. (2017) used the methods of structured and abstract argumentation to formalise the theory of informal mathematics developed in Lakatos's *Proofs and Refutations* (1976) as a set of rules for turn-taking in a dialogue game. This work shows that formally specified and fully implemented argumentation tools can be brought together and applied to a specific, demanding, domain of human reasoning.[1] Dauphin and Cramer (2018) produced a similar model of natural-deduction style arguments, explanations, and the "*prima facie* laws of logic" such as may be debated in work on mathematical foundations. These prior efforts focus on developing rules that give a plausible codification of mathematical process. Our concern is different, but complementary. We are interested in a better understanding of what is actually said in mathematical arguments, and on the reasoning that is conveyed. Accordingly, we will adapt a general-purpose argument modelling approach, *Inference Anchoring Theory*, which is described in the following section.

[1] The dialogue game defines ordered operations on a shared information state represented in the Argument Interchange Format (AIF) (Lawrence et al. 2012), which is then interpreted by The Online Argument Structures Tool (TOAST) (Snaith and Reed 2012) and passed on to DungOMatic (Snaith et al. 2010) to calculate the grounded extension, which in this case represents the currently accepted, collaboratively constructed, proof or theory under discussion.

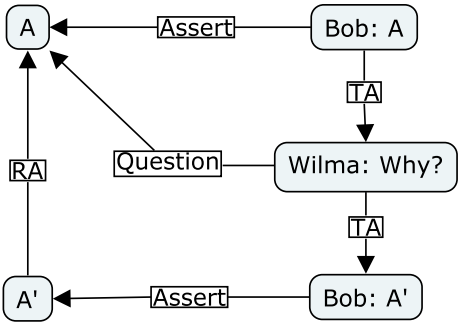



**Fig. 1** IAT diagram for the conversation '*A*'/'*Why?*'/'*A*'

### 2.2 Inference Anchoring Theory

Inference Anchoring Theory (IAT) is used to model the logical relationships between the propositional contents of utterances made in dialogues (Budzynska and Reed 2011). As noted by Reed et al. (2017), the inspiration for developing IAT lies in earlier work on representing dialogue in the Argumentation Interchange Format.

IAT is grounded in a notion of *dialogical relations* that formalise the informal "conventions and norms that dictate the flow of dialogue" (Snaith and Reed 2016). Per Budzynska and Reed (2011), these dialogical relations are also referred to as "*transitions*," a term that is meant to recall the notion of transitions between operating states in a finite state machine. Indeed, when the norms have been fully codified in a *dialogue protocol*, the transitions are exactly described by a finite state machine.[2] Content relationships are typically identified by matching locutions against known argument schemes, e.g., an 'Argument from Positive Consequences' is associated with two transitions, 'challenging' and 'substantiating' (Walton et al. 2008). Budzynska et al. (2014a) describe Inference Anchoring Theory in terms of three components:

(i) relations between locutions in a dialogue, called *transitions*;
(ii) relations between sentences (propositional *contents* of locutions); and
(iii) *illocutionary connections* that link locutions with their contents.

(Budzynska et al. 2014a), emphasis added

In Figs. 1 and 2, below, "TA" stands for a *default transition*, "RA" stands for *application of rule of inference*, and "CA" stands for *default conflict*. That is to say, there is no explicit formal dialogue protocol attached to these two examples.

Figure 1 is a typical example of an IAT analysis. Figure 2 illustrates a feature that was not directly mentioned in the list (i)–(iii), above; specifically, this figure uses an *'implicit' speech-act* to anchor propositional content on a transition rather than a locution. Here, when a speaker asserts '*A*' and their interlocutor says '*No*', the

[2] In such a setting the formal argumentation-theoretic techniques and tools mentioned in Footnote 1 can be applied, though IAT models are not required to be fully formal in this regard.

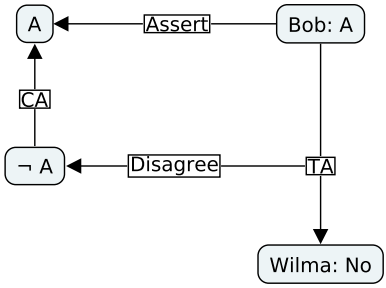



**Fig. 2** IAT diagram for the conversation '*A*'/'*No*'

logical content '¬*A*' is attached to the transition, rather than to the negating word. The basic rationale is that the locution '*No*' cannot be made sense of without the preceding context. There has been some debate about what to do about this. Botting (2015) says that the choice to anchor arguments on transitions is a conceptual mistake. However, for the creators of IAT, the reason illocutionary acts can be rooted on dialogical relations follows

> ...directly from pragma-dialectical analysis which views the speech act of assertion [...] as occurring at the 'sentence' level, and the speech act of argumentation as occurring at a 'higher textual level.' (Budzynska and Reed 2011)

Visser et al. (2011) describe the theoretical considerations in more detail. The pattern common to both Figs. 1 and 2 is that allowable inferences are governed by dialogue norms. In Fig. 1, for instance, we would not immediately know that '$A'$' is intended to *support* '*A*' without Wilma's intermediate question which explicitly requested such support. Given the context, the intended inference is clear. Thus, both examples serve to illustrate that

> the connection between locutions in a dialogue has an inferential component beyond any that may hold between the contents of those locutions (Reed and Budzynska 2010).

In short, IAT studies "the way in which the rules of dialogue influence the construction of argument" (Budzynska et al. 2016).

Although the specific example in Fig. 2 is very simple, the following general observation on dialogue norms is useful for thinking about how the conversation might continue from the point it has reached so far:

> [T]here is an asymmetry between the *production* of arguments, which involves an intrinsic bias in favor of the opinions or decisions of the arguer whether they are sound or not, and the *evaluation* of arguments, which aims at distinguishing good arguments from bad ones. (Mercier and Sperber 2011, p. 72)

If the conversation were to continue, Wilma would typically have the burden of justifying her rejection of '*A*', which might be done with counterarguments that would dig into the details of '*A*' looking for flaws (*ibid.*, p. 67); in addition, she might

begin to make a case for an alternative position, '*B*'. These considerations point to the direction we will be taking with IATC.

Our main strategy will be to supplement IAT with an explicit register for content. Alongside (i)–(iii), above, we introduce:

(iv) a model of non-propositional content, namely of the mathematical objects under discussion, and the relations between them.

We will describe the implications of this addition in detail in Sect. 3, along with some other adaptations to IAT that we have found useful in mathematical settings. One of the implications is that in the current work we do not need to emphasise transitions—of either the explicit or implicit variety—since a more explicit treatment of content gives us another way to manage context relationships.

### 2.3 Lamport's Structured Proofs

*Structured proofs*, as described by Lamport (1995, 2012), inhabit the middle ground between formal and informal mathematics, and provide a useful point of reference for our work on IATC. Structured proofs offer a notational strategy that is a "refinement of [...] natural deduction" (Lamport 1995). While the proofs represented using this system are not required to be strictly formal, the language of structured proofs has evolved together with Lamport's work on a formal language and corresponding proof checking system, the "Temporal Logic of Actions+" (TLA$^+$), which is used to model concurrent systems Lamport (1999, 2014).[3] Structured proofs are, specifically, structured as a strict hierarchy of lemmas. An example appears later on in this paper, in Fig. 6, which we will use to illustrate the similarities and differences with IATC.

For now, we comment that while the use of strict hierarchies is not representative of the way proofs are usually constructed in day-to-day practice, Lamport has proposed that structured proofs can assist in proof development, e.g., by helping to bring errors to the surface. However, they do not necessarily make the job of the reader easier: Lamport (2012, p. 20) quotes a referee who had read one of his structured proofs:

> The proofs [...] are lengthy, and are presented in a style which I find very tedious. [...] My feeling is that informal proof sketches [...] to explain the crucial ideas in each result would be more appropriate.

[3] Only a few of the keywords available in the latest version of TLA$^+$ appear in the structured proof notation. Per Lamport (2015), the full list of TLA$^+$ keywords is as follows. Those which are also used in structured proofs are decorated with underlining: `assume` ... `prove` ..., `boolean`, `by`, `case`, `choose`, `constant` (synonymously, `constants`), `corollary`, `def`, `define`, `domain`, `else`, `except`, `extends`, `have`, `hide`, `if`, `instance`, `lambda`, `lemma`, `let` ... `in` ..., `new`, `omitted`, `pick`, `proposition`, `recursive`, `subset`, `suffices`, `take`, `theorem`, `unchanged`, `union`, `use`, `variable`, `witness`.

Unlike structured proofs, IATC is intended to express the typical processes by which proofs are generated in standard practice, rather than make the process of proving and reading proofs easier. It would nevertheless be compatible with our aims to include formal statements in TLA$^+$ (or some other language) in IATC's content layer.

## 3 Inference Anchoring Theory + Content

IATC has many things in common with IAT, but should not be seen as a strict addition to the earlier theory. Adding explicit models of content and discussions about content prompts several adaptations. In this section we describe these adaptations, and introduce the IATC modelling language.

Several important requirements arise from the features of the mathematics domain. As we saw above, IAT is concerned with anchoring propositions to utterances and with mapping the logical relationships that obtain between them. However, various mathematical objects—Larvor (2012) mentions "diagrams, notational expressions, physical models, mental models and computer models"—are more comfortably thought of as non-propositional in nature. Discussions about proofs have been theorised formally using the notion of *proof plans*, which are constructed and transformed using explicit heuristics and tactics (Bundy 1988). However, Fiedler and Horacek (2007, pp. 63–64) have suggested that existing work with proof plans cannot be straightforwardly adapted from machine-oriented to human-oriented contexts, because proof plans are, from a potential human reader's perspective, overly detailed, with insufficient structural abstraction. By contrast, a language like IATC is charged with expressing "strategic arguments that are meaningful to humans" (Fiedler and Horacek 2007, p. 68). Nevertheless, as important as strategic reasoning is, low-level mathematical content seems to be even more fundamental.

We see the first-class role that content plays in mathematical discourse when new terms are introduced and referred to, for example. Thus, the editor's introduction to Karttunen (1976) notes the following:

> ...informal notational practise [sic] of mathematicians, who will write an existentially quantified formula (say, $(\exists e)(\forall x)(xe = ex = x)$, as one of a set of postulates for group theory) and thenceforth use the variable bound by the existential quantifier as if it were a constant as when they will write the next postulate $(\forall x)(\exists x^{-1})(xx^{-1} = x^{-1}x = e)$. [punctuation modified]

Karttunen's concept of "discourse referents," illustrated in the quote above, underlies Discourse Representation Theory (Kamp and Reyle 1993) and its extensions. While the developers of IAT acknowledge the generality of Structured Discourse Representation Theory (SDRT), in particular, they criticise it for making "assumptions of context-independent semantics" (Budzynska et al. 2016). Nevertheless, DRT has been successfully applied to model some aspects of mathematical discourse, and we will discuss that work further in Sect. 5, and contrast it with our orientation here.

For now, we emphasise that IATC differs from IAT in its approach to context. Specifically, IATC sets the notion of dialogical relations to one side, and instead

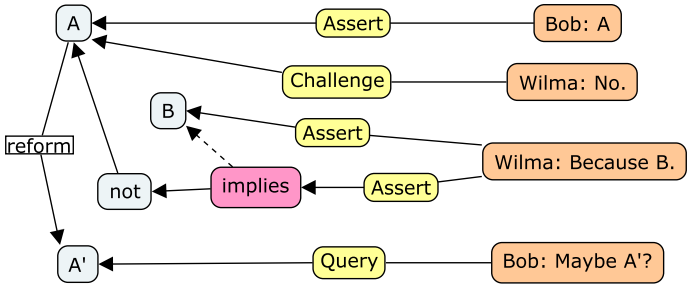



**Fig. 3** Simple IATC diagram exhibiting an assertion, a refutation, a counterexample, and a reformation. (Color figure online)

connects locutions to each other directly in the content and intermediate (meta-discussion) layers.

Before we describe the language in detail, we present a simple example, Fig. 3, which reanalyses and extends the '*A*'/'*No*' dialogue from Fig. 2. The first two dialogue moves in these two examples are identical.

Here, rather than connecting '*No*' to '*A*' with a transition, we connect it directly to the previously modelled content, *A*, via a '`Challenge`' illocution. From there, we continue to use the content and intermediate layers to explicitly model interconnections. For example, '*B*' does not simply conflict with *A*, but rather presents a warrant for "not *A*", modelled here using the two-parameter '`implies`' relation.

With these changes in place, dialogue relations could in principle be reintroduced. For example, '*Because B*' could be seen to 'substantiate' the previous utterance, '*No*', as a communicated reason for rejecting *A*. Nevertheless, in the current work we continue to leave these links out, on the basis that we do not yet have a detailed theory of the norms of mathematical dialogue. The Lakatosian model developed by Pease et al. (2017), for example, only covers a limited subset of the rules and norms involved, specifically, those dealing with conjectures, lemmas, and the production and evaluation of counterexamples. By interconnecting contents in the content layer and through intermediate relations, we are able to make an explicit model of the logical structure of mathematical arguments. Such models could potentially inform a subsequent analysis of the associated dialogue structures.

For example, the long-range `reform` connection from *A* to $A'$ in our content analysis would suggest a corresponding long-range transition from Bob's first to his last statement in the dialogue. However, that would still neglect Bob's so-far implicit reasoning to the effect that $A'$ is (potentially) not vulnerable to objection *B*. If the dialogue continued from this point, detailed relationships between the constituent contents of '$A'$' and '*B*' may need to be discussed, and an IATC analysis would be able to unpack these and account for the details.

In line with these design decisions, and inspired by the specific features of mathematical dialogue and exposition, IATC introduces a range of extra machinery to the IAT framework to model the relationships between mathematical objects and propositions, along with an array of dialogue moves related to the strategic aspects of proof. Unlike IAT, we make no attempt to cover argumentation in law, natural

**Table 1** Inference Anchoring Theory + Content, part 1: Performatives

| Performatives (`perf[...]`) | |
|---|---|
| `Assert` ($s$ [, $a$ ]) | Assert belief that statement $s$ is true, optionally because of $a$ |
| `Agree` ($s$ [, $a$ ]) | Agree with a previous statement $s$, optionally because of $a$ |
| `Challenge` ($s$ [, $a$ ]) | Assert belief that statement $s$ is false, optionally because of $a$ |
| `Retract` ($s$ [, $a$ ]) | Retract a previous statement $s$, optionally because of $a$ |
| `Define` ($o$, $p$) | Define object $o$ via property $p$ |
| `Suggest` ($s$) | Suggest a strategy $s$ |
| `Judge` ($s$, $v$) | Apply a heuristic value judgement $v$ to some statement $s$ |
| `Query` ($s$) | Ask for the truth value of statement $s$ |
| `QueryE` ($\{p_i(X)\}$) | Ask for the class of objects $X$ for which all the properties $\{p_i\}$ hold |

science, or interpersonal mediation, fields in which the norms that govern inference can be vastly different. (Precedent, for example, may be acceptable in a legal argument but not in one about ethics.) In mathematical argumentation, many of the conventions are embodied in the objects under discussion and the things that can sensibly be said about them. Details of our notational apparatus are given in Tables 1 and 2. We collect reference examples of short texts marked up with these codes in an Appendix.

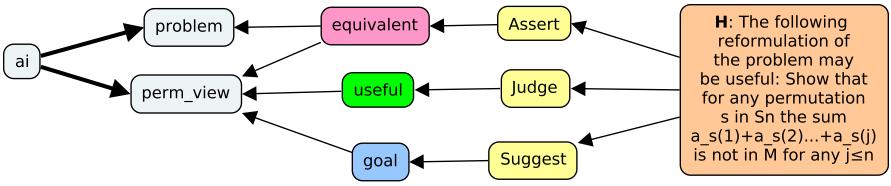
Our method for producing this set of tags was as follows. Two of us (with first degrees respectively in Mathematics and Information Systems, both with more than 10 years experience studying argumentation and social machines) performed close content analysis (Klaus 2004) together on the first 100 comments in MiniPolymath 1. Our analyses resulted in an initial tag set, including both typical illocutionary performatives and mathematics specific performatives, like `Define` and `QueryE`, as needed (see "Appendix" for examples). Several of the typical illocutionary connections (`Assert`, `Question`, `Challenge`, `Agree`) could be carried over from the schemes commonly applied in IAT. Our initial tag set was discussed and iteratively developed over the same 100 comments by all co-authors, with any recurring differences discussed, allowing us to align our results. A third co-author (with a first degree and Ph.D. in Mathematics) then further developed and refined the tag set by performing close content analysis on the entire MiniPolymath 3 conversation and on sections of MiniPolymath 1. Again, this was conducted alongside discussion with the other co-authors throughout the process. A fourth co-author (with a first degree in Mathematics) later extended the tag set with additional informal logical relationships, such as `analogy`, and specific content-focused relationships, such as `sums`, which played a role in the further examples we treated in Sect. 4. These extensions were again reviewed by all co-authors.

Our discussions concerned issues such as whether to label a statement such as '*it would be good to approach the problem in this way...*' as simply a suggested `strategy` or, additionally, as a `value[...]` judgement about the strategy. Shortly, in Fig. 4, we will show an example tagging in which the multiple layers of interpretation are included. However, perfect agreement about how to treat such cases is not intended; the IATC framework is designed to account

**Table 2** Inference Anchoring Theory + Content, part 2: inferential structure, heuristics and value judgments, reasoning tactics, and content-focused relations

| | |
|---|---|
| Inferential structure (`rel[...]`) | |
| `implies` (*s*, *t*) | Statement *s* implies statement *t* |
| `equivalent` (*s*, *t*) | Statement *s* implies statement *t* and vice versa |
| `not` (*s*) | Negation of *s* |
| `conjunction` (*s*, *t*, ...) | Conjunction of statements *s*, *t*, ... |
| `has_property` (*o*, *p*) | Object *o* has property *p* |
| `instance_of` (*o*, *m*) | Object *o* is an instance of the broader class *m* |
| `indep_of` (*o*, *d*) | Object *o* does not depend on the choice of object *d* |
| `case_split` (*s*, {$s_i$}) | Statement *s* is equivalent to the conjunction of the $s_i$'s |
| `wlog` (*s*, *t*) | Statement *t* is equivalent to statement *s* but easier to prove |
| Heuristic value judgments (`value[...]`) | |
| `easy` (*s* [, *t*]) | Statement *s* is easy to prove; optionally, easier than statement *t* |
| `plausible` (*s*) | Statement *s* is plausible |
| `beautiful` (*s*) | Statement *s* is beautiful (or mathematically elegant) |
| `useful` (*s*) | Statement *s* can be used in an eventual proof |
| Reasoning tactics (`meta[...]`) | |
| `goal` (*s*) | Used with `Suggest` to guide other agents to work to prove the statement *s* |
| `strategy` (*m*, *s*) | Indicate that method *m* might be used to prove *s* |
| `auxiliary` (*s*, *a*) | Statement *s* requires an auxiliary lemma *a* |
| `analogy` (*s*, *t*) | Statement *s* and statement *t* should be seen as analogous in some way |
| `implements` (*s*, *m*) | Statement *s* implements the method *m* from a previously suggested strategy |
| `generalise` (*m*, *n*) | Method *m* generalises method *n* |
| Content-focused structural relations (`struct[...]`) | |
| `used_in` (*o*, *s*) | Object *o* is used in statement (or object) *s* |
| `reform` (*s*, *t*) | Statement *s* can be reformed into statement *t* |
| `instantiates` (*s*, *t*) | Statement *s* schematically instantiates statement *t* |
| `expands` (*x*, *y*) | Expression *x* expands to expression *y* |
| `sums` (*x*, *y*) | Expression *x* sums to expression *y* |
| `cont_summand` (*x*, *y*) | Expression *x* contains *y* as a summand |

for flexibility in interpretations. The additional tags in Table 2 were not at first divided into the present categories, but repeated analysis quickly revealed structural content relations, as well as inferential structure, as natural categories, intuitively corresponding to the mathematical and logical contents of the MiniPolymath discussions we examined. By far the most difficult categorisation to make was between value judgements and reasoning tactics. For example, the difference between deeming a statement `useful` and suggesting it as a `goal` could depend




```
perf[Assert](rel[equivalent](problem, perm_view))
perf[Judge](value[useful](perm_view))
perf[Suggest](meta[goal](perm_view))
struct[used_in](ai, problem)
struct[used_in](ai, perm_view)
```

**Fig. 4** IATC markup of the statement "The following reformulation of the problem may be useful: show that for any permutation $s$ in $S_n$, the sum $a_s(1) + a_s(2) \ldots + a_s(j)$ is not in $M$ for any $j \leq n$." A larger portion of the dialogue is analysed in graphical and textual form in Fig. 12 and Table 3. (Color figure online)

completely on how polite or how bold the person making the utterance wished to be!

Our performatives have *slots*, which are filled by *statements* or *objects*. Statements may be represented in various ways: in unparsed natural language, as symbolic tokens that serve as shorthand for such statements, or in some representation language. The other relations are clustered into segments treating *Inferential Structure*, *Heuristics and Value Judgments*, *Reasoning Tactics*, and *Content-Focused Structural Relations*. The associated grammatical categories are given the following abbreviations in our linear notation: '`rel`', '`value`', '`meta`', and `struct`'. For example, the expression '`perf[Assert](rel[has_property](`$o$, $p$`))`' denotes the assertion of the statement "object $o$ has property $p$." IATC allows direct, explicit, statements about objects, propositions, and statements. For example, '`perf[Assert](used_in (`$o$, $s$`))`' denotes the assertion of the statement "object $o$ appears in statement $s$."

We have two notational strategies that call attention to features of discourse or content that are taken as understood, but not explicitly stated. Performatives may be marked as "unspoken" when the contents are only broadly implied. Several examples of this notational strategy appear in Sect. 4.1. Similarly, content-focused structural relations are sometimes introduced without an attached performative, whenever they have been noticed by the analyst. Figure 4 includes examples of this latter usage. This figure represents the analysis of a short excerpt from a real mathematical dialogue, showing its diagrammatic and textual representations in IATC. The discussion ("MiniPolymath 1") concerned Problem 6 from the 2009 International Mathematical Olympiad. The text analysed in Fig. 4 is a portion of the fourth comment made in the discussion (Tao et al. 2009, 20 July, 6:50 am). An expanded excerpt is discussed in Sect. 4.3 along with more details of our IATC analysis of MiniPolymath data. Here, colour coding highlights the

correspondence between the graphical and textual grammar elements. One statement has been analysed into three performatives:

- The speaker `Asserts` that the problem has an *equivalent* reformulation. "The following **reformulation of the problem** may be useful**:** Show that **for any permutation** $s$ **in** $S_n$, **the sum** $a_s(1) + a_s(2) \ldots + a_s(j)$ **is not in** $M$ **for any** $j \leq n$."
- The speaker `Judges` the reformulation to be (potentially) *useful*. "**The** following **reformulation** of the problem **may be useful**: [...]"
- The speaker `Suggests` that the reformulation describes a *goal* that could be worth pursuing: "[...] **Show that** [...]"

In addition, mathematical objects (several symbols, $a_i$) are analysed as component pieces of tagged content ('`problem`' and '`perm_view`'). Note that bold lines at left in the figure are a shorthand for the '`used_in`' relation. Subsequent statements in the dialogue will be able to link back to these objects: the analysis of an expanded extract appears in Fig. 12.

The relations given in Tables 1 and 2 have been sufficient to describe the reasoning in a range of examples, however we do not claim that this list of relationships would treat all mathematical texts. Nor do these relationships describe mathematical texts at the level of formality found in proof checking systems, or the level of detail found in some other theorisations of discourse. Thus, in the future IATC should not be limited to the set of tags presented here. For example, we have found uses for the value judgments '`easy`', '`beautiful`', and '`useful`', but it is quite plausible that future work would find use for values such as '`efficient`', '`generative`', or something else. Similarly, useful additions may be found in the other grammatical categories. The evidence from our examples in Sect. 4 is that these major grammatical categories—performatives, inferential relations, meta-level reasoning, value judgments, and content relations—are themselves stable.

We have described, and illustrated with simple examples, the way content and strategic relationships can be used to mediate contextual relationships, but context is also representable in IATC in another more explicit way. Although IATC does not require proofs to be structured in a tree-like hierarchy, nested structure is introduced as follows. In general, language elements in Table 2 that have a *statement* slot can also have that slot filled by a (possibly disconnected) *subgraph*. In this way, structure corresponding to a "lemma" can be indicated. A lemma, in this sense, is understood to be the reasoning that '`implements`' a '`strategy`', or, alternatively, a specific section of reasoning that '`implies`' some conclusion. This representation strategy is similar to the "partitioned networks" introduced by Hendrix (1975, 1979). An example will appear in Sect. 4.1.

To summarise, IATC resembles IAT in many ways, but with changes that are required when content, and discussions about content, are explicitly modelled. These features are necessary to express details of mathematical reasoning. For example, one proposition that can be extracted from the statement in Fig. 4 has the schematic form "The reformulation $P$ is equivalent to the original question $Q$." IAT would have no way to extract $P$ and $Q$ from the assertion, but IATC can do so: they are represented as '`problem`' and '`perm_view`' in the figure. Later moves can

then connect to these pieces of content, and we already see such structure forming in our analysis of the above short excerpt.

IATC retains and extends IAT's approach to modelling contents and inferences, by adding non-propositional contents and more complex logical and heuristic relations. Illocutionary connections are also retained, with some mathematics-specific additions. However, IATC sets aside the notion of transitions, not because we view dialogue norms as unimportant, but because they are difficult to model at this stage. In IAT, relations between propositional contents roughly mirror the norms involved. The corresponding notion for IATC would be heuristics that account for the production of new expressions, and which take preceding expressions and background knowledge into account. We will have more to say about such heuristics in Sect. 4, nevertheless, many considerations must be deferred to future work.

## 4 Examples

In this section, we use three examples to showcase what IATC has to offer as a tool for analysis. We illustrate

- how IATC expresses the reasoning structures that arise in proof construction,
- how it might be used to support computational models of mathematical reasoning,
- and how it helps to uncover the salient elements of mathematical discourse.

To illustrate the points above, we have selected and analysed three examples that exhibit informal, expository, and discursive features of mathematical reasoning. The presentation here is a novel and self-contained synthesis and expansion of remarks made in previous papers (Corneli et al. 2017a, b; Pease and Martin 2012). The three examples collectively show the richness of mathematical argument, and were selected to match the three aims indicated above:

- Section 4.1: A carefully spelled out informal solution to a tricky but non-technical mathematical problem serves to illustrate the thought processes involved in successful mathematical problem solving. The example shows how IATC captures this sort of thinking.
- Section 4.2: A discussion of the relationships between, and merits of, different mathematical questions exhibits a level of abstraction above that needed in an individual proof. We explore the ramifications for explicit representations of the reasoning involved.
- Section 4.3: A multi-participant dialogue that develops a challenging but not highly technical proof casts light on processes of mathematical collaboration and mathematical reasoning. An analysis of this material using IATC allows us to explore the process of proof-construction in detail.

In each of the following subsections, we give more details of the context of each example, before presenting our analysis and comments.

What is the 500$^{\text{th}}$ digit of $(\sqrt{2}+\sqrt{3})^{2012}$?
Even this, eventually, a computer will be able to solve.
For now, notice that total stuckness can make you do desperate things. Furthermore, knowing the origin of the problem suggests good things to try. The fact that it is set as a problem is a huge clue.
Can we do this for $(x+y)$? For $e$? Rationals with small denominator?
And how about small perturbations of these? Maybe it is close to a rational?
$m^{\text{th}}$ digit of $(\sqrt{2}+\sqrt{3})^n$?
$(\sqrt{2}+\sqrt{3})^2$?
$(2+2\sqrt{2}\sqrt{3}+3)$
$(\sqrt{2}+\sqrt{3})^2+(\sqrt{3}-\sqrt{2})^2=10$
$(\sqrt{2}+\sqrt{3})^{2012}+(\sqrt{3}-\sqrt{2})^{2012}$ is an integer!
And $(\sqrt{3}-\sqrt{2})^{2012}$ is a very small number. Maybe the final answer is "9"?
We need to check whether it's small enough. $(\sqrt{3}-\sqrt{2})^{2012} < \left(\frac{1}{2}\right)^{2012} = \left(\left(\frac{1}{2}\right)^4\right)^{503} = \left(\frac{1}{16}\right)^{503} < .1^{503}$, so we're in luck.
The answer is indeed 9.

**Fig. 5** A "magic leap" challenge problem and its solution, presented by Timothy Gowers as part of a public lecture at the University of Edinburgh, November 2, 2012. (Reproduced from notes taken at the lecture.)

### 4.1 Making the Reasoning Explicit in the Solution to a Challenge Problem

In this section we aim to show that IATC is a natural modelling tool for informal mathematics. Whereas Robinson (1965, p. 23) had sought to

> reduce complex inferences, which are beyond the capacity of the human mind to grasp as single steps, to chains of simpler inferences, each of which is within the capacity of the human mind to grasp as a single transaction,

an alternative path of enquiry seeks to describe the heuristic process of proving theorems in more cognitively plausible terms. In particular, one relevant question to ask is how (human) mathematicians avoid large searches (Gowers 2017). IATC can contribute to the further development of this effort, by giving a uniform but expressive way to outline the process of developing proofs. Researchers working on mathematical software meant to exhibit human-style reasoning may find this expressiveness useful.

Our chosen example is a "magic leap" problem presented in a public lecture by Timothy Gowers, describing joint work with Mohan Ganesalingam (2012). The reasoning was communicated by a combination of speech and marks on a chalkboard, and is reproduced in Fig. 5. This example has been modelled in IATC by Corneli et al. (2017b). The problem initially appears difficult to solve without a computer algebra system, but a simple algebraic solution is available once the correct strategy is found. As such, an important part of the reasoning involved in solving the problem is to find the correct strategy. The steps involved in this part of the reasoning process are heuristic rather than deductive. We redescribe the analysis here.

PROOF SKETCH: What is the 500th digit of $(\sqrt{2}+\sqrt{3})^{2012}$? Even this, eventually, a computer will be able to solve. The fact that this has been set as a problem is a huge clue. Can we do this for $x+y$? For $e$? Small rationals? And how about small perturbations of these? Maybe it is close to a rational?

⟨1⟩1. For $n$ large enough and $m$ small enough in comparison, the $m$th digit of a sufficiently small rational $r$ to the $n$th power is equal to 0. **1**

PROOF:

CASE: $r < 1/10$

⟨2⟩1. $(1/10)^n = .1^n$ has 0 in $n-1$ places in its decimal expansion. **1.1**

⟨2⟩2. $r < 1/10$ implies $r^n$ has zeros in at least $n-1$ places in its decimal expansion, so we simply need to select $m < n$. □ **1.2**

⟨1⟩2. Can we compute the $m$th digit of $(\sqrt{2}+\sqrt{3})^n$? **2**

PROOF:

CASE: $n = 2$

⟨2⟩1. $(\sqrt{2}+\sqrt{3})^2 = 2+2\sqrt{2}\sqrt{3}+3$ ∗ **2.1**

⟨1⟩3. Step 2.1 fails to give us an answer by direct computation, but if we eliminate cross-terms, we can see that $(\sqrt{2}+\sqrt{3})^2$ is "close to" an integer. **3**

PROOF:

⟨2⟩1. $(\sqrt{2}+\sqrt{3})^2+(\sqrt{2}-\sqrt{3})^2 = 10$ **3.1**

⟨2⟩2. $(\sqrt{2}+\sqrt{3})^2 = 10-(\sqrt{2}-\sqrt{3})^2$ **3.2**

⟨2⟩3. $(\sqrt{2}-\sqrt{3})^2$ is small **3.3**

PROOF:

⟨3⟩1. $(\sqrt{2}-\sqrt{3})^2 \approx (1\frac{1}{3}-1\frac{1}{2})^2$ by continued fraction approximation. **3.3.1**

⟨3⟩2. $(1\frac{1}{3}-1\frac{1}{2})^2 = (\frac{8}{6}-\frac{9}{6})^2 = 1/36$ □ **3.3.2**

⟨1⟩4. Generalising Step 3.1, $(\sqrt{2}+\sqrt{3})^{2012}+(\sqrt{2}-\sqrt{3})^{2012}$ is an *integer*! **4**

PROOF:

⟨2⟩1. Terms with odd exponents cancel by the binomial theorem. □ **4.1**

⟨1⟩5. And $(\sqrt{2}-\sqrt{3})^{2012}$ is a *very* small number. Maybe the final answer will be "9"? We need to check whether it's small enough. **5**

PROOF:

⟨2⟩1. $(\sqrt{2}-\sqrt{3})^{2012} < (1/2)^{2012}$ by Step 3.3.1 **5.1**

⟨2⟩2. $(1/2)^{2012} = ((1/2)^4)^{503}$ **5.2**

⟨2⟩3. $((1/2)^4)^{503} = (1/16)^{503}$ **5.3**

⟨2⟩4. $(\sqrt{2}-\sqrt{3})^{2012} < (1/16)^{503} < .1^{503}$, and so has at least 502 zeros in its decimal expansion by Step 1 **5.4**

⟨2⟩5. The answer is indeed 9. □ **5.5**

**Fig. 6** The solution to the challenge problem as a Lamport-style structured proof

For comparison with the IATC analysis, Fig. 6 reproduces the proof in Lamport's style. Figures 7, 8, 9 and 10 present portions of the IATC tagging of the solution that was presented in Gowers's lecture. Figure 7 illustrates an initial exploration of the question, and Fig. 8 establishes a '`strategy`' based on that exploration ("The

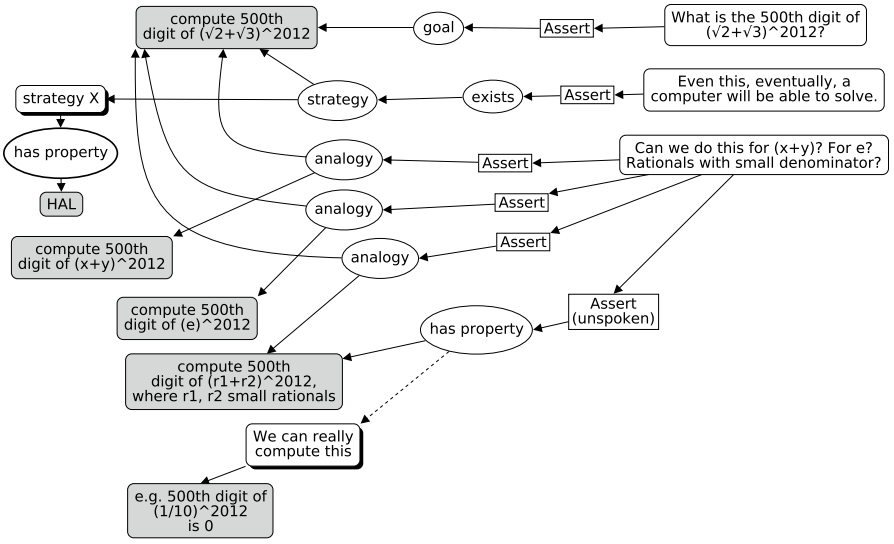



**Fig. 7** IATC tagging for the first portion of the challenge problem

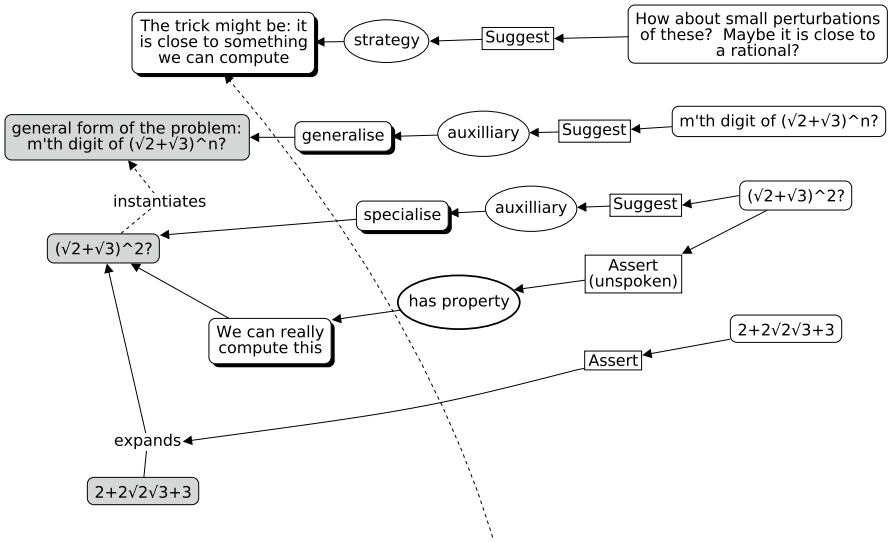



**Fig. 8** IATC tagging for the second portion of the challenge problem

trick might be: it is close to something we can compute"). Figure 9 opens the door to applying the strategy. The central part of the proof that '`implements`' the strategy is highlighted in Fig. 10.

The introduction to the proof, expanded in Fig. 7—and condensed into a "PROOF SKETCH" in Fig. 6—contains interesting examples of heuristic reasoning. This part of the solution centres on the probing question "Can we do this for $\mathfrak{x}$?", where $\mathfrak{x}$

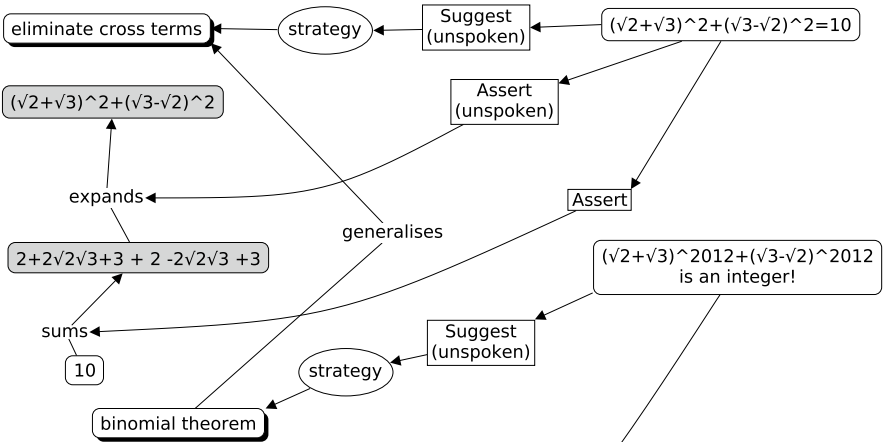



**Fig. 9** IATC tagging for the third portion of the challenge problem

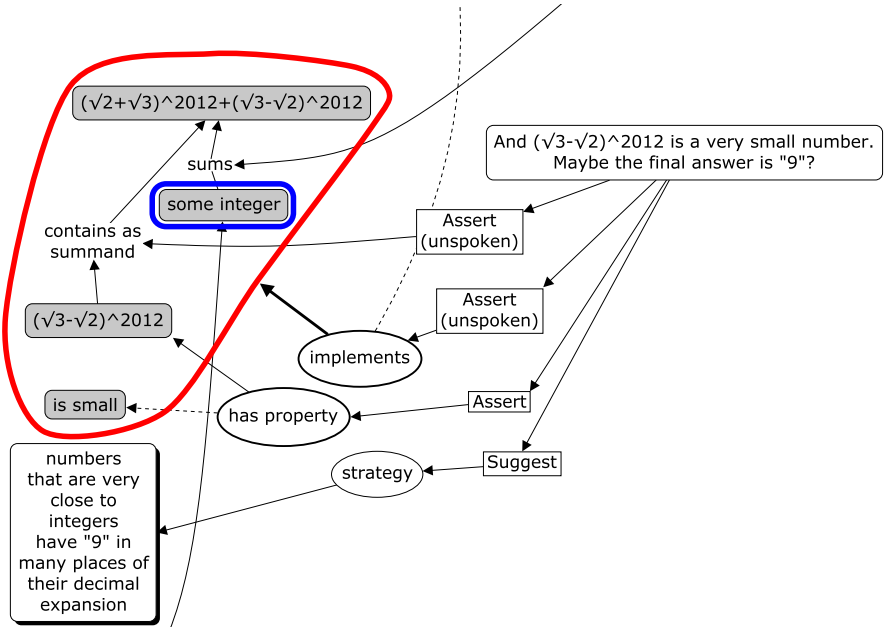



**Fig. 10** Nested structure (in red) `implements` the `strategy` suggested earlier: "The trick might be: it is close to something we can compute." The intermediate conclusion reached in this phase of reasoning (highlighted in blue), when taken together with a further computational check, subsequently `implies` that the answer is "9". (Color figure online)

ranges over several examples: $x + y$, $e$, and small rationals, and where 'this' denotes "find the 500th digit of $\mathfrak{X}^{2012}$." In the IATC representation, each tentative proposal to "do this..." stands in `analogy` with the original problem statement. Although Fig. 7 contains only `Assert` performatives, a more complete representation would

also include `Query` performatives, since the analogies are not only proposed: their validity is also queried, much as we saw in the example treated in the previous section.

Step 1 in the structured proof works out one of the ideas from the proof sketch at a level of detail that was not present in the lecture, which instead progressed directly on to the material treated in Step 2. As Fiedler and Horacek (2007, p. 69) noted, "The analysis of human proof explanations shows that certain logical inferences are only conveyed implicitly, drawing on the discourse context and default expectations." There is no hard and fast rule that can tell us how much of the implicit material we need to explicate, but one rule of thumb that naturally arises from our representation strategy is that coherently related discussions should correspond to connected graphs in the expansion. Thus, for example, Fig. 7 includes an implicit "unspoken" `Assertion`; the proof is made fully explicit in Step 1 of the Lamport-style proof, but never appeared in the original lecture. Again, in a standard IAT representation, unspoken assertions would typically be represented as 'implicit' speech acts rooted on transitions, whereas in IATC, we see how these unspoken assertions play a role in the argument via their expansion and subsequent interconnections in the content layer.

Indeed, nowhere in the explicitly communicated reasoning is the key `strategy` fully and explicitly stated. The basic strategy of the proof is that *the quantity of interest may be sufficiently close to something we can compute*. In the IATC representation (Fig. 8), this is understood to be `Suggested` by the following statements from the proof sketch, "And how about small perturbations of these? Maybe it is close to a rational?" Step 1 of the structured proof shows that rationals do, in fact, match the strategy's preconditions. The IATC representation is less explicit on this point, since it sticks more closely to the reasoning expressed in the lecture. This example shows that even relatively explicit statements may need further interpretation to be represented meaningfully in IATC. Specifically, the way the proof progresses only makes sense if we recognise the '`strategy`' implied by what might otherwise appear to be a throwaway comment early on.

Step 2 in the structured proof concerns another analogy. This time, a special one which, the IATC analysis notes, symbolically `generalises` the initial question (Fig. 8). That is, rather than considering $(\sqrt{2}+\sqrt{3})^{2012}$ we now consider $(\sqrt{2}+\sqrt{3})^m$. (NB. an edge connecting the '`generalise`' node to the problem statement has been omitted.) However, the concept of generalisation remains implicit in the corresponding portion of the structured proof. Indeed, Step 2 is not a good match for the requirements of structured proof at all, since it is not a real lemma, and its "proof" fails (indicated by "*"). Including failed proof steps is not a problem for IATC. In Fig. 9 the process of solving the problem proceeds apace, without pausing to remark on a failed lemma, now that something more interesting has been discovered.

Meanwhile, Step 3 in the structured proof implements the main strategy for resolving a special case of our generalised problem, namely showing that $(\sqrt{2}+\sqrt{3})^2$ is close to an integer, establishing a pattern that leads to the conclusion. Again, Step 3.3 offers considerably more detail than was present in the original lecture.

Step 4 subsequently generalises the method that was used in Step 3, and applies it to the expression we were originally interested in. Figure 10 diagrams out the reasoning that underlies this step. The long-range dashed edge in this figure connects with the node "The trick might be: it is close to something we can compute" pictured in Fig. 8. The collection of nodes highlighted in red `implement` that strategy. Notice, though, that the computation is not done explicitly: it's unimportant which integer the number of interest is close to. Collectively, the fact that $(\sqrt{2}+\sqrt{3})^{2012}+(\sqrt{3}-\sqrt{2})^{2012}$ sums to "some integer" and the fact that $(\sqrt{3}-\sqrt{2})^{2012}$ is sufficiently small `implies` the result. Step 5 shows the details of the final computational check.

Several objections could be raised about the structured proof presented in Fig. 6, most notably to the inclusion of a failed lemma in Step 2. However, as a source of information about the intuition behind the proof, this failure is valuable. While objections to the IATC treatment are also possible, it is clear that this method helps to make explicit features of the proof process that remain implicit in the structured proof. In particular, analogies, strategies, and relationships between methods are made explicit. While the structured proof augments the lecture with more technical details, IATC provides a more faithful model of the reasoning expressed in the lecture itself.

### 4.2 Towards Computable Models of Mathematical Reasoning Via IATC: A Q&A Example

Contributors to discussions about mathematics on MathOverflow do more than just talk about proofs.

> The presentation is often speculative and informal, a style which would have no place in a research paper, reinforced by conversational devices that are accepting of error and invite challenge. (Martin and Pease 2013)

IATC allows the argumentation aspects of mathematical dialogues to be represented as explicit graphical structures, which gives a plausible basis from which to develop an explicit computational model of the reasoning steps that are implied in mathematical argumentation. Corneli et al. (2017a) showed how IATC could be used to create graphical models of the discussion that develops around a question posted on MathOverflow. Here we will remark further on implications for computational modelling. The question, which was given the title "Group cannot be the union of conjugates" (Chandrasekhar et al. 2010), is as follows:

"*I have seen this problem, that if G is a finite group and H is a proper subgroup of G with finite index then* $G \neq \bigcup_{g\in G} gHg^{-1}$. *Does this remain true for the infinite case also?*"

In the most straightforward reading, two superficially similar group-theoretic propositions seem to be at stake:

(*P1*) "*If G is a finite group, H is a subgroup of G and the index* $[G : H]$ *is finite, then G is not equal to the union of* $gHg^{-1}$"; and,

(*P2*) "*If G is an infinite group, H is a subgroup of G and the index* $[G : H]$ *is finite, then G is not equal to the union of* $gHg^{-1}$."

The question thus implicitly outlines an argument by analogy:

- (*P1*) is true
- (*P2*) is similar to (*P1*)
- Therefore, (*P2*) is (potentially) true as well

The essence of the question is to ask whether the mathematical facts align with this schematic argument. As it turns out, this question is answered in the affirmative. Shortly after the question was asked, one discussant make the terse comment "the case of infinite *G* readily reduces to the case of finite *G*"; months later, another discussant supplies an explicit proof of (*P2*).

In the mean time, other discussants had proposed and addressed several alternative formulations of the question. An important distinction hinges on the interpretation of the phrase "*infinite case*." An alternative proposition that incorporates some of the suggested revisions is as follows:

(*P2′*) "*If G is an infinite group, H is a proper finite index subset of G and the index* $[G : H]$ *is infinite, then G is not equal to the union of* $gHg^{-1}$."

In this case an argument by analogy would not match the facts: a counterexample is supplied to show that proposition (*P2′*) is false.

The dialogue is an interesting example of mathematical reasoning in which proof certainly plays a role, but is nevertheless of secondary interest compared with asking interesting questions, and thinking about how different questions relate to each other. What would be necessary to represent this sort of dialogue computationally? Expressing propositions like (*P1*) in IATC is straightforward, though, as we noted, the content layer is not directly modelled in this representation language. The following expression represents this proposition in IATC, introducing additional invented pseudocode representations (in italics) in the content layer.

```
perf[Assert](
 rel[implies](
  rel[conjunction](finite_group(G),
                   subgroup(H,G),
                   rel[has_property](index(H,G),is_finite))
  rel[not](equal(G,
                 union_over(conjugates(H,g),elements(g,G))))))
```

Processing such expressions to build a model of a dialogue will require adding numerous stanzas like this one, each rooted on an IATC performative, into one graph database that records the relationships between the statements and their constituent parts. Individual expressions like the `implies` relationship would need to be addressable, in order for an `analogy` between two implications to be

proposed. Definitions for predicates like `finite_group` and special constructions like `union_over` could be supplied in an accompanying knowledge base. In further rounds of computational processing, the analogies between (*P1*) and (*P2*), and between (*P1*) and (*P2′*), could be checked using graph-processing methods described by Sowa and Majumdar (2003). New heuristics would be needed if the aim was to demonstrate the truth or falsity of the various propositions, not just to recreate the surface analogies. Moreover, as we've seen, mathematical dialogues are not just concerned with verifying statements, but may also consider the qualities that make a particular question interesting in a given context. Heuristics that can be used to select interesting problems are not prevalent in current mathematical software.

As a limited proof of concept showing the plausibility of adding a computational deduction and verification layer on top of IATC representations, Corneli et al. (2017b) give a detailed expansion of one step of a mathematical proof using simple rules for transforming the underlying graph structures. It is worth emphasising that the representations of reasoning afforded by language elements in Tables 1 and 2do not themselves encode the meta-level reasoning associated with such graph transformations.

### 4.3 MiniPolymath Revisited

The data that underlie this section were generated in a series of online experiments in collaborative problem solving convened by mathematician Terence Tao (2009; 2011). We use IATC to expand on a previous analysis of this data presented by Pease and Martin (2012), showing how IATC can advance the theory of mathematical argument through the detailed analysis of real world examples, as per Carrascal (2015).

In their 2012 paper, Pease and Martin analysed the third MiniPolymath project in broad strokes, with each blog comment comprising a single unit to be tagged. They developed a typology of five intuitive comment types, based on the mathematical content of each comment: *examples*, *conjectures*, *concepts*, *proofs*, and *other*.

In order to assign comments to these categories, both authors performed close content analysis on all comments posted between the time Tao posted the problem to his blog (8pm, UTC on July 19th, 2011) and the time he announced that a solution had appeared (9.50pm, UTC on July 19th, 2011). The discussion comprised 147 comments over 27 threads. Ten comments were assigned to more than one category.

Our present IATC analysis of the same data is designed to give a more complete picture of the linguistic, dialectical, and inferential structure of the comments that fall within the five intuitive categories mentioned. There are three main differences between the two analyses. First, in comparison with the earlier broad-stroke analysis, the IATC analysis is richly detailed, with a unit defined as any quantum of commentary with taggable content. Secondly, our focus in the earlier analysis was purely on mathematical content, and on the *type* of mathematical content in particular. This contrasts with our present analysis, in which we provide a more fine-grained representation of mathematical content in the taggable units, and furthermore take into account linguistic, dialectical, and inferential structure. Third, the IATC analysis

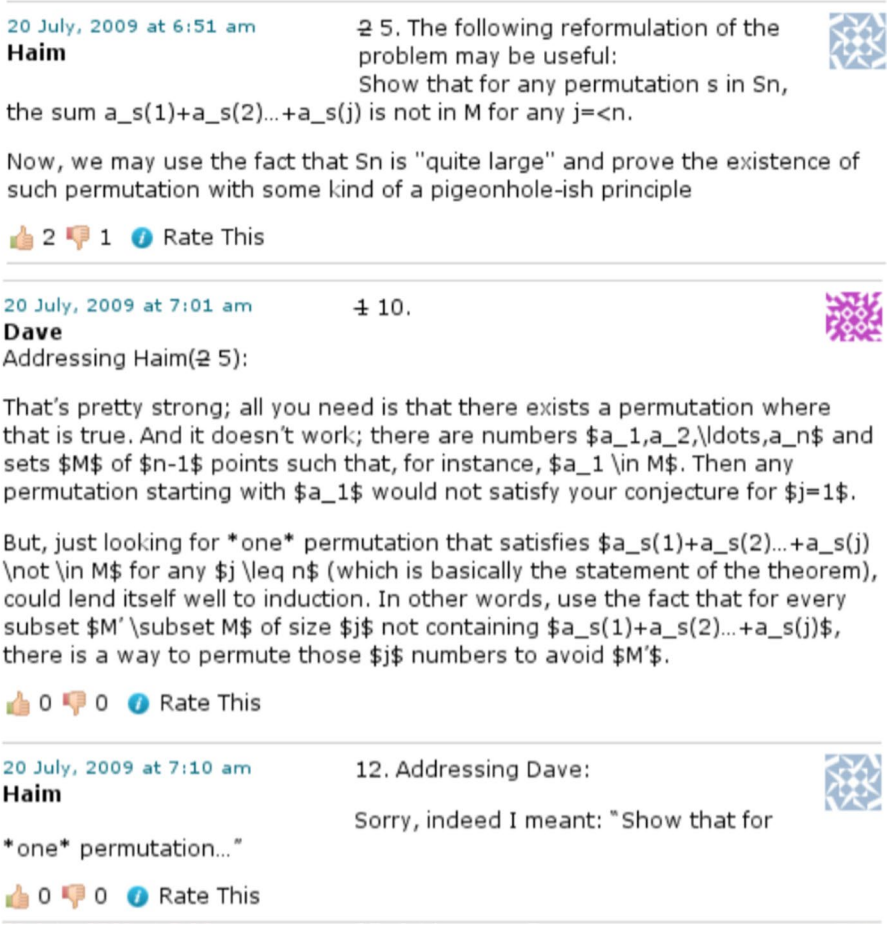



**Fig. 11** Screenshot of a portion of the MiniPolymath 1 dialogue. (Color figure online)

takes into consideration the entire MiniPolymath 3 conversation, including the comments that came after Tao had announced that a proof had been found.

The new analysis, accordingly, adds depth to our earlier analysis. Crucially, the new perspective will be more relevant to argumentation theorists, and supports a detailed understanding of what went on in the process of constructing the collaborative proof. The earlier typology provided an initial way to sort the content, whereas the IATC tag set developed along with our analysis via the iterative, discursive method discribed in Sect. 3. Though they cover the same data and show some correlations, as described below, the latter categorisation was not derived from the earlier one.

Figure 11 presents an excerpt from the MiniPolymath 1 dialogue (MPM1) as it originally appeared on Tao's blog. Figure 12 and Table 3 give the IATC analysis of this excerpt in diagrammatic and textual form. The first portion of Fig. 12

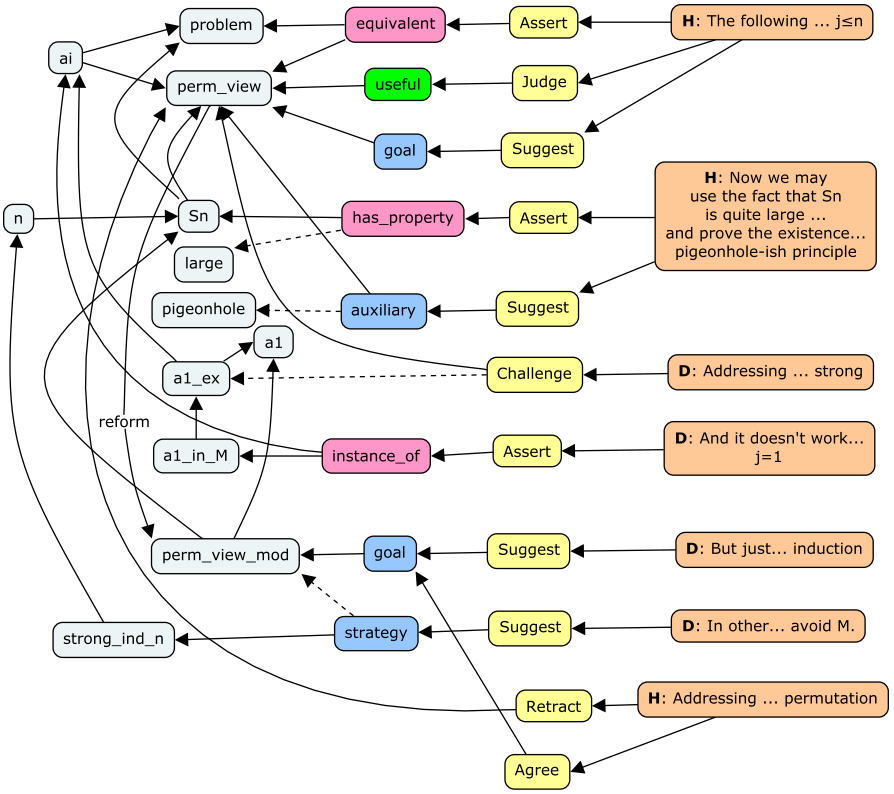



**Fig. 12** IATC analysis of MPM1 excerpt (graphical form). (Color figure online)

repeats the contents of Fig. 4. The longer excerpt shown here illustrates complex contextual interconnections forming in the content layer.

Our main example in this section is MiniPolymath 3 (MPM3), which we tagged into IATC in its entirety. (This work was carried out by one co-author with a first degree and PhD in Mathematics, in consultation with others as described in Sect. 3.) As an indicative sample, the first three comments and their tags are shown in Fig. 13. Figure 14 shows how tags from IATC's five grammatical categories were distributed over time. Thus, for example, we see '`value`' tags used early in the discussion as strategies are being considered, and again later in the discussion when solutions are being vetted. Figure 15 gives another view of the timeline, showing how the comments were categorised into the 5-part typology from Pease and Martin. In the initial categorisation developed for that paper, comments were allowed to be in multiple categories at once. Here, to facilitate a clean mapping to IATC, we redid the categorisation with the requirement that each comment should fit into exactly one main category. We arrived at a nearly equal division of comments among the five categories: *example* (20.3%) *conjecture* (21.2%), *concept* (19.5%), *proof* (19.5%), and *other* (19.5%). (This

**Table 3** IATC analysis of MPM1 excerpt (text form). (Color table online)

| |
|---|
| (**Haim**) The following reformulation of the problem may be useful: "Show that for any permutation $s$ in $S_n$, the sum $a_{s(1)} + a_{s(2)} + \ldots + a_{s(j)}$ is not in $M$ for any $j \leq n$." |
| (**1**) `perf[Assert](rel[equivalent](problem, perm_view))`,<br>(**2**) `perf[Judge](value[useful](perm_view))`,<br>(**3**) `perf[Suggest](meta[goal](perm_view))`,<br>(**4**) `struct[used_in](ai, problem)`,<br>(**5**) `struct[used_in](ai, perm_view)` |
| (—) Now, we may use the fact that $S_n$ is "quite large" and prove the existence of such permutation |
| (**6**) `perf[Assert](rel[has_property](Sn, large))`,<br>(**7**) `struct[used_in](Sn, perm_view)`,<br>(**8**) `struct[used_in](Sn, problem)` |
| (—) with some kind of a pigeonhole-ish principle. |
| (**9**) `perf[Suggest](meta[auxiliary](meta[strategy](perm_view,pigeonhole), rel[has_property](Sn, large)))` |
| (**Dave**) Addressing Haim(2 5): That's pretty strong. And it doesn't work; there are numbers $a_1, a_2, \ldots, a_n$ and sets $M$ of $n-1$ points such that, for instance, $a_1 \in M$. Then any permutation starting with $a_1$ would not satisfy your conjecture for $j = 1$. |
| (**10**) `perf[Challenge](perm_view, a1_ex)`,<br>(**11**) `perf[Assert](rel[instance_of](a1, a1_in_M))`,<br>(**12**) `struct[used_in](a1_ex, a1_in_M)` |
| (—) But, just looking for **one** permutation that satisfies $a_{s(1)} + a_{s(2)} + \ldots + a_{s(j)} \notin M$ for any $j \leq n$ (which is basically the statement of the theorem), could lend itself well to induction. |
| (**13**) `perf[Suggest](meta[goal](perm_view_mod))`,<br>(**14**) `struct[reform](perm_view, perm_view_mod)`,<br>(**15**) `struct[used_in](Sn, perm_view_mod)`,<br>(**16**) `struct[used_in](a1, perm_view_mod)` |
| (—) In other words, use the fact that for every subset $M' \subset M$ of size $j$ not containing $a_{s(1)} + a_{s(2)} + \ldots + a_{s(j)}$, there is a way to permute those $j$ numbers to avoid $M'$. |
| (**17**) `perf[Suggest](meta[strategy](perm_view_mod, strong_ind_n))`,<br>(**18**) `struct[used_in](n, strong_ind_n)`,<br>(**19**) `struct[used_in](n, Sn)` |
| (**Haim**) Addressing Dave: Sorry, indeed I meant: "Show that for **one** permutation..." |
| (**20**) `perf[Retract](perm_view)`,<br>(**21**) `perf[Agree](meta[goal](perm_view_mod))` |

"*Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A windmill is a process that starts with a line $\ell$ going through a single point $P \in S$. The line rotates clockwise about the pivot P until the first time that the line meets some other point Q belonging to S. This point Q takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of S. This process continues indefinitely. Show that we can choose a point P in S and a line $\ell$ going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.*"

```
rel[structural](problem, S)
rel[structural](problem, P)
rel[structural](problem, l)
rel[structural](problem, windmill)
rel[structural](problem, pivot)
rel[structural](P, pivot)
rel[structural](P, windmill)
rel[structural](l, windmill)
```

1. *Could you start off with a random point in the plane and prove it doesn't work, if you can't prove that then the opposite holds.*

```
perf[query](random_test_false)
perf[assert](rel[stronger](rel[not](prove_rtf),
                           rel[not](random_test_false)))
rel[structural](random_test_false, prove_rtf)
```

2. *Connecting the dots: At the point where the pivot changes we create a line that passes through the previous pivot and a new pivot – like a side of a polygon.*

```
perf[define](pivot_seq, ps_def)
rel[structural](pivot_seq, pivot)
```

2.1. *Nice. We need only to consider the times when to points are connected – this gives us a path, and after some time this path will come back to some already visited point. So there is a cycle. If only we could find a cycle which spans all the points, the question is solved... That may be some useful simplification.*

```
perf[assert](rel[has_property](pivot_seq, has_cycle))
perf[suggest](meta[goal](cycle_spans_S))
perf[judge](value[useful](pivot_seq))
rel[structural](S, cycle_spans_S)
```

**Fig. 13** IATC tags for the problem and first three comments in MiniPolymath 3

replication work was carried out independently by one of the coauthors with a first degree in Mathematics.)

Figure 16 illustrates the correspondence between IATC tags with the earlier typology. Aligning the bulkier 5-part categorisation with the IATC tagging shows

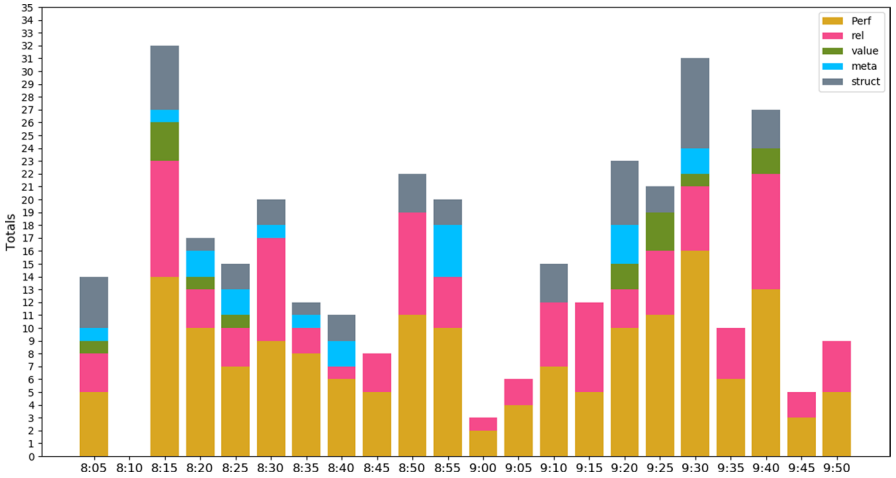



**Fig. 14** Timeline of the MiniPolymath 3 dialogue showing the IATC grammar categories used in the tagging. Comments are binned into 5 min intervals. The first interval is 8:05–8:09 and the last is 9:50–9:59, inclusive. (Color figure online)

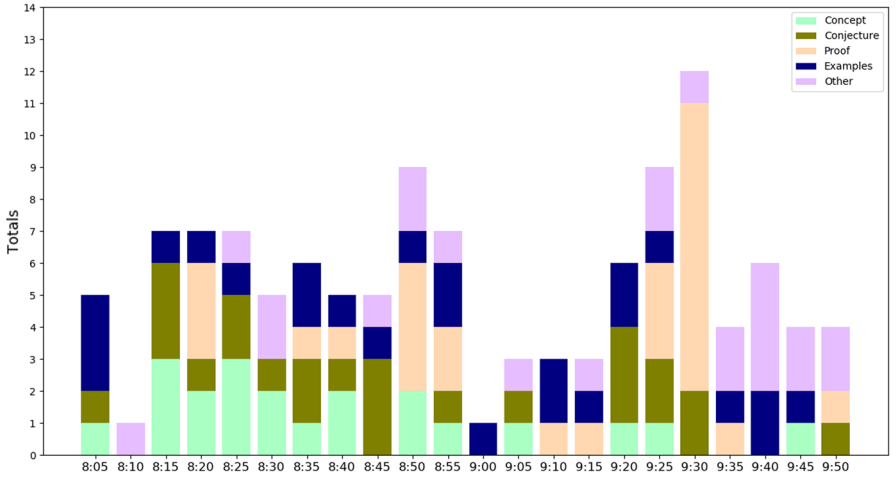



**Fig. 15** Timeline of the MiniPolymath 3 dialogue showing comments categorised into five categories: Concept, Conjecture, Proof, Examples, and Other. (Color figure online)

that these five intuitive labels are mapped in very different ways to the more detailed IATC tag set.

We observe certain regularities: for example, `Assert` is present in all five types of comments, but is used most frequently within proof-related comments. Annotations from the '`struct`' grammatical category are most prevalently associated with conjecture-related comments. (NB. In this tagging exercise we only considered the '`used_in`' facet of the '`struct`' category, so '`structural`'

is here a synonym for 'used_in'.) It is not surprising that the performative Challenge is used most frequently in examples, since, intuitively, an example is likely to be put forward as a counter-example. The most prevalent use of Agree is in comments that are categorised as "other". Retract is frequently used in this category as well, as is stronger (here, a synonym for 'implies'). These usages reflect social values as well as mathematical semantics. E.g., one can express support for an idea by underscoring one's belief in an implication, as in the comment "Yes, it seems to be a correct solution!" (Tao et al. 2011, July 19, 9:35 pm).

One might suspect that Suggest should be used only within conjectures, but in the current categorisation it is used somewhat more frequently along with concepts. This is partly explained by the fact that Suggest can be used to introduce either a goal or a strategy. Sometimes goals represent conceptual tidying, as in "I guess there is an odd/even number of point distinction to do" (Tao et al. 2011, July 19, 9:31 pm).

Furthermore, despite our self-imposed constraint to map each comment only to the most salient of the five categories, in practice a comment may simultaneously introduce a concept along with a conjecture that applies that concept. For example the straightforward concept of "restriction[s] on how the next pivot is chosen" appears along with the more speculative conjecture "Can we start with a complete graph and all cycles on that graph and just discard the ones that don't follow the restrictions to converge on the ones that do?" (Tao et al. 2011, July 19, 8:56 pm). The need to introduce concepts also applies in the case of more outlandish conjectures, such as "It might be fun to use projective duality" (Tao et al. 2011, July 19, 8:23 pm). However, a concept may suggest a vague method without raising a conjecture as such, e.g., "I'm thinking spirograph rather than convex hull" (Tao et al. 2011, July 19, 8:44 pm).

In sum, the IATC analysis of MiniPolymath 3 shows in detail how individual contributions to the dialogue are comprised. In aggregate, this analysis exposes the structural anatomy of a successful collaborative proof. It should be noted that not all the contributions to MPM3 were equally relevant to the final solution. By entering the structures in an explicit graphical model in the manner described in Sect. 4.2, graph theoretic analysis could establish, e.g., the centrality of the various concepts used in the content layer, and who introduced them into the conversation.

## 5 Conclusion

We have sought to advance the study of mathematical practice from an argumentation-theoretic perspective. We introduced Inference Anchoring Theory + Content, offered a brief comparison with IAT, which it builds upon, and used three examples to showcase IATC's capabilities. We showed that:

– IATC offers a more faithful representation of everyday mathematical practice than does, e.g., Lamport-style structured proof.

**Fig. 16** Pie charts showing the relative proportion of IATC tags used to code MPM3, across five intuitive kinds of comments. E.g., Comment 1 has been categorised as a *Conjecture*. The IATC stanza `perf[assert](rel[stronger](rel[not](prove_rtf), rel[not](random_test_false)))` associated with this comment (see Fig. 13) therefore adds these values to the usage counts within the *Conjecture* pie chart: '`Assert`' +1, '`stronger`' +1, and '`not`' +2. (Color figure online) ▶

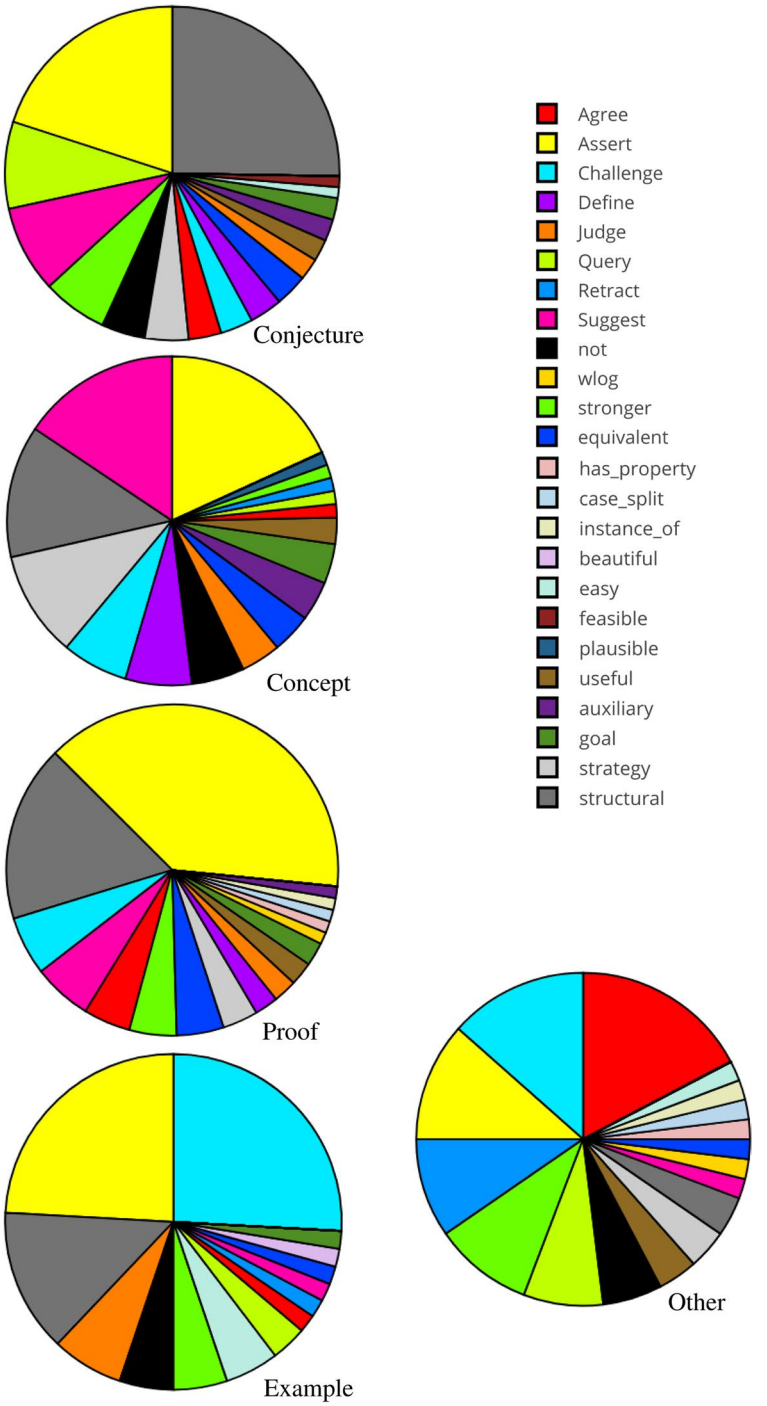
– IATC has the potential to support computational reasoning about mathematics by bringing structural relationships between pieces of mathematical content to the surface.
– IATC can recover salient elements of discourse within comments, as well as the way these contents connect across comments.

Some limitations to the approach should be considered when applying the framework. We emphasise that these are limitations and not necessarily flaws in the overall design. In general, the limitations could be addressed with extensions to the language.

– IATC does not yet handle everything that is said in mathematical dialogues. We saw above that IATC nevertheless helps disambiguate the "other" category bracketed by Pease and Martin (2012).
– There are places where IATC representations remain bulky, pushing much of the actual reasoning into whatever representation system handles the content layer.
– One related limitation is that implications and assumptions that mathematicians consider "obvious" are typically elided from their discourse, often for valid expository reasons, and that, therefore, unpacking the contextual relationships between statements typically requires a mathematically trained annotator.
– We introduced a graphical way to segment dialogues, but IATC does not currently have the ability to express context shifts – although it can compare contexts with '`analogy`'.

Corneli et al. (2018) survey other relevant frameworks that might form extensions for a future version of IATC. More general-purpose formalisms like the W3C's "PROV" (Groth and Moreau 2013) would allow us to say something about the provenance and evolution of concepts, but would have nothing to say about the mathematics-specific features that interest us.

In Sect. 3, we mentioned that Discourse Representation Theory (DRT) has informed several earlier efforts to model mathematical discourse. We are aware of three PhD theses—by Clauss Zinn (2004), Mohan Ganesalingam (2013), and Marcos Cramer (2013)—which have made use of somewhat similar mathematics-specific interpretations of DRT. Zinn and Cramer focused on proof checking, while Ganesalingam looked at mathematical communication from a linguist's perspective. However, he opted to focus exclusively on mathematics in the "formal mode," leaving informal communication about matters such as "interestingness" to one side, because they bring with them a host of additional complications (Ganesalingam 2013, pp. 7–8). From a linguistic point of view, DRT is useful in a mathematical setting, in the first instance, because of its core ability to express "legitimate


Agree
Assert
Challenge
Define
Judge
Query
Retract
Suggest
not
wlog
stronger
equivalent
has_property
case_split
instance_of
beautiful
easy
feasible
plausible
useful
auxiliary
goal
strategy
structural
Conjecture
Concept
Proof
Example
Other

antecedents for anaphor" (Ganesalingam 2013, p. 50). In Ganesalingam's work, this basic feature is extended to allow sidelong references to definite descriptions (such as 'the set of natural numbers') by "introducing generalised anaphors which can have presuppositional material attached to them" (Ganesalingam 2013, pp. 25, 237). Specifically, this allows one to infer from statements such as "$x$ is prime" that $x$ is in fact a member of the set of natural numbers (p. 25).

The associated requirement of combining semantics and pragmatics (van der Sandt 1992, p. 336) is reminiscent of our treatment of unspoken assertions and unstated features of content in our IATC-based analyses. To continue the comparison, Ganesalingam's adaptations of DRT overcame limitations, having to do with quantifier scoping, that constrained earlier type-theoretic analyses (Ganesalingam 2013, pp. 81–82). This is broadly similar to our use of nested structure in Sect. 4.1. Indeed, Sowa (2000) shows that several different approaches to nested structure (including DRT) are all mutually equivalent from a logical point of view. As indicated by van der Sandt (1992), pragmatics is relevant for DRT-based models because it can inform the context-specific resolution of Discourse Representation Schemes. This is related to the question we highlighted in Sect. 4.2: how to model the transitions between discourse moves in mathematics? IAT accounts for similar issues by making reference to dialogue norms, but we have seen that for mathematical dialogues, detailed content- and context-specific issues need to be taken into consideration at each stage. The models of content evolution used by Ganesalingam and Gowers (2017) to keep track of proof generation were structurally similar to the DRT-based models developed by Ganesalingam (2013): in this case, the evolution was governed by a limited set of reasoning tactics. Our work with IATC highlights features of mathematical reasoning, like `analogy`, that more general heuristics will need to account for.

There are other resources available which could further expand IATC's offerings in this regard. For example, a recent special issue of *Argument & Computation* (Harris and Marco 2017) includes papers detailing the usefulness of rhetorical structures for argument mining. Mitrović et al. (2017), in that volume, indicate the SALT Rhetorical Ontology (Groza 2012) as relevant prior work. SALT contains three categories—coherence relations, argument scheme relations, and rhetorical blocks—each of which unfolds with considerable further detail. These three categories can be seen as somewhat analogous to IATC's grammatical categories. Mitrović et al. (2017) and Lawrence et al. (2017) point to foundational work of Fahnestock (1999, 2004) on the argumentative function of rhetorical figures, particularly in science writing. IATC might be profitably connected to such analyses. Furthermore, the integration of rhetoric into argument mining highlights the relevance of structures that are rather different from the IAT-style transitions that have been used in work summarised by Budzynska et al. (2015). White's (1978, p. 6) pithy assertion that "logic itself is merely a formalization of tropical strategies" can serve as an additional provocation to develop structural analyses of this sort.

Nevertheless, whether mathematical content is modelled using ideas from logic, rhetoric, or other sources, considerable further work will be required to effectively describe the processes that are employed in forming and responding to mathematical arguments. A small case study included as an appendix to Pease et al. (2017)

(and, incidentally, based on MiniPolymath 3) illustrates the plausibility of Lakatos's model—however that model is clearly far from complete as a theory of mathematical production. Pease et al. were concerned with mathematical content only insofar as it fills slots for some 20 dialogue moves that are based on Lakatos's strategy for arguing about lemmas and counterexamples. For example, *MonsterBar*$(m, c, r)$ gives a reason $r$, contradicting the justification $m$ for the counter-conjecture not-$c$. At no point does this theory touch the supposed mathematical ground of axioms and rules of inference. That the reason $r$, for example, may have been formed inductively, or deductively, or in some other way, goes undiscussed. IATC would allow us to expand the structure that appears within statements like $r$. Whereas Pease et al.'s formalisation of Lakatosian reasoning as a dialogue game offers a computational model of certain dynamical patterns in mathematics, our current work has focused on kinematics. The efforts can be seen as complementary: Bundy (2013) has argued that the right representation can considerably simplify reasoning.

One promising approach to modelling process combines argumentation and multi-agent systems (Modgil and McGinnis 2007; Maghraby et al. 2012; Robertson 2012). However, most approaches to modelling specifically *mathematical* agents have had significant limitations. Thus, for example, Fiedler and Horacek (2007) have described the difficulty of squaring argumentation-theoretic work with the methods of formal proof. Ganesalingam and Gowers's (2017) project aimed at simulating a solitary individual rather than a population. However, Furse (1990) had already called into question the robustness of approaches to modelling mathematical creativity that only model a solitary creative individual. Pease et al. (2009) describe an implementation effort that made use of a multi-agent approach, drawing on argumentation theory concepts and a Lakatosian model of dialogue. However, the mathematical applications of that system were limited to straightforward computational aspects of number theory and group theory, which suggests a "knowledge bottleneck" (Saint-Dizier 2016; Moens 2018).

As indicated in a report of the National Research Council (2014, p. 90), "knowledge extraction and structuring in the context of mathematics" is in demand on an increasingly industrial scale. IATC allows methods of argumentation to interface with those of knowledge representation; both aspects are relevant to knowledge extraction. Formalisation of IATC would assist in its applicability: "IKL Conceptual Graphs" defined by Sowa (2008) would provide a natural foundation. IKL, the *IKRIS Knowledge Language* (Hayes 2006; Sowa 2008), deals elegantly with context and has been used as a representational formalism in a project with aims comparable to our own: the Slate project (Bringsjord et al. 2008), which centred on an argumentation tool that could support a mixture of deductive and informal reasoning.[4] Previous work on mathematical usage can also inform future efforts in knowledge modelling with IATC (Trzeciak 2012; Wells 2003; Wolska 2015; Ginev 2011).

Mathematical Knowledge Management, particularly in the "flexiformal" understanding developed by Kohlhase (2012) and Kohlhase et al. (2017), presents another paradigm that could eventually be integrated with IATC.

[4] http://www.jfsowa.com/ikl/IKLslate.pdf.

Flexiformality combines strict formalisations of those parts of mathematics for which that makes sense with opaque representations of constants, objects, and informal theories. Iancu (2017) built on Kohlhase's work, and focused on "co-representing both the narration and content aspects of mathematical knowledge in a structure preserving way" (pp. 3–4). However, modelling narrative in Iancu's sense is more relevant to the "frontstage" presentation of mathematics in a single authorial voice than to the "backstage" production of mathematics (cf. Hersh 1991)). Section 4.2 illustrated one such example from backstage: mathematicians need to be able to choose between different mathematical problems.

IATC offers a step forward for research into both the communication and production of mathematics, and can play a role in future work on knowledge extraction and simulation. Potential applications include, among others, the development of a new generation of mathematics tutoring software and digital assistants that engage their users in thought-provoking dialogues.

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## Appendix: Reference Coding Samples

This "Appendix" collects sample texts and IATC codings to supplement Tables 1 and 2 in Sect. 3, which introduced the available codes. Texts are sourced from the examples discussed in Sect. 4. In general, one utterance may expand to multiple statements in IATC; accordingly, texts may appear here multiple times. Bold face is used to illustrate the portion of the text, at right, that justifies the tag that appears, at left. Numbering refers to the tree-ordering of MiniPolymath 3 comments, unless another source is indicated.

| Performatives | |
|---|---|
| perf[**assert**](rel[implies](rel[not](prove_rtf), rel[not](random_test_false))) | 1. Could you start off with a random point in the plane and prove it doesn't work, **if you can't prove that then the opposite holds** |
| perf[**agree**](cycle_partition) | 2.2.1.1. **I believe this is true.** It proves that it's enough to find a cycle that visits each vertex at least once |

| Performatives | |
|---|---|
| perf[**challenge**](problem, equi_tri_stuck) | 3.1.1. Say there are four points: an equilateral triangle, and then one point in the center of the triangle. No three points are collinear. **It seems to me that the windmill can not use the center point more than once! As soon as it hits one of the corner points, it will cycle indefinitely through the corners and never return to the center point.** I must be missing something here |
| perf[**retract**](perf[challenge](problem, equi_tri_stuck)) | 3.1.1.2.1. **Ohhh...I misunderstood the problem. I saw it as a half-line extending out from the last point, in which case you would get stuck on the convex hull.** But apparently it means a full line, so that the next point can be "behind" the previous point. **Got it** |
| perf[**define**](pivot_seq, ps_def) | 2. Connecting the dots: **At the point where the pivot changes we create a line that passes through the previous pivot and a new pivot—like a side of a polygon** |
| perf[**suggest**](meta[goal](cycle_spans_S)) | 2.1. Nice. We need only to consider the times when two points are connected—this gives us a path, and after some time this path will come back to some already visited point. So there is a cycle. **If only we could find a cycle which spans all the points, the question is solved**. That may be some useful simplification |
| perf[**judge**](value[useful](pivot_seq)) | 2.1. Nice. We need only to consider the times when two points are connected—this gives us a path, and after some time this path will come back to some already visited point. So there is a cycle. If only we could find a cycle which spans all the points, the question is solved. **That may be some useful simplification** |
| perf[**query**](random_test_false) | 1. **Could you start off with a random point in the plane and prove it doesn't work**, if you can't prove that then the opposite holds |
| perf[**queryE**](additional_condition_on cycles(X)) | 2.1.1.1. For example, the restriction on how the next pivot is chosen (geometrically: comment 9). **Are there any other restrictions?** Can we start with a complete graph and all cycles on that graph and just discard the ones that don't follow the restrictions to converge on the ones that do? |
| **Inferential structure** | |
| perf[assert](rel[**implies**](rel[not](prove_rtf), rel[not](random_test_false))) | 1. Could you start off with a random point in the plane and prove it doesn't work, **if you can't prove that then the opposite holds** |
| perf[assert](rel[**equivalent**](problem, forall_exists_problem), cycle_partition) | 2.2.1.1. I believe this is true. It proves that **it's enough to find a cycle that visits each vertex at least once**. There are no "rho" processes with an initial segment that doesn't repeat |

| Inferential structure | |
|---|---|
| perf[assert](rel[implies](rel[not](prove_rtf), rel[**not**](random_test_false))) | 1. Could you start off with a random point in the plane and **prove it doesn't work**, if you can't prove that then the opposite holds |
| perf[question](rel[implies](rel[**conjunction**](G_infinite_group, H_subgroup_of_G, H_finite_index_in_G), G_not_equal_to_union_of_cosets)) | (Sect. 4.2, Question) I have seen this problem, that if $G$ **is a finite group and** $H$ **is a proper subgroup of** $G$ **with finite index** then $G \neq \cup_g \in GgHg^{-1}$. Does this remain true for the infinite case also? |
| perf[assert](rel[**has_property**](pivot_seq, has_cycle)) | 2.1. Nice. We need only to consider the times when two points are connected—**this gives us a path, and after some time this path will come back to some already visited point. So there is a cycle.** If only we could find a cycle which spans all the points, the question is solved. That may be some useful simplification |
| rel[**instance_of**](S, convex_plus_point) | 3.1. Yes. Can we **do it if there is a single point not on the convex hull of the points?** |
| perf[assert]( rel[**indep_of**]( disj_path, M )) | (MPM1, 31.) Quick thought following on David Speyer's first comment: The problem asks us to prove that no set of size $(n-1)$ can disconnect two diagonally opposing vertices in the $n$-cube. By Menger's theorem, this is equivalent to proving that there are n internally vertex-disjoint paths between these two vertices. So, now we are faced with **a constructive problem, independent of the set** $M$**: Construct** $n$ **vertex-disjoint paths from** $0^n$ **to** $1^n$ **in the** $n$**-cube** |
| perf[assert](rel[**case_split**](IS, IS_A, IS_B)) | 2.1.1.1.1. The line must sweep out a full rotation (and only one full rotation) of $2\pi$ during the traversal of $S$. I feel like this is intimately related to proving that there is a starting angle for any point $P$ in $S$ such that all of $S$ is then traversed. I'm trying to show this by induction. Base case ($\|S\| = 2$) is obvious. **Let** $\|S\| = n$, **take** $S' = S \cup \{Q\}$, **and start with some windmill traversal of** $S$. **Case A:** $Q$ is unreachable. Therefore we just traverse $S$, taking $2\pi$ to do so by induction. **Case B:** $Q$ is reachable at some angle. [...] |
| perf[assert](rel[**wlog**](problem, zero_angle), one_turn) | 11.2.3.1. Only the starting point matters. By the problem statement, it appears that the initial angle is irrelevant to the existence of a pivot point $P^*$ from which all of $S$ is traversed. Every point in S is a pivot point, but only with a specific range of starting angle (e.g. those consistent with the cycle generating $S$). The union of these intervals must necessarily be $[0, 2\pi)$, and thus **we can assume WLOG that the starting angle is 0** (and thus we single out a specific point—or points in the case of $\|S\| = 2$) |

| Heuristic value judgments | |
|---|---|
| perf[judge](value[**easy**](S_is_conv)) | 3. **If the points form a convex polygon, it is easy** |
| perf[judge](rel[not](value[**plausible**](equi_tri_stuck))) | 3.1.1. Say there are four points: an equilateral triangle, and then one point in the center of the triangle. No three points are collinear. It seems to me that the windmill can not use the center point more than once! As soon as it hits one of the corner points, it will cycle indefinitely through the corners and never return to the center point. **I must be missing something here** |
| perf[judge](value[**beautiful**](proof_sugg)) | 14.2.1. **Very nice!** Don't we run into problems with a convex hull though? Take a square with a point in the middle ($M$) and pass the diagonal of the square (not through $M$)—it seems to me M is never visited (though I may be wrong here). I think we should be more specific in our initial choice of line, maybe? |
| perf[judge](value[**useful**](pivot_seq)) | 2.1. Nice. We need only to consider the times when two points are connected—this gives us a path, and after some time this path will come back to some already visited point. So there is a cycle. **If only we could find a cycle which spans all the points, the question is solved. That may be some useful simplification** |
| **Reasoning tactics** | |
| perf[suggest](meta[**goal**](cycle_spans_S)) | 2.1. Nice. We need only to consider the times when two points are connected—this gives us a path, and after some time this path will come back to some already visited point. So there is a cycle. **If only we could find a cycle which spans all the points, the question is solved**. That may be some useful simplification |
| perf[suggest](meta[**strategy**](cycle_spans_S, process_of_elim)) | 2.1.1.1. For example, the restriction on how the next pivot is chosen (geometrically: comment 9). Are there any other restrictions? **Can we start with a complete graph and all cycles on that graph and just discard the ones that don't follow the restrictions to converge on the ones that do?** |
| perf[suggest](meta[**auxiliary**](problem, forall_split)) | Ok. I think the solution might involve this observation, **with the observation that every point participates in a "splitting" line (one with $n$ / 2 points on one side)** |
| perf[assert](meta[**analogy**](compute 500th digit of (sqrt(2)+sqrt(3))^2012, compute 500th digit of (x+y)^2012)) | (Sect. 4.1) **Can we do this for $x + y$? For $e$? Rationals with small denominator?** |
| perf[assert](rel[**implements**](#SUBGRAPH, the trick might be)) | (Sect. 4.1) $(\sqrt{2}+\sqrt{3})^{2012} + (\sqrt{3}-\sqrt{2})^{2012}$ **is an integer! And** $(\sqrt{3}-\sqrt{2})^{2012}$ **is a very small number.** Maybe the final answer is "9"? |
| rel[**generalise**](binomial theorem, eliminate cross terms) | (Sect. 4.1) $(\sqrt{2}+\sqrt{3})^{2012} + (\sqrt{2}-\sqrt{3})^{2012}$ **is an integer!** |

| Content-focused structural relations | |
|---|---|
| rel[**used_in**](pivot_seq, pivot) | 2. Connecting the dots: At the point where the pivot changes we create a line that passes through the previous **pivot** and a new **pivot**—like a side of a polygon |
| rel[**reform**](H_finite_index_in_G , H_infinite_index_in_G) | (Sect. 4.2, Second comment on question) Yes, the statement is out of focus: $gHg^{-1}$ is intended (**and "infinite index case"**). The natural starting point is to ask whether the proof for finite index breaks down |
| rel[**instantiates**]((sqrt(2) + sqrt(3))^2,general form of the problem) | (Sect. 4.1) $(\sqrt{2}+\sqrt{3})^2$ |
| perf[assert](rel[**expands**](2 + 2sqrt2sqrt3 + 3, (sqrt(2)+sqrt(3))^2)) | (Sect. 4.1) $2+2\sqrt{2}\sqrt{3}+3$ |
| perf[assert](rel[**sums**](2 + 2sqrt(2)sqrt(3) + 2 + 2 - 2sqrt(2)sqrt(3)+3,10)) | (Sect. 4.1) $(\sqrt{2}+\sqrt{3})^2+(\sqrt{2}-\sqrt{3})^2=10$ |
| perf[assert](rel[**cont_summand**]( (sqrt(2) + sqrt(3))^2012 + (sqrt(2)-sqrt(3))^2012,(sqrt(3)-sqrt(2))^2012) | (Sect. 4.1) And $(\sqrt{3}-\sqrt{2})^{2012}$ is a very small number. Maybe the final answer is "9"? |

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# Patterns of Design

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## Abstract

In a straightforward meta-level shift of focus, we use design patterns as a medium and process for capturing insight about the process of design. We survey mainstream design genres, and draw conclusions about how they can help inform the design of intelligent systems.

## CCS Concepts

• **Computing methodologies** → **Knowledge representation and reasoning**; • **Software and its engineering** → **Patterns**; • **Human-centered computing** → *Collaborative interaction*;

## Keywords

Design Processes, Design Patterns, Conceptual Blending, Artificial Intelligence, Human-Centered Artificial Intelligence, Creativity

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## Introduction

What is design, and why does it matter? The word derives from the Latin verb for marking (*dēsignō*). The etymological perspective tells us that design is linked with designation, and, more fundamentally, with significance. Etymology also shows us that design is close to 'programming', which comes from the ancient Greek word for a written public notice or edict (Figure 1). Knox [45] describes the modern evolution of the meaning of 'design', which has variously denoted:



**Figure 1**

- "an art of giving form to products for mass production,"
- a practical theory of "planned obsolescence,"
- combinations of "science, technology and rationalism" addressed to "human and environmental problems,"
- surfaces for "the luring of consumers for the purpose of gaining their money,"
- the deeper problem of "designing the consumers themselves."

Johansson-Sköldberg et al. [44] discuss five related contemporary theoretical perspectives on "design and designerly thinking," encompassing the *creation of artefacts*, *reflexive practice*, *problem-solving*, *reasoning and sensemaking*, and *the creation of meaning*. Design may be, by now, the essential discipline needed for survival in the Anthropocene era, in which humanity is at work on a "concrete and discrete project of global immune design" [80, p. 451]. At any rate, it no longer belongs to the "pipe-smoking boffin" or even the "solitary style warrior" [30, p. 2]. In practice "various experts are in constant close co-operation" and indeed "no group covers a wide enough field" [8, p. 20].

Landscape designer Rolf Roscher suggests that 'belief' and 'landscape' are related in two ways:

> The 'specific': where belief is derived from a place. [ . . . ] The 'transported': where a landscape is created as a metaphor for a set of beliefs. [75, p. 124]

A now-popular account by the UK's Design Council [24] takes on a somewhat similar two-part form. They propose: "In all creative processes a number of possible ideas are created ('divergent thinking') before refining and narrowing down to the best idea ('convergent thinking'), and this can be represented by a diamond shape." They then suggest that in the process of design, "this happens twice – once to confirm the problem definition and once to create the solution" (Figure 2).



**Figure 2**

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| | | | |
|---|---|---|---|
| **Section 1:** | Design Patterns<br>*Decision support for complex situations* | Learning Design<br>*Strategies for learning while doing* | Business Design<br>*Innovations that have a chance of succeeding in the marketplace* |
| **Section 2:** | Research Design<br>*Developing a worthwhile problem to work on* | Concept Design<br>*New ways to think about things* | Product Design<br>*Address problems using prior knowledge and common sense* |
| **Section 3:** | Experience Design<br>*Transformations of participants* | Knowledge Base Design<br>*Strategies for making "meaning" available to computers* | Designing Intelligence<br>*Strategies for building intelligent systems* |

**Table 1: Primary Contents: A survey of mainstream design genres.**

Accordingly, the Design Council describes the overall process in terms of four phases: *discovery*, *definition*, *development* and *delivery*. Other divisions are possible. Tim Brown [12], one of the chief proponents of "design thinking," describes design in terms of three "spaces"—*inspiration*, *ideation*, and *implementation*—noting that "Projects will loop back through these spaces – particularly the first two – more than once as ideas are refined and new directions taken" (p. 89). Some design thinking acolytes translate this into a more methodical process, based on the six steps *empathize*, *define*, *ideate*, *prototype*, *test*, and *implement*, with additional feedback loops as appropriate. Peffers et al. [72] present a somewhat similar six-phase model; Vaishnavi and Kuechler [86, p. 130] use five phases, and give detailed design patterns that are relevant at each phase. Nessler [65] divides the Design Council's double diamond into no fewer than sixteen sub-phases, and Mann [56] points out that design is often a highly iterative process so that the basic diamond shape could be repeated many times on the way to an ideal solution.

In the design patterns literature, individual patterns are models for specific design processes, and pattern languages are models of more complex design processes with moving, optional, parts. Kohls [46, 47] had described design patterns as being like a journey or route map, but in light of the remarks above they can also be understood in a more ecological way. Moran [63, p. 131] had already remarked, "From the point of view of methodology, it is not so important how good each pattern is, but only that each one is transparent and open to criticism and can be improved over time." Design patterns emerge from an interaction between interpretation and application. The terrain is shaped by patterned behaviour.

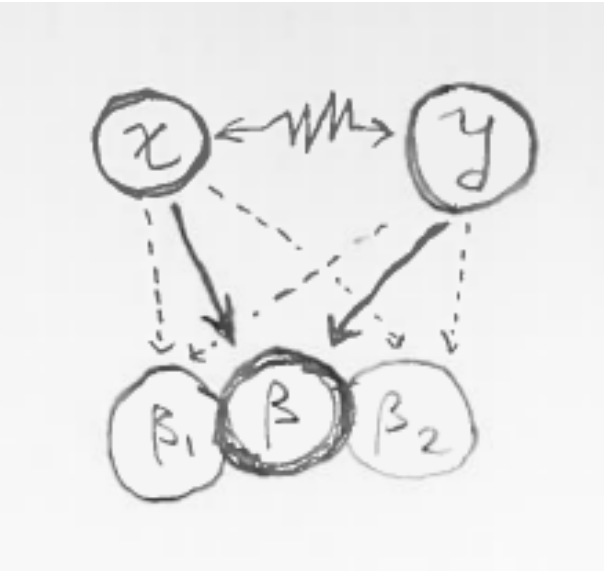
By analogy with solution-oriented phases of design work—where machine intelligence is employed in Computer Aided Drafting (CAD) [50], procedural architecture [70], etc.—we think that technological support could make a big difference to conceptual aspects of problem-oriented work. Several recognised technical efforts notwithstanding, many teams continue to do early-stage design with simple, flexible, analogue tools, such as 3M's PostIt Notes, relying on methods that date back to classical dialectic [29, 68]. These tools and methods are useful for the problems to which they are applied—but these simple, useful, familiar ways of working do not readily incorporate computational intelligence. By engaging computational intelligence in the early stages of design, more perspectives could be taken into account, and more complex problems addressed. The benefits must be bootstrapped using both technical and informal means [64, pp. 67–68].

Nevertheless, there is an important caveat. Alexander [2] argued that once we have a well-defined problem, computers can be used for optimisation, but that this was never the core issue in design. Nevertheless, experimental work has applied computers to design problems both autonomously and together with users [18, 19]. "Creative design" is seen as a testbed for cutting edge AI research [51]. Considering the complexity of today's challenges, and the opportunity that widespread distribution of computing power provides, another effort to understand the key concepts of design is called for. We focus on surveying mainstream genres of design, as summarised in Table 1, with a broader theoretical discussion and practical conclusions in Sections 4 and 5 respectively.

## 1 Patterns in the creative process

We begin with a recursive move: we describe design patterns *using* a design pattern. This first pattern helps show how all of the patterns are variations on a central theme. We make use of a simple design pattern template inspired by the foundational work of Alexander and Poyner [6].

Design Patterns




**Assuming** You work in a context that has reasonably stable features; in which the wholes have a somewhat modular form; in which documentation can and will be read and used.

**If** Forces $x$ and $y$ are frequently seen to be in conflict, and the conflict can be resolved better by $\beta$ than any alternatives ↯ the relation between $x$, $y$, and $\beta$ is not already well understood.

**Then** Describe the relation as a design pattern, so that it is accessible and its fitness may be judged.

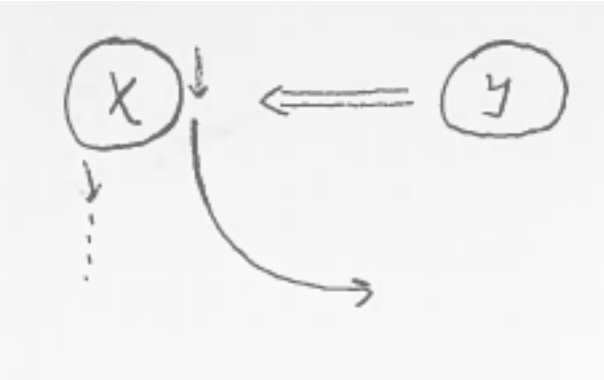
**Example** This section can serve as a first example of a *design pattern*, and the paper as a whole serves as an example of a *pattern language*.[1]

**Alternatives** TRIZ is an approach to engineering design centred on common conflicts (e.g., an aeroplane must be both big and light) [84].

*Comment.* An important implication is that: *whereas design is concerned with the form or structure of objects and processes, the way we think is also structured, and can be shaped by design.* However, design patterns should not be understood as a silver bullet. Alexander reflects, "To caricature this I could say that one of the hallmarks of pattern language architecture, so far, is that there are alcoves all over the place; or that the windows are all different" [36, p. 189]. The way in which patterns are deployed matters. For example, in a programming context, Graham [37] has argued that "regularity in the code is a sign [ . . . ] that I'm generating by hand the expansions of some macro that I need to write."

Here we have opted to use a pattern template that is simpler than the ones often used in pattern languages for programming. Other fields often have standard templates of their own. We collect examples in Tables 2 and 3, later on below.

### Learning Design




**Assuming** You are working with or within a process that is robust enough to incorporate feedback; there is time and liberty for repeating practice.

**If** You, or someone you know, needs to keep doing $x$ ↯ continuing to do $x$ unchanged is likely to incur a big cost, or else, there could be a big benefit to changing ↯ change will not be instantaneous.

**Then** Incorporate suitably-structured feedback into the process to adapt it as it goes.

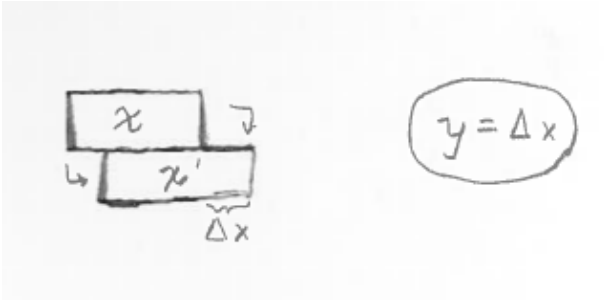
**Example** Writers Workshops [31].

**Alternatives** *Self-directed* learning tends to integrally involve Research Design.

*Comment.* Learning is said to come in three main varieties: classical conditioning, instrumental conditioning, and skills-based practice (cf., e.g., [20, pp. 160–161]). The questions *who is learning* and *what are their native capacities* are is closely linked with how they learn, and with the corresponding learning designs that are applied. There are often one or more thresholds of response, with different consequences on either side. At a population level, this can lead to stratification.

Watt [88, pp. 7-8] observes that "The way that people learn declarative knowledge (such as facts and rules) is simply different from the way that people learn procedural knowledge (such as skills)." Although some regularities can be observed, learning is not an entirely predictable process, or it would be referred to instead as "control." In pragmatic terms, learning is related to the broader concepts of sociality and co-evolution. Deleuze's [21] remark that "There is no more a method for learning than there is a method for finding treasures" serves to highlight the difference between *design* and *method*.

### Business Design




**Assuming** A new way of doing things (a method) or a new connection between disparate regions (with a corresponding opportunity for arbitrage) opening the way to desirable products or exchanges.

**If** You know that there is a market for $x$ ↯ You reckon that you can deliver $x'$, which has some differences.

**Then** Figure out how people perceive $x'$; be prepared to market $x'$ in a different. way from $x$, or to adjust the process and produce $x''$ (etc.) instead

**Example** Honda could not compete with Harley Davidson in the US, but their Super Cub was fun to ride off-road: they made a niche for dirt bikes.

**Alternatives** Markets; central planning.

*Comment.* Designing a business means thinking about a product or service in context. A perfectly good product might not sell if it's marketed the wrong way. A great idea might not be economical if it's produced in the wrong way. According to Ronald Coase, firms come into being when in-house communication is more efficient than market transactions. Simon [78, Chapter 2] contends that the key difference between businesses and markets has to do with the way they process information, namely via hierarchy in the business setting. Businesses may be thought about in systematic or even quasi-organic terms: although they remain somewhat modular, the parts are not strictly interchangeable. Along with supply, demand, competition, and turnover, one of the important parameters is how the business is funded: as with a home mortgage, responsibilities

[1] We use the "↯" symbol to demarcate the conflicting forces in each pattern. Accompanying diagrams are intended to illustrate these conflicts pictorially.

| Pattern Name (Scope, Purpose):<br>Intent:<br>Also Known As:<br>Motivation:<br>Applicability:<br>Structure:<br>Participants:<br>Collaborations:<br>Consequences:<br>Implementation:<br>Sample Code and Usage:<br>Known Uses:<br>Related Patterns:<br>*Gang of Four pattern template* | Course Aims:<br>Course Structure:<br>Learning Outcomes:<br>Assessment Plan and Submission Dates:<br>Timetable:<br>Course Tools and Requirements:<br>Indicative Reading List:<br>*standard course syllabus outline* | Key Partners:<br>Key Activities:<br>Key Resources:<br>Value Propositions:<br>Customer Relationships:<br>Channels:<br>Customer Segments:<br>Cost Structure:<br>Revenue Streams:<br>*Business Model Canvas,*<br>*from Alexander Osterwalder* |
|---|---|---|

**Table 2: Sample templates for** Design Patterns**,** Learning Design**, and** Business Design

to funders can come with considerable constraints on future development [36, Chapter XIV]. As software eats the world, "business" may increasingly tend to rely on statistical A/B testing. (Notice as well the similarity to Learning Design: this is in some sense a discretised version of that pattern.)

## 2 Parameters of design processes

Each of the patterns we present has a number of parameters that can change the way it works in specific instances. For example: *Who is creating the design in question? What are their capacities and constraints? How will the designs be used? How fluid is the overall situation?* In this section, our aim is to outline strategies that address basic questions that will come up time and time again in various design settings. *What is the problem? How might it be addressed?* and *How can the solution be developed?* Given the general nature of these questions, there are many ways to go about generating an answer. These constitute the essential parameters of design.

### Research Design




**Assuming** A knowledge domain that is at least implicitly organised; some liberty of choice to select and explore problems, strategies, and priorities.

**If** You need something worthwhile to work on. ↯ You don't know in advance what will turn out best. ↯ It might not even be clear what's possible or practical

**Then** This can be tackled at the meta-level—make it personal. Ask how known problems relate to each other through you. What do you need to learn to participate more fully? What's at stake? (Cf. Learning Design, Business Design.)

**Example** Edward Jenner, inoculated with smallpox as a child (a risky process!), later developed vaccination (safer and more effective) by drawing on local knowledge and careful experiments, subsequently carrying out impactful dissemination.

**Alternatives** Individual grants are even more about "the person" than standard grants.

*Comment.* Russell [76, p. 110] suggested that the essential thing is "the substitution of observation and inference for authority." According to Kuhn [48, p. 5] "normal science," makes some compromises *vis à vis* that requirement. Further varieties include "basic", "applied" and "use-inspired" research [83]; another typical division is between research that is "exploratory," "descriptive," or "explanatory" [85].

### Product Design




**Assuming** An at least somewhat decomposable problem; empathy for those affected; previous problem-solving experience.

**If** You are confronted with a problem $y$ ↯ You have a repertoire of solution strategies $x_1$, $x_2$, etc., none of which apply directly to $y$.

**Then** "Reverse" the difficult parts of $y$, one by one, until you find an easier variant of the problem that can be solved using one of your existing strategies. Think about how the solution

would be perceived emotionally by those affected, and then adjust the solution based on your assessment.

**Example** Scaltsas [77] analyses the famous myth of the Trojan horse using this model (Figure 3). *The reversal*: What if Troy was no longer impenetrable: say the Trojans opened the gates? *Emotional assessment*: They would be happy to do this if they were receiving a gift. *The adjustment*: What if the gift disguised a trap? Compare: homelessness might be addressed (symptomatically) by creating small-scale shelters: would the public and users find it acceptable if shelters were built into advertising hoardings [40, p. 59]?

**Alternatives** TRIZ similarly says to move from a specific problem to a generic one that has a solution. The overall structure of this pattern is also similar to the basic situation in case based reasoning. Something along the lines of anti-unification, which leads to the least-general generalisation, may be relevant to making the reversal step useful.

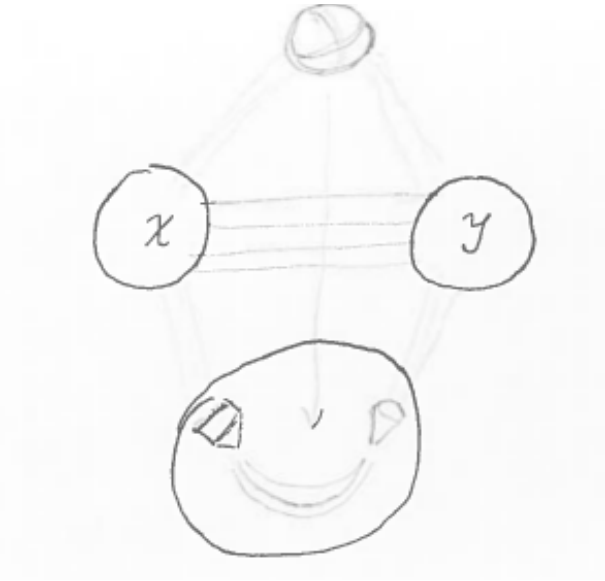
*Comment.* Scaltsas's "BrainMining" approach attempts to work with the human tendency to *anchoring bias*, which in the case of a difficult problem can readily result in "stuckness." The same bias might tend to make every problem "look like a nail" relative to existing solution strategies. The emotional assessment phase moderates that tendency. However, if the emotional assessment component ends up dominating, the process could begin to look like "design by committee." The focus on solutions in this pattern could come with a tendency to develop symptomatic treatments rather than etiological understanding.

This pattern could be seen as the mirror image of Research Design, and they can be used together. Whereas that pattern assumed a stock of problems that must be navigated and selected from or added to, this one assumes a stock of existing solution strategies.



**Figure 3**

Concept Design




**Assuming** the capacity to think about one thing in terms of another, to find common themes, and to keep a potentially complex array of models in mind; also, a certain background understanding of the way the world works, and some sense of the distinction between fiction and fact.

**If** you need to think or communicate about $x$ but you don't understand it completely ↯ you understand the phenomenon $y$ reasonably well ↯ $x$ has some abstract features in common with $y$.

**Then** form a blended structure $\beta$ that integrates $x$ and $y$ by first identifying and incorporating their common features; combine additional structure from both sides, possibly recruiting additional extraneous structure to round out the picture.

**Example** "Thatcher, Thatcher, milk snatcher" blended the then-Secretary of State for Education and Science with a policy whose implementation she oversaw, creating an image at human scale.

**Alternatives** With minor changes to the above, two different phenomena, both understood only partially, can be thought about by integrating them together.

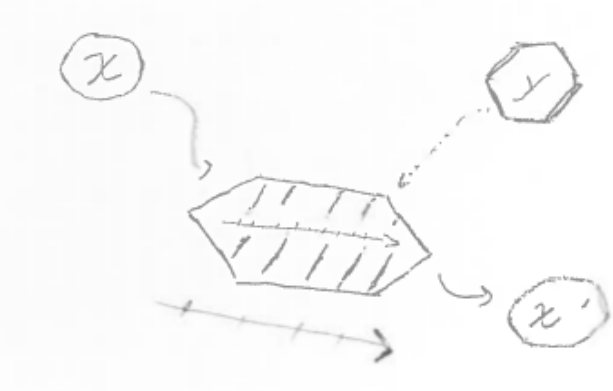
*Comment.* The central goal in concept blending or concept integration, as described by Fauconnier and Turner [28], is to *achieve human scale*. This is similar to the process of making things personal that comprised the core of the solution in Research Design. Blending has influenced our analysis of all of the patterns we present here. This can perhaps be seen most clearly in the accompanying diagrams, each of which presents a different variant on the $x$, $y$, $\beta$ scheme. Whereas Alexander and Poyner [6] focused on resolving contradictions, this takes on further dimensionality in the blending process: first, finding an analogy between the two sides, then finding a context in which they fit together, and, moreover, in which this becomes meaningful. In contrast to the metaphor of "atoms" that Alexander and Poyner used, Fauconnier and Turner [27] write that "the most suitable analog for conceptual integration is not chemical

composition but biological evolution." They contrast their model of figurative thought with formal approaches, in which "identity is taken for granted," "analogy [ . . . ] is typically not even recognized" and in which, accordingly, difference is hard to conceptualise [28, Chapter 1].

## 3 Relationships with technology

This section presents three patterns with a more speculative feel, ultimately drawing together the themes discussed so far.

### Experience Design




**Assuming** organisation of activity is to be maintained over time, or created anew; emotions can be aroused and senses engaged.

**If** new whole persons must be created ↯ relevant experience is not immediately accessible or likely to happen on its own.

**Then** Build experiences that shape the person, so that they enact patterns of structure, order, hierarchy, and category—or their opposites, chaos, play, unpredictability—as well as associated contrasts such as in/out, reveal/conceal, inversion/reversion. If the experience should result in transformation, resolve contradictions between who the person is and who they must become, using significant lived symbolism. The experiences should be immersive: time may pass differently for those engaged in them; in the extreme it may be as if an entire lifetime had passed.

**Example** In the pattern *Entrance Transition*, Alexander et al. describe a building entrance as a place to shed the persona associated with the street (Figure 4). "The experience of entering a building influences the way you feel inside the building. If the transition is too abrupt there is no feeling of arrival, and the inside of the building fails to be a sanctum" [5, p. 594]. Elsewhere, Alexander remarks that "The buildings that I build very often have a dreamlike reality. I don't mean by that they have a fantasy quality at all, in fact quite the reverse. They contain in some degree the ingredients that give dreams their power" [9].

**Alternatives** Handelman [38] gives a typology of "public events" and a thorough analysis that inspires this pattern. In particular, *events that present* are juxtaposed with *events that model*. The former "are mirrors held up to social order, reflecting and expressing the compositions that their composers desire for society" (p. xxix). An event of the latter sort "generates

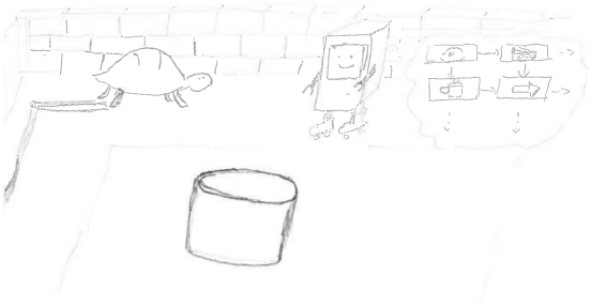


**Figure 4**

and produces controlled change within itself, change which has a directed and direct effect on the world beyond the event" (pp. xxi–xxii). He also describes a third variety, *events that re-present*, which "do work of comparison and contrast in relation to social realities [ . . . ] by offering propositions and counter-propositions, within itself, about the nature of these realities" (p. 49).

*Comment.* In typical usage, "experience design" would imply *user experience design*. Human beings tend to be able to make sense of phenomena such as being distinctly separate yet intimately related [38, pp. 155–156] which would presumably fail to register for current machines. Notice as well that despite the power of conceptual blending as described in Concept Design, it doesn't directly explain which concepts will be experienced as significant.

### Knowledge Base Design



**Assuming** Some agent who must navigate an unfamiliar space.

**If** the agent needs to make sense of possible behaviours in this space ↯ the agent has their own background of meaningful behaviours in some other space.

**Then** form bridges between the two ways of thinking, e.g., by making the goals and beliefs that apply in the new space explicit.

**Example** Two paradigmatic examples are depicted in the image above. The first is the Logo turtle, a virtual robot that followed instructions written in the Logo language, originally developed by Seymour Papert, to run on a custom computer

| | | |
|---|---|---|
| Who benefits, and how do they benefit:<br>Description for a general audience:<br>Track record:<br>Objectives & Methods:<br>Research Programme:<br>Project Gantt Chart:<br>Justification Of Resources:<br>Pathways To Impact<br>(Knowledge, People, Society, Economy):<br>Academic beneficiaries:<br>National importance:<br>*"Standard Research" from the Engineering and Physical Sciences Research Council* | Compress what is diffuse.<br>Obtain global insight.<br>Strengthen vital relations.<br>Come up with a story.<br>Go from Many to One.<br>*Sub-goals of blending, from Fauconnier and Turner* | Theory of evaluation:<br>utility theory, statistical decision theory<br>Computational methods:<br>choosing optimal alternatives<br>for choosing satisfactory alternatives<br>The formal logic of design:<br>imperative and declarative logics<br>Heuristic search:<br>factorization and means-ends analysis<br>Allocation of resources for search:<br>Theory of structure and design organization:<br>hierarchic systems<br>Representation of design problems:<br>Social design:<br>Bounded rationality<br>Data for planning<br>Identifying the client<br>Organizations<br>Time and space horizons<br>Designing without final goals<br>*Curriculum for design from Herbert Simon* |

**Table 3: Sample templates for** Research Design, Concept Design, **and** Designing Intelligence

for educational use designed by Marvin Minsky [11]. Physical turtle robots had been developed previously, typically with sensors. In standard Logo implementations, there were no sensors, and moreover, the turtle did not have access to a global coordinate system: it follows strictly local instructions, step-by-step [34]. (Subsequent work with StarLogo expanded upon these basic features [74].) The other example comes from another educational program, ChipWits, released in 1984. The eponymous *ChipWits* are virtual robots that "inhabit maze-like worlds of connected rooms, each filled with an odd assortment of junk" [7]. Unlike the Logo turtle, these robots have sensors, which they can use to detect useful items and avoid threats. In addition, rather than being programmed in a restricted dialect of LISP, ChipWits are programmed in an Icon Based Operating Language, hooking actions together on circuit boards (Figure 5). The visual language makes it more clear that alongside the maze-like worlds, there is an abstract space of possible programs for navigating these worlds. This seems to help the programmer identify with the agents' mental states [88]. A natural evolution of ChipWits would add facilities for meta-programming, whereby simulated robots could self-program by interacting with their environments. Devlin et al. [25] tackle a similar problem using examples and code synthesis; DeepMind's breakthroughs in videogame-playing used self-programming with non-symbolic representations (i.e., pixel-level inputs and neural networks) [61].

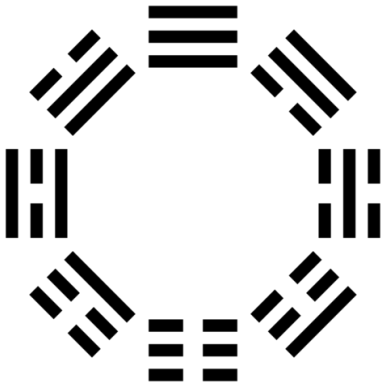
**Alternatives** Word co-occurrence and ordering are two ways by which meanings are transmitted in language [39], using associations which have developed over time [20, pp. 63–64]. Contemporary computational models often exploit this property. For instance, McGregor [59] uses a geometric approach to build models that locate sub-categories within categories, in such a way that projections onto lower dimensional subspaces reveal the salient relationships between terms. Thus, predators and pets, as well as canines and felines, are found within the category of animals; in one subspace "wolf" and "lion" are nearby, in another, "wolf" and "dog" are nearby (pp. 38–39). Character-level analyses have the advantage that they do not require prior knowledge of the language [73].



**Figure 5**

*Comment.* Simon [78, p. 111] says that "Everyone designs who devises courses of action aimed at changing existing situations into preferred ones." Clearly, some representation of the state of world is a prerequisite. Simon considers memory as external to the agent.

Designing Intelligence



**Assuming** a notion of "intelligence" that is recognisable to humans.

**If** designing a system where intelligence is needed, of whatever form or scale ↯ which cannot be abstract but must be embodied in some social, physical, or software system.

**Then** Notice when the patterns we have described are blocked, so that, e.g., learning doesn't happen, businesses stagnate or crumble, research is ineffective, products are not useful, concepts are unclear, experiences are not meaningful, the world is incomprehensible. This can suggest conflicts around which new Design Patterns can be created, or hint at how existing patterns can be refined.

**Example** Marvin Minsky's [60] *Society of Mind* presents a high-level design in which the small component systems—agents—contribute different aspects to the system as a whole. His core rubric is: "Each mental agent by itself can only do some simple thing that needs no mind or thought at all" (p. 17).

**Alternatives** The patterns we have presented point out some directions in what is clearly a vast possibility space. According to Alexander [3, p. 10] what is essential is to "build on the structure that is there, do not destroy it or interfere with it, but rather enhance it and elaborate it and deepen it," in contribution to a larger whole. Consider that animal and even vegetable intelligences can learn [32], and, moreover, engage in niche creation.

*Comment.* Andy Clark [16] remarks on the special form this takes for humans:

> "Against the enabling backdrop of the homeostatic machinery that keeps us within our windows of organismic viability, the shape and contents of the rest of our mental lives are determined by prediction-driven learning as it unfolds in the ecologically unique context of our many designer environments [ . . . ] that enforce exploration and novelty-seeking in ways hitherto unknown among terrestrial animals."

Good [35] sketches a neural model comprised of relatively stable assemblies and more frenetic subassemblies, closer to the senses, which seems analogous to the above: "If assemblies correspond to conscious thoughts, it might well be that subassemblies correspond to unconscious and especially to preconscious thoughts" (p. 58).

## 4 Discussion

Galle [33] meditates on design patterns as potential "atoms of conceptual structure." He notes that, with few exceptions—such as Moran's classic proposal for an "Architect's Adviser"—design patterns were ignored in the knowledge-based systems literature. Many of the historically-early support tools emphasised the physical properties of objects and their combinations.

With this in mind, we can contrast *conceptual design*, as we understand this term, with the perspectives on "making" advanced by Ingold [42]. Ingold follows Deleuze and Guattari [22, p. 408] in highlighting interactions between maker and medium, e.g., "surrendering to the wood, and following where it leads." This example is cited in opposition to simplified narratives of form-giving considered as a "technical operation which imposes a form on a passive and unspecified matter" [79]. The passive and active processes might be diagrammed as follows:

$$\frac{clay}{brick}\,\text{form} \qquad\qquad \frac{axe \quad wood}{split\ wood}\,\text{technique}$$

A clue that these authors are not actually refuting "hylomorphism" in the way they claim comes directly from the choice of examples, and the fact that *húlē* originally means wood. In any case, Ingold's broader concern is with theories and thinking that he deems to be insufficiently aware of process, including applications of causal thinking to situations which are more complex. He makes the case that co-evolution is more widespread than we tend to acknowledge. Following Alexander and Poyner [6, p. 318], let us fly right into the heart of the debate with another diagram rather like the two above:

$$\frac{if}{then}\,\text{because}$$

The associated issues seem to become clearer if, instead of "because," we understand "assuming," as per our usage in the foregoing sections. In design patterns, the links between "if" and "then" seem to depend on complex articulations, not on single causes. Consider these examples, adapted from Aristotle (*Physics*, Book II, Part 9):

- *If* you want to make a house, *then* you need a roof, *assuming* the house is for humans on open ground.
- *If* you want to make a saw, *then* use hard material for the blade, *assuming* the saw is driven by hand-power.

When phrased this way, it is as if we have been explicitly invited to think of exceptions to the rule. Moreover, when the exceptions have something in common, they can be captured in design patterns. Consider:

- Both a *sheep pen* and a *cave dwelling* do not need a roof because they Reuse A Natural Covering.
- Both a *water jet cutter* and a *plasma cutter* can Have The Power Source Do The Hard Work.

From a design perspective there will be further exceptions, e.g., a sheep's wool can protect it against rain, not against predators; a plasma cutter can only cut conductive materials, and so on.

Simon [78] described "goals" as the interface between internal and external organisation, and something similar is going on in the diagrams above. Goal structure, whether situated in form, technique, cause, or articulation through reasoning or embodied action, relates systems' internal and external structure. This helps explain why the brick-making and wood-splitting scenarios feel different. The grasp of the hands on the axe handle is intimately related to the mind's grasp on the chop. Part of the goal structure of the activity of chopping wood has been solidified in the shape of the axe itself. However this is not fully determining: the axe could, under different circumstances, be used as a weapon [20, p. 72–74]. More broadly,

> "The existence of top-down causality implies that the evolution of any given assemblage will be partly autonomous and partly influenced by the environment created by the larger assemblage of which it is a part" (*ibid.*).

DeLanda points out that the term *assemblage*

> [ . . . ] fails to capture the meaning of the original *agencement*, a term that refers to the action of matching or fitting together a set of components (*agencer*),

> as well as to the result of such an action: an ensemble of parts that mesh together well. (*ibid.*, p. 1)

This notion of an evolving "*agencement*" nicely characterises the status of the proto-patterns *Reuse A Natural Covering* and *Have The Power Source Do The Hard Work*. In contrast to the brick-making and wood-cutting examples, these two example proto-patterns are *creative*, insofar as they involve concepts "not present in [the] statement of the problem and the general knowledge surrounding it" [57]. Let's consider this more deeply.

Smith [81] describes *concepts*, as they are treated within Deleuze's analytics, as existing in a state of becoming that requires both self-consistency and internal variability. Moreover, new concepts only arise when we are forced to think! Alexander and Poyner [6] say something quite similar: *design* is only needed when there is a conflict between tendencies that cannot be resolved in a more direct way. Nevertheless, where could new concepts possibly come from if not some broader or restructured context surrounding the problem? For instance, one class of inventions could be accounted for in terms of Simondon's notion of *autocorrelation*, which is involved in the literal *Reinvention of the Wheel* as built around a hub that contains free-rolling ball bearings [15, pp. 10-11]. Another distinct option would be to go on a journey and collect new material (Figure 6). The journey metaphor is preferred by Kohls in his model of design patterns [46, 47]. Notice that with a long-enough journey, it may be natural for the set of assumptions themselves to change, hinting at something akin to Peircean abduction [26]. Combinations of the two pattern-schemas recover Alexander's abstract model of "harmony" [3, p. 38].

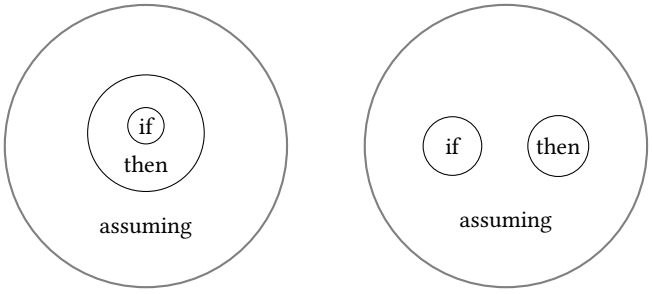



**Figure 6: Two different pattern-schemas**

Elsewhere, Alexander [1, p. 134] observed a distinct meta-level phenomenon that is similar to autocorrelation, namely the "structural correspondence between the pattern of a problem and the process of designing a physical form which answers that problem."

As always, the precise details depend on context—and also on how "context" is understood. Surveying developments in 18th and 19th Century science, Georges Canguilhem [13] pointed out that:

> With the success of the term *milieu* [over the related notions of *circumstances* and *ambience*] the representation of an indefinitely extendible line or plane, at once continuous and homogeneous, and with neither definite shape nor privileged position, prevailed over the representation of a sphere or circle, which are qualitatively defined forms, and, dare we say, attached to a fixed center of reference." (Translation in [14].)

Both Alexander and Deleuze have sought to recover certain circumstantial and vital aspects of being, without descending wholesale into vitalism (*viz.*, the belief that "living organisms are fundamentally different from non-living entities *because they contain some non-physical element or are governed by different principles than are inanimate things*" [10], emphasis added). In fact, both authors take the concept of "life" and extend it to the inorganic [4, 23]. Ingold [41] discusses a related perspective. "Creativity" is the essence of this leap. Here, we have traced connections between creativity and conceptual design, with examples, leading to the following:

## 5 Conclusions

Artificial intelligence pioneer John McCarthy [58] wrote: "The key to reaching human-level AI is making systems that operate successfully in the common sense informatic situation." Conceptual design, e.g., via developing pattern catalogues, offers opportunities for feedback and evolution of a humanistic, social, approach to *Intelligence Augmentation*—and, perhaps eventually to Artificial Intelligence as McCarthy described.

The linked problems of representing design knowledge so that it is useful for collaborative design in distributed communities, or usable at all by artificially intelligent computer systems—though of longstanding interest [87]—still needs further effort. Experiments like Oxman's Think-Maps [69, 71] and other examples surveyed by Galle [33] have the air of being technical demonstrations, and are not in widespread use. Pattern repositories like the one described by Inventado and Scupelli [43] do not make significantly more intensive use of computer technology than the Portland Pattern Repository which was hosted on the world's first wiki. The usefulness, for common sense reasoning purposes, of logic-based representations has been debated [62, 66]: evidence suggests that there is always more to the picture. For example, technologies based on the conceptual graphs of Sowa [82], deployed in architectures inspired by the society of mind, have seen industrial use [55], while associated efforts to automate natural language understanding are still ongoing [49, 52, 54]. Fauconnier and Turner [28, p. 109–110] suggest that complex mental phenomena like blending should be studied with human data not simulations.

"Crowd creativity" manages to integrate many of these themes. Here, designs are produced by an evolutionary process with humans in the loop [53, 67, 89, 90]. However, current workflows miss a reflexive component. Design patterns could usefully be incorporated into these processes, to serve as "a living language" that supports design and guides reflection [5, p. xvii]. Corneli et al. [17] outlined one approach for evolving design patterns in a collaboration. Future work could make use of more sophisticated ways to integrate feedback in "biomechanical" [35, p. 34] social systems. The expected outcome would be that citizens would be able to more fully engage with processes that matter. This is just one of many possible designs that these patterns could inform (Figure 7).

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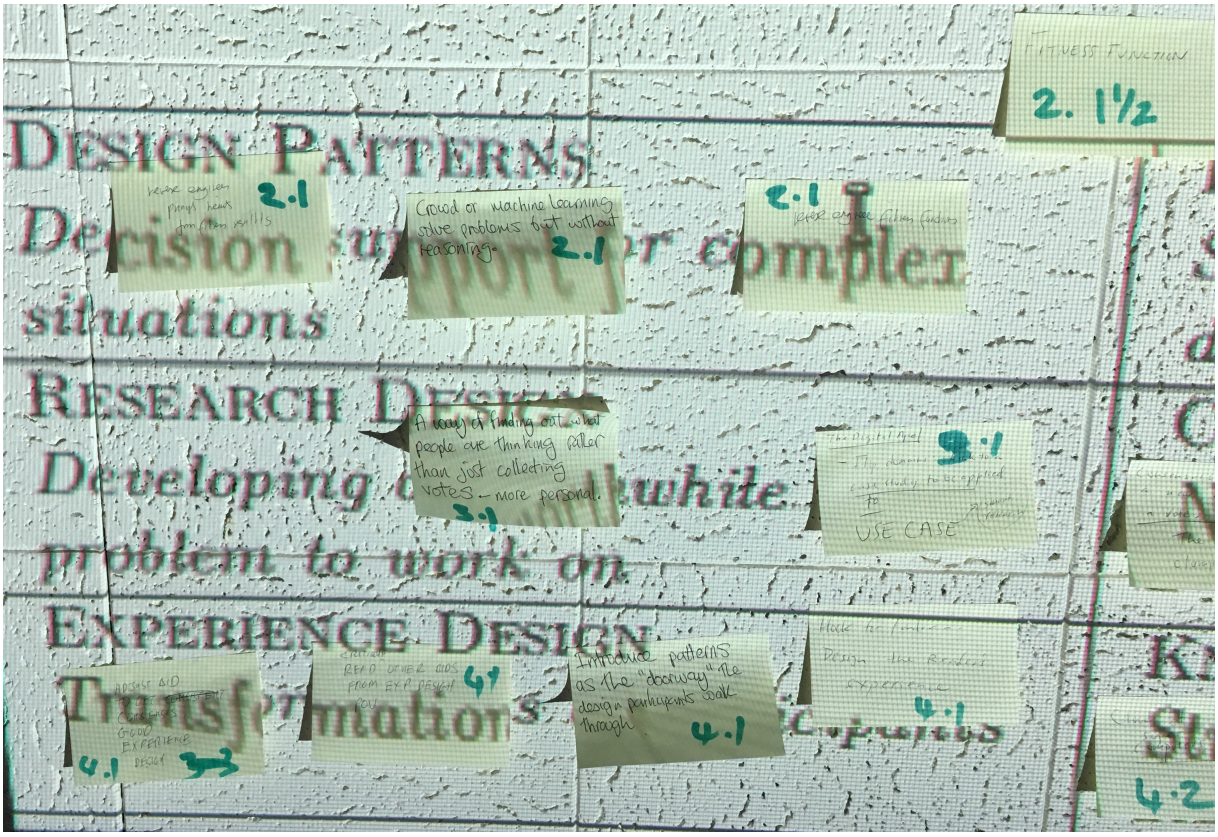



**Figure 7: A practical exercise in which we used PostIt notes and a projection of Table 1 to design a research project**

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# Chapter 10
# Evaluation of Creativity

Alison Pease and Joseph Corneli

**Abstract** This chapter develops a meta-evaluation of progress markers in Computational Creativity. We rely on an analysis of interview data with people who have applied several standard metrics. We use an existing meta-evaluation framework to distil findings in a format that will support comparison with future research on this topic.

## 10.1 Introduction

The goal of this chapter is to discuss the role of evaluation in computational creativity. Early writings on this area were developed by Ritchie (2001) and Pease et al. (2001). Pease et al. cast creativity in functional terms.[1] On this view, creativity is to be found amongst the "input, output and the process by which [the output] is achieved." Specifically, "Creativity may be seen as output minus input." With this understanding, creativity is fundamentally – but not merely – generative. It can be abstractly understood as happening in two phases or stages: "generation" and "evaluation." In effect, the generative stage generalises the input, and the evaluation stage assesses what is new and useful in this output. Ritchie (2001) calls the (salient aspects of the) input an *inspiring set*, and focuses on thinking about creativity within domains where there are known product-evaluation strategies: specifically, ways to

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[1] This approach is reminiscent of Marr's (1982) description of *computational theories*: "a computational theory is commonly understood roughly speaking as at least a fairly precise characterization of inputs to the system, outputs of the system, and a spelled out hypothesis concerning the functional mapping between both ends."

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determine the *novelty* and *quality* of generated artefacts. Pease et al. (2001) follow Boden (2004) and embark on an explicit examination of "process" itself as potentially creative.[2]

The basic challenge, as outlined by Pease et al. (2001), is "to find a framework which is both practically useful and theoretically feasible." Theoretical feasibility means, at least in part, that an evaluation strategy should "reflect human evaluations of creativity" – in other words, that the framework should be "faithful" to everyday notions of creativity.

A decade after this challenge was framed, Jordanous (2011) examined the uptake and quality of various evaluation strategies that had been proposed in the intervening years. Drawing on the work of Pease et al. and other literature, Jordanous subsequently (2014) put forward a total of five "meta-evaluation" criteria: *Correctness*, *Usefulness*, *Faithfulness Usability*, and *Generality*. This high-level framework for meta-evaluation was complemented by preliminary empirical work that applied the framework to assess previous evaluation work. The core method used in the meta-level study was to involve third party experts, who audited evaluations that had been carried out by others.

In the present work, we make meta-evaluation more down to earth. Our primary source of empirical data is a series of interviews with people who have applied key evaluation strategies in computational creativity. Furthermore, these interview subjects were closely involved with the systems that were being evaluated, and their perspectives were, accordingly, largely practically-based. Importantly, the interviews did not focus on evaluation of the system or its outputs, but rather on meta-evaluation: that is, *evaluation of the evaluation strategies that were employed in the course of practical work on the project*. Since the interview subjects did not create the evaluation strategies they employed, we can largely avoid any potential for creator bias. We see practitioner perspectives as particularly relevant for process-oriented evaluation.

The outline of the chapter is as follows. Section 10.2 discusses the way evaluation has evolved since the early days of computational creativity research, in the context of broader progress in the field. Here we take the view that evaluation methods have evolved alongside the main systems. Section 10.3 describes several evaluation frameworks that are currently in use. Section 10.4 evaluates these frameworks, drawing on interviews with practitioners. We discuss our findings in terms of the meta-evaluation metrics proposed by Jordanous (2014), and develop a set of questions for further discussion by researchers in computational creativity who are concerned with evaluation and meta-evaluation. Section 10.5 presents a summary of the chapter.

[2] Boden's description of creativity in terms of processes that explore and enrich a conceptual space would later be presented in more explicit computational terms by Wiggins (2006).

## 10.2 A Short Historical Survey of Evaluation in Computational Creativity

The definition of the field of study that was originally adopted by the Association for Computational Creativity (and that remained current up until 2013) was spelled out as follows:[3]

> *Computational Creativity is the study and simulation, by computational means, of behaviour, natural and artificial, which would, if observed in humans, be deemed creative.*

The implied focus on human behaviour was subsequently dropped in an adapted definition proposed by Colton and Wiggins (2012). This version has been widely quoted in recent literature:[4]

> *The philosophy, science and engineering of computational systems which, by taking on particular responsibilities, exhibit behaviours that unbiased observers would deem to be creative.*

Nevertheless, in practice, researchers in computational creativity often work on domains that are typical of human creativity, particularly in the arts. This has had a strong effect on evaluation strategies. Computational models of artistic creativity are often connected with *artefact-centred evaluation* – where, at least in principle, it does not matter how the artefacts are created, or by whom. Ritchie's papers (2001; 2007) are foundational for this style of work.

Criticising the tendency to do "process-blind" evaluation, Colton (2008) observed that although artefact-centred evaluation seems to present a level playing field, in practice, it is not representative of the way we usually think about art. This is because everyday assessments of creative work by humans tends to take into account history-rich issues of process and identity.

Much more broadly, perceptions of creativity may be highly context-sensitive. Anna Jordanous (2013) surveyed the literature and found fourteen wide-ranging components of creativity that are frequently referred to by authors writing on this subject (Table 10.1). This work has subsequently been refined and validated via card-sorting studies and principal component analysis by van der Velde et al. (2015) who showed that, indeed, terms similar to the ones that Jordanous identified seem to be what people have in mind when they think about "creativity." Jordanous and Keller (2016) revisit and extend the original corpus-based research.

These concepts have been used by Jordanous and others to evaluate creativity in conjunction with the three-part assessment framework also proposed by Jordanous (2012): the so-named "standardised procedure for evaluating creative systems" (Method 10.1).

(In practice, the specific criteria used in Step 1 of Method 10.1 may be conveniently selected from Table 10.1.)

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[3] Anna Jordanous, What is Computational Creativity? April 10, 2014 http://www.creativitypost.com/science/what_is_computational_creativity

[4] Jordanous and Keller (2016); Besold et al. (2015); Emmerson and Landy (2016); Green and Kaufman (2015); Toivonen and Gross (2015), among others.

| | |
|---|---|
| C1. **Active Involvement and Persistence** | C8. **Originality** |
| C2. **Dealing with Uncertainly** | C9. **Progression and Development** |
| C3. **Domain Competence** | C10. **Social Interaction and Communication** |
| C4. **General Intellect** | C11. **Spontaneity/Subconscious Processing** |
| C5. **Generation of Results** | C12. **Thinking and Evaluation** |
| C6. **Independence and Freedom** | C13. **Value** |
| C7. **Intention and Emotional Involvement** | C14. **Variety, Divergence and Experimentation** |

Table 10.1: Fourteen frequently-used criteria for creativity gathered from the literature in Jordanous's Ph.D. research

Step 1. *Identify a definition of creativity that your system should satisfy to be considered creative.*
Step 2. *Using Step 1, clearly state what standards you use to evaluate the creativity of your system.*
Step 3. *Test your (potentially) creative system against the standards stated in Step 2 and report the results.*

**Method 10.1:** The Standardised Procedure for Evaluating Creative Systems (SPECS), from Jordanous (2012)

Although this work provides a thorough exploration of the "social construct" of creativity, Jordanous's survey does not attempt to state what creativity "is" in a way that is derived either empirically or from first principles. One may wonder whether creativity is anything other than a construct. Furthermore, there is nothing intrinsically tied to the concept of "creativity" in the structure of the three steps of Method 10.1, which could be straightforwardly adapted to assess any other qualitative feature of the system's behaviour.

The connection between computational creativity and artificial life has been taken up by some researchers (Saunders and Bown, 2015). This research trajectory is open to empirical criteria (e.g., viability, empowerment, complexity) – and tends to characterise "creativity" and "aesthetics" in these terms (Guckelsberger and Salge, 2016; Javaheri Javid et al., 2016).[5] The relationship between explicit, computable, metrics like these and the more subjective evaluation criteria outlined in Table 10.1 remains an open problem.

To be sure, Table 10.1 subsumes the key concepts in artefact-centred evaluation: C8 (Originality) and C13 (Value). Several of the criteria, such as C1, "Active Involvement and Persistence," focus on the context and workflow through which creativity happens. Other similar sets of criteria that can be found in the literature are:

- van der Velde et al. (2015)'s *originality*, *emotional value*, *innovation*, *intelligence*, *skill*;
- Colton's (2008) "tripod": *skill*, *appreciation* and *imagination*;

[5] Guckelsberger and Salge (2016) are particularly concerned with meanings that are created by agents, after Von Uexküll (1982).

- and Colton's more recent (2013) "spider": *skilful*, *appreciative*, *imaginative*, *intentional*, *adaptive*, *reflective*, *inventive*, *innovative*.

While criteria like "originality" continue to be largely *artefact-centred* and criteria like "intelligence" can be seen as *agent-centred*, the overall thrust is towards *process-centred* and *context-based* assessment of creativity.

Increased attention to context and process has had a noticeable effect on thinking in the field. For example, it has cast light on the importance of the prior (or ancillary) process whereby the system that is evaluated is itself created (Colton et al., 2014). Via a reflexive loop, the impetus to bracket software creation and software's creativity together has elicited increasing interest in software that writes software (Charnley et al., 2016). Thinking about automated program synthesis from a computational creativity point of view may require attention to the question: "how do people creatively write code?" (Colton, quoted in Moss (2015)). Software systems are themselves approached in increasingly contextual terms (Johnson, 2014; Jordanous, 2016); and creative contexts may themselves be analysed using software-like rules (Corneli, 2016). Within increasingly diverse and process-rich workflows, evaluation becomes correspondingly complex. Software is seen as requiring greater and greater levels of creativity and independence (Ventura, 2016). As part of the shift away from after-the-fact product-based evaluations, the locus of evaluation is increasingly embedded within the creative process itself (Bou et al., 2015).

### *10.2.1 Other Perspectives*

The pragmatic approaches outlined above can be contrasted with other more philosophical points of view. Henri Bergson (1911) thought of "creativity" and "life" as nearly synonymous. Indeed, the connection between computational creativity and artificial life has been taken up by some researchers (Saunders and Bown, 2015). This research trajectory is open to empirical criteria (e.g., viability or complexity) and would tend to characterise creativity in these terms (Guckelsberger and Salge, 2016) (as opposed to using more abstract notions of 'fitness', as per, e.g., Gabora and Tseng (2014)). Related perspectives are developed in the anthropological work of James Leach (2004). Leach contrasts the "dominant Euro-American… 'appropriative mode'" (p. 152), in which creativity is "primarily intellectual" (p. 154), with the cultural views of creativity held by the Reite people of Papua New Guinea. Among the Reite, creativity is centred on the contingent "necessity of keeping the human world in existence" through action in accordance with traditional mythic knowledge (p. 165). The cultural background in the mainstream of computational creativity research seems to match the 'appropriative mode' described by Leach: the focus is on artefact creation and the mental processes involved therein, even when artefact quality is not the main concern. Nevertheless, as sketched above, some alternative perspectives are beginning to emerge in the field.

## 10.3 Evaluation Frameworks

The evaluation options listed in Table 10.2 are characteristic of the strategies that are available to, and used by, researchers for evaluation of computational creativity.

| Measure | When to apply | Who applies | Outcome |
|---|---|---|---|
| **Ritchie** | *product-based*: apply when the system has produced a set of artefacts | ≈ 100 novices or domain experts | A set of ratings that can be presented as 12 proportions |
| **Jordanous** | *process or product-based*: apply when the system can be used or its workflow clearly described | ≈ 7 domain experts discuss process and product, using some subset of the 14 components listed in Table 10.1 or some other similar criteria | Interview data |
| **Colton et al** | *focus mostly on process*: apply at each development epoch | system developers | Diagrams describing the phases of system development |

Table 10.2: Three evaluation methods characteristic of product-based, holistic, and developmental evaluation

The modus operandi of the evaluation strategy proposed by Ritchie (2007) (as an adaptation of Ritchie (2001)) is to establish whether the system can generate artefacts that are appropriate to a genre, while at the same time offering novelty within that genre. Human judgements of the artefacts are used, but evaluators are never asked "is the system creative?" – unlike some other product-based evaluations (Lamb et al., 2015). Derived measures provide a framework for further assessment (Method 10.2): however, again, Ritchie does not specify threshold values of these measures that are required for the system to be deemed "creative."

Step 1. *The system generates a pool of artefacts.*
Step 2. *These artefacts are rated according to their* ***typicality*** *and* ***novelty****.*
Step 3. *Derived metrics are computed from the ratings in Step 2.*

**Method 10.2:** The evaluation method used by Ritchie (2007)

As we saw in Section 10.2, the focus for Jordanous (2012) is to examine how the system is perceived. In effect, the question is whether users can project features like C1–C14 (or some subset of them) onto the system. Jordanous (2011) describes several example applications of Method 10.1 along these lines.

Lastly, Colton et al. (2014) take a high-level view that focuses on the system's evolution over time. In this framework, the features present in any given snapshot of the system are described using a schematic terminology advanced by Colton et al. (2011). The focus in this approach to evaluation (Method 10.3), however, is on es-

tablishing which entity is responsible for creating a new feature: namely, the developer, or the system itself?

Step 1. *The system's development history is divided into versions, and versions are divided into components.*
Step 2. *The logical workflow between components is diagrammed, and processes within components are characterised in terms of the* ***generative*** *and* ***administrative*** *acts which comprise them, following the typology proposed by Colton et al. (2011) (Framing, Aesthetic, Concept, and Example, at "ground," "process," and "meta" levels).*
Step 3. *Judgements are made about the system's autonomy and creativity based on the extent to which it takes responsibility for generative acts.*

**Method 10.3:** The evaluation method used by Colton et al. (2014)

## 10.4 Evaluating the Evaluation Frameworks

In the present work, we examine the three methods outlined in the previous sections in light of new and pre-existing criteria for "meta-evaluation." Our study draws on interview data with people who have had direct experience with the evaluation methods described above. In some cases these people have adapted, or abandoned, the existing strategies, in favour of other strategies that were deemed more suitable.

### *10.4.1 Methods*

We arranged four hour-long interviews ($N = 4$). Participants were recruited via personal connections and via an open call at a recent Computational Creativity (CC) conference. The main inclusion criterion was that participants had applied some evaluation metrics to software in CC. All participants possessed doctorates in technical fields and were working as post-doctoral research assistants on multi-site projects in Computational Creativity. All participants were male. The interviews were semi-structured, following a protocol that was shared with participants. This followed the outline below:

1. Warm up: Academic background, familiarity with CC and evaluation.
2. Fact gathering: Which criteria were applied, why, and by whom? Who developed the software that the methods were applied to?
3. Ease of application: How much time and expertise was needed? Who was involved?

4. Usefulness of application: Did the results tell you anything useful?
5. Faithfulness to notion of creativity: What did the criteria measure?
6. Evaluation of the criteria: What properties should evaluation criteria in CC have? Are the current measures accurate? Do you have thoughts about improving the criteria or developing new ones?

These interviews were carried out face to face in a room with two interviewers and one interviewee. At the beginning of the interview participants were given an information sheet about the study, a consent form to sign, and were shown a copy of the interview guide so they could see the sorts of questions that they would be asked. All participants were offered a break partway through the interview. The interviews were audio-recorded.

After conducting the interviews, we transcribed them, and analysed the resulting text via a data-driven methodology based on grounded theory. Specifically, we iterated through the phases below:

I. We identified short textual 'codes' which describe key themes in the data and went through all of our data, identifying and revising our codes.
II. We grouped the data by collecting the codes of similar content into categories.
III. We grouped similar categories into concepts, which we used to formulate our resulting theory. We worked together by each independently reading a portion of the data and conducting analysis identifying factors related to evaluation.
IV. We compared these analyses and discussed any recurring differences, allowing us to align our results.

We used the Dedoose system – an online, computer-assisted, qualitative data analysis software tool – to perform our analysis.[6]

### *10.4.2 Findings*

The four interviews were an average of 9,400 words each, and the overall transcript was 37,601 words long, in total. We identified a total of 38 codes, categories and concepts, across 355 excerpts. The codes were arranged hierarchically as shown in Figure 10.1.

### *10.4.3 Discussion*

Participants mentioned that because they worked on a pre-defined project they did not always have the freedom to develop the software or their ideas about creativity and evaluation: thus, the resulting discussion should be read with that in mind.

[6] http://www.dedoose.com/

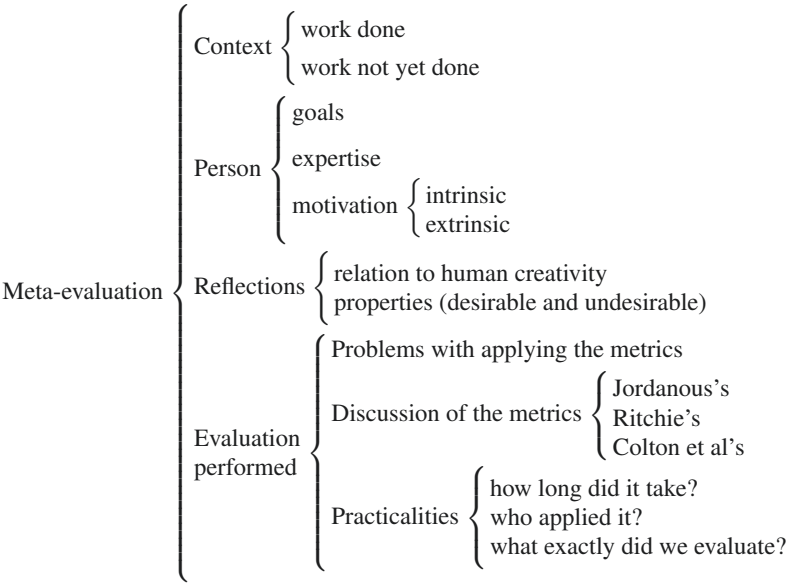



Fig. 10.1: A hierarchical view of our emergent theory

Participants commented on the difficulty of measuring progress in CC, describing the attempt to formulate and formalise evaluation criteria as a "great big meta problem" and a "strong, difficult problem." Perhaps given that they worked within relatively large consortia (of around 25 people) in which different tasks were carried out by different people, participants commented that one of the roles of an evaluation framework should be to provide a way of evaluating progress of the team, rather than just that of the system.

All three frameworks use words which are purposefully left open to interpretation by the researcher, such as "novel," "valuable" (Ritchie, 2007); "domain competence" (Jordanous, 2013); and "framing," "aesthetic" (Colton et al., 2014). Participants pointed out that even given a domain-specific interpretation of these, the subjective nature of the terms can mean that interpretation is left open to participants in empirical work. In particular, if the empirical work is carried out in another language, further ambiguities and linguistic issues can arise, e.g., due to terminological collisions in translated texts given to study participants.

A meta-evaluation criterion for assessing evaluation frameworks was suggested: namely, to determine whether the evaluation measure can identify negative and positive instances of creative systems. Independent means of determining whether a system is creative are not available, in general, especially in a domain-independent sense. However, especially for systems that approach human-like creativity within a domain, there may be a workable consensus about approximately where the systems lies on the spectrum between "creative" and "not creative." It was suggested

that frameworks which determine when a system is *less* creative than another might provide useful direction for research.

Some of the participants felt that the experiment-based frameworks (Ritchie, 2007; Jordanous, 2013) required more understanding of computational creativity than they possessed, and that the FACE model would be a better fit given their own background in computer science: "I felt the other two criteria, Jordanous and Ritchie, required a bit more understanding of computational creativity in general."

#### 10.4.3.1 Experiment-based Frameworks

All participants had some prior familiarity with the ideas of both Ritchie (2007) and Jordanous (2013). They typically chose to apply the metrics in a very flexible way: picking and choosing aspects which suited them and omitting others. For example, a sub-selection of the 14 items in Table 10.1 might be seen by a given research team as both particularly relevant to their domain, and a concrete measure that they could ask participants to evaluate in experiments. In particular, there was a focus on novelty and value, which were identified by Ritchie and present as two of Jordanous's components (originality and value). Our respondents reported that other aspects of both frameworks, such as proportion calculations (Ritchie, 2007), ranking of the components by domain experts, and interviews based on system demonstrations (Jordanous, 2013) were not carried out.

Selection of components was partly based on pragmatic considerations: "we thought they were easy to evaluate – at least in terms of empirical evaluation – that's why we chose them." Components were broken down into smaller, more interpretable, domain-specific units, which were seen as easier and more objective for study participants to judge. Judgements were largely, though not always, made on output which had been generated by the system, rather than on a description of the system's process. In one case, in which one member of the research team was also an expert in the creative domain under consideration, that individual went through all of the 14 components and provided a rating based on knowledge of the system's processes as well as products.

The level at which output was evaluated was one interesting issue that shaped the selection of metrics: in particular, the creativity of a small part of a larger artefact might be difficult to assess. This was important for continuity of variable; a question could become meaningless or muddled if asked at the wrong level of output.

One participant selected originality and preference from Jordanous's 14 components model, since these were seen as relevant to the domain in which they were working. While these were closely connected to Ritchie's criteria, Ritchie's approach was not applied, since the system only generated one artefact per run (rather than a population) and thus the notion of proportions could not be established. In this case, the evaluation was done by an expert who was not involved in building the system, and followed system development over one and a half years. The dependence of Ritchie's criteria on a set of artefacts was seen as problematic, both in theory and in practice. The participant commented that we don't use the idea of

proportions to evaluate human creativity: for instance, we don't know how many versions of a sonata Beethoven wrote before the final one, and nor is this especially relevant for our evaluation of his creativity. Furthermore, given that the system only produced one artefact per run (with a given set of initial conditions), there was no natural "set" of output artefacts to compare.

One participant liked the component-based approach because of its tactfully indirect way of approaching creativity. It was seen as more productive to ask participants questions concerning preferences, for example, rather than to ask whether they thought a particular artefact was "creative." In part, this was because of sociological factors in how the latter question is answered – participants in one study were students and it seemed likely that they would worry if they appeared to be judging the creativity of their professor, for example. Less value-laden questions about personal preference, or perception-based features, were seen as simpler and more likely to receive an honest answer. Additionally, questions concerning preferences were seen as more consistent. Translation problems arising in one study seemed to bias the results, requiring that study to be re-run.

Another participant felt that 14 components were too many for the purpose at hand, and that the whole set was tending towards an evaluation of general artificial intelligence. This contrasted with his own primary goal of building a tool for the purpose of aiding human experts in their creative work. For this, which he felt that five or six of the components would have been sufficient, commenting that "you get lost" with so many components.

It was felt that the components correlated well with human creativity, with one participant commenting that he "really liked" the components: "It was complete, it was sound," and another commenting on the importance of using them in combination: "It's not that these individual measures are measuring creativity, but maybe incrementally you're moving towards something that is creative." One participant particularly liked the emotional evocativeness of the components, commenting: "you get astonished, if you get moved, if you get a close personal experience with the work . . . I like it a lot."

#### 10.4.3.2 System Development Framework

The FACE model was selected and applied only by system developers. Participants took from two days to a week to apply this model. By and large it was applied retroactively to describe work which had already been completed.

The FACE model was seen as rather complicated, and the diagrammatic notation was described as "difficult," "unintuitive," "a lot of subscripts and superscripts and parentheses." The concepts represented were seen as necessary, but the language as lacking clarity. One participant suggested using icons rather than algebraic notation: "I could imagine a version of it that has exactly the same information, but displayed with icons, for example, that have a different colour or shape, or where it fits together in an obvious way, because it does look like a big kind of algebraic mess." Participants did not all agree on this point, with others saying that "It was not so

complicated," and it was "Easy – once you understood what the symbols mean." Along these lines, the question of how well it would scale arose, "I don't know how good it scales, on representing a very large system or – many phases of software development, like, ah, if the software evolves, ah, very much – these diagrams can be very very complicated or if the size of the system is very big, the boxes become very big."

Some participants questioned whether the FACE model actually measured progress in CC. One argued that it measures progress towards autonomous behaviour, rather than the creativity of the system, where progress could be made in the one but not in the other: "you identify the parts in which your software can be improved by making it more automatic. But it is not certain that replacing part of the system by a piece of code you make the system more creative." A further comment was that: "you're not measuring the creativity of the system: for me, you're trying to analyse, somehow, the steps that you put into your system and – in a critical way – that allows you, that helps you to analyse, in a critical way, what the system does and how you can improve it." One participant commented that while the model is a useful development tool, it doesn't help to decide whether the *output* is creative.

Broadly speaking the FACE model was seen as a useful tool for developers: "the FACE model measures or represents the steps that you follow to meet the specification of the system." The question of who should apply it arose: "the problem was, because it was myself that applied the model on something I developed"; and, "maybe by comparing the result of the model done by another person with what the system really does you can infer something interesting – maybe you can identify some creative results, or something that's missing there." Nevertheless, the complexity of the system was seen to weigh against its usefulness in practice, at least without sufficient training: "I can imagine a software engineer . . . or professional software developer, having really troubles understanding the notations."

While all participants had done the evaluation at the end of the system development it was generally felt that it should be used in formative evaluation: "maybe it would have been good to develop a FACE model of the system early in the project, and use that as guidance," and "I really think the FACE model could be used kind of prescriptively." It was also suggested that the model could be useful for other members of the team, not just software developers.

#### 10.4.3.3 Recommendations for Improving Evaluation Effectiveness in CC

Broadly there are two ways for project managers to improve the usefulness of evaluations. (1) Tailor your evaluation to the staff and expertise that you have; and (2) Tailor your staff to the sort of evaluation that you want to carry out. The second option might be a consideration when hiring new staff or for providing training to existing team members. One concrete suggestion would be to hold a tutorial in conducting experiments either within a given research consortium, or more broadly, e.g., in a conference like the International Conference on Computational Creativity

(ICCC). Such training might include qualitative and quantitative methods, ethics, and so on. Further issues to consider are as follows:

1. Consider wider roles that an evaluation framework may play:
    a. Providing a way of evaluating progress of a whole team, rather than just that of the system-developers.
    b. Evaluating progress in aspects that are not related to the system, such as philosophical advances.
    c. Considering the end-goal of a project or piece of work. For instance, if domain experts will interact with the system then there should be communication between developers and experts, and it may be useful to consider methodologies in Human-Computer Interaction, such as user-centred approaches.
2. Framework-specific recommendations:
    a. Process-based model. Provide more documentation, such as a user manual and further examples. Put on a training session. Provide domain-specific versions and examples. Develop the language further, with an emphasis on clarity of notation.
    b. Component-based model. Suggest concrete domain-specific suggestions of how each of the components can be measured, to combine qualitative with quantitative approaches.
    c. Proportion-based model. Suggest concrete domain-specific suggestions of how value and novelty can be measured.

#### 10.4.3.4 Related Work

Jordanous (2011) noticed that, at least in the early days of the field, rigorous evaluations have not always been carried out. As evaluation became more prevalent, in a subsequent paper she outlined several dimensions for *meta-evaluation* (Jordanous, 2014, p. 4):

- **Correctness**: how accurately and comprehensively the evaluation findings reflect the system's creativity.
- **Usefulness**: how informative the evaluative findings are for understanding and potentially improving the creativity of the system.
- **Faithfulness**: how faithfully the evaluation methodology captures the creativity of a system (as opposed to other aspects of the system).
- **Usability**: the ease with which the evaluation methodology can be applied in practice, for evaluating the creativity of systems.
- **Generality**: how generally applicable this methodology is across various types of creative systems.

Jordanous (2014) used this framework to re-assess previous data on evaluation by engaging external (meta-)evaluators. In that study, the meta-evaluators had not ap-

plied the various evaluation methods themselves, but were shown results of several comparable system evaluations.

We can briefly consider the findings described above through the lens of Jordanous's meta-evaluation criteria.

Correctness — The important difference between building a system that is independently creative and one that supports a user's creative work was noted by several participants. What is understood by "creativity" in these two cases is likely to be different, or at least approached differently.

Usefulness — The difference between interrelated perceptions of usefulness, creativity, autonomy, and preference was noted. For instance, evaluation metrics that were useful for meeting an initial product specification might have little direct connection with those geared towards building a creative system. The degree to which goals have been specified in advance contributed to the salience usefulness (or not) of the measures.

Faithfulness — One participant suggested that several of Jordanous's components could "expand on what's between the spaces of being useful and being creative." The subjective nature of direct creativity evaluations was noted, and participants tended to prefer objective measures (whether of small-scale perceptual data, or intersubjectively verifiable holistic outcomes).

Usability — Evaluation metrics were generally selected that were feasible to apply, given the background of the people applying them and resources available. Adaptations were often made that simplified the original methods. In this respect we can point to one interesting negative finding, which is that none of the participants opted to combine the several methods into one coherent analysis.

Generality — The level of output (e.g., small components of larger work) as well as the scale of the system (e.g., one involving many complicated interconnected modules) both appear to present challenges for evaluation. Another related issue is the level of analysis: if an evaluation metric is applied by a larger team, the demands placed on it will be different from a metric that can be applied in an ad hoc fashion by one individual. Similarly (see "Correctness" above), co-creativity and autonomous creativity may be evaluated differently.

## 10.5 Summary

We have carried out a meta-evaluation of progress markers in Computational Creativity. We began with an historical survey of evaluation in this field, focusing on three evaluation metrics: component, process, and proportion-based models. We carried out interviews with people who had applied these metrics, and built an emergent framework for analysis, using a method based on grounded theory. We provided an extended discussion of our findings, supported by quotes from research subjects. Based on this material, we provided recommendations for improving evaluation effectiveness in CC. We then connected our findings to a pre-existing meta-evaluation

framework in order to provide a summary that findings in future studies can be compared to.

| |
|---|
| 1. **Perspective of the evaluator.** Personality and background are relevant to metric development and application. For example, research questions like "What is a joke?" lead naturally to evaluation questions like "Is X joke-like?" and "Is X a good joke?" Experiment-based methods tend to be selected by people with a social science background. FACE is designed to be carried out by system developers. |
| 2. **Increasing the user friendliness of evaluation methods.** How might we provide guidance to (potential) users of the metrics; e.g.:<br>• domain-specific guidance, fleshing out what value/novelty/framing and so on mean in specific domains?<br>• guidance relative to the level of the artefact that will be evaluated: holistic or smaller chunks; relative to what context? |
| 3. **Ways to develop or improve existing measures.** How did you come up with your metrics? Do you have suggestions for ways to:<br>• "Focus on the negative," i.e., identify ways in which the system fails to be creative (e.g., not working at all, or producing the "wrong" thing, etc.).<br>• Use metrics that focus on something other than creativity (for instance, preference) and then draw inferences about creativity. |
| 4. **Non-system-centred criteria.** All evaluation criteria currently centre on the system and/or its output. What about other ways of assessing progress in CC (e.g., progress at the level of philosophical theories)? Keep in mind that ICCC has several different kinds of papers apart from "system" papers. |
| 5. **Additional questions for discussion.**<br>• What are your motivations for using the evaluation method you developed, and what do you think the motivations of others who use them are?<br>• Machine learning has really taken off in recent years. What would be the conditions that would make the study of "machine creativity" similarly relevant? |

Table 10.3: Questions for future research

## 10.6 Conclusion and Future Perspectives

Our meta-evaluation of the FACE, Jordanous, and Ritchie models is more than just a side-by-side comparison. It also gives a holistic perspective on the way evaluation has been applied in computational creativity. Since specific methods are applied inconsistently, a higher level of analysis may be useful.

In future work, we plan a second round of interviews with the researchers who devised the main evaluation criteria discussed above. Our questions are inspired by the analysis of the first set of interviews (Table 10.3). We plan to analyse the new dataset using the same meta-evaluation criteria we applied in Section 10.4.3.4, which will allow us to compare the findings. As a brief meta-meta-evaluation, we can comment that the criteria proposed by Jordanous (2014) seem useful for this purpose.

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# Chapter 6
# Social Aspects of Concept Invention

Joseph Corneli, Alison Pease, and Danae Stefanou

**Abstract** This chapter surveys frameworks that describe social creativity. It focuses on mathematics, but includes work which heads in a more general direction: first, by examining social creativity in music, and then turning to a description of several new results that use formal, qualitative, and simulation techniques to theorise social creativity on computers. The chapter includes a pilot study examining the salience of various frameworks for the analysis of mathematical text.

## 6.1 Introduction

This chapter surveys theoretical frameworks that lie on the spectrum between real-world social creativity and formal theory. We will focus primarily on creativity in mathematics, with brief forays into musical creativity and more general domains to round out the picture. Some of the settings that the frameworks describe are immediately recognisable as places where real-world social creativity happens (but these tend to be less formal); some are more convincingly computational, in the sense that they describe stand-alone simulations (however, these tend to be less obvious examples of what we would understand by social creativity). Relative to our broader research goals, the existing frameworks and efforts all have limitations, which we plan to address through integration.

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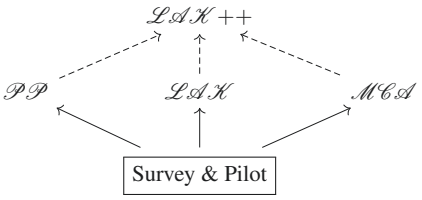



Fig. 6.1: Survey of theoretical frameworks describing social creativity, focusing on mathematics (Section 6.2); $\mathscr{LAK}$: Lakatosian creativity (Section 6.3.1); $\mathscr{PP}$: Patterns of Peeragogy (Section 6.3.2); $\mathscr{MCA}$: Meta-Cellular Automata (Section 6.3.3); and $\mathscr{LAK}++$: Summary of findings and plans for future integration efforts (Section 6.4)

Figure 6.1 presents an overview of the chapter in graphical form. Section 6.2 examines a variety of existing frameworks, and includes a small pilot study that applies these frameworks to analyse mathematical texts. In Section 6.3, we describe, in overview, three directions of research that we have developed:

- $\mathscr{LAK}$ is a formal, implemented system that models social creativity in mathematics at a high level.
- $\mathscr{PP}$ models the "recognisably social domain" of peer learning and peer production using design patterns (but does not have a formal, computational, implementation).
- $\mathscr{MCA}$ is a highly abstract simulation of social creativity that uses a simple evolutionary computing model.

Some take-away points for future integration efforts and steps towards "$\mathscr{LAK}++$" are presented in Section 6.4. Further details on the original work that is summarised in a necessarily abbreviated format here can be found in Pease et al. (2017), Corneli et al. (2015), and Corneli and Maclean (2015).

We see this broad survey of approaches to be highly relevant for our long-term project of building computational systems that can engage meaningfully in varied and diverse mathematical discourses. This chapter can therefore be seen as groundwork for future projects in natural language processing (NLP) and natural language generation (NLG) for mathematics.

## 6.2 Social Creativity in Mathematics

We will begin with a catalogue of examples of social creativity in mathematics, together with a survey of existing approaches to their socio-linguistic and "meta-mathematical" interpretation. In Section 6.2.4, we will take a brief look at music as a comparison case that helps to flesh out a theory of social creativity – showing the limitations, for example, of an entirely linguistic approach. Nevertheless, mathem-

atics is seen as a reasonably typical domain. Thurston (1994) offered the following explanation for why social creativity is an important part of mathematical practice:

> *We are not trying to meet some abstract production quota of definitions, theorems and proofs. The measure of our success is whether what we do enables people to understand and think more clearly and effectively about mathematics.*

A long-term goal in our research is to build a systematic, computational account of the way mathematical understanding develops. The examples in which we ground this work can be broadly classed as follows:

(*A*) Live, in-person mathematical dialogues show how mathematical sociality works in the everyday practice of mathematicians and students. However, recorded examples are somewhat limited, and often contain many extra-mathematical features. These examples are useful for theory development, but can be difficult to systematise.

(*B*) At another extreme, some recent developments in computer mathematics have produced proofs written in a manner that resembles natural language, while others provide basic examples of computational social intelligence. We would ultimately like to have a system that can incorporate and extend these feature sets.

(*C*) Perhaps the most central set of examples for this work are an existing catalogue of records from social experiments that incorporate computer-mediated dialogue. These include the Polymath and MiniPolymath projects, which have focused on computer-mediated problem solving at the research and advanced pre-college student level, respectively. For our purposes, the transcripts of these online dialogues are particularly useful, since as researchers we have access to the same material as research subjects (cf. Stahl (2010)).

These thematic areas can be tied together by considering how a relatively informal mathematical discussion corresponds, in an appropriate sense, to some set of formal and computationally meaningful objects, such as *proof plans* (Bundy, 1998). Figure 6.2 presents a general schematic that depicts the overall research situation, translating the sources of examples *A*, *B*, *C* to a parallel set of interlinked research perspectives, **I**-*Discourse*, **II**-*Formalisation* (relative to a hypertextual model), and **III**-*Model* (suitable for embedding in a formal system). We envision the possibility of participants in a discussion directly modifying the state of a running computation (**I**→**II**), and of the computer intervening in the discussion as a participant (**II**→**I**).

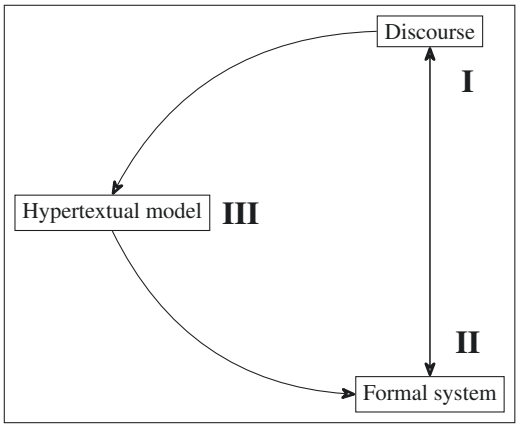



Fig. 6.2: Schematic relationship between (I) discourse, (II) formalisation, and (III) model

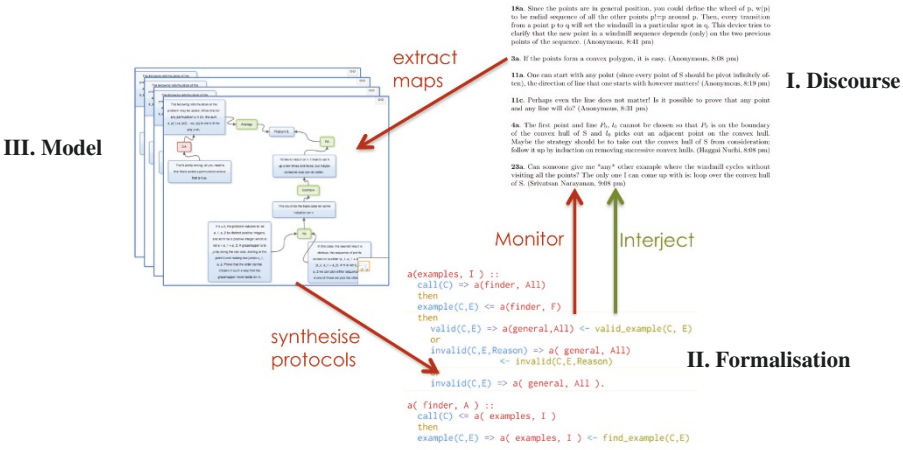



Fig. 6.3: The Online Visualisation of Argument (OVA) system as a bridge between the Minipolymath3 dialogue and a formalisation of the discourse in the Lightweight Social Calculus (graphic by Dave Murray-Rust, used with permission)

Figure 6.3 shows one possible instantiation of this schema, with discourse drawn from one of the MiniPolymath dialogues (see comments above on the general characteristics of Data Source C, above), formalised using the Lightweight Social Calculus, and modelled using the Online Visualisation of Argument (OVA) toolset.

In the typical case, moves in **I** drive moves in **II**, mediated implicitly or explicitly by **III**. The situation as a whole resembles the general case of *heterogeneous reasoning* as described by Barwise (1993), and instantiated, for instance, in his Hyperproof system. In Hyperproof, **I** would be a graphical puzzle with certain implicit rules governing its structure, and **II** the corresponding logic problem. In a typical mathematics setting, **I** may include some explicit pictorial features, but its basic structure is (implicitly) that of a graph or hypertext that depicts a dialogue. This structure is made explicit as an *argument* in **III**. **III** has the important feature that it reflects all of the structures in **I** that have a recognisable mapping in **II**, and, symmetrically, there are no logical- or language-level features in **II** that do not have an analogue in **III**. As Barwise (1993) puts it: "*Inference, as we understand the term, is the task of extracting information implicit in some explicitly presented information.*"

In the broader features of the schematic introduced above, we are inspired by *Inference Anchoring Theory (IAT)* (Reed and Budzynska, 2011; Budzynska and Reed, 2011; Budzynska, 2013), which develops connections between a dialogical context and logical argument by way of illocutionary schemes. In this manner, IAT takes the earlier notion of *dialogue games* from Carlson (1982) and others[1] in a more explicitly computational direction. Prakken (2006) writes: "[W]*hereas logic defines the conditions under which a proposition is true, dialogue systems define the conditions under which an utterance is appropriate*."

Our strategy in what follows examines the parallel between dialogue and the interpretation of dialogue, logic and models of logic, and contextualised discourse and its hypertextual models. We will think of the sets of examples *A* ("live discussion"), *B* ("computer mathematics"), and *C* ("computer-mediated discussion") that were introduced earlier along the schematic lines of analysis suited to perspectives **I**, **II**, and **III**.

Much previous research has often focused on just one of these components, rather than on their interconnections, although there is also precedent for studying just these interconnections. Our overall philosophical perspective on *sociality* derives from that of Mead (1932), witnessed here by the key feature that both **I** and **II** are understood to be evolving works-in-progress. If either is completed, the conversation ends, or the proof is resolved. This is similar to the Meadian (multi)-perspective in which both "organism" and "environment" undergo change together. In this metaphor, **III** corresponds to the sensory interface between "organism" and "environment." Alternatively, from a linguistic point of view, **III** could be seen as a Rosetta stone that defines translations between domains. More formally, it is the existence of satisfactory mappings between discourse on the one hand and the logical description on the other that allow us to think about connections between **I** and **II** in

[1] *NB*. Dialogue games are also known as *dialogue systems*.

terms of the *infomorphisms* of Barwise and Seligman (1997). Barwise and Seligman would call **III** the *core* of the information channel.

The work presented here is a preliminary step in a larger programme that aims to identify *dialogical* moves that can be associated with meaningful computational moves.[2] In the mathematics domain, the relevant computations are typically *logical*.[3] Other dualities (e.g., content and expression, genotype and phenotype, organism and environment, etc.) may potentially be treated similarly: the key issue in each case will be their *pragmatic* interplay – i.e., the effect of "context" on the outcome of events.

The role of context is well-studied in linguistics and the philosophy of language, since merely decoding statements is not usually sufficient to determine the intended meaning (Sperber and Wilson, 2002). Context has also been studied in computing, for instance by Sowa (1995), although attempts to formalise the concept are not conclusive – with Hirst (1997) in particular arguing that "*the notion of 'context' can be defined only in terms of its effects in a particular situation.*" The notion of context that we rely on here is embedded in **III**. This can be illustrated, for example, with a switch that converts a computer keyboard between the QWERTY and the Dvorak layouts. Whereas both keyboard layouts can model the same discourse (as represented by a flow of characters), flipping the switch influences the practical (stateful, and contextually-meaningful) characteristics of the model.

In the survey that follows we trace the pragmatic and computational features of mathematical discourse. We are inspired by earlier work of Louwerse and Crossley (2006), who examined a specialised linguistic task,[4] and found *n*-grams that were strongly associated with pre-defined speech acts. This survey that we develop here constitutes necessary preliminary work that can help to chart a course for future NLP-based analyses. We emphasise that our present effort is much more theoretical high-level than the work found in contemporary linguistic analyses of mathematical texts, such as that of Ganesalingam (2013) and Wolska and Kruijff-Korbayová (2004). Although mathematics is a fairly specialised domain that is communicated with an array of technical languages, mathematical language is also tremendously expressive. Classifications of speech acts in expressive languages tend to be correspondingly broad and nuanced, as has been pointed out by Leech and Weisser (2003) and Stolcke et al. (2000).

A secondary aim is to develop an approach that can be replicated to model social creativity in other areas. A claim to extensibility derives in part from the generality of the core theories that we build on, as described in Section 6.2.1. Following the survey in Section 6.2.2, Section 6.2.3 presents a pilot study that applies the collected frameworks to code various examples of mathematical dialogue at differing levels of formality. A cursory discussion of music in Section 6.2.4 draws out some additional

[2] Dialogic is more general than *dialectic*, which aims to converge on a common ground (Sennett, 2012, pp. 18–20).

[3] Other discourses appeal, for example, not to reason, but to the emotions, or to arguments based on character (*viz.*, *logos*, *pathos*, *ethos*), per Burke and Zappen (2006).

[4] http://groups.inf.ed.ac.uk/maptask/

issues related to indeterminacy, non-verbal communication and problem selection, and further explain the role of context.

### 6.2.1 Core Theories

The primary ingredients we will use to build a formal theory of social creativity are *Information Flow* from Barwise and Seligman (1997), *Unified Concept Theory (UCT)* from Goguen (1999, 2006) (see Chapter 1), and the *Method of Proofs and Refutations* from Lakatos (1976). We review and contextualise these here, outlining some adaptations that are necessary for our project.

We briefly introduced the notions of infomorphisms and information channels above. The more formal definition of an information channel revolves around a *core*:

$$\mathscr{C} = (\text{tokens}(\mathscr{C}), \text{types}(\mathscr{C}), \models_{\mathscr{C}})$$

where $\models_{\mathscr{C}}$ is a binary relationship between tokens and types; $t \models_{\mathscr{C}} T$ stands for the statement "$t$ is classified as being of type $T$ in $\mathscr{C}$." The core is complemented by an indexed collection of classifications and maps

$$\{A_i = (\text{tokens}(A_i), \text{types}(A_i), \models_{A_i})\}_{i \in I}$$

$$\{\check{f_i}\colon \text{tokens}(\mathscr{C}) \to \text{tokens}(A_i), \hat{f_i}\colon \text{types}(A_i) \to \text{types}(\mathscr{C})\}_{i \in I}$$

with the property that

$$\forall i \in I, c \in \text{tokens}(\mathscr{C}), \alpha \in \text{types}(A_i)\colon \check{f_i}(c) \models_{A_i} \alpha \text{ iff } c \models_{\mathscr{C}} \hat{f_i}(\alpha)$$

The elements $c \in \text{tokens}(\mathscr{C})$ are called *connections* between the various tokens $\check{f_i}(c)$. A good example would be English sentences (tokens) and their classification as dialogue moves (types – e.g., "attack", "conflict", "question", etc.), which are mapped into a graphical representation that shows the pattern of dialogical response together with the associated illocutionary points (e.g., assertions, inferences).[5] The result could be placed in correspondence with a computational model of the argument's logical structure.

*Blends* of various kinds can be easily spotted in the literature on mathematical conceptualisation and problem solving and in the informal logic of proof. However, the settings we are interested often require a dynamical and empirical approach to blending that is not present in straightforward *concept blending* in Unified Concept Theory (Goguen, 1999, 2006). The need for a somewhat more sophisticated blending approach was outlined in a discussion of dynamical systems and creativity in the context of music presented by Borgo and Goguen (2005). When thinking about music, Goguen (2004) diverges from his earlier work on concept blending "*because*

[5] See Reed et al. (2010) and Budzynska and Reed (2012) for an expanded discussion of this sort of example.

*musical structure is inherently hierarchical, and hence cannot be adequately described using only atomic elements and relation instances among them*." However, many aspects of the model were salvaged, as Borgo and Goguen turned to analogous *structural blending*, which draws upon the idea of semiotic systems with both level ordering and priority ordering: "*Levels express the whole-part hierarchy of complex signs, whereas priorities express the relative importance of constructors and their arguments.*"

Andersen (2002) points out that semiotic structures can also express dynamics through the use of recursion, although Borgo and Goguen did not pursue this. We are interested in *dynamical blending* because of the importance of modelling not just hierarchical content, but evolving and emergent works-in-progress. Not only does discourse and its formal representation change over time, the very terms by which this representation is made may change as the context changes. Andersen (2002) writes that the basic metaphor for thinking is *travel*. But rather than considering a simple path that avoids obstacles, we might envisage a skier descending a slope covered with moguls. Through continued use, the landscape shifts, and the classifications of paths in terms of their homotopic features or their desirability may change. The "*relations between relations*" (Kockelman, 2011) that define semiotic systems can be hooked together and react back on themselves, as our representations, relations, and the world we live in evolve over time.

Lakatos's model of mathematical discourse provides an example of the need for dynamic approaches to meaning. A Lakatosian dialogue begins with a *conjecture* and an *example* and then – following structured rules, in a way that depends on the problem and the discussion to date – may yield a revised conjecture, or a revised example. These changes in meaning correspond to updates in a developing mathematical theory. It is through an informal and often hands-on process of arguing about conjectures and counterexamples that the underlying formal structure of a given mathematical domain is made clear.

In the work of Pólya (1981), there is another kind of constructive blend that develops in the course of decomposing a given problem into several interrelated *auxiliary problems*. The relationship between problems is described in terms of new variables, whose values are unknown. As the problem solution progresses, these unknowns are determined using the given data. Any given solution will traverse various intermediate states between the given data and the goal as it is worked out. There may be a degree of indeterminacy in the solution strategy (e.g., multiple possible paths to the goal). At another level, *problems* (and, indeed *solutions*) can be combined, like the differential operators in quantum mechanics. Deleuze (1968) considers related issues from a broad philosophical perspective,[6] drawing particularly on Albert Lautman's earlier discussion of dialectics in mathematics; see Larvor (2011) for a relevant survey.

Stahl (2006, 2010, 2013) has built an extensive research programme that studies mathematics in a social context, developing the theme of "*group cognition*," and drawing on online interactions between students as a source of data. His notion of

[6] On auxiliary problems, see esp. (Deleuze, 1968, p. 239).

"*adjacency pairs*" is a broader category than "Lakatosian moves." Contemporary strategies from natural language processing, including the field of *argument mining* (Peldszus and Stede, 2013) which is gaining traction within the broader field of *discourse mining* (e.g., Stab and Gurevych (2014); Webber et al. (2012)), will ultimately be very relevant for tracing these logic-level "n-grams". In the current work we aim to sketch both argumentative and pre-argumentative structures: that is, we will consider the constructive features of *informal logic* (cf. Hitchcock (2006)).

## *6.2.2 Survey of Analytic Frameworks*

In this section we will discuss a range of frameworks, under the headings "**I**", "**II**", and "**III**" depending on whether they have more to do with *dialogue*, *logic*, or *pragmatics*, respectively. These three classes parallel the three types of examples (dialogical, computational, and computer-mediated) and three fundamental strategies (discourse, formalisation, and modelling) that we introduced in connection with Figures 6.2 and 6.3. We apply the frameworks surveyed here to analyse concrete examples in Section 6.2.3.

### 6.2.2.1 (I) Dialogue

Walton (2003) considers the basic patterns in dialogue to be: ***persuasion***, ***negotiation***, ***information seeking***, ***deliberation***, and ***quarrel***. In some earlier writings these were augmented with a sixth category *inquiry*, and with the subtype *pedagogy* and the mixed form *debate* (Walton, 1997). Aberdein (2007) indicates that "*Many other familiar argumentational contexts may be represented in terms of Walton's six basic types of dialogue by such hybridization and subdivision*."

In a mathematics-specific context, Aberdein offers a discussion of proof dialogues instantiating Walton's dialogue types. Thus, we have *proof as inquiry*, *proof as persuasion*, *proof as information seeking*, *proof as deliberation*, *proof as negotiation*, *proof as eristic/debate*. Each of these is described in terms of an *initial situation*, the *main goal*, the *goal of the prover* and the *goal of the interlocutor*. Depending on which type of proof is under discussion, Aberdein indicates that the *main goals* are:

1. ***Prove or disprove a conjecture*** (inquiry),
2. ***Resolve difference of opinion with rigour*** (persuasion),
3. ***Transfer of knowledge*** (information seeking)
4. ***Reach a provisional conclusion*** (deliberation),
5. ***Exchange resources for a provisional conclusion*** (negotiation),
6. ***Reveal deeper conflict*** (eristic).

In general "*an utterance is appropriate if it furthers the goal of the dialogue in which it is made*" (Prakken, 2006). Prakken here is concerned especially with

*persuasion dialogues*. He explains that in this context, some of the typical dialogue features are: ***arguments***, ***questions***, ***claims***, ***challenges***, ***conceding***, and ***retraction***.

In the context of informal mathematical discussions, Pease et al. (2014) recast persuasion in terms of *Lakatosian moves* relevant to conjectures and proof:

1. A ***conjecture*** is what is argued for.
2. ***Surrender*** consists of abandoning a conjecture in light of a counterexample.
3. ***Piecemeal exclusion*** defends a conjecture by dealing with exceptions through the exclusion of a class of counterexamples.
4. ***Strategic withdrawal*** uses positive examples of a conjecture and generalises from these to a class of object, then limiting the domain of the conjecture to this class.
5. ***Monster barring*** argues that a 'counterexample' can be ignored because it is not a counterexample, as it is not within the claimed concept definition. Using this method, the original conjecture is unchanged, but the meaning of the terms in it may change.
6. ***Monster adjusting*** is similar to monster barring: in the monster-adjusting case, an object that is proposed to be a counterexample is altered or reinterpreted so that it is no longer a counterexample. The object is still seen as belonging to the domain of the conjecture.
7. ***Lemma incorporation*** distinguishes between *global* and *local* counterexamples. The first would be counterexamples to the main conjecture, and the latter is a counterexample to one of the proof steps or lemmas.

The Speech-Act Annotated Corpus (SPAAC) presented by Leech and Weisser (2003) examines domain non-specific forms of discourse. Speech acts are conveniently organised into five superordinate categories: ***expressive***, ***interpersonal management***, ***dialog control***, ***mainly initiating***, ***mainly responding*** — plus several acts classified as "other" (***external to dialog goals***, ***unspecified***).

#### 6.2.2.2 (II) Logic

In the previous section, we made note of several frameworks for describing the flow of language at the (relatively high) level of discourse. One of these, the Lakatosian framework, can also be thought of as controlling the logical flow within a discussion. Specifically, the Lakatosian moves discussed above can be thought of as operators whose outcome depends on the way in which a new example is construed:

$$\text{conjecture} \oplus \text{counterexample} \mapsto \text{surrender this conjecture} \tag{6.1}$$

$$\text{conjecture} \oplus \text{exception} \mapsto \text{exclude a range of examples} \tag{6.2}$$

$$\text{conjecture} \oplus \text{examples} \mapsto \text{limit scope} \tag{6.3}$$

$$\text{conjecture} \oplus \text{monster} \mapsto \text{revised statement} \tag{6.4}$$

$$\text{conjecture} \oplus \text{monster} \mapsto \text{revised example} \tag{6.5}$$

$$\text{conjecture} \oplus \text{counterexample} \mapsto \text{revised approach} \tag{6.6}$$

In particular, the "revised approach" in 6.6 may be associated with changes to lemmas or to the conjecture itself. There is no reason why 6.1–6.6 should limit the range of possible "production rules". For example, one would expect there to be a set of discourse moves that take place at the "managerial" level and that move the discourse between conjectures in ways that are not explicitly connected with (counter)examples. There are various "computational" heuristics that underlie and motivate the discourse moves mentioned above.[7]

Bookkeeping that explicitly describes mathematical objects and the relationships between them is "logical" rather than merely "dialogical". For example, Pease and Martin (2012) introduced a typology of comments in mathematical discussions. Comments of these kinds move the conversation forward in a different way: ***concepts***, ***examples***, ***conjectures***, ***proof*** (and ***other***).

These Lakatos-inspired reflections remain abstract in the sense that they "tag" mathematical conjectures, examples, and lemmas as such, but do not consider other kinds of (often entirely mundane) mathematical objects. Corneli (2014) describes the following kinds of mathematical object types that are either supported explicitly in the PlanetMath.org mathematics website, or planned for some future version system: ***article*** [$A$], ***link*** [$\ell$], ***project*** [$\mathbb{X}$], ***post*** [$T$], ***solution*** [$S$], ***review*** [$R$], ***update*** [$\sharp$], ***question*** [$Q$], ***correction*** [$C$], ***fork*** [$\prime$], ***outcome*** [$\star$], ***problem*** [$P$], ***collection*** [$L$], ***classification*** [$M$], ***conjecture*** [$\mathbb{J}$], ***group*** [$G$], ***user*** [$U$], ***request*** [$W$], ***heuristic*** [$H$], and ***ephemera*** [$\mathbb{E}$]. In a system like PlanetMath, the underlying software logic depends on the specific implementation that describe how these elements connect and interact (Corneli, 2011). Object types are considered to exist within a "grammar" that constrains behaviours within the system (Corneli, 2014, pp. 74,160). This sort of explicit modelling may mirror the way such objects are discussed informally in text-based discussion threads, e.g., "the ideas underlying this question suggest the following broader conjecture …".

Whether the development of a mathematical theory is construed as an evolving, dynamically-constructed tapestry of structure blends, or a growing network of interrelated terms and objects, we require a range of more or less mechanical operations to carry out the reasoning steps. One recent computational example has been developed by Ganesalingam and Gowers (2013), who describe an automatic problem solving program ROBOTONE which, they claim, works roughly in the same way that people do (in the limited domain of textbook problem solving). While this claim may be debated, as it stands the system offers a candidate mapping between "human operations" and "machine operations." In ROBOTONE, "*An individual move is an operation that transforms a specific problem state into another state in a sound fashion; thus individual moves correspond to application of tactics to a specific*

[7] For example, Minsky (2008–2009) mentions: *reasoning by analogy*, *dividing the problem into parts*, *changing the problem's description*, *focusing on a more specific example*, *making a simpler version*, *trying to identify what makes the problem hard*, *imagining what an expert would do*, *stopping what you're doing if you're stuck*, *retrieving the knowledge that tells you how to solve the problem*, and *asking for help*.

*LCF-style prover state.*"[8] Ganesalingam and Gowers describe, in total, 27 different moves, in the broad categories ***Deletion***, ***Tidying***, ***Applying***, ***Suspension***, and ***EqualitySubstitution***. The basic heuristic is to look at the problem state and apply the first possible move from an ordered list.[9]

#### 6.2.2.3 (III) Pragmatics

The way people solve problems has been studied extensively, with the best-known work being that of Pólya (e.g., 1990a; 1990b; 2014; 1981). Empirical work continuing in this tradition has been developed by Alan Schoenfeld (e.g., 1985; 1987; 2010) and others. Pólya's high-level problem-solving heuristics follow a well-known outline:

1. ***Understand the problem (unknown, data, condition)***,
2. ***Find the connection between the data and the unknown to obtain a plan of the solution***,
3. ***Carry out the plan***,
4. ***Examine the solution obtained***.

These can be broken down into further steps; in particular, via the famous heuristic: "If you can't solve a problem, then there is an easier problem you can solve: find it."

The empirical aspects of Schoenfeld's work on problem solving connect with earlier protocol-based methods for doing research in mathematics education, e.g., Lucas (1980), and Lucas et al. (1980), who built on earlier work in process coding by Kilpatrick (1967) as well as early research by Schoenfeld himself (1979). However, Schoenfeld found that the codings used by Lucas et al. (1980) were overly complex, even while they focused only on tactics rather than strategy. His primary coding is divided among stages that are clearly informed by Pólya: ***Read***, ***Analyze***, ***Explore***, ***Plan***, ***Implement***, and ***Verify***. These are supplemented by ***New information and local assessments***, and ***Transition***. In problem solving, Schoenfeld argues (1985, p. 4) "*it is what the person does rather than what the person produces*" that is important. Somewhat along these lines he insists that *omissions* (e.g., neglecting to plan) should be coded for, along with explicit behaviour.

One of Schoenfeld's central research strategies was to put people in small groups or pairs and have them talk through mathematical problems together. An alternative would have been to use speak-aloud protocols – but a dialogical format helped to alleviate situational pressure. "[W]*hen two students worked on the problem it was typical for one student to turn to the other and say something like, 'I have no idea what to do. Do you?'*"

---

[8] LCF stands for *Logic for Computable Functions* (cf. Milner (1972); Gordon et al. (1979); Gordon (2000)).

[9] Gordon (2000) explains why activities related to *deletion* are given particular priority: "*The steps of a proof would be performed but not recorded, like a mathematics lecturer using a small blackboard who rubs out earlier parts of proofs to make space for later ones.*"

This not only produced more natural records of a thought process, but also helped to avoid answers that were formal for the sake of being formal (Schoenfeld, 1985, p. 279–281). Nevertheless, a social setting is not a guarantee of problem-solving success, as becomes clear from several dialogues that record attempts to solve this problem:

> *Three points are chosen on the circumference of a circle of radius R, and the triangle containing them is drawn. What choice of points results in the triangle with the largest possible area? Justify your answer as best you can.* (Schoenfeld, 1985, p. 319)[10]

In short: "*If one makes major strategic mistakes, then matters of tactics are of little importance*" (Schoenfeld, 1985, p. 289).[11] In connection with this observation, Pólya's (1981) expansion of the *planning* step is interesting. He writes that in order to form a plan, one should proceed in this way:

1. ***Identify the goal*** (what do you want?)
2. ***Identify the conditions*** (what do you have?)
3. ***Decompose the problem*** (introducing ancillary problems and new unknowns as needed)
4. ***Connect the data to the problem***
5. ***Determine any unknowns***

Schoenfeld (1985, p. 15) introduces a "macro-level" framework that supplements these increasingly detailed frameworks from Pólya. The macro framework characterises thinking in terms of four categories: ***resources*** ("genetic epistemology"), ***heuristics*** (per Pólya), ***control*** (comprising both "decision making" and "metacognition"), and ***belief systems*** (or "mathematical world view"). He points out that this framework is "*far from comprehensive*" and that it generally excludes both cognitive and social details (Schoenfeld, 1985, p. 16). *Mutatis mutandis*, if we consider a social framework then we must find a way to embed features like heuristics, control structures, and belief systems within it.

Pease et al. (2018) are working on *explanation* in mathematical texts. Explanations can be people-centred (paralleling Schoenfeld's macro-level framework) or domain-centred. Items from the following list can function either as an *explanandum*

[10] After a brief period of time spent *reading*, the first group of students spends the rest of the time *exploring* in a rather meandering fashion, even though they had a suitable conjecture within the first few moments of discussion (Schoenfeld, 1985, p. 294). Schoenfeld's commentary (1985, pp. 288–289) is as follows: (1) The students neglected to assess the potential utility of calculating the area of the equilateral triangle; (2) the discussants gave consideration to several interesting alternative problems that might have added insight, but did not pursue them vigorously; (3) progress was not monitored or assessed during the solution attempt. A second group described a "hunch" that the answer was an equilateral triangle, but spent most of the available time *implementing* a somewhat related demonstration to show the size of the largest inscribed right triangle. They had no satisfactory answer when asked by the investigator how what they found would relate back to the original problem.

[11] Given Schoenfeld's choice of words it is perhaps worth noting that "tactics-based proof systems" may or may not concern themselves with strategic matters; strategy does not appear to play a particularly significant role in the system described by Ganesalingam and Gowers (2013), for example.

or an *explanans*. For example, we might attempt to explain why we don't understand something, or we could point to the fact that we don't understand something *as* an explanation.

1. ***abilities*** (what can/can't we do, e.g., we can reduce the problem to $P$)
2. ***knowledge*** (what do/don't we know, e.g., $X$ is wrong)
3. ***understand*** (what do/don't we understand, e.g., do you see why this is a contradiction?)
4. ***value/goals*** (what do/don't we want, e.g., $X$ is a good idea)
5. ***initial problem*** (e.g., the initial problem $P$ is harder than problem $Q$)
6. ***proof*** (e.g., $A$ is not a useful approach)
7. ***assertions*** (e.g., $M$ is subset of $P$)
8. ***specific cases/instances*** (e.g., there will always exist instance $X$ that satisfies condition $C_1$)
9. ***arguments*** (e.g., let us suppose $X$. Then $Y$.)
10. ***representation*** (e.g., by reducing the problem to $P$)
11. ***property*** (e.g., we don't know if it has property $P$)

### *6.2.3 Pilot Study*

In this section we present a novel application of the material surveyed above, namely an examination of the ways in which these concepts model real mathematical texts. This pilot study helps to clarify the salience and range of application of the concepts in our survey.

Referring to the foregoing material, we have extracted a large array of potential textual annotations at different levels of specificity. These are presented in overview in Table 6.1. There is clearly a degree of redundancy among the frameworks surveyed above, which we have attempted to express with a numbering scheme (*1a.*, *1b.*, etc.) that elides some of the redundancies present in the 13 frameworks mentioned above. Even with this compression, we arrive at a position that is at least as complex as that taken up in the process coding work of Lucas et al. (1980): indeed, our situation is bound to be somewhat more complex, since these authors considered only individual problem solvers working with a speak-aloud protocol, and not people working in groups. Nevertheless, in an initial survey it seems best to take a comprehensive view, rather than risk leaving something out for the sake of concision. We can now ask: Which components of this broad canvas of available frameworks are the most relevant? In order to address this question, we have intensively annotated several brief passages, drawing as needed from all of the above frameworks. In subsequent phases of research, we expect to be able to deal with longer passages (including informal dialogues leading to full proofs). In the current study, the authors sought direct consensus on which tags best apply to which portions of text. The texts, together with the added tags, are presented in Figures 6.4–6.8. The sources for these texts are described in Section 6.2.3.1.

| | | |
|---|---|---|
| *0.* | The general-purpose *SPAAC Classified List of Speech Acts* | ("SA") |
| *1a.* | Walton's *patterns of dialogue* | ("WD") |
| *1b.* | Aberdein's interpretation of these in the context of *proof dialogue* | ("AD") |
| *2.* | Prakken's specialisation to *persuasion dialogues* | ("PD") |
| *3.* | Pease et al.'s *Lakatosian moves* | ("LD") |
| *4.* | Pease and Martin's *types of mathematical comments* | ("MC") |
| *5.* | PlanetMath/Planetary's *types of mathematical objects* | ("MO") |
| *6a.* | Pólya's *stages of problem solving from "How to Solve It"* | ("PS") |
| *6b.* | Schoenfeld's refinements to this in the form of his *process coding* | ("SS") |
| *7.* | Pólya's *stages of planning from "Mathematical Discovery"* | ("PP") |
| *8.* | Schoenfeld's framework describing *factors in mathematical thinking* | ("SF") |
| *9.* | Pease, Aberdein, and Martin's *components of explanation* | ("CE") |
| *10.* | Ganesalingam and Gowers's *LCF-style tactics* | ("RO") |

Table 6.1: List of frameworks, with two-letter abbreviations

#### 6.2.3.1 Data Sets

We are particularly interested in the MiniPolymath problems that were been posed, discussed, and solved online, on Terrence Tao's blog and the Polymath blog Tao et al. (2009, 2011). The problems – drawn from Mathematics Olympiads[12] – are interesting in that they are challenging enough to spur considerable discussion, but not so challenging as to go unsolved when people put their heads together.[13] There have been four such discussions to date – however, the experiment itself would be relatively easy to replicate. MiniPolymaths do not yet have the scope of the full "Polymath" projects, which dealt instead with research topics and in some cases resulted in published papers (cf. Barany (2010); Gowers and Nielsen (2009)). Nevertheless, unlike Lakatos's (1976) *Proofs and Refutations*, MiniPolymath data is "real" (and available online) – as opposed to "rationally reconstructed in fictional form."

We also consider brief excerpts from Schoenfeld's collected data set (several dialogues between students are presented in full in his book); an excerpt from a monologue presented in a public lecture by Timothy Gowers (2012); and a proof written by the ROBOTONE prototype system described by Ganesalingam and Gowers (2013), which operationalises the line of thinking from the lecture.[14] We emphasise that single author works addressed to an arbitrary audience *can* be coded using the frameworks we've described, although this is not the main intended application. Nevertheless, single author works provide a natural point of comparison, as we think about what the "social dimension" brings to mathematics.

[12] https://www.imo-official.org/

[13] Tao notes, regarding MiniPolymath 1: "*Of the 500-odd participants in the Olympiad, only a half-dozen or so managed to solve this problem completely.*"

[14] http://people.ds.cam.ac.uk/mg262/robotone.pdf

Figures 6.4–6.8 show the selected texts on the left, verbatim except for footnote-style anchors which cross-reference our consensus codings of text fragments. Details appear as marginal annotations to the right.

| | |
|---|---|
| NATE: Well, my first thought is to see if the hypotheses seem reasonable.[ss3] The hypothesis that $s = a_1 + \ldots + a_n$ not lie in $M$ is certainly necessary, as the last jump that the grasshopper takes will land on $s$.[pp2] The grasshopper's other steps will land on a partial sums $a_{\sigma(1)} + \ldots + a_{\sigma(k)}$ for some permutation $\sigma$, but we get to choose the permutation. Thus it seems plausible that we can avoid a given set of $n-1$ points.[ss2] | [ss3]explore<br>[pp2]conditions<br>[ss2]analyze |
| THOMAS: Quick observation.[sa2] The grasshopper must make a first step.[sf2] This is always possible, since the $a_i$ are distinct and $\lvert M\rvert = n-1$; that is, there is always an $a_i$ not in $M$.[pp2] However, let's say $M$ matches all but one of the $a_i$. Then the first step is uniquely determined. Still, according to the claimed theorem, a second step must still be possible.[pp3] | [sa2]inform<br>[sf2]heuristic (simplify)<br>[pp2]conditions<br>[pp3]decomposition |

Fig. 6.4: Excerpt from MiniPolymath1

| | |
|---|---|
| HAGGAI NUCHI: The first point and line $P_0$, $l_0$ cannot be chosen[ce7] so that $P_0$ is on the boundary of the convex hull of $S$ and $l_0$ picks out an adjacent point on the convex hull.[mc2] Maybe the strategy should be to take out the convex hull of $S$ from consideration; follow it up by induction on removing successive convex hulls.[ld7] | [ce7]assertion<br>[mc2]example (monster)<br>[ld7]lemma incorporation |
| HAGGAI NUCHI: More specifically, remove the subset of $S$ which forms the convex hull to get $S_1$; remove the new convex hull to get $S_2$, and repeat until $S_n$ is convex. Maybe a point of $S_n$ is a good place to start.[ss5] | [ss5]implement |
| SRIVATSAN NARAYANAN: Can we just assume by induction that we have proved the result for all the "inner points" $S_2 \cup S_3 \cdot \cup S_n$.[ad5] The base case would be that $S = S_1$, i.e., it forms a convex polygon.[mc4] | [ad5]negotiation<br>[mc4]proof |

Fig. 6.5: Excerpt from MiniPolymath3

### 6.2.3.2 Results

As illustrated above, several short texts have been marked up with codes corresponding to the frameworks introduced in Section 6.2.2. Figures 6.6–6.8, respectively, re-

| | |
|---|---|
| DK: [reads the question][ss1] | [ss1]read |
| BM: Do we need calculus for this?[sf1] So we can minimize, or rather maximize it.[mc1] | [sf1]resources<br>[mc1]concept |
| DK: My guess would be more like [indiscernable] my basic hunch would be that it would be –[ld1] | [ld1]conjecture |
| BM: An equilateral –[ld1] | [ld1]conjecture |
| DK: 60, 60, 60[mc1] | [mc1]concept |
| BM: Yeah.[sa2] | [sa2]ackn |
| DK: So what choice of points has to be where on the triangle[pp1] – these points are gonna be.[pp2] | [pp1]goal<br>[pp2]conditions |
| BM: Try doing it with calculus – see if you can – just draw the circle[ss3] – see what we'll do is figure out the right triangle –[ss8] | [ss3]explore<br>[ss8]transition |
| DK: Yeah, or why don't we find – or why don't we know the – some way to break this problem down into[sf2] – like, what would a triangle be for half the circle?[sf2] | [sf2]heuristic (decompose)<br>[sf2]heuristic (symmetry) |

Fig. 6.6: Excerpt from Schoenfeld [pp. 324–325]

| | |
|---|---|
| Gowers: What is the $500^{\text{th}}$ digit of $(\sqrt{2}+\sqrt{3})^{2012}$?[ss1] | [ss1]read |
| Even this, eventually, a computer will be able to solve.[mc5] | [mc5]other (phatic) |
| For now, notice that total stuckness can make you do desperate things.[sf2] Furthermore, knowing the origin of the problem suggests good things to try. The fact that it is set as a problem is a huge clue.[sf4] | [sf2]heuristic (total stuckness)<br>[sf4]belief systems |
| Can we do this for $(x+y)$?[ss3] For $e$? Rationals with small denominator?[sf2] | [ss3]explore<br>[sf2]heuristic (compute!) |
| And how about small perturbations of these?[sf3] | [sf3]control |
| Maybe it is close to a rational?[ld1] | [ld1]conjecture |
| $m^{\text{th}}$ digit of $(\sqrt{2}+\sqrt{3})^n$?[sf2] | [sf2]heuristic (formal gen.) |
| $(\sqrt{2}+\sqrt{3})^2$?[sf2] | [sf2]heuristic (simplify) |
| $(2+2\sqrt{2}\sqrt{3}+3)$[mo20] | [mo20]ephemera |
| $(\sqrt{2}+\sqrt{3})^2+(\sqrt{3}-\sqrt{2})^2=10$[sf2] | [sf2]heuristic (compute!) |
| $(\sqrt{2}+\sqrt{3})^{2012}+(\sqrt{3}-\sqrt{2})^{2012}$ is an integer![ce11] | [ce11]property |
| And $(\sqrt{3}-\sqrt{2})^{2012}$ is a very small number.[ss7] | [ss7]local assessments |
| Maybe the final answer is "9"?[ld1] | [ld1]conjecture |
| We need to check whether it's small enough.[ss5] | [ss5]implement |
| $(\sqrt{3}-\sqrt{2})^{2012}<(\frac{1}{2})^{2012}=((\frac{1}{2})^4)^{503}=(\frac{1}{16})^{503}<.1^{503}$, so we're in luck...[ss6] | [ss6]verify |

Fig. 6.7: Partial transcript from Timothy Gowers's Maxwell Institute Lecture, November 2, 2012

produce our tagging of: a single-threaded discussion in Minipolymath 1, a portion of one of the threads in the multithreaded discussion in Minipolymath 3, a short passage from a student problem solving dialogue, the full solution to a challenge problem presented by mathematician Timothy Gowers, and a portion of a textbook-style problem solved by Ganesalingam and Gowers's program ROBOTONE.

The results of this exercise are summarised in Figure 6.9, which shows the total count of the tags used from each of the several tag sets available. For example, a grand total of two tags from tag set 0 ("SA") were used: "inform" was applied to

| | | |
|---|---|---|
| 1 | ROBOTONE: If $A$ and $B$ are open sets, then $A \cap B$ is | |
| 2 | open.[ss1] | [ss1]read |
| 3 | Let $x$ be an element of $A \cap B$.[mc2] | [mc2]example (arbitrary inst.) |
| 4 | Since $x \in A \cap B, x \in A$ and $x \in B$.[pp3] | [pp3]decomposition |
| 5 | Since $A$ is open and $x \in A$, there exists $\eta > 0$ | |
| 6 | such that $u \in A$ whenever $d(x,u) < \eta$.[pp2] | [pp2]conditions |
| 7 | Since $B$ is open and $x \in B$, there exists $\theta > 0$ | |
| 8 | such that $v \in B$ whenever $d(x,v) < \theta$.[pp2] | [pp2]conditions |
| 9 | We would like to find $\delta > 0$ s.t. $y \in A \cap B$ | |
| 10 | whenever $d(x,y) < \delta$.[pp1] | [pp1]goal |

Fig. 6.8: Partial transcript of ROBOTONE's proof that the intersection of two open sets in a metric space is open

code the short statement "Quick observation" in Figure 6.4, Line 10, and "ackn" is applied to code the one-word speech "Yeah" in Figure 6.6, Line 8.

In Figure 6.9, the tags attached to dialogical and monological texts have been distinguished, with contributions from single-author texts added as an "increment" above the tags used in dialogues.

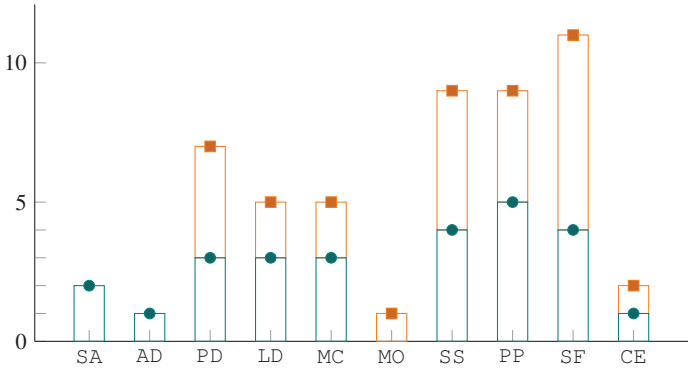



Fig. 6.9: Count of tags from the several schemes

### 6.2.3.3 Discussion

As noted above, the aim of this pilot study is to clarify the salience and range of application of the concepts in our survey. Given the small sample size, a limited number of applications for a specific tag set does not necessarily mean that it will be irrelevant in future studies: rather, even a few usages (as from the AD and MO tag sets) give an indication of the types of text where we would expect to see more replete usage of the corresponding tag sets.

Before discussing the findings, we will comment on some of the things that this pilot study did not attempt. Note that although they are available in Ganesalingam and Gowers (2013), we did not tally the explicit tactics used by ROBOTONE (RO). Rather than focusing on the LCF-style details, the coding used in Figure 6.8 gives an informal view of the processes involved in "expanding the definition." The study did not include Walton's patterns (WD) which in this context would be redundant with Aberdein's interpretation (AD); and similarly, we do not include tags denoting Pólya's stages of problem solving (PS), preferring Schoenfeld's slightly more detailed rendering (SS).

Table 6.2 presents a summary showing exactly which tags were selected. This table is ordered and divided into three segments to show whether the tag set is primarily associated with (I) dialogical, (II) logical, or (III) pragmatic discursive manoeuvres (this is according to the subdivisions of Section 6.2.2). The annotation "*" means that all of the tags in a given category were used, up to redundancy within the categories I, II, and II. Tags that have been used but that are redundant in this sense are enclosed in hard brackets (e.g., "conjecture" is part of LD, MC, and MO, but it is present in this only as a representative of LD). Similarly, we have tagged *examples*, including counterexamples, into MC. The annotation "°" indicates that all but one of the tags was used: specifically, *plan* from the SS tag set is not used; note, however, that several specific tags related to *planning* from the PP tag set were applied.

| | |
|---|---|
| SA | ackn, inform |
| AD | negotiation |
| PD | question |
| LD | [conjecture], lemma incorporation |
| MC* | concept, example (arbitrary inst.), example (monster), [proof], other (phatic) |
| MO | ephemera |
| SS° | read, analyze, explore, implement, verify, local assessments, transition |
| PP | goal, conditions, decomposition |
| SF* | resources, heuristic (compute!), heuristic (decompose), heuristic (formal gen.), heuristic (simplify), heuristic (symmetry), heuristic (total stuckness), control, belief systems |
| CE | property, [assertion] |

Table 6.2: Summary of tags used, by category

The most popular tag set was Schoenfeld's "macro-level" framework (SF), with 11 applications overall. Many of these deal with the application of specific heuristics, such as: *if you can compute something, do it!*; *it's a good idea to decompose a problem into sub problems*; and *try a simple case*. Note that a discussion about a heuristic, as in Lines 14-16 of Figure 6.6 – tagged "*heuristic (decompose)*" – is

different from the application of a heuristic, as in Lines 10-17 of Figure 6.4 – tagged as "*decomposition*" from the Pólya planning (PP) tag set.

The "Pólya-Schoenfeld" tag set (SS) and Pólya planning tag set (PP) were both used frequently, with nine applications each, although not all of the tags were used. The Lakatos (LD) and comment-type (MC) tag sets were used with moderate frequency, with five applications each. We only used two tags from the SPAAC Classified List of Speech Acts (SA), despite this being a general-purpose lexicon, and having the largest set of available tags. (SPAAC-style general purpose conversational moves seem particularly likely to happen in in-person, real-time dialogues, which are not well represented in our data set.)

We included only one tag, *ephemera*, selected from our list of types of mathematical objects (MO). In the current data set, discussants typically referred to *concepts*, rather than to to concrete objects like *articles* where the definitions of these concepts could be found.

We also include only one tag, *negotiation*, from Aberdein's interpretation of proof dialogues (AD). Aberdein's discussion of the goals associated with proof includes "prove or disprove the conjecture" (corresponding to the tag *inquiry*) as one of six different intentional states. In the texts we examined, this was a goal that all of the participants shared: accordingly, using this tag would convey no information. We interpreted Lines 13-15 of Figure 6.5 to be (part of) a brief *negotiation* about the need for a "backward" operation in an induction proof.

Finally, we include one tag, *question*, from Prakken's framework (PD).[15] Lines 1-2 of Figure 6.5 – marked as *assertion* using a component of the explanation framework (CE) – might have, synonymously, been marked as a *claim*. The tag "*assertion*" is therefore recorded in hard brackets in Table 6.2. In addition to directly redundant tags, tagging these sample texts also makes more clear the relationships and interdependencies among the dimensions. For example, as noted above, PP supplies a dimension that expands on the idea of *planning* that is described abstractly in SS. MC includes examples, which are a necessary input for most of the elements of LD. The tag sets are also associated with different senses: not only "use versus mention", but also *use-to-explain* or *use-to-question*.

In terms of our earlier decomposition: SA, AD, PD, and LD were associated with *dialogical* features; MC and MO, together with an interpretation of LD, with *logical* features; and SS, PP, SF, and CE with *pragmatic* features.

Frameworks that are only used for one (AD, PD, and MO) or two (LD, CE) tags are nevertheless interesting and worthy of further attention and potential refinement. Other terms from these frameworks (for example, Prakken's *challenges*) are likely to appear in longer texts. Similarly, more instances of *negotiation* and *deliberation* about sub-problems and proof strategies would be likely to appear in more wide-ranging discussions (like the research-level Polymath projects). Note, as well, that a

[15] Note: for reasons of typographical convenience, the question tag is not explicitly recorded in Figures 6.4–6.8 – rather, we count each paragraph that includes at least one question mark as a question. We have also counted the question-like sentence on Lines 13-15 of Figure 6.5 – which is punctuated with a full stop – as a question.

framework that has proved to be a particularly popular source of tags may be calling out to be separated into several different "levels" (après Borgo and Goguen (2005)).

### *6.2.4 Direct Extensions*

Many standard problem solving approaches can be implemented in a social way (Corneli, 2014, p. 193). Indeed, according to the Meadian hypothesis, the emergence of agency may be seen as social, as Martin and Gillespie (2010) explain.[16] Although the sample size in the pilot study presented above was too small to make strong conclusions, it nevertheless offers interesting clues about what "social thinking" brings to mathematics. In the first instance – as Tao noted – the MiniPolymath problems are not easy, and yet, working as a group, participants have been able to solve them at a (time-wise) competitive rate. Why might that be?

People in discussions are presumably more likely to *negotiate*. Indeed, the entire Lakatosian framework might be considered in terms of negotiation and an exchange of resources, rather than simply persuasion and the resolution of differences. Relatedly, discussants may be more likely to *plan*. They are, certainly, more likely to *discuss*, and as a result may possibly consider a wider range of *examples*. That said, at least in the texts we studied, single authors were more likely to ask *questions*, even if this was only to create a simulated conversation. A didactic text written by someone who knows what they are doing may be more likely than a meandering dialogue to simply get on with it and do *problem solving*. Single authors may tend to rely more on *heuristics* than on *negotiation* to overcome difficulties.

Compared with the 42 primary tags used by Lucas et al. (1980), the 27 tags that we used are less focused on problem solving *per se*, although there is a significant overlap in the terminology and sources used. The problem solving aspects may be addressable using our combined "pragmatic" tag set (that is, `SS`, `PP`, `SF`, and `CE`, the segment of our tag collection which bears the closest similarity to Lucas et al. (1980)) as the core of an information channel, and using these elements to map between "subjective" dialogical moves and "objective" state changes in a logical setting. To treat research-level discussions, this set of tags would need to be expanded in order to deal with problem identification, positing, and selection.

With ROBOTONE, we get the logical mapping for free, albeit in a monological rather than dialogical format; thus, *example (arbitrary inst.)* in Figure 6.8 corresponds to the moves – elided in that figure – `expandPreUniversalTarget` followed by `peelBareUniversalTarget`, and *decomposition* to the move `expandPreExistentialHypothesis`; etc. While ROBOTONE seems to be

[16] "[I]*t is only by acting toward ourselves as others do … that we recognize and understand ourselves as objects and authors of our own activity. As we learn to coordinate our acting with the acting of others, we differentiate and develop our selves and our abilities to self-determine. Eventually, we not only understand the perspectives (i.e., action orientations and possibilities) of numerous particular others, but also those perspectives explicit and implicit within the broader, more generalized social, cultural practices in which we are immersed*."

reasonably proficient at basic problem solving, it currently has no explicit social intelligence.

It would not be unreasonable use the moves described in the Lucas et al. (1980) tag set as a guide to problem solving behaviours – without yet requiring social intelligence. After all, this earlier tag set was designed to analyse single-author texts. Working from the other direction, systems like Singh's (2005) EM-ONE do possess at least rudimentary social intelligence. It would be natural to extend EM-ONE with a set of narratives related to the SA, AD, and PD (dialogical) tag sets. The biggest challenge for future work seems to be the development of the domain-specific pragmatic knowledge base related to MC, MO, and LD, and mappings between this register and the others.

Naturally, in different domains it will make sense to consider different types of annotations. There are different "relations between relations" to be considered, both within disparate fields of human endeavour, and in different situations within a given field. Music makes a useful initial point of comparison.

### *6.2.5 Additional Frameworks from Music Theorists*

Our survey in Section 6.2.2 focused on mathematics, but included the domain-independent SPAAC tag set and the work of theorists Walton and Prakken which described arguments in some generality. A look at music can help broaden this perspective: many of the ideas in musical collaboration would apply in other forms of collaboration.

Cook (2001) proposes to think of a musical score as a "script," rather than as text that is complete in itself. In this way, the space between score and performance is understood as an open and non-linear one, and perhaps more easily conceivable as a continuous exchange between processes and products, comparable to our comments above on the relationship between discussion and proof. Bruce Ellis Benson's work on improvisation-as-dialogue identifies 11 types of performance situations that could cover virtually any type of music (Benson, 2003, pp. 26-30). Benson's typology ranges from fully notated works (types 1 to 3) to the deliberate subversion of expectations associated with a particular compositional or performative tradition, via real-time improvisation (type 11).

Types 7 to 11 in Benson's typology are well illustrated by examples in jazz and freer forms of improvisation, and are particularly useful in investigating open problem-solving spaces, where structure and meaning are formed and communicated in real time. On that front, Borgo and Goguen (2005) attempted a typology of real-time transitions in their joint work on free jazz, in an analysis of a recording by Sam Rivers.[17]

1. ***sudden/unexpected segue*** (an unprepared, immediate change with unexpected continuation)

[17] This work would match types 10 and 11 in Benson's categories.

2. ***pseudo-cadential segue*** (an implied cadence with sudden and unexpected continuation)
3. ***climactic segue*** (a peak moment that stimulates unexpected change and continuation)
4. ***feature overlap*** (one feature of the antecedent section is sustained and becomes part of the consequent section)
5. ***feature change*** (a gradual change of one feature that redirects the flow, usually subtly)
6. ***fragmentation*** (a gradual breaking up, or fragmenting, of the general texture and/or rhythm)
7. ***internal cadence*** (a prepared cadence followed by a short silence then continuation with new material)

In their experimental work with free improvisers, Canonne and Garnier (2012) relied on musicians' own subjective accounts of short improvised sessions in three-person teams, and found significant evidence of intersubjective structural patterns emerging in the interactions between participants. This led to a distinction between stable and oscillating behaviours, corresponding to coordinated musical sequences (representable as fixed points in phase space) and discoordinated sections (representable as spaces without fixed attractors). Canonne and Garnier identified four main strategies that improvisers used to convey intentions while playing: ***Stabilization***, ***"Wait and see"***, ***"Playing along"***, and ***Densification***. The decision to employ these strategies, as well as the end-result of each strategy employed, depends on the overall musical situation, understanding of other musicians' individual intentions and objectives, and on team preferences, based on, e.g., competence, range of instruments used, etc. As a result, "misrepresentations" and, by extension, "contrasting evaluations of a given situation" may occur. Improvisers therefore often resort to "meta-pragmatic" evaluations of their strategies while playing, e.g. repeating a strategy until intention is successfully conveyed, or re-evaluating a strategic goal in real time on the basis of new information gathered from another player's response (Canonne and Garnier, 2012, pp. 202–203). Similar issues, and more, would apply in any setting where people are not solving predetermined problems.

There is a range of further challenges for and limitations in the above annotation paradigms. Firstly, given the largely non-discursive, or at least non-verbal, nature of musical performance, any dialogic pattern reflecting decisions *vis-à-vis* structure or other parameters is usually represented post-hoc, and mostly through the aid of recordings. Other domains (including mathematics) have many non-verbal features that can limit the range of applicability of text-based or discursive methods. Secondly, musical situations that correspond to Benson's types 1-6 involve a high level of consensus as to what constitutes musical meaning, and what formal or structural prerequisites are necessary for its production. They are thus closer to what Hall (1992) describes as "low-context" (LC) situations – whereas types 8-11 involve highly subjective, "high-context" (HC) processes. The latter are prone to contrasting evaluations, and pose a range of challenges for modelling.

Contexts that change over time or that depend on the observer's perspective suggest the need to develop a theory of information flow that takes on emergent proper-

ties. At least among musicians, decisions reached verbally may be reconfigured and in some cases entirely reversed based on playing and listening sessions between discussions; analogous behavious may be found in mathematical settings. The frameworks introduced in this section suggest a range of domain independent ways of moderating group dynamics. Nevertheless, more work is necessary to arrive at a more conclusive representation of how individual aesthetic preferences and collective behaviour are negotiated within groups.

## 6.3 Related Work: Social Creativity on Computers

In this section, we give an overview of some of our efforts to go beyond the work surveyed above, following the plan laid out in Figure 6.1. Section 6.3.1 presents work that formalises the relationships between moves in the Lakatosian framework `LD`, described above. Section 6.3.2 describes more informal theorisation of peer learning and peer production, which is nevertheless able to express high-level topics in problem identification, positing, and selection. Section 6.3.3 describes an evolutionary computer system that is much more low-level, but which illustrates the potential of simulation studies for research in creativity. These three short sections are not closely related to one another; rather they provide a selection of approaches that will inform future work.

### *6.3.1 A Formal Representation of Lakatosian Creativity*

Lakatos (1976) describes a notion of social creativity in mathematics which springs from the interaction of two conflicting theories, leading to the synthesis of a third theory that resolves inconsistencies. In Pease et al. (2017), we explore the connections along the pipeline running from philosophical theory to formal expression of mathematical arguments as dialogue games. This allows us to express linguistic structures of reasoning in terms of formal structured argumentation, abstract argumentation and argumentation semantics. Finally, coming full circle, we show that such implementations can provide value to mathematical communities. This work lays a foundation for a formally sound and linguistically coherent theorisation of collaborative intelligence and social creativity.

This interpretation of Lakatos through the lens of dialogue game theory (Hamblin, 1970) and, in particular, as a dialogue game ranging over structures of argumentation, allows us to keep track of the mutually consistent and mutually inconsistent portions of a mathematical dialogue. The practice of interaction between mathematicians is mediated by language, so our choice to rely on argumentation theory is, in the first instance, governed by the need to handle challenges presented by reasoning about linguistic expressions. On the other hand, in order to manipulate these structures computationally, there is a need for formal and ontological clarity.

We have adopted the Argument Interchange Format (AIF) (Chesñevar et al. (2006); Rahwan et al. (2007)) to handle the data. The understanding of Lakatos is refined and characterised as a set of update semantics on AIF structures. At a high level, the dialogue game is constrained to follow the flowchart in Figure 6.10.

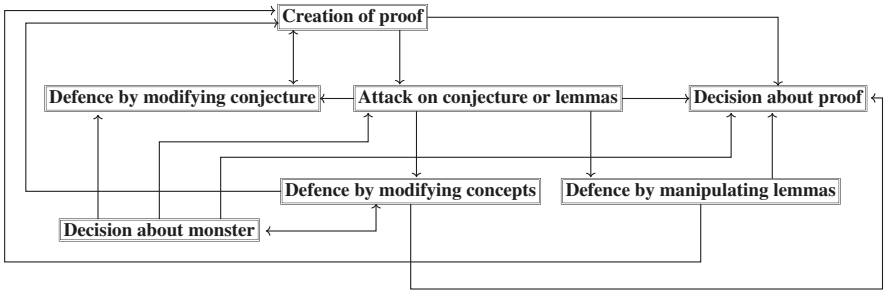



Fig. 6.10: Lakatos's informal logic of mathematical discovery represented as a flow chart

Whereas the Lakatosian moves in Section 6.2.2 were treated as a set of independent dimensions, in this formalism we see the constraints on their use. Previous work by Bex et al. (2013) has shown how AIF can be interpreted as a nonmonotonic system, with mappings built from AIF to the ASPIC+ structured argumentation formalism.[18] Prakken (2010) has shown how abstract argumentation systems in the style of Dung (1995) can be induced from ASPIC+, from where the argumentation semantics can be computed to provide labellings of the acceptability status of each argument. In our treatment, the abstract argumentation framework corresponds precisely to the theory that has been collaboratively created by the participants in a Lakatosian dialogue.

In addition, each of these formal steps is also available in implementation: details are described in Pease et al. (2017). Finally, an appendix to that paper shows how the model can also be retrospectively applied to examples of extant mathematical discussions from MiniPolymath (Tao et al., 2011). This demonstrates the depth of Lakatos' insight into mathematical creativity, and suggests that the formal characterisation is honest both to his original work and to the practical workflows used by mathematicians.

### *6.3.2 Patterns of Peeragogy*

Corneli et al. (2015) use the descriptive language of *design patterns* to take stock of the common processes found in successful collaborations, a typical context for

[18] *ASPIC+* is follow-on work from the *Argumentation Service Platform with Integrated Components* (ASPIC) project: http://www.cossac.org/projects/aspic

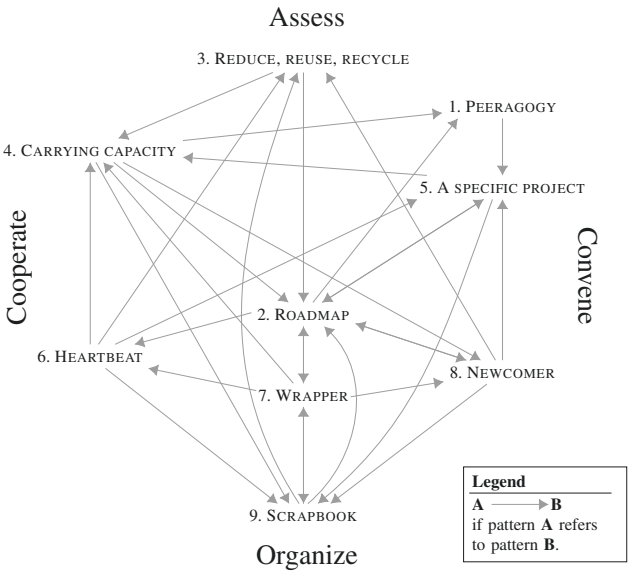



Fig. 6.11: Connections between the patterns of peeragogy

everyday social creativity. Their focus is on *peer produced peer learning* projects, in which the structure of the collaboration is not fixed in advance, and must be created alongside the project's other products. The central pattern describes the ways in which project participants collaborate to build a shared ROADMAP that gets them "from *here* to *there*" – possibly integrating multiple different "*heres* and *theres*." The eight additional design patterns describe additional "social glue," holding together and structuring work on the project (Figure 6.11). For example, the NEWCOMERS pattern describes interactions between experienced project members and beginners. The degree to which social glue and out-of-domain discourse is required varies according to a project's scale and scope. For example, in the Polymath project, discussants needed to determine a large-scale overall plan, including which problems to focus on, while in MiniPolymath a single problem is set in advance, and participants only needed to develop a suitable approach to solve it. Future work could formalise this material further, possibly by adopting the dialogue game treatment described above.

### *6.3.3 The Search for Computational Intelligence*

Corneli and Maclean (2015) aim to present a minimal convincing computational simulation of social creativity that exhibits emergent results. Cellular automata are

taken as an underlying domain, thought of as "multi-agent systems based on locality with overlapping interaction structures" (Flache and Hegselmann, 2001). The novel approach that we described as a *search for computational intelligence* permits local adjustments to the behaviour of cells in the automaton. Change at this level is embedded in a higher-order evolutionary system called a *meta-cellular automata* or "MetaCA." That is, MetaCAs co-evolve rules for local behaviour based, in part, on the currently observed behaviour. From a philosophical perspective, this work is aligned with the notion from Mead (1932) of the social as emergent co-evolution: "an adjustment in the organism and a reconstitution of the environment." Our initial simulations used a modified version of Goguen's Unified Concept Theory to carry out evolutionary steps that evolve local behaviours via concept blending over successive generations. However, we found that still-more sophisticated interactions between phenotype and genotype data (after Baldwin (1896)) were needed to produce familiar "edge of chaos" (Mitchell et al., 1993) effects. Expanding on the semantically simple domain of cellular automata, future work could encode, e.g., mathematical problems in a cellular program (Sipper, 1997) and involve a group of agents in finding solutions to these problems as a society.

## 6.4 Summary

In this first part of this chapter, we developed a catalogue of different ways to theorise mathematical conversations, and presented a brief pilot study that marked up several sample texts using this framework, in order to gain perspective on the salient dimensions. In the second part of the chapter, we surveyed recent work that explores several additional directions of analysis and design. Drawing on the frameworks and methods for theorising social creativity that we have described, we are in the process of synthesising a protocol which can both account for the behaviours observed in a mathematical text and dialogues, and also admit the possibility of computational support (Martin et al., 2017). In the move from the Lakatosian theory $\mathscr{LAK}$ to a "$\mathscr{LAK}++$" that brings in themes from social simulation and the design of social machines, we have the opportunity to be both more descriptive of everyday social reality in mathematics and other domains, and to explore more of what is computable.

## 6.5 Conclusion and Future Perspectives

One of the issues hinted at in our survey of frameworks was the 'black box' of mathematical content. By adding structured identifiers and suitable representations, we will be able to understand the substantive mathematical relationships, not just schematic relationships. For example, when performatives are attached to text fragments, a history of the evolution of core concepts in the discussion becomes avail-

able. Some segments may be tagged as explicitly Lakatosian; others might be seen as "peeragogical".

In developing "$\mathscr{LAK}++$", our strategy makes use of:

I. A (relatively) minimal set of performatives for the argumentation structure in the dialogue, such as *suggesting*, *asserting*, *retracting*, and *challenging*.
II. A set of relations between the mathematical structures under discussion, such as: *has_property*, *sub_conjecture*, *equivalent*, *stronger*, and *weaker*. This allows a model of *information flow* between structures, in the sense of Barwise and Seligman (1997).[19]
III. A set of meta-objects representing the conjectures to prove, and strategies to use to prove them, such as *goal*, *strategy* and *difficult*.

At an abstract level, a locution $E$ that presents *an example of a concept* might map to a hypertextual representation that connects the example $E$ to the concept $C$ which it exemplifies, which could then be mapped in a formal model of the dialogue to the statement `example(C,E)`. The *Lightweight Social Calculus* described by Murray-Rust and Robertson (2014)[20] that was mentioned in Figure 6.3 is one suitable target language for this sort of relation.

Note that we do not require every conversation to happen within the same scope or the same formal model, since information channels can themselves be chained together and given multiple and emergent interpretations. In particular, we would like to build relationships between high-level features of discourse and the implicative and other structures that are carried by mathematical objects themselves.

Thus, for instance, we would summarise the utterance "*The following reformulation of the problem may be useful: Show that for any permutation s in $S_n$, the sum $a_s(1)+a_s(2)+a_s(j)$ is not in M for any $j =< n$*"[21] as

$$assert(equivalent(\text{main_problem}, \text{any_permutation})).$$

This statement might be flagged up (using a higher-level model) as an example (i.e., an example of a broader class of equivalent problem formulations).

The outline above reflects the view that much of an argument's structure is carried by relations between the mathematical objects under discussion. We do not want to have to represent the whole of mathematics in order to reason about individual proofs. Thinking in terms of the relationships between objects will offer a level of detail that allows us to represent most of the important structure of specific arguments.

Reflecting back over the work described in this chapter, some of the ways in which the methods outlined in I–III could develop are clear:

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[19] These relationships go beyond those explicitly dealt with in the Lakatosian theory and show recursive (and other) relationships between theories. Cf. the comparison with intuitionistic logic in Restall (2005).

[20] Referred to in earlier work by Robertson (2005) as the *Lightweight Coordination Calculus*.

[21] Tao et al. (2009)

I′. As remarked in Section 6.3.2, in more complex domains, more kinds of moves are needed: for instance, *problem positing* as well as *problem solving*. We intend to progressively expand the approach outlined above to deal with more actively social problem scenarios, like Polymath.

II′. The ability to reason computationally about mathematical and musical structures was a core part of the COINVENT project. We have carried out preliminary experiments that integrate the agent-based model described in Section 6.3.3 with the simple structure annotations like those mentioned in II as CASL representations of mathematical and musical objects. Continuing to develop this work could lead to a convincing and useful simulation of social creativity in our target domains.

III′. In this work, we are not necessarily restricted to a Lakatosian or even an argumentation-based approach, although these continue to be relevant in many situations. For example, Confalonieri et al. (2015) explored an argumentation-based approach to evaluating the meaning, interest, and significance of concept blends, using a simplified version of the Lakatosian dialogue game from Pease et al. (2017). However, scenarios with "multiple right answers" (Corneli et al., 2015) require more elaborate coordination methods.

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# Towards Open-World Scenarios: Teaching the Social Side of Data Science

**Joseph Corneli**[1] and **Dave Murray-Rust**[1] and **Benjamin Bach**[1]

**Abstract.** This article reflects on current challenges we encounter in teaching data science to graduate students. A common critique of data science classes is that examples are static and student group work is embedded in an 'artificial' and 'academic' context. We look at how we can make teaching data science classes more relevant to real-world problems. Student engagement with real problems—and not just 'real-world data sets'—has the potential to stimulate learning, exchange, and serendipity on all sides, and on different levels: noticing unexpected things in the data, developing surprising skills, finding new ways to communicate, and, lastly, in the development of new strategies for teaching, learning and practice.

## 1 Introduction

At first sight, data science is a hands-on technical activity, concerned with 'hard' knowledge such as statistics, machine learning, visualization, etc. But practicing data science requires an array of 'softer' skills, including understanding of the context and implications of data, communication, or collaboration. This array of requirements is reflected in common texts and references, which attempt to introduce students to the complex world of professional practice [33]; which highlight the *"need for this material to be offered more broadly"* (not just to engineering and science students) [4]; and which contrast data science teaching with *"traditional statistics courses* [...] *focused on describing techniques and their mathematical properties rather than solving real-world problems or answering questions with data"* [20].

Faced with the challenge to deliver a course on data science to graduate students in a design-oriented Master's program, we wanted to account for both 'hard' and 'soft' skills. Students came from a range of backgrounds; some with little or no prior programming experience, others with an undergraduate degree in computer science. Additionally, about 80% of the students had recently arrived from non-English-speaking countries. They brought along different cultural expectations related to communication, collaboration, and pitching.

As in other courses we've encountered, our syllabus progressed from rather closed tasks to more open ones. The first few lessons covered tutorial material on programming with Python.[2] In a second stage, we taught more applied data science problems on a specifically curated toy dataset. In a set of pre-existing csv files (detailing Titanic survivors, tips spending, etc.[3]) we systematically introduced errors such as incorrect data formatting, empty cells, spelling-errors, and non-integer values. Finding and treating these errors, as well as answering several analytical questions about the given data set was part of a second assignment. Eventually, we would connect students with larger set of data sources (Wikidata, open data from the BBC, historical databases, Twitter data, sensor data, smartphone app usage data) to develop their own approaches to analysis and visualization with less supervision.

Working together on this course lead us to discuss many challenges with current course formats and to think about methods to improve teaching the social factors involved in data science. In this short paper we reflect on our experience teaching with the above model and how we can in future improve the teaching strategy and the student experience, by including more room for serendipity in the course. We are interested in how students can encounter and cope with uncertainty, interact with people from different disciplines, and find joy in developing their skills and in noticing how these skills can shape the world around them. *How can serendipity play a role in teaching data science? How can we foster and combine engagement, discovery, and learning? How can we teach data science as a social, iterative, and mindful engagement?* The concept of serendipity can be a narrative for this kind of open-world experience: we give up some control, and this creates a real risk of failure. For example, one way to introduce serendipity into the classroom is to involve students in real-world collaborations, but this poses considerable challenges.

After enumerating and reflecting on some of these challenges (Section 2), Section 3 then surveys literature on alternative learning approaches and Section 4 talks about the role of serendipity in professional practice, comparing that with the student experience. Finally, in Section 5 we put forward our conclusions, and sum up some of the ways this work may evolve in the future.

## 2 Challenges in Teaching Data Science

Many current teaching setups for data science can be classified as *closed-world*, *guided*, and relatively *controlled*. These characteristics make teaching and assessment relatively straightforward, but they give an impression of data science as simply being an area of expertise, rather than a professional practice.

This section reviews the challenges we considered while developing and teaching a new course, Data Science for Design (DS4D). The following list reflects our discussions as co-developers of the course, along with our previous experience teaching data science and visualization classes, and facilitating peer and online learning experiences [10, 11, 12], as well as extended discussions with colleagues about their teaching experiences. The list is not complete: it may serve to stimulate feedback and discussion from other scholars and teachers.

C1: **Toy datasets:** the term 'toy datasets' is denotes the opposite of real-world datasets, lacking significant characteristics from the latter, e.g., size, complexity, messiness, relevance, context, etc. Toy datasets are usually small, curated, clean, and contain ground truth students

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[2] `http://swcarpentry.github.io/python-novice-inflammation`

[3] `https://github.com/mwaskom/seaborn-data`

are required to find. While they make assignments and assessment straightforward, they (i) require some effort from the side of the teacher (retrieval, curation, etc...), (ii) might be of little personal interest to the students, (iii) might match with available solutions from other and past courses, and (iv) might allow students to cheat by passing around their solutions.

C2: **Real-world datasets:** one way to overcome issues with toy datasets is to provide real-world data to students. Yet, real-world datasets come with their own set of challenges: (i) some may be difficult to obtain, (ii) some may be too messy to be used in a course, (iii) students might fail to comprehend the data at all, or (iv) might lack the respective knowledge to drive an analysis and interpret their findings, (v) many steps are required before analysis can take place, e.g. obtain data, transform, clean, etc.; and lastly, (vi) real-world data puts strains on evaluation and balancing difficulty.

C3: **Motivation:** Both C1 and C2 have ramifications for student motivation. Toy datasets might be too simple or just not interesting; real-world data might be too specific and not relevant to students. Allowing students to choose datasets themselves partly solves the problem but requires more preparation from the side of the teacher in terms of access, provision, description, and evaluation. However, motivation is key in learning and it exhibits multiple facets that may offset the difficulties: interests, skills, social setting, personal relevance, ideas for approaches, etc.

C4: **Complexity:** If different students use different real-world data sets, then they are likely to have widely different experiences in the course. *How do we adapt problem complexity to manageable levels?* Can a course help students learn to cope with complexity and uncertainty, phenomena they will encounter in the real world?

C5: **Relevance:** *How do students know to whom and which real-world problem their skills will be relevant?* This aspect reflects a common critique of university teaching and academia more broadly. Who is the "client"?

C6: **Soft-skills:** Since data science practitioners are not simply engaged with technical work, students need an opportunity to develop and practice relevant soft skills: problem definition, collaboration, collaboration, placing their contribution in context, understanding when and how data science can be applied, communicating their findings and discussing technical decisions with stakeholders, etc.

C7: **Method evaluation:** Eventually, every course must assess students learning outcomes. While data science is a wide field, learning outcomes will differ across courses, levels, and course audiences. *What are the learning outcomes of a course and their priority? How to evaluate each of them?* While it might be easy to evaluate technical 'hard' skills (relatively, depending on the choice of data and the methods taught), due to their nature, 'soft' skills are somewhat harder to evaluate. It would be an over-simplification to assume every student must exhibit all skills equally well.

C8: **Interdisciplinary audience:** Though not a problem in every data science course, our course was offered to related disciplines within the university and hence attracted people without programming experience and strong mathematical backgrounds. We believe interdisciplinarity in a course benefits students with technical skills and students with background in other disciplines. We believe data science is a broad methodology and serves a wider knowledge of "dealing with data".

It is probably impossible to address all of these challenges fully in a single course. Any good curriculum will balance different types of courses and learning opportunities: lectures, tutorials, projects, dissertations, presentations, etc. This gives rise to two focal questions:

- *Which structures can be implemented in individual (data science) courses in order to help weave together a consistent set of projects, skills, and engagement across courses within a curriculum?*
- *How to provide relevance and motivation in usually closed-world teaching in the context of open-world challenges?*

## 3 Open-World Teaching

Open-world courses, contrary to closed-world courses, are more like real-world scenarios; they can be characterized by the explicit interaction with *course-external* entities (data, collaborators, domains, etc.), less guidance, and a grain of uncertainty.

There are many ways to involve students in real-world contexts that may help address some of the above mentioned challenges. This section gives an overview of the variety of approaches that might inspire an adaption to data science classes.

**Universities and Society**—Various formulations of the relationship between institutions of higher learning and the wider community have been proposed and pursued. E.g., according to "the Wisconsin Idea", originated in 1905 at Wisconsin's large public "land grant" university,[4] the university must "*assume leadership in the application of knowledge for the direct improvement of the life of the people in every sphere*" [14, p. 88]. Research that adds to the store of knowledge is another fundamental obligation (*ibid.*, p. 550). Harvard takes a less interventionist stance: the university does not have a formal mission statement,[5] while its undergraduate programme states that its mission is *"to educate the citizens and citizen-leaders for our society* [via] *the transformative power of a liberal arts and sciences education."*[6]

**Teaching and Research**—Learning by doing research is a widespread educational practice, with various schemes available, though entrance to these is often competitive. Student involvement in research may go along with a shift from "teaching" via instruction to "peer learning" [3]. For example, recently gifted high school students have coauthored mathematics papers using online collaboration tools, with some help from mentors [19]. *Problem-based learning* involves open ended problems but, typically, a structured programme of approach [32]: it has been tried in data analytics teaching [28].

**Public Action**—In her proposal for a "new liberal arts" [8], Elizabeth Coleman makes contemporary social challenges the core of the curriculum. Rather than being insulated from these problems for four years, students would organize their work around challenges having to do with the environment, health, energy, economics and equity, governance, and so on. *Public action* would be adopted as a key criterion of successful performance. The relationship between students and members of the wider community is foregrounded, and practice-based education is the order of the day. As part of this effort a new Center for the Advancement of Public Action was announced [7] at Bennington and subsequently built at a cost of $20 million [26].[7]

**Field-work and Collaboration**—But indeed since its foundation, Bennington College had emphasized "the concrete approach" and *"engaged students in projects 'involving continuous periods in the laboratory, library, or field' under the supervision of a professor"*[39, p. 263]. *"College administrators called for education that prompted students to actively engage their social and cultural worlds"* emphasizing *"social participation and cooperation"* (*ibid.*). Similar views were expressed by other mid-20th Century thinkers (e.g., [13]).

[4] https://www.wisc.edu/wisconsin-idea/

[5] https://www.harvard.edu/about-harvard/harvard-glance

[6] https://college.harvard.edu/about/mission-and-vision

[7] Coleman's late-2000s proposal echoed aspects of an earlier contentious restructuring of Bennington College under her leadership in the 1990s [17, 24], most notably in calling for increased community engagement.

**Teaching to Develop Deeper Understanding**— Kenneth Burke, at Bennington in the 1940s and 50s, proposed a "'synoptic' project for 'unifying the curriculum'" [39, p. 265]. What Burke names as the "question that ultimately concerns us most" is one that can be studied by a data scientist as well as by a literature scholar: "What is the nature of a symbol-using animal?" (*ibid.*, p. 266). Other authors from the same era, working from widely different disciplinary standpoints but all influenced by ideas in cybernetics were similarly concerned with the synthesis of meaning and form (e.g., Alexander [1], Korzybski [25], Simondon [34], and von Uexküll [38]). Although our work is data-focused in name, we can nevertheless be concerned with the entire Data-Information-Knowledge-Wisdom (DIKW) hierarchy [31]—and the way meaning is made and used. Indeed, the learning outcomes in DS4D—*Data*, *Programming*, *Communication*, and *Professionalism*—are well-aligned with the terms of this hierarchy. (Furthermore, all of these issues are important insofar as we are not just teaching data science, but teaching science *per se*.)

## 4 Serendipity in Practice

Serendipity is linked to scientific discovery [30]. Furthermore, with today's data-driven scientific methods, *"Instead of waiting for the happy accidents in the lab, you might be able to find them in the data"* [23]. Investigators make unanticipated discoveries, find unexpected correlations, notice outliers, strange trends, etc.

Thinking about the role of serendipity in data science goes back (at least) to John W. Tukey and his definition of *Exploratory Data Analysis* (EDA) [37]. The core idea of EDA is the 'grand tour', a walkthrough of the facets and dimensions of a dataset, using a sufficiently large array of charts and data visualizations; glancing over multiple charts at once in a way that both gives a general overview over the different aspects of a data set (time, space, relations, distributions, dimensions, etc.), and also allows for serendipitous discoveries—answering questions that 'one did not know one was interested in' and which one would never have been found through a purely statistical approach. Following EDA, numerous visualization interfaces have been designed with serendipity in mind (e.g., [36, 16, 15]).

As a defined area of study "data science" dates to William S. Cleveland's more recent (2001) proposal to "enlarge the major areas of technical work of the field of statistics" [6]. Among the key elements of the proposal are the importance of work in multidisciplinary teams, and new methods for model building. He suggested that "*A basic premise is that technical areas of data science should be judged by the extent to which they enable the analyst to learn from data*" (p. 21). He remarks that "*data are the heat engine for invention*" and that "*Creative researchers, faced with problems posed by data, will respond with a wealth of new ideas that often apply much more widely than the particular data sets that gave rise to the ideas*" (*ibid.*, p. 22). He also highlighted that data science teaching should "encompass more than the university setting" and convey to non-statisticians "how valuable data science is for learning about the world" (*ibid.*, p. 24).

In the context of practices related to teaching and learning—including learning on the job—the implications of serendipity go far beyond discoveries through EDA, to the development of new professional and skill-development practices.

Consider hackathons, which bring together people with different skills, ideas, and perspectives; given a challenge these (usually interdisciplinary) teams will attempt to develop solutions in a very limited time frame. The posed challenge may require team members to develop new skills, to work with new people, and to engage with new problems. A similar idea was adopted by the IEEE VIS conference, which started a series called the *VAST Challenge*[8], which provided datasets with a specific question and a quest to solve. Participants in the challenge were entering into competition on building visualization interfaces that would visualize the data and allow people to solve the quest. Without specifically trying, hackathons can serendipitously address some of the challenges mentioned in Section 2. We can notice some common themes in open world "solutions", as found in hackathons, lab work, or data science practice, and the various teaching strategies surveyed above.

**Collaboration:** People with different skills may be able to find suitable opportunities for skilled practice and learn from others' skills. They may need to learn skills that foster serendipitous outcomes, taking advantage of opportunities to share early insights [9]. For example, through collaboration within and beyond the group, partnerships are formed, such as meeting talents and future employers.

**Topics:** In contrast to toy data sets, which are deliberately kept simple and self-contained with little connection to external knowledge, open research questions allow the possibility of serendipitous discoveries through the activation of domain knowledge and interests otherwise 'hidden' in learners. Specifically, data collaborators might help students make new connections that they would not think of on their own.

**Contextualisation and interpretation:** Discoveries need to be interpreted and put in context [2]. For example, learners can come up with data and insights, but only external data collaborators with the appropriate domain knowledge are able to interpret and contextualise findings from the data, eventually turning them into true discoveries. Working with domain experts helps learners to find value in their findings, and to understand any serendipitous implications of those findings.

**Motivation:** Learners may exercise more creativity, motivation, and interest by addressing a problem that they have chosen or helped shape, rather than a problem that got handed down to them. More broadly, Taleb advises: "Work hard, not in grunt work, but in chasing [potentially high-payoff] opportunities and maximizing exposure to them" [35, p. 110].

**Skills:** The talent for making serendipitous discoveries can be cultivated, and consists, in part, in learning how to pay attention to details [22]. With practice, people can get better at making interesting observations. In particular, one important skill is to discover a more interesting problem than the one you were initially working on: many new inventions were conceived by people working on some unrelated project; communication with end users can be a particularly valuable source of inspiration [22, 18].

**New models, methods, organisations, and theories:** As Cleveland highlights "Creative researchers, faced with problems posed by data, will respond with a wealth of new ideas" [6, p. 22]. Serendipity can apply to the discovery of new ways to think about things, not just to the discovery of facts that fit a given frame of reference.

## 5 Discussion and Future Work

In order to realize the concepts described in Section 4, which mechanisms for emphasising the open-world approach in data science class-rooms are needed? Again, most program curricula involve a variety of learning scenarios: open project work, lectures, tutorials, and so on, many of which contain elements of open-world teaching. E.g., writing a Master's thesis typically follows some coursework and requires students to formulate research questions, give presentations,

[8] `http://www.vacommunity.org/VAST+Challenge+2017`

plan their project, etc. Our hunch is that thinking about integrating different elements into *one single* structured course might help thinking about applying this structure to one coherent *open-world program curriculum*.

Echoing the data science pioneer Cleveland, we can say that universities are driven by an invention-engine, though they also achieve the preservation and translation of cultural values. As they learn data science, students have the opportunity to *"insert* [themselves] *into that machinery"* [29]. Accordingly, as data science teachers we are inviting students into the "power-house [. . . ] of knowledge construction" [21]. We think that open-world class-projects can enhance the visibility of universities, classes, and teaching programs, and potentially make them more attractive to people pursuing continuing education.

A clear limitation of this paper is that it is based on our own experiences and discussions with colleagues. We surely need to widen the discussion, to bring in more ideas about teaching; and, eventually, we hope to provide an empirical evaluation of the methods outlined here. We hope our reflections might stimulate a pro-social approach to teaching a technical topic, one that gives soft skills due attention. We see the future of data science as inextricably wrapped up with the development of *humanistic intelligence* [27], i.e., intelligent systems with humans in the loop.

Increasingly, basic discoveries can be made using smart tools, and these tools are making inroads into interpretation of their findings: "Cognitive computing technologies can be configured to make cross-domain linkages [rather than] rely on serendipity" [5]. However, as yet, autonomous intelligent systems typically cannot deliver sophisticated, contextual, interpretations.

In spite of, and indeed, *a fortiori* because of the pace of technical advances in artificial intelligence, we need to keep in mind that teaching and doing data science requires not just technical solutions but also the cultivation of human capacities. Coleman mentions capacities for civic engagement, discrimination between core and peripheral issues, collaboration and innovation [7]. Moreover, and centrally, by expecting the expected and bringing open-world problems into the classroom, we may give students the opportunity to develop their own critical sensitivities [29].

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# Modelling the Way Mathematics Is Actually Done

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## Abstract

Whereas formal mathematical theories are well studied, computers cannot yet adequately represent and reason about mathematical dialogues and other informal texts. To address this gap, we have developed a representation and reasoning strategy that draws on contemporary argumentation theory and classic AI techniques for representing and querying narratives and dialogues. In order to make the structures that these modelling tools produce accessible to computational reasoning, we encode representations in a higher-order nested semantic network. This system, for which we have developed a preliminary prototype in LISP, can represent both the content of what people say, and the dynamic reasoning steps that move from one step to the next.

***CCS Concepts*** • **Computing methodologies** → *Philosophical/theoretical foundations of artificial intelligence*;

***Keywords*** Arxana, knowledge representation and reasoning, inference anchoring theory, conceptual dependency, mathematics, natural language, formal proof, exposition

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## 1 Introduction

Building a computer system that can understand mathematics at a high level was one of the intriguing challenges identified early on in the field of artificial intelligence [62, 69]. Simply understanding mathematics at all is a challenge faced by schoolchildren and others everywhere. Mathematics as it is practiced professionally is a compelling combination of formal reasoning and informal exposition. Teaching the craft of mathematics involves helping students learn how to think like experts [11], however our understanding of expert thinking in mathematics is not yet systematic. Mathematical AI could make both education and research more effective.

And yet, most mathematicians are completely uninterested in "computer mathematics" per se. The vast majority do not write their theorems in any of the many systems available for specifying and checking proofs formally. Nevertheless, the power of computational approaches has been amply demonstrated in formal mathematics [26] and in quasi-formal domains ranging from Chess and Go to logistics.

Mathematical AI could ultimately be as indispensable in conceptual domains as machine learning is in subsymbolic domains. And indeed, machine learning is likely to be useful for building mathematical AI—but in order to get to the point where such an application of machine learning techniques is possible, we would need representations of mathematics in which meaningful patterns can be found. This is the challenge towards which we address this paper.

At the moment, natural language representations of mathematics are largely obscure to text processing techniques, although inroads have been made in understanding the linguistic features of some aspects of mathematics [15, 16, 21, 22]. Mathematical formalisms take this to an extreme, since they simplify and tightly constrain the ways in which their users can write things down. We note that machine learning methods have been applied to corpora of formalised mathematics to good effect [30, 31, 33, 44]. However, mathematics as it is done by most mathematicians has a very different structure – and there is a lot more of it available, as summarised in Table 1.[1]

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[1] Compare: 1253 articles in the Mizar Mathematical Library, http://mmlquery.mizar.org/. Mizar is one of the oldest proof checking systems. The Archive of Formal Proofs associated with another prover, Isabelle, comprises some 362 entries (https://www.isa-afp.org/statistics.shtml).
[2] https://mathoverflow.net/questions
[3] https://math.stackexchange.com/questions
[4] https://arxiv.org/archive/math
[5] https://tools.wmflabs.org/enwp10/cgi-bin/list2.fcgi?run=yes&projecta=Mathematics

| | |
|---|---|
| 791,310 | questions about mathematics in relevant Q&A forums on Stack Exchange[2,3] |
| 292,516 | mathematics papers on the Cornell e-print arXiv[4] |
| 15,983 | Wikipedia articles on mathematics[5] |
| 25,436 | books about mathematics in the Library of Congress[6] |

**Table 1.** Sources of mainstream mathematical writing

The Stack Exchange corpus alone is much larger than the 160000 games in AlphaGo's initial training set [34], but the data looks completely different. Go games are comprised of sequences of black and white stones placed on an otherwise uniform $19 \times 19$ grid. The space of possible configurations for mathematics is much more complicated. Machine learning has also made impressive progress in textual domains – and incidentally, this remark includes DeepMind [27], the company behind AlphaGo – but the biggest successes, such as machine translation, throw away fine structural details that would be essential for machine understanding of mathematics. Coarse models of non-mathematical dialogue are state of the art for machine learning [71].

To begin to model mainstream mathematics in a way that exposes its structure to machines, we need the tools that are in some ways closer to the scholarly apparatus of literary criticism than to the formal logic used by Russell and Whitehead to write *Principia Mathematica.* Indeed, we must stress that our focus is on the way mathematical reasoning is communicated, rather than on the physical, embodied, and cultural intuitions that underlie mathematical thought. These intuitions, we believe, would be difficult to model computationally [63]. However, when they are communicated they are referred to a distinct set of reasoning patterns – albeit not only deductive, but also abductive, inductive and heuristic. Peirce said: "all mathematical reasoning is diagrammatic" [54, p. 47] – and our approach pursues this line of thinking.[7] In the AI subfield Knowledge Representation and Reasoning (KRR), the keywords "ontology" and "semantic network" denote specific types of diagrams that computers can read. We will describe a more general strategy for representing and reasoning about mathematical texts. Our approach necessarily begins *in media res* (cf. [56, p. 4]). More specifically, we put inspiring developments in formal proof, embodiment, linguistics, and machine learning each to one side for now: not because these topics are unimportant, but precisely because they are well-studied elsewhere.

In overview: we select a representation model called Inference Anchoring Theory+Content [12, 58] as the foundation of our strategy. We are not aware of any other representation framework that is both flexible and expressive enough to connect the formal and informal reasoning that is present in mainstream mathematical texts and dialogues. Our survey in §3 will point to structured proofs [40] and Lakatos Games [53] as two pieces of exemplary work that do not possess this bridging property. We also draw on Conceptual Dependence theory [45, 61, 64], and broadly, on a reflexive strategy for representation, because we are not only aiming to represent reasoning, but also to simulate it computationally – whereas IATC by itself is purely about representation. Herein we present a preliminary proof-of-concept operationalization of *flexiformalism* [36], the key tenets of which are:

(1) Formal models should show the correspondence with informally-given problems;
(2) Proofs are fundamentally informative communication;
(3) Formality is not all-or-nothing.

In outline: §2 characterizes mathematical texts in terms of their distinct formal and expository registers, and offers a high-level characterization of the current effort. §3 surveys related work, with a focus on the available modelling languages that bear on our representational goals. §4 illustrates, informally, how mathematical text can be modelled in one of these languages, and gives an initial description of our strategy for representing non-deductive reasoning. §5 adds more detail to the modelling approach and shows how existing AI strategies can be applied to make non-deductive reasoning explicit. §6 describes a prototype software implementation that we have developed in LISP, and shows how it can be applied to make the kind of diagrammatic reasoning we describe in the paper actionable on a computer. The system implementation itself is available as literate program [13]. The application to mathematics should be understood to be at a preliminary stage of development, the discussion in the current paper focuses on describing how it can be done; our argument is supported by initial experimentation. §7 presents our conclusions and returns to the four themes we set aside earlier to give an outlook on future work.

## 2 Background

Since our purpose is to understand mathematical reasoning as practiced by mathematicians, we began by studying examples of mathematical texts. In this section, we will summarize some high-level observations.

One of the first features which leaps to the eye when reading mathematics is that much of the text is written in a fashion which employs a precise technical vocabulary, logical

[6] https://www.loc.gov/books/?fa=partof%3Acatalog|subject%3Amathematics&all=true

[7] Peirce advanced this definition: "By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, performs experiments upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses this in general terms" [54, pp. 47–48].

idioms, and symbolic notation to state mathematical propositions. We will refer to this as the *formal register* of mathematical discourse.[8] One example of an utterance in the formal register is "Every integer equals the sum of four squares." Symbolic paraphrases can readily be found, in whatever formal system one chooses. For instance, the four-square theorem could also be written $(\forall n \in \mathbb{N})(\exists m_1, m_2, m_3, m_4 \in \mathbb{N})\, n = \sum_{i=1}^{4} m_i^2$. Furthermore, in mathematical writing it is understood that nothing essential is lost in translating between the verbal and symbolic statements.

While statements in the formal register are an important part of mathematical writing, a text consisting only of formal statements would be frustrating to read because, while it may contain all the technical information, it would offer no guidance to the reader as to where the statements came from, why they are interesting, and how to go about understanding them. To offer this guidance, a mathematical text will also contain expository statements such as the following:

> Next, we will prove the four-square theorem using an algebraic identity similar to the one we just used to prove the two squares theorem.

This *expository register* has several salient features which work together and which distinguish it from the formal register.[9] While the subject matter of expository statements is also mathematical objects and propositions, the language involves not only description, but also narration and argumentation. In the example given above, we see a narrative structure in which a proof is introduced by comparing the technique to that used in a previous proof.

Whereas in formal logic, only the strictest deductive reasoning is allowed, mathematical exposition also makes use of inductive and abductive reasoning, and even looser reasoning by analogy. For instance, our example above makes an analogy between two "similar" algebraic identities. In the following sequence of questions, we see evidence of an inductive mode of reasoning in which the questioner examines several examples, searching for a suitable generalization of the problem under consideration.

> Can we do this for $x + y$? For $e$? Rationals with small denominator?

A related point is that, whereas formal statements only make use of the truth values and predicates of formal logic, the expository register also makes use of loose, heuristic, judgements of plausibility. Oftentimes, these are expressed using adjectives like "superficial" and "deep" which are not formally defined but refer to approximate notions which inform heuristic choices.

The expository register is also metamathematical, discussing not only objects and propositions, but also proofs and strategies. When setting out to prove a theorem, one may first assess different possible proof strategies using heuristic judgements like "depth" or "difficulty," or by making analogies with known techniques to prove abstractly similar results. In this way, informal reasoning frequently serves to guide the construction of a formal proof. Quite often, even if known, the exact deduction of a mathematical result might be lengthy and/or opaque. In this case, it is helpful to the reader to augment formal reasoning with informal reasoning which is shorter and easier to understand.[10]

In this essay, our main focus will be to construct a theory of mathematical exposition which will account for the features noted above, and implement it computationally.

### 2.1 Framing the current effort

For a high-level view of how mainstream mathematics relates to other challenge problems for computational understanding, one can compare previous research on the blocks world [70], board games, and story comprehension (Table 2). These require increasingly sophisticated patterns of *inference*, *thinking*, and *reasoning*, which we understand, after David Moshman [49], to be defined as follows:

- **Inference**—going beyond the data—is elementary and ubiquitous.
- **Thinking** is the deliberate application and coordination of one's inferences to serve one's purposes.
- **Reasoning** is epistemically self-constrained thinking [...] with the intent of conforming to what [are deemed] to be appropriate inferential norms.

We assert that, understood as a computational challenge, mainstream mathematics lies somewhere in between board games and story comprehension. It deals with rules (axioms) and strategies (as described, e.g., by Pólya [55]), but it also

[8] Carnap defined the *formal mode of speech* similarly: "A theory, a rule, a definition, or the like is to be called formal when no reference is made in it either to the meaning of the symbols (for examples, the words) or to the sense of the expressions (e.g. the sentences), but simply and solely to the kinds and order of the symbols from which the expressions are constructed" [10, p. 1].

[9] Carnap opposes his "formal mode" (see Note 8, above) to what he calls the *material mode of speech*. This conception is related to but distinct from what we have termed the expository register. Carnap's material mode includes "those logical sentences which assert something about the *meaning*, *content*, or *sense* of sentences or linguistic expressions of any domain" [10, p. 285]. He points out that the material mode of speech is "transposed" insofar as "in order to assert something about an object $a$, something corresponding is asserted about an object $b$ which stands in a certain relation to the object $a$" (p. 308). In particular, "in order to say something about a word (or a sentence) we say instead something parallel about the object designated by the word (or the fact described by the sentence, respectively)" (p. 309).

[10] Lamport [40, p. 20] quotes a referee who had read one of his structured proofs (see Section 3.1.3, below): "The proofs ... are lengthy, and are presented in a style which I find very tedious. I think the readers ... are going to be more interested in understanding the techniques and how they can apply them, than they will be in reading the formal proofs. A problem with the proofs is that they do not clearly distinguish the trivial manipulations from the nontrivial or surprising parts. ... My feeling is that informal proof sketches ... to explain the crucial ideas in each result would be more appropriate."

**Table 2.** Comparison between computation in different domains

| *Level* | **Blocks World** | **Board Games** | **Story Comprehension** |
|---|---|---|---|
| elements | blocks on a table | game pieces on board | episodes from everyday life |
| inference | follow instructions | rules & strategy | analogy |
| thinking | consistency | prediction of winning | costs and benefits |
| reasoning | (trivial) | multiple strategies | ethical dilemmas |

must make sense of informal descriptions, loose parallels, and multiple points of view.

Since a city is not a tree [2], tree-shaped guides – like information scientists' ontologies, or other taxonomic descriptions – will miss its interacting sub-systems. Similarly, we claim that a formal proof will miss out the ontogenetic aspects of proving (cf. [5, p. 219]).

In the following section we draw on several strands of related work to formulate a novel approach to representation of and reasoning about mathematics.

## 3 Survey of related work

In the theoretical side of our work we will draw on two primary frameworks: *conceptual dependence theory*, which originated in early work in story understanding by Schank and his coauthors [61], and *inference anchoring theory*, which is a relatively recent development in the field of argumentation [6]. In what follows we contextualise our use of these two theoretical models, but do not attempt a full survey of story understanding and argumentation theory, both of which are subjects of considerable ongoing attention.[11]

The practical aspects of our work are rooted in a graphical style of reasoning. In broad terms we reason about triples, rather like those found in the Semantic Web's Resource Description Framework (RDF). Triples stand for labelled relations between objects, as in a classical semantic network. Because we need to reason about both the contents and structure of graphs in detail, relations are themselves treated as first-class objects: that is, in Semantic Web terminology, the triples are "reified" [47, §4.3]. We have experimented with different strategies for representing this data: for instance, one earlier prototype used a MySQL database that replicated simple triple store features, and we have experimented with more complex storage mechanisms. In the work presented here, we opted to use a representation strategy more closely coupled with LISP. We are broadly inspired by the genre of *annotative programming*, in which programs take the form of hypertext and dependencies are indicated by links, as pioneered in the Flare Programming Language[12] and Nelson's ZigZag [50]. In a case of parallel invention, another platform called AtomSpace has been created based on similar design decisions [23, 24]. Both AtomSpace and our system, Arxana (described in Section 6) treat relations as first-order objects, support the creation of programming structures in their graphs, and generally prefer expressibility to computational efficiency. AtomSpace directly incorporates a range of advanced features, such as attentional weighting and relations between an arbitrary number of nodes, which are not present in Arxana, but which could be simulated within our simpler "first order" relational system. AtomSpace seems a considerably more mature piece of software, e.g., it comes with both Scheme and Python bindings and a range of optimisations, and it has been used in interesting applications [43]: however, its greater complexity makes it less immediately useful as an illustrative tool for prototyping. Some of its features may represent future development targets for our system.

Having presented an overview of the landscape within which our work sits, the following subsections continue the survey, focusing in on several key landmarks that will guide our work later in the paper.

### 3.1 Models of mathematical reasoning

Bundy has argued that finding the right representation is the key to successful reasoning [8, p. 16]. Since we cannot simply build on formal theorem proving for our current purposes, we need to look further afield for foundations. We are concerned in particular with building representations *of* reasoning [12]. This helps guide our search for useful paradigms. McCarthy [48] posited several high-level features of a hypothetical "language of thought", highlighting in particular the ways in which such a language would not be like spoken language. We quote two key points:

- Much mental information is represented in parallel and is processed in parallel.
- Pointers to processes while they are operating may be important elements of its sentences.

Diagrammatic languages are in general well aligned with the first point. Most modern programming languages have a notion of concurrency, addressing the second. There have been several efforts to build "graphical" programming languages (e.g., [1]), but they are not directly relevant to our application. Here, we will review the features of several representation languages that we can build upon.

[11] Briefly, computational research on stories continues in many venues; one recent paper with some similarity to Schank's work is [57]. The intersections of argumentation with computer science have been explored, for example, in the biennial Computational Models of Argument conference since 2006.
[12] http://flarelang.sourceforge.net/

#### 3.1.1 Inference Anchoring Theory + Content

Our modelling approach builds on *argumentation theory*; in particular Inference Anchoring Theory (IAT), which was devised by Budzynska et al [6, 7] to model the pro-and-contra logical structure of dialogical arguments. IAT has since been adapted to model mathematical arguments with an extension we call "IAT+Content" (IATC) [12, 58]. The primary features of this adaptation are: (1) to introduce explicit relationships between pieces of mathematical content; and (2) to describe a range of intermediate relations that model the way these objects fit together in discourse. These intermediate relations include inferential structure, judgements of validity or usefulness, and reasoning tactics. Whereas, as its name suggests, IAT anchors logical inferences in dialog norms, IATC simultaneously connects logical inferences to contentful purposes and constraints (see Moshman's definitions of *thinking* and *reasoning*, quoted above).

#### 3.1.2 Conceptual Dependence

Conceptual Dependence was initially introduced as a tool for understanding natural language [60]. However, it is also used to represent knowledge about actions [45, 64]. An example shown in Figure 1 shows how "John sold Mary a book" can be represented as a combination of smaller actions.

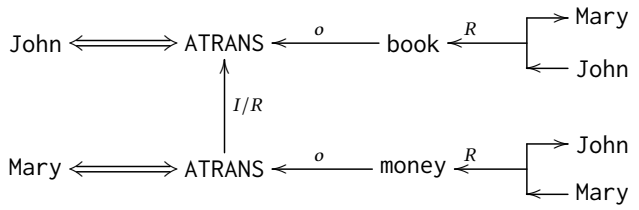



**Figure 1.** CD example (adapted from Figure 3 in [45])

CD has been used to reason about a range of informally-constructed scenarios. For instance, the system might be presented with a description of a restaurant in the form of a script, and a CD representation of a story such as:

> John went to a restaurant. He sat down. He got mad. He left.

Schank and Abelson's Script Applier Mechanism (SAM) could produce a paraphrase and make inferences such as concluding that John was upset because the waiter did not come [4, pp. 306 et seq.]. Furthermore, by augmenting the knowledge base with goals and plans, it became possible to explain actions. For instance:

> Willa was hungry. She picked up the Michelin guide.

Here one could explain why she picked up the guide as follows: The first sentence suggests the goal of finding food, and the restaurant script describes a way of satisfying this goal – but applying the script requires the user to know the location of a restaurant, hence the guide.

#### 3.1.3 Structured proofs

Lamport's notion of a "structured proof" [38] is an interesting example of a "missing link" between informal and formal mathematics. In essence, a structured proof combines a high-level summary (or "proof sketch") with a mode of writing out details in such a way that each step is made clear and each assumption and assertion is clearly labelled. Structured proofs are trees, and, as such, are directed to the formal register. As first introduced, they made use of just a few keywords: `assume`, `prove`, `let`, and `case`. In a more recent reassessment [40], Lamport states "The way I now write proofs has profited by my recent experience designing and using a formal language for machine-checked proofs." In line with this, additional keywords `new`, `suffices`, and `pick` are imported from his "Temporal Logic of Actions+" (TLA$^+$) language [39], along with a shorthand for equational proofs.[13] At least these basic keywords, or equivalents, should be supported in the lower, content-addressing, layers of our language.

#### 3.1.4 Lakatos Games

Arguments are similar to games, insofar as both typically have winners and losers: this is made precise with the notion of a *dialogue game* [9, 46].

Lakatos [37] developed an abstract, dialectical, model of mathematical creativity which has recently been formalized as one such dialogue game [52, 53]. In the "Lakatos Game", assertions are made and then progressively refined through discussion and debate. For instance, a statement may be sharpened so that it applies to a restricted class of objects ("Strategic Withdrawal"), or a putative counterexample may be shown not to be a counterexample at all ("Monster Barring"). Lakatos's theory is limited, first, in its theoretical commitments, insofar as Lakatos's mathematics is closely linked to the Hegelian notion of thesis/antithesis/synthesis [42]. Secondly, even though Lakatos's theory is now implemented computationally (per [53]), the model leaves out concrete details of mathematical objects and relationships among them. The trace of a Lakatos Game does include content – but the relationships between elements are mediated by a controlled vocabulary of 20 dialog moves, whose sequence is structured into a game tree.[14] We would like to broaden our understanding to include, for example, dialogs that do not follow strict

[13] More recently, a second version of this language called TLA$^{+2}$ has been made available [41]. In full, TLA$^{+2}$'s keywords are: `action`, `assumption`, `axiom`, `by`, `corollary`, `def`, `define`, `defs`, `have`, `hide`, `lambda`, `lemma`, `new`, `obvious`, `omitted`, `only`, `pick`, `proof`, `proposition`, `prove`, `qed`, `recursive`, `state`, `suffices`, `take`, `temporal`, `use`, `witness`.

[14] Part of an example from [53, §A.1]: *Conjecture*('statement holds in the convex polygon plus point case') →*GlobalCounter*('equilateral triangle plus point', 'statement holds in the convex polygon case') →*MonsterBar*('equilateral triangle plus point', 'statement holds in the convex polygon case', 'line is alternating') →*PDefinition*('equilateral triangle plus point', 'line is alternating', '"line" extends in both directions') →*MonsterAccept*('equilateral triangle plus point', 'line is alternating').

dialectical outlines, as well as broader informal patterns of reasoning.

### 3.2 The search for the 'quantum of progress'

The Polymath project aimed to answer the question "is massively collaborative mathematics possible?" This project was convened by Timothy Gowers in 2009, and he developed 15 rules to address the "questions of procedure" that he anticipated would arise in the project.[15] The basic premise was that people would work together on a shared blog to discuss a given mathematical problem in the open, and jointly work out a solution. Within these ground rules, Gowers carefully explicated the imperative to share comments that are "not fully thought out." good contribution to the project would comprise what he called a "quantum of progress." Continuing with Gowers's physics metaphor, it should be clear that we need to better understand the "fields" that generate these quanta! That is, in order to build a functional representation of mathematical reasoning, we need to model both what's there at any given step, and also how it evolves – noting that such evolution is generally heuristic. To date, twelve Polymath projects and four MiniPolymath projects (focused on collective solutions to pre-college level competition problems) have been convened.[16] For researchers studying mathematical practice, these provide a useful source of data.[17]

To add to the body of thought on "proof and progress in mathematics" [68], it is interesting to compare Polymath with Gowers's joint work with Ganesalingam on a "human-oriented theorem proving" system that they call ROBOTONE [20]. Work on this system motivated Gowers's public presentation of a mathematical challenge problem that we will take up as our central example in Section 4. However, Ganesalingam and Gowers's system was only able to solve more straightforward "textbook" problems, in which the next step is always more or less "obvious." ROBOTONE has a number of heuristics, ranked by attractiveness, that it uses to transform its goal step-by-step in a logically-sound manner. ROBOTONE's progress is distinctly mechanical. The novel and interesting aspect of the system is that it generates proofs in natural language and also makes its internal processing explicit – so it straightforward to diagram its operations in IATC.[18] (However, we should emphasize that ROBOTONE's inferences are not "driven" by natural language as its main motivation or mediator.)

Sussman gives an example of a LISP program that models "'how to' as well as 'what is'" [66] – namely, a program that he wrote with with Richard Stallman that can replicate the process of analyzing a circuit [67]. The program explains *why* it comes up with the answers it does. The "why question" is even more important (and interesting) when nontrivial "creativity" is involved in the reasoning process [25]. Mathematical creativity may prove to be both possible and interesting to model, and it may be possible to devise computational systems that offer thought-provoking answers to the "why question." However, in the case of Ganesalingam and Gowers's ROBOTONE, described above, the answer to the "why question" will always be more or less the same [20, pp. 261–261]:

> the program can [...] be regarded as repeatedly applying a single tactic, which is itself constructed by taking a list of subsidiary tactics and applying the first that can be applied.

This is unlikely to work outside of a limited domain of "textbook" problems – for example, it would stumble over Polymath-style dialogues, or even "challenge problems" from outside the textbook genre – such as the example we describe in the following section.

## 4 Example: a diagrammatic model of mathematical reasoning via IATC

An informal proof is in general not a tree, even if a structured proof – or the trace of moves made in a dialogue game – does rather satisfactorily *describe* the proof as a tree. In the limit of full formality, tree-shaped descriptions are machine checkable – and by the Curry-Howard correspondence, proofs-as-programs are even runnable. However, by default these objects give very little information about the proof's (or program's) genesis. Indeed, as we shall see, we soon run into problems even with a still-more-general network model. Figure 2 presents an IATC analysis of Gowers's walkthrough solution to the problem "What is the 500th digit of $(\sqrt{2}+\sqrt{3})^{2012}$?" In IATC, utterances expand to *performatives* (here, only `Assert` and `Suggest` are used).[19] The performatives, in turn, expand to nodes that convey inferential structure, reasoning tactics, and heuristic judgements (e.g., `implies`, `strategy`, and `generalize`, respectively) – as well as content-level relations (e.g., "`contains as summand`"). Intermediate nodes typically have targets in the content layer. For example, the assertion that "'$(\sqrt{3}-\sqrt{2})^{2012}$' `has property` 'is small'" is typical in this regard.

The `implements` node in Figure 2, and the `implies` node somewhat lower down, both with bold outbound arrows pose some difficulty – insofar as these outbound arrows do not have another node as a target. What the `implements` node is saying is that the subgraph it is pointing at implements a heuristic that was proposed earlier, namely "The trick might be: it is close to something we can compute." Without such

[15] http://gowers.wordpress.com/questions-of-procedure/

[16] http://michaelnielsen.org/polymath1/index.php?title=Main_Page

[17] E.g., Footnote 14 draws the data that the Lakatosian framework models from one of the MiniPolymath dialogues, and Rino Nesin tagged two MiniPolymath dialogues into IATC in detail as an initial check on the completeness of the language as it was being developed.

[18] http://metameso.org/ar/robotone-example.pdf.

[19] Austin [3] characterises performative utterances as follows: (a) they do not 'describe' and accordingly have no truth value; and (b) the utterance itself performs an action.

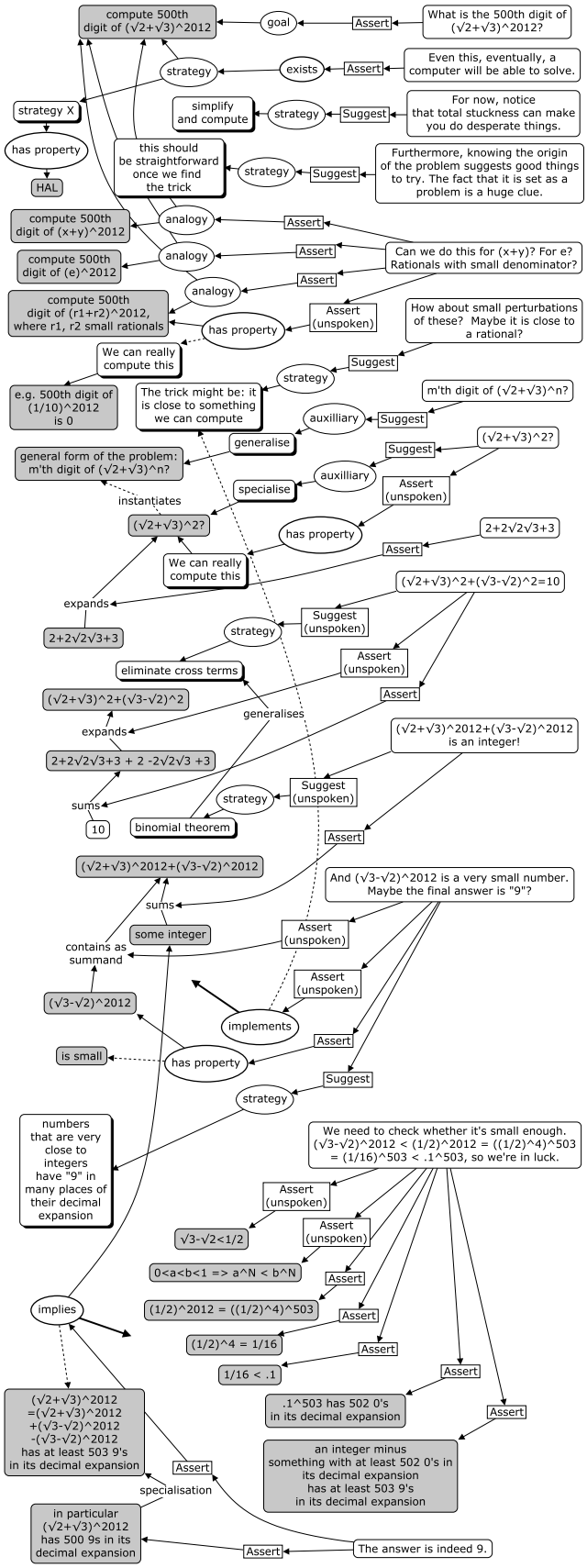



**Figure 2.** A mathematical challenge problem

certified implementations, we would see a proliferation of heuristic suggestions, without any sense of which ones are actually used later on. Again, this assertion is made about a subgraph, and not a specific piece of content. Recall McCarthy's assertion about the language of thought: "Pointers to processes while they are operating may be important elements of its sentences." Whereas a structured proof would just give the indicated subgraph a name and push it down the hierarchy into a lemma, we can see from the diagram that such a representation strategy has limitations. The inbound link to "some integer" (from the subsequent `implies` reasoning step) could be represented formally as a reference to the lemma's conclusion, but the link to an heuristic introduced much earlier on is entirely informal. The example here is simple, but reasoning about graph structure has the potential to become highly complex. To cope with these complications, we need *hypergraphs*, not just graphs. Specifically, we will rely on one of the standard generalizations of the hypergraph concept: the ability for an edge to point to another edge.[20] The basic reason, again, is that we need a way to lasso the embedded "lemma" as an object of interest, and point to it, without wholly separating it off from the rest of the proof (see Section 6).

## 5 Towards functional models of mathematical reasoning

By combining ideas from IATC (for representation) and CD (for reasoning) it should be possible to have the system look at the data in Figure 2 and answer questions like:

- "Why are we interested in integers at this point?"
- "Why is 9 seen as a likely answer once we know that $(\sqrt{3}-\sqrt{2})^{2012}$ is small?"

The premise here is that the dynamics at work are based on performatives being chosen based upon current valuations, goals, and other contextual features. We expand this idea in the following subsections.

### 5.1 Kinematics (IATC+CD)

The basis of CD theory is a classification of actions into broad classes called *primitives*, each of which are equipped with a number of places for arguments (referred to as *slots*) which hold objects that fulfil roles specific to that class of actions. When a primitive's slots have been filled in, the result takes the form of the graph.

We emphasize that the primitives of CD were designed to represent everyday activities in the physical world and hence might not be the best fit for mathematical texts. Therefore, we will instead use the IATC performatives as our action primitives. Like the standard CD primitives, these performatives have slots. The statements that are introduced via these performatives also have slots, e.g., '`has_property` (*o*,

[20] https://www.slideshare.net/delaray/knowledge-extraction-presentation: "Mom resents the fact that John disapproves of Jane and Jim's marriage."

*p*)' asserts that object *o* has property *p*. The specification of IATC includes 9 performatives and 15 intermediate relations [58]; a subset of the specification is given in Appendix A.

Assertions that are fully unfolded generally bottom out in the content layer: assertions can be made directly about objects and propositions, or about relationships between the same, as in Figure 2's "`contains as summand`" relation; content can be addressed in other intermediate assertions, such as "$(\sqrt{3}-\sqrt{2})^{2012}$" `has_property` "is small". The performatives rooted on the utterance

> And $(\sqrt{3}-\sqrt{2})^{2012}$ is a very small number. Maybe the final answer is "9"?

are represented as s-expressions in Listing 1. (We defer further comment on the somewhat technical matter of pointing at a specific subgraph to Section 6, but highlight here that that from the point of view of IATC, it just means that we can substitute "subgraph" for "statement" in slots that ask for a statement.)

```
(Assert
 "contains as summand"
 "(sqrt(2)+sqrt(3))^2012+(sqrt(3)-sqrt(2))^2012"
 "(sqrt(3)-sqrt(2))^2012")

(Assert (has_property "(sqrt(3)-sqrt(2))^2012"
                      "is small"))

(Assert (implements #SUBGRAPH
                    "the trick might be: it
                     is close to something
                     we can compute"))

(Suggest (strategy "numbers that are very close
                    to integers have \"9\"
                    in many places of their
                    decimal expansion"))
```

**Listing 1.** Assertions from Figure 2 modelled with CD-like code, using IATC performatives as our primitives

### 5.2 Dynamics

Static graphical representations of mathematical reasoning do not by themselves offer *functional* models of mathematical reasoning. We need to be able to reason about structure: answering "why" questions, as above, or introducing new reasoned statements into a dialogue. Here we are inspired by Sowa and Majumdar's treatment of reasoning by analogy [65]. For example, early in the solution to the challenge problem from Figure 2, Gowers asks:

> Can we do this for $x + y$? For $e$? Rationals with small denominator?

Inspecting the context, we see that the anaphor "this" refers to *computing the 500th digit of X*. As we remarked in Section 2, the quoted sequence of questions embodies an inductive search for a suitable generalization of the question; specifically, this search arrives at "the $m$th digit of $(\sqrt{2}+\sqrt{3})^n$". Sowa and Majumdar [65, §2] show that inductive reasoning and analogical reasoning can achieve similar results. Using a typed formalism we could be specific about the difference between the two sides, while retaining a structural similarity on the two sides of the analogy.

Given sufficiently detailed representations – sometimes requiring background theories – we expect analogical reasoning to serve as the main explanatory mechanism. Sowa and Majumdar describe three ways to do analogical reasoning about conceptual graphs. There is in fact a (high-order) analogy between what goes on in building representations via parsing and reasoning about the representations so-described. Lytinen [45] summarises Schank and Birnbaum [59] on semantic parsing, again, referring to three aspects of the parser: Making this analogy more precise, Sowa and Majumdar refer to a "graph grammar"; this topic is developed in more detail by Ehrig et al [18]. We will describe an example relevant to our setting in the following section.

Originally, the various systems related to CD were implemented in different software packages such as SAM, PAM, and VAE [4, 65]. Rather then adapt these packages and interface them, we will take a different approach: computing with hypergraphs.

## 6 Our prototype, *Arxana*

### 6.1 System overview

In order to store graphical representations of knowledge in memory and query them, we will make use of a system called *Arxana* which we have been developing in LISP [13]. The basis of this system is higher-order nested semantic networks. By "higher order," it is meant that links can point to other links. By "nested," it is meant that the nodes which make up a network may contain other networks, which themselves may have nodes containing yet other networks, etc. In addition, all the links are bidirectional and no fundamental distinction is made between links and nodes. We call the basic unit from which we will construct our networks a *nema*. Nemas (or nemata) are data objects which serve to encode both links and nodes and are characterized by the following components: **Identifier**, **Source**, **Sink**, and **Content**. Source and sink are pointers to other articles and content is a place in which to store a LISP object, which might be some text, or an expression, or maybe a number or maybe something else. In particular, the content of an article can even be another network constructed out of more articles. By choosing suitable conventions, one can build up more complex functionality from the basic features. For instance, one can introduce typing by designating a certain nema as the "home nema" for that type and interpreting a link whose source is the home

nema and whose sink is some other nema as stating that the sink nema is of that type.

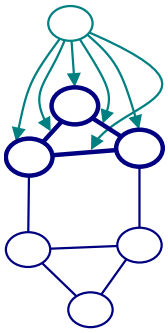


**Figure 3.** Cone over a subgraph

Likewise, as shown in Figure 3, one can encode subhypergraphs by designating a specific nema as a label, and then indicating the components of the hypergraph of interest by links from that label. Here, the lighter-colored nema at the top of the diagram together with the arrows emanating from it point out a bold-faced triangle comprised of three nodes and the lines between them. We call this structure a *cone*.[21] We could use a similar construction to make the `implements` relation discussed in Section 4 explicit, namely by pointing to all of the constituent elements of the relevant subgraph.

In order to make these representations above actionable, we use two main facilities: *hypergraph matching* and *hypergraph programming*. Given a hypergraph, we can query it for subhypergraphs which match some suitable criterion. A query consists of a hypergraph whose articles contain predicates and a match consists of a mapping of hypergraphs from the query onto a subhypergraph such that, for every article in the query, the predicate it contains is true of the contents of the corresponding database article.

Like a cons cell, a nema comes with two links, *source* and *sink* (analogous to car and cdr) which point to other nemata. However, we extend the model in two ways: firstly, each nema also contains an additional place, called *content*, in which one can store an arbitrary LISP object which could be a text string, a program, or even a network. By embedding programs inside the hypergraph, reasoning steps can be stored close to where they are used. In particular, we can store hypergraph grammar rules that are used to transform graphs (e.g., on a link).

### 6.2 Application to mathematical reasoning

Relative to our current purposes, the first key function of Arxana is to make graphical representations like the ones in Figure 2 accessible to the computer.

Here, we focus in on expanding and making sense of the s-expressions from Listing 1. Note that the utterances recorded in Figure 2 do not contain everything needed for this task. In particular, some of the expansions that shown in the figure are noted to be "unspoken". Background information about mathematics needs to be programmed into a knowledge base, i.e., the system needs a range of facts, so that, for example, "is small" is known to be synonymous to "close to zero". Relatedly, the lengthy strings embedded in Listing 1 need to be parsed further – for example "the trick might be: it is close to something we can compute" breaks down to two content-level statements: "it is close to something" and "something is computable", along with a meta-level bid to find suitable objects with the corresponding properties.

When we expand the "`implements`" and "`#SUBGRAPH`" we obtain a node whose content is a proof certificate and a cone below that node, respectively, as illustrated in Figure 2 and described above. What's missing from Figure 2, however, is a link between this certificate and the new strategy the same set of performatives `Suggests`. A detailed IATC+CD analysis reveals that this missing link, $F$, emerges from the reasoning process itself. We demonstrate one part of this derivation below, leaving out a few steps related to the first s-exp from Listing 1, which a computer algebra system could quickly make sense of. The result of this process is a second certificate that is provided by the last lambda expression in Listing 2 and verifies the reasoning behind the suggested strategy. The corresponding cone (illustrated in Figure 4) is constructed from eight links labelled $A, B, C, D, E, F, \alpha, \beta, \gamma$. The links $A, B, C, D, E, F$ point to semantic subnetworks whose serializations are given at the top of Listing 2 and the links $\alpha, \beta, \gamma$ point to lambda expressions which implement inference rules. These lambda expressions are of the form `(lambda (plex) (replace (car (search LHS plex)) RHS))` where `LHS` and `RHS` are subexpressions specific to the inference rule and are given in the middle of Listing 2.

Verification proceeds by using the Arxana facility for running embedded programs. When it runs, the certificate first gathers the various premises and conclusions from the cone. Starting with the premise $C$, it apples the rule "`weaken-equiv`" which it finds by following link $\gamma$ of the cone. Implementing this rule is accomplished by hypergraph search and replacement. It then proceeds similarly with the remaining hypotheses, using inference rules to extend the chain of reasoning and finishes by checking whether the final result agrees with the conclusion pointed to by link $F$.

At the higher level, one would invoke an inference engine such as mini-kanren (from *The Reasoned Schemer* [19]) to unify the conclusion from the hypotheses. Analogous questions about restaurants were answered by Schank et al.'s Lisp programs, however their work predates Prolog: nevertheless, CD inferencing and parsing would port over to a logic language naturally.

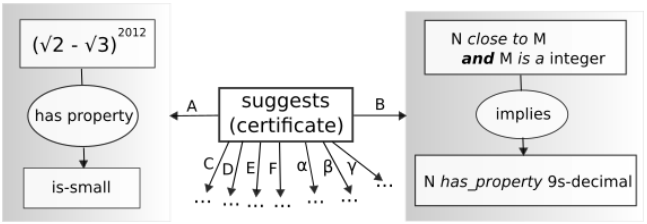



**Figure 4.** The reasoning itself suggests a key strategy

[21] From https://ncatlab.org/nlab/show/cone: "In homotopy theory, the cone of a space $X$ is the space got by taking the $X$-shaped cylinder $X \times I$, where $I$ may be an interval object, and squashing one end down to a point."

```
;; assertions
(A (has_property "(sqrt(3)-sqrt(2))^2012" is-small))
(B (implies (and (close-to N M)
                 (isa M integer))
            (has_property N 9s-decimal)))

;;; existing knowledge base
(C (equivalent-to (N has_property is-small)
                  (N close-to 0)))
(D (isa 0 integer))
(E (implies (N isa integer)
            (N has_property computable)))

;;; conclusion
(F (viable-strategy "(sqrt(3)+sqrt(2))^2012"
                    (has_property N 9s-decimal)))

;;; inference rules
(simplify-has_property+implies
 :lhs (((y) (a has_property) (x) (b implies) (z))
       ((x src y) (y flk a) (x snk z) (x flk b)))
 :rhs (((y) (a has_property) (z))
       ((z src y) (y flk a))))
(transitivity
 :lhs (((y) (a implies) (x) (b implies) (z))
       ((x src y) (y flk a) (x snk z) (b flk z)))
 :rhs (((y) (a implies) (z))
       ((z src y) (y flk a))))
(weaken-equiv
 :lhs (((b equivalent-to)))
 :rhs (((b implies))))

;;; computing the certificate
(lambda (V)
  (apply
   (lambda (F A B C D)
     (equal
      (simplify-has_property+implies
       (graph-union
        B
        (transitivity
         (graph-union
          D
          (simplify-has_property+implies
           (graph-union
            A
            (weaken-equiv C)))))))
      F))
   (mapcar
    (lambda (X)
      (get-sink
       (car
        (triples-given-beginning-and-middle V X)))
      '(A B C D)))))
```

**Listing 2.** Key elements for reasoning about Listing 1

## 7 Conclusions and Future Work

The formal register of mathematical discourse has been studied by de Bruijn, Ranta, Ganesalingam, and others. Once we have formal representations, it is possible to reason effectively about them using tools from Frege and others. In this paper we have focused, instead, on a computational theory of the expository register. We have drawn upon both classic and recent AI research which has considered situations with which everyday mathematics shares common features.

Quite to the contrary of the sentiment conveyed by the misattributed quote "Mathematics is a game played according to certain simple rules with meaningless marks on paper," David Hilbert, an early and important proponent of mathematical formalism, emphasized the role of intuition and experience, and discussed conjectural thinking and the fallibility of mathematical reasoning [14, 28].[22] Our work continues roughly in this spirit: but rather than referring intuition to logical and mathematical foundations, we have outlined a novel, if preliminary, operationalization of flexiformal representation and reasoning. Specific items of future work will re-integrate the themes we set aside in the introduction to this paper. They include:

***Formal proof***

- Demo the system walking through the steps of a proof like the challenge problem we discussed, or a MiniPolymath dialogue, which would necessitate refining both representations and reasoning.
- Integrate external SMT solvers, proof checkers, proof assistants, and computer algebra systems (e.g., '$X$ `sums` "some integer"' and '$X$ `contains as summand` $Y$' could be treated by different external systems).

***Embodiment and cognitive science***

- Build on CD theory to reason about embodied intuitions in geometric problems, integrate with Lakoff and Núñez's conceptual metaphors [51]; connect this with formal approaches in geometry.
- Relate reasoning about hypergraphs to work by Silvia de Toffoli [17], Mateja Jamnik [29], and other contemporary scholars working on reasoning with diagrams.

***Linguistics and NLP***

- Integrate parsers to generate IATC+CD automatically, initially via intermediate semantic markup [35].
- The IATC specification (presented in part above) is to be understood as preliminary, and the language should be expanded and improved based on corpus studies.
- Use statistical methods on the relevant corpora, expanding on the work of Kaliszyk et al [32] who ascertained the frequency of various schematic usages like "let $X$ be a $Y$" in a specific corpus of proofs.

[22] https://www.cs.nyu.edu/pipermail/fom/2005-April/008889.html.

***Machine learning***

- Integrate with knowledge bases of mathematical terms and frequency data (as above).
- Model Stack Exchange dialogues, in parallel with the work done on Reddit discussions [71].
- Build a system with multiple agents that "converse with each other to sharpen their wits" [69].

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Auxilliary material, including computer programs, data sets, and full versions of diagrams quoted above are available via this URL: http://metameso.org/ar/.

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## A Partial specification of IATC

| | |
|---|---|
| `Assert` (*s* [, *a* ]) | Assert belief that statement *s* is true, optionally because of *a*. |
| `Agree` (*s* [, *a* ]) | Agree with a previous statement *s*, optionally because of *a*. |
| `Challenge` (*s* [, *a* ]) | Assert belief that statement *s* is false, optionally because of *a*. |
| `Retract` (*s* [, *a* ]) | Retract a previous statement *s*, optionally because of *a*. |
| `Define` (*o*, *p*) | Define object *o* via property *p*. |
| `Suggest` (*s*) | Suggest a strategy *s*. |
| `Judge` (*s*) | Apply a heuristic value judgement *s* to some statement. |
| `Query` (*s*) | Ask for the truth value of statement *s*. |
| `QueryE` ($\{p_i(X)\}$ . *i*) | Ask for the class of objects $X$ for which all of the properties $p_i$ hold. |
| `has_property` (*o*, *p*) | Object *o* has property *p*. |
| `strategy` (*m*, *s*) | Method *m* may be used to prove statement *s*. |
| `beautiful` (*s*) | Statement *s* is beautiful. |

# Towards mathematical AI via a model of the content and process of mathematical question and answer dialogues

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**Abstract.** This paper outlines a strategy for building semantically meaningful representations and carrying out effective reasoning in technical knowledge domains such as mathematics. Our central assertion is that the semi-structured Q&A format, as used on the popular Stack Exchange network of websites, exposes domain knowledge in a form that is already reasonably close to the structured knowledge formats that computers can reason about. The knowledge in question is not only facts – but discursive, dialectical, argument for purposes of proof and pedagogy. We therefore assert that modelling the Q&A process computationally provides a route to domain understanding that is compatible with the day-to-day practices of mathematicians and students. This position is supported by a small case study that analyses one question from Mathoverflow in detail, using concepts from argumentation theory. A programme of future work, including a rigorous evaluation strategy, is then advanced.

**Keywords:** Q&A, argumentation, mathematics

## 1 Introduction

In this paper, we outline a computational approach to modelling mathematical dialogues, and show how it can be used to model *Q&A dialogues* in particular. We argue that a strongly empirical approach – along these lines – can support the development of robust, knowledge-rich, mathematical artificial intelligence. Mathematical dialogues convey the processes through which new mathematics is created and existing mathematics is taught. We claim that mathematical question and answer (Q&A) dialogues are a practically and theoretically important subclass. In particular, there is now a large corpus of mathematical questions, answers, and accompanying discussion available in online Q&A forums.

The Q&A forums that we consider here are "social machines" – defined by Tim Berners-Lee to be a class of systems "in which the people do the creative work and the machine does the administration" [4]. Question-and-answer websites differ from other popular social machine formats, like general purpose forums, wikis, and mailing lists – with which they nevertheless share some features – in the relatively explicit semantics that they support (and require).

This has to do with both content and form. The Stack Exchange network of Q&A sites has a network-wide norm of focusing on questions whose answers are not primarily opinion-based: in other words, questions which have answers that can be considered "right" or "wrong," or that can otherwise be compared with each other in objective (as opposed to purely subjective) terms. Stack Exchange sites treat a wide range of technical and non-technical subjects, ranging from computer programming, to travel, to advice on academic careers, the internal logic of science fiction universes, and beyond. It has two specialist sites devoted to mathematics: Mathoverflow, for research-level Q&A (often dealing with new, open, conjectures, and for which background approximately equivalent to a strong mathematics degree is a minimum barrier to entry), and math.stackexchange.com, which focuses on non-research mathematics (e.g., at school, university, or postgraduate level). Whereas a mailing list, for example, would permit more open-ended discussions, the questions discussed on these websites tend to have right and wrong answers, and the discussion focuses on exposition of the correct answer (or answers).

Technical Q&A often embody knowledge about "how to" as well as "what is." In this regard, Q&A is similar to computer programming [38] – and it is no coincidence that the most popular site on the Stack Exchange subject is devoted to programming concepts. As a commentary on the medium and its affordances, it is useful to note that, by number of questions, the math.stackexchange.com website is the second-most popular site in the Stack Exchange network.[4]

Mathematical Q&A has some interesting things in common with other kinds of mathematical discussions, such as the discussions that take place among professional mathematicians working on a paper together, or among novice problem-solvers leading to personally-new insights. Both these types of dialogues have been studied extensively – in some cases using online discussions as a ready source of data [3, 31, 35, 37].

Proof dialogues often contain Q&A sub-dialogues: for example, a discussant may state "I'm sorry, I don't understand what you did in this part of the proof" or "I don't understand why this works, but it seems to." Similarly, Q&A dialogues contain elements of *mathematical argumentation*, typically sufficient to convince the querent and subsequent readers. Following Walton [42], Aberdein has outlined a range of purposes that "proof" may serve: inquiry, persuasion, information seeking, deliberation, negotiation, and debate [1]. Q&A dialogues seem to serve many similar purposes. Martin and Pease [26] developed an empirically-founded typology of questions on Mathoverflow (expanding on Mendes and Milic-Frayling's earlier study [28]). The three popular types of questions observed were "Is it true that…?", "What is this?", and "Could I have an example please?" (we summarise this earlier work in more detail below).

We support our case for a strongly empirical approach to mathematical AI by building a proof-of-concept model of a Q&A dialogue, using a technique from argumentation theory [5] with suitable adaptations for the mathematics domain. We have stated the motivations for this approach elsewhere [27], and demon-

[4] `http://stackexchange.com/sites#questions`

strated its overall salience [32]. In brief, we show in [32] that it is possible to build meaningful computational models of proof dialogue. However, this earlier work dealt with the high-level logic of argument structure. Here we demonstrate extensions that provide a much more detailed model. We will summarise the relevant background below, and in our small proof-of-concept case study, introduce enough formalism to walk through an example in detail. The outline of the paper is as follows.

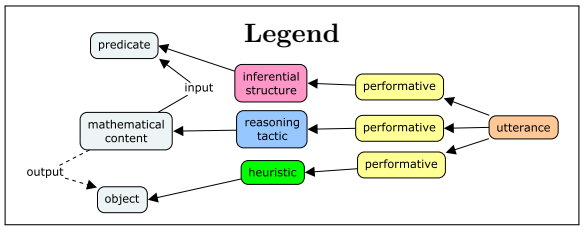
- We present a brief overview of relevant background in Section 2.
- Section 3 develops the main case study.
- Section 4 offers discussion and an outline of future work.

## 2 Background

### 2.1 Overview

The approach we take in this paper is grounded in a part of argumentation research called *Inference Anchoring Theory* (IAT), which was designed to model the inferential structure of dialogues, by connecting statements with their logical import [5]. In a recent paper we have described a broadly IAT-inspired theory that uses a constrained set of rules called a *dialogue game* to model mathematical discourse [32]. The specific set of rules were adapted from Lakatos's *Proofs and Refutations* [21]. The "Lakatos game" developed in [32] is a formalised approximation of the "informal logic" that people use when arguing about concepts on the way to a shared proof. The dialogue game shows which assertions are being used to support or to argue against a given conjecture, and it shows when assertions are in conflict with each other. This leads to a sociologically interesting, but rather atypical, idea of a proof as a developing set of mutually coherent statements that support a given conjecture, and that have yet to be successfully refuted. This conception of proof is at odds with the way mathematicians (and, especially, formal mathematicians) would describe proofs, i.e., that proofs are derivations from axioms by way of valid inference rules. Philosophers of mathematics have expressed doubt as to whether mathematical worldviews as different as these can be brought into alignment with each other in a routine way [39]. Nevertheless, mathematics has a logical structure, embodied in its theories and objects, which is typically not subject to debate. For instance, two numbers are either co-prime, or not. Accordingly, we have been developing a new strategy for representing mathematical discussions, in which 'The Cayley graph of group $G$', for example, is modelled as an object; the relationship between the proposition $P$ and the proposition '$P$ is difficult to prove' is made explicit; and in which Lakatos-style conjectures, refutations, and repair are modelled. We call this framework *IAT+Content* or IATC. *Inferential structure*, like `implies`, describes statements about pieces of *mathematical content*; meta-level *reasoning tactics*, like `goal` or `auxiliary`, are used to strategise proof development; *heuristics* guide the proof or manipulate content. New nodes are brought

into being in connection with IAT-style *performatives* labelled `agree`, `assert`, `challenge`, `define`, `query`, `retract`, and `suggest`. The Legend at right provides a schematic summary of the features of IATC.




While the proto-language is not complete (and not yet implemented, like the work in [32]), we hope to show that it can be used to model real-world mathematical dialogues. In the current paper, we focus on Q&A dialogues. These were summarised above; some further detail on this specific domain follows.

### 2.2 Q&A

The strategies used by Mathoverflow "contributors [to] communicate and collaborate to solve new mathematical 'micro-problems' online" have been studied previously [40], and a typology of collaborative acts was proposed: 1. *provide information*; 2. *clarify the question*; 3. *critique an answer*; 4. *revise an answer*; 5. *extend an answer*. The study focused on understanding, quantitatively, how these different activities contributed to answer quality. Some more specific activities within this framework are noted in an example,[5] e.g., 1→ referencing a related Q&A post, which IATC might model as an '`auxiliary`' problem; 2→ stating that the question is harder than the related post, which IATC would model with a (reversed) '`easy`' heuristic value judgement. This framework is useful as a high-level check on the completeness of IATC.

The difficulty of questions in math.stackexchange.com has been studied [24]. However, this was done, not primarily by examining the content of questions, but by devising a "competition-based" score that estimates a given question's difficulty using the estimated expertise of discussion participants (learned via a Bayesian model). In particular: "the expertise score of the best answerer is higher than that of the asker as well as all other answerers." Then, the difficulty of a question is estimated to be "higher than the expertise score of asker $u_a$, but lower than that of the best answerer $u_b$." These authors defer a detailed analysis of question content to future work.

**Mathoverflow.** Related work by some of us examined the production of mathematics on Mathoverflow [26]. A typology of questions was developed, as follows:

- **Conjecture 36%** — asks if a mathematical statement is true. May ask directly "Is it true that" or ask under what circumstances a statement is true. (This corresponds to the purpose of 'inquiry' noted by Walton.)
- **What is this 28%** — describes a mathematical object or phenomenon and asks what is known about it. (This also corresponds to 'inquiry'.)

[5] `http://mathoverflow.net/q/12732`

– **Example 14%** — asks for examples of a phenomenon or an object with particular properties. (This may be 'inquiry' or 'information seeking'.)
– **Formula 5%** — ask for an explicit formula or computation technique.
– **Different proof 5%** — asks if there is an alternative to a known proof. In particular, since our sample concerns the field of group theory, a number of questions concern whether a certain result can be proved without recourse to the classification of finite simple groups.
– **Reference 4%** — asks for a reference for something the questioner believes to be already documented in the literature
– **Perplexed 3%** — ask for help in understanding a phenomenon or difficulty. A typical question in this area might concern why accounts from two different sources (for example Wikipedia and a published paper) seem to contradict each other.
– **Motivation 3%** — asks for motivation or background. A typical question might ask why something is true or interesting, or has been approached historically in a particular way.
– **Other 2%** — closed by moderators as out of scope, duplicates, etc.

Answers are also examined, although in less detail. Responses typically present information known to the respondent, and readily checked by other users, but not necessarily assumed to be known by them. Some specific findings:

– **Existing research literature 56%** — over half of the questions in the sample refer to existing literature
– **Errors 37%** — many questions (and answers) contain errors; these are acknowledged politely, and corrected when pointed out.
– **Examples 34%** — the use of specific examples gives some evidence of broadly "Lakatosian" reasoning (i.e., per [21]).

## 3 Case study

In this section we will use the IATC formalism to analyse one example Q&A dialogue in detail.[6] We quote the text of this dialogue verbatim, and present the analysis graphically. We selected one of the questions from Mathoverflow that was part of the sample described above: the example was classed as an "Is it true that…?" question; examples and references are supplied. As they appear on the site, both the specific question and the top-rated answer are quite succinct. Below, we will replay the conversation in order. As we will see, most of the interesting argumentation takes place in comments.

> (Original question, 18:05) I have seen this problem, that if $G$ is a finite group and $H$ is a proper subgroup of $G$ with finite index then $G \neq \bigcup_{g \in G} ghg^{-1}$. Does this remain true for the infinite case also. [→ Figure 1]

The first follow-up comment to be submitted observes that this question doesn't quite make sense as written, and suggests a correction.

[6] `http://mathoverflow.net/q/34044`

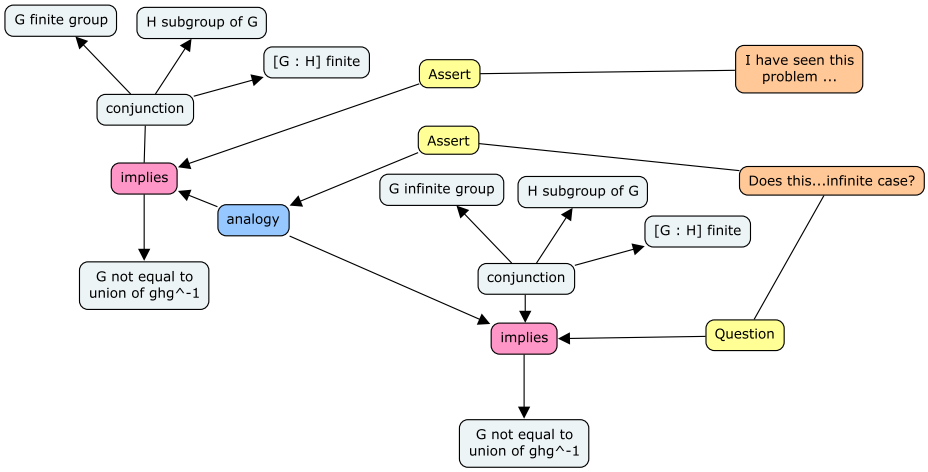



**Fig. 1.** Original question, diagrammed

> (First comment on question, 18:15) There's something I don't understand here: do you perhaps mean $gHg^{-1}$ instead of $ghg^{-1}$? [→ Figure 2]

Meanwhile, it seems an answer was already being composed, since the following text appeared on the site one minute after the above clarification.[7]

> (Answer, 18:16) Not in general. Every matrix in $\mathrm{GL}_2(\mathbf{C})$ is conjugate to an invertible upper triangular matrix (use eigenvectors), and the invertible upper triangular matrices are a proper subgroup. [→ Figure 3]

Even though an answer has been given, suggestions for fine-tuning the question continue in the comments.

> (Second comment on question, 18:20) Yes, the statement is out of focus: $gHg^{-1}$ is intended (and "infinite index case"). The natural starting point is to ask whether the proof for finite index breaks down. [→ Figure 4]

Quite a lot happens in the foregoing short comment. A change in the problem's set of hypotheses is suggested. The analogy to the known "inspiring" theorem for finite groups is deemed not particularly relevant, and its hypotheses are changed as well. This then suggests a strategy for proving the (revised) problem. (*NB.*, to save room, in the diagram that follows, we have elided some of the structure that accumulated earlier: the earlier nodes and links are still assumed.)

[7] At this point, the discussion becomes multi-threaded, since comments can now attach to the answer as well.

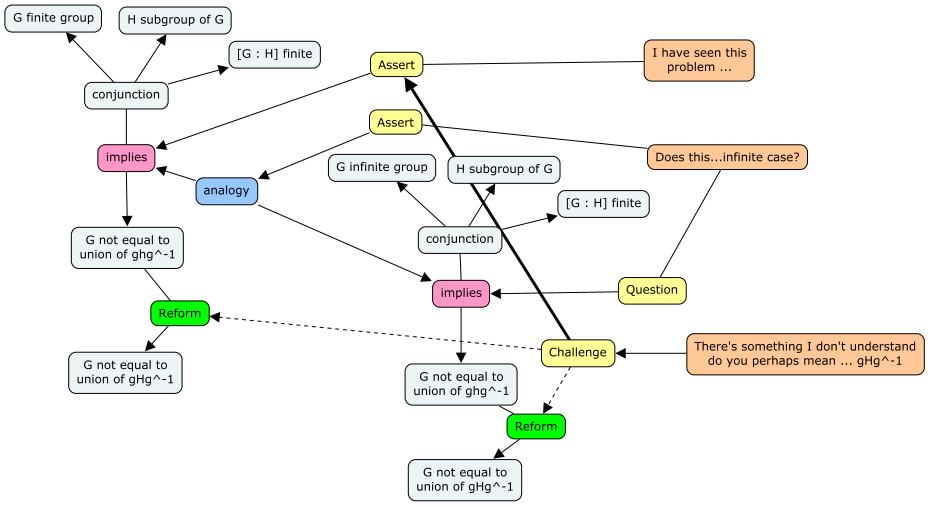



**Fig. 2.** First comment on question, diagrammed

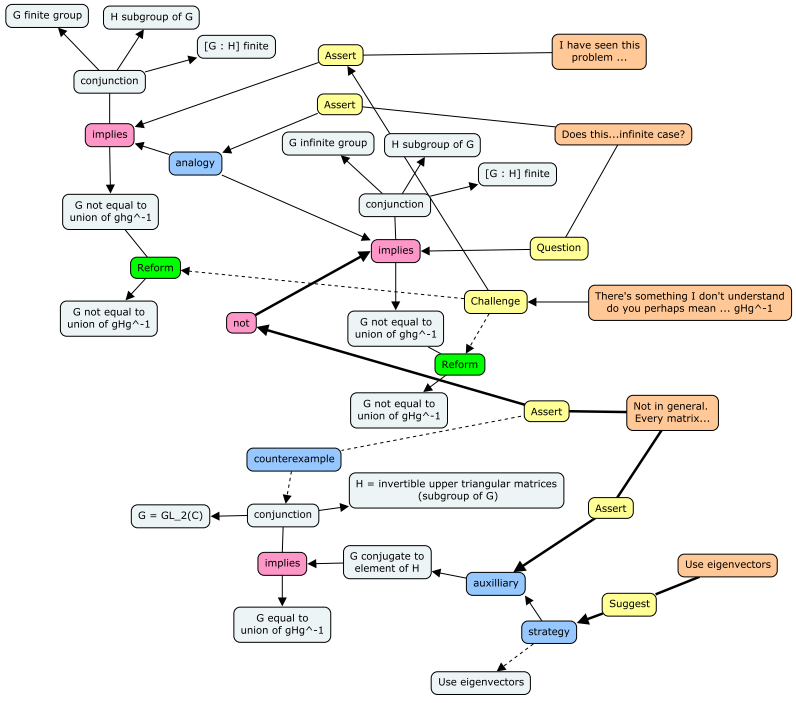



**Fig. 3.** Answer, diagrammed

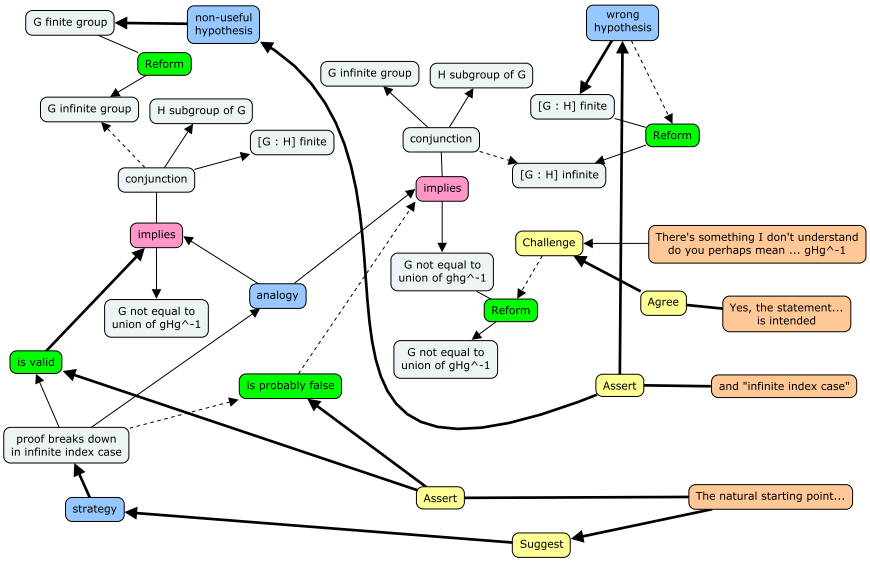



**Fig. 4.** Second comment on question, diagrammed

In the following comment, both the foregoing refinements and the earlier proposed answer are taken into account.

> (Third comment on question, 18:24) If $G$ is a finite group then all its subgroups have finite index. What the statement should say is that if $H$ is a proper finite index subset of $G$ then $G \neq \cup_{g \in G} gHg^{-1}$ (the case of infinite $G$ readily reduces to the case of finite $G$). As Keith shows, this is not always true for subgroups of infinite index. [→ Figure 5]

Here, the earlier assertion that $[G : H]$ is the "wrong hypothesis" is challenged. Essentially, this comment is looking for the most precise way to phrase the problem statement. "$[G : H]$ finite" is not wrong, but necessary if we want the implication to hold. The counterexample of invertible upper triangular matrices – "Keith's counterexample", diagrammed in Figure 3 – does indeed have infinite index in $\mathrm{GL}_2(\mathbf{C})$.[8] It is not, therefore, precisely a counterexample to the three-part conjunction in that diagram that it appears to refute; it does serve as a counterexample to the revised three-part conjunction in Figure 4. One should keep in mind that the statement "not in general" that prefaces the answer was addressed not to the specific conjunction in Figure 3, but instead to the OP's considerably more vague question "Does this remain true for the infinite case

[8] Intuitively, upper-triangular $2 \times 2$ matrices have one element that is zero, so the subgroup has one codimension in $\mathrm{GL}_2(\mathbf{C})$, namely, a copy of $\mathbf{C}$. The fact that the index is infinite follows (but an algebraic proof is also straightforward). For an example of an infinite group and a subgroup with finite index, consider $\mathbf{Z}$ and $2\mathbf{Z}$: in this case, the group is equal to the union of cosets.

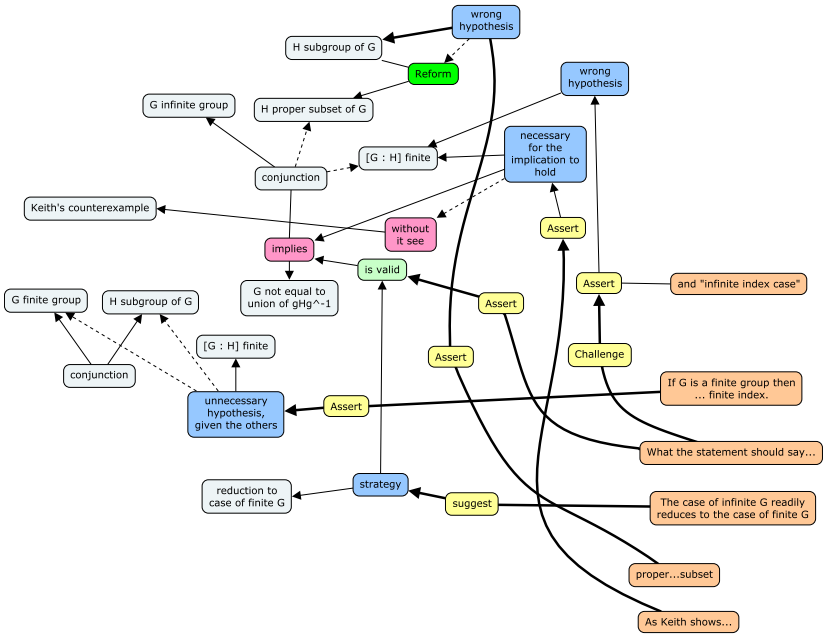



**Fig. 5.** Third comment on question, diagrammed

also?" Once the question has been clarified by other contributors, the logic of the answer works.

Subsequent to this, further terminological issues arise in comments on the answer, which again have to do with understanding exactly what is being asked, e.g., adding another condition of "discreteness" to the conjunction, for an answer with an even more "finite feel." New examples are proposed, both with reference to the literature and by straightforward adaptation of the answer already presented. In addition, four alternative answers are supplied, without attracting further comment (but with argumentation provided "in advance", so to speak). We will not draw diagrams for this material, because the illustrative presentation above is sufficient for our current purposes.

## 4 Discussion and Future Work

The case study that was examined above shows both the basic promise of the argument-theoretic approach to modelling mathematics, and some of the difficulties that would have to be overcome in 'scaling up' this approach. Bundy argues that the right representation is the key to successful reasoning [6, p. 16]. While both representation and reasoning are crucial (and, for 'functional inference', it is important to be strategic about representations *of* reasoning) in a material sense, the logically-prior step is that of building the representation. One

alternative to hand-coding is to search for catalogues of existing data, and more specifically, of reasoning, that have already been represented in some preliminary and readily digestible form. Learning from text was a key part of IBM's approach in building the version of Watson that competed in *Jeopardy* [11]. The Austin-based company, Cycorp, known for its large hand-coded AI system, also emphasises learning from documents in more recent work [34]. More broadly, "Knowledge Extraction from Text" is a well-known domain of computing research (with workshops running since 2013). The requirements for knowledge extraction from mathematical text, in particular, are being worked out in specialist efforts in linguistics and NLP [13, 10, 8] (and see in particular the survey in [14]).

Issues associated with mining Wikipedia and scientific literature have been explored (e.g., [7], [16], [17]) with successful scientific and technical proofs-of-concept at various scales. The technical issues associated with mining mathematical literature are increasingly well-understood [29] – and implementation work is ongoing.[9] *Argument mining* is, however, a relatively new area [33, 22]. In the current state of the art, discourse structure may have to be laboriously hand-coded.[10] We may be aided in our specific pursuit by the fact that much of mathematics is relatively "formulaic." For example, the first 100 most frequent schematic constructions (like "let $X$ be a $Y$") were shown to cover half of the sentences on the detailed but mathematically informal ProofWiki website [18].

Even so, as we've seen in our case study, modelling mathematical arguments is far from a trivial task. The initially quite simple language of '`implies`' and '`analogy`' from Figure 1 is complemented by a range of much more complex relations in Figure 5, like '`necessary for the implication to hold`'. The '`Reform`' relation has been applied to individual pieces of mathematical content, but in some cases the relevant transformations would have to happen on the graph of statements, for example, if we were to specify the '`analogy`' between two problem statements; reasoning from analogy can be complicated [25, 36]. The precise representations need to be worked out; and, as indicated above, we also want our representations to be capable of simulating the reasoning evolved in an effective manner.

Looking to the future, given a suitably-represented knowledge base of mathematical facts from whatever source, one interesting line for research (with a Q&A feel) would build on the work of Nuamah et al [30], who describe a strategy for automatically assembling the answers to queries when the answers are not directly stored in the database. As a simple example, it may be the case that there is no prerecorded answer to the following question:

> *Does the country x that has the highest GDP per capita (GDP/c) out of all countries in South America have a higher GDP/c than the country y that has the highest GDP/c out of all countries in Africa?*

[9] `https://www.authorea.com/users/5713/articles/51708-understanding-a-dataset-arxiv-org/_show_article`

[10] `https://research.googleblog.com/2017/05/coarse-discourse-dataset-for.html`

Nevertheless the answer may be computed from information that is available in the database (namely, GDP/c for all countries, together with the association of countries to continents). We assert that at a high level the logic of 'functional inference from heterogenous data' would be similar if the data was not CIA Fact Book-style, but, instead, mathematical facts – or, indeed, facts about mathematical arguments.

Broadly, the proposal that we suggest pursues the slow refinement of mathematical knowledge from knowledge about mathematical arguments. We can compare and contrast this with a famous proposal in the computer mathematics community, the QED manifesto [2]. This proposed "a project to build a computer system that effectively represents all important mathematical knowledge and techniques." We agree with this general aim. However, the QED manifesto relied on the idea of "the use of strict formality in the internal representation of knowledge and the use of mechanical methods to check proofs of the correctness of all entries in the system." If relied on as the only method, formal mathematics may be a premature optimisation. More specifically, before achieving full formality, it may be necessary to be 'capable of being in uncertainties' [20]. Later reassessments of the QED proposal [43, 15] have had to deal with the fact that, by in large, it has not worked out as hoped.

In this paper, we have been inspired by the observation that Q&A is *both* a useful source *of* explicitly represented reasoning, *and* a useful (i.e., 'effective') modality *for* developing additional explicit reasonings. Q&A is a popular way for humans to learn from each other – including, in particular, about technical subjects like mathematics and computer programming. There are large existing 'traces' of Q&A dialogues available online, and many users actively engaging with these dialogues on a daily basis (both as querents and respondents). Q&A dialogues are, accordingly, a likely source of explicit knowledge – and they are, also, if our ansatz is correct, a potential modality for automated knowledge-building that could realise Turing's vision of machines that 'converse with each other to sharpen their wits' [41]. Naturally, dialogues between humans and automated agents/agencies would be a potential application. Systems like this could support activities ranging from automatic tutoring to mixed initiative proof and program-construction.

Argumentation techniques for agent research are surveyed in [9]. This work should be compared with Ganesalingam and Gowers's natural-language generating automated problem solver [12]. The kinds of natural language that are employed in Q&A dialogues is quite different from that employed in the textbook problems that are the focus of [12]. Nevertheless, at the level our proof-of-concept illustration above, we seem to have a good handle on the semantics of mathematical dialogues, and we should be able to support textbook reasoning as a special case. It should also be possible to map dialogue semantics to a theorem proving system (an LCF-inspired system like the one used by Ganesalingam and Gowers, would be one starting point but others should be considered as well).

Lastly, to evaluate systems working in this area – whether driven by argumentation, agents, theorem proving, simple machine learning heuristics, some com-

bination, or by some other approach entirely – we propose the simple knowledge-based computational benchmarking task SEMATCH, which is defined as follows. This challenge problem requires a system to match existing *questions* and *answers* selected from the Stack Exchange network. This can be solved in suitably-blinded forward and backward directions (Q↦A or A↦Q); that is, the program that is being tested will be presented with a sample of questions and answers, and will not be told which question matches which answer: it will then try to recover that information by reasoning about the sample's content. Accordingly, SEMATCH can be applied to routinely evaluate heuristics, without necessarily requiring a system that is capable of generating new answers or new questions. As a benchmarking problem for programs that reason about natural language, one could compare SEMATCH with the Winograd Schema Challenge [23], which is similarly open-ended as to methods of solution. However, the Winograd Schema Challenge currently requires expert intervention to generate new test questions, which follow a certain prescribed form. In the case of SEMATCH, a large corpus of ground-truthed data exists. Additional benchmarks related to the same ground-truthed data can be straightforwardly devised, e.g., to correctly sort the answers to a given question in order of empirical user ratings, based on their contents. Ultimately, after workouts with evaluation metrics like this, along with generative experiments, a system may be devised that can answer specific sets of domain problems correctly, and have its answers validated by human experts.

## Acknowledgements

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# Lakatos-style Collaborative Mathematics through Dialectical, Structured and Abstract Argumentation

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**Abstract**

The simulation of mathematical reasoning has been a driving force throughout the history of Artificial Intelligence research. However, despite significant successes in computer mathematics, computers are not widely used by mathematicians apart from their quotidian applications. An oft-cited reason for this is that current computational systems cannot do mathematics in the way that humans do. We draw on two areas in which Automated Theorem Proving (ATP) is currently unlike human mathematics: firstly in a focus on soundness, rather than understandability of proof, and secondly in social aspects. Employing techniques and tools from argumentation to build a framework for mixed-initiative collaboration, we develop three complementary arcs. In the first arc – our theoretical model – we interpret the informal logic of mathematical discovery proposed by Lakatos, a philosopher of mathematics, through the lens of dialogue game theory and in particular as a dialogue game ranging over structures of argumentation. In our second arc – our abstraction level – we develop structured arguments, from which we induce abstract argumentation systems and compute the argumentation semantics to provide labelings of the acceptability status of each argument. The output from this stage corresponds to a final, or currently accepted proof artefact, which can be viewed alongside its historical development. Finally, in the third arc – our computational model – we show how each of these formal steps is available in implementation. We demonstrate our approach with a formal, implemented example of real-world mathematical collaboration. Finally, we offer reflections on our mixed-initiative collaborative approach.

*Keywords:* Automated theorem proving, automated reasoning, abstract argumentation, argumentation, collaborative intelligence, dialogue games, Lakatos, mathematical argument, structured argumentation, social creativity, philosophy of mathematical practice

# 1. Introduction

The simulation of mathematical reasoning has been a driving force throughout the history of Artificial Intelligence research [98, 86, 58, 87]. However, despite significant successes in 'computer mathematics' (e.g., [18, 40, 42, 45]) computers are not widely used by mathematicians apart from their quotidian applications like running word processing tools, email programs, web servers and web browsers, and (sometimes) computer algebra systems. An oft-cited reason for this is that current computational systems cannot do mathematics in the way that humans do. Despite – or perhaps because of [69] – their profound rigour, machine proofs are often thought to be unclear, uninspiring and untrustworthy, as opposed to human proofs which can be deep, elegant and explanatory [21, 41]. In order to help to close the gap between machine-constructed proofs and human-constructed ones, we consider two key areas of focus: informal and social aspects of proof discovery in the human context. We propose that theories and tools from the field of argumentation can be used to more closely align AI systems with the human context in these two areas.

## 1.1. Informal aspects of proof

Evaluation metrics in the Automated Theorem Proving (ATP) community are focused on soundness, and the power of a solver to prove a wide selection of difficult problems with specific resource limits.[1] Qualities of the resulting proof other than soundness are rarely considered. This stands at variance with the practices of the mathematical community, in which a lack of soundness might be forgiven if a proof is interesting or complex. Indeed, an error in a proof may be neither "perturbing," nor "surprising," if it is judged to be the right sort of error (one which is not critical to the integrity of the proof) [20].[2] Instead, one of the main criteria by which a proof is judged in the human context is its understandability. A well-written proof can provide insight as to why a theorem may be true, point to new conjectures, form connections between different fields and suggest solutions to open problems [44, 75, 106]. Fields medal winners Gowers and Thurston, respectively, have said: "We like our proofs to be explanations rather than just formal guarantees of truth" [41, p. 3], and "reliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas" [94, p. 10]. Thurston emphasises that informal conversations between mathematicians can often convey ideas more quickly and comprehensibly than a written proof [94, p. 6]. Hersch has suggested that "The standard style of expounding mathematics purges it of the personal, the controversial, and the tentative, producing a work that acknowledges little trace of humanity, either in the creators or the consumers" [47, p. 131].

Lakatos offered similar insight into proof-understanding [55]. Building on Pólya's

[1] "Wide", "difficult" and "resources" are all defined appropriately: see, for instance [92].

[2] Indeed, Aschbacher – one of the main mathematicians involved in the development of the proof of the Classification of Finite Simple Groups (one of the main achievements of twentieth century mathematics, on which many other results depend) – commented that "the probability of an error in the [CFSG] proof is one" [9], related in [20].

distinction between informal, unfinished mathematics-in-the-making and formal, finished mathematics [76], he argued that a theorem and proof which are presented in isolation from their development are "artificial and mystifyingly complicated", analogous to a "conjuring act" [55, p142]. In order to make results understandable, they should be presented alongside the "struggle" and "adventure" involved in the story of their development. This insight is echoed by Ernest, who criticises the practice of presenting mathematics learners with the "sanitized outcomes of mathematical enquiry": "The outcome may be elegant texts meant for public consumption, but they also generate learning obstacles through this reformulation and inversion" [67, p. 67]. Bundy points out that this practice also obscures understandability for research mathematicians: "Mathematicians find informal proofs more accessible and understandable [than formal proofs]" [21, p. 2].

In contrast to the concerns about understandability voiced by mathematicians and philosophers of mathematics, understandability is not traditionally a concern for ATP. A handful of exceptions have focused on making an existing machine proof more comprehensible [29, 30, 36]. MacKenzie [60] has argued that rather than treating machines as oracles and giving them responsibility for verifying the reliability of both hardware and software, there needs to be a continued interaction between computer systems and our collective human judgment: "The finished product of formal verification – the 'proof object' – may thus be less important than the process of constructing it." [61, p. 2348]. Constructing or verifying proofs which are written in a classical logical formalism does not align with mainstream mathematical activity, since proofs are typically neither constructed nor presented in this way.

Accordingly, our objective to model mathematical dialogues connects closely with the theory of defeasible argument (reasoning that is rationally compelling but not deductively valid [52]). The structure of classical proof theoretic systems and formal theorisations of defeasible argument differ [99]. Defeasible argument is used during the initial construction of a proof, and as the proof is refined or changed over time to reflect conceptual changes in the underlying theory, or to rectify deductive errors discovered after a proof is commonly accepted – all themes that Lakatos emphasised [55]. In practice, we may have an argument whose conclusion states that *for all* $x$, $P(x) \rightarrow Q(x)$, whose logical validity rests on a particular interpretation of $P$ and $Q$. In some cases $P$ or $Q$ might not be clearly defined, and can be subsequently defined in different ways by different people, sometimes rendering the initial argument invalid. Whether a consensus ever occurs and whether we could be sure that the consensus is final, is an open and somewhat contentious question.

We propose that applied argumentation theory can improve the understandability of output from, and input to, ATP systems, and other computer-mediated, -moderated, or -motivated proof systems. Doing this will help to close the cultural gap between human and machine mathematics. One way to go about this is to keep track of informal proof development, presenting the errors, conflicts and deadends involved, alongside a finished or current proof artefact.

## 1.2. The social dimension of human mathematics

The social dimension is typically neglected in automated reasoning, which usually consists of two approaches: *autonomous theorem proving*, in which a single system proves theorems, or *interactive theorem proving*, in which there is one system and one user interacting with it. New models of working in a social context have gained traction with the notion of the social machine – a new paradigm identified by Berners-Lee [13] for viewing a combination of people and computers as a single problem-solving entity. Martin and Pease propose a research agenda for mathematics social machines as "a combination of people, computers, and mathematical archives to create and apply mathematics, with the potential to change the way people do mathematics, and to transform the reach, pace, and impact of mathematics research" [63, p. 1] . Epstein outlines a more general vision of collaborative intelligence, in which she revisits original goals in AI in the light of its history, successes and failures, and recommends a new form of synergy between people and computers [34].

These ideas suggest a third approach in automated reasoning – *mixed-initiative proving* – in which proof discovery occurs via interaction of multiple participants (both human and computer) working together towards a common goal. In this paper we build on these ideas and use techniques from argumentation to provide a way of formalising social aspects of mixed-initiative proof via dialogue theory.

## 1.3. Argumentation

The relationship between the study of mathematical practice and argumentation theory is not well-explored, though has already borne fruit. Toulmin applied his argumentation structure to Theaetetus's proof that there are exactly five platonic solids [95]; Aberdein showed that Toulmin's structure can represent more complex mathematical proofs [2, 3]; and Alcolea has shown that it can also be used to represent meta-level mathematical argument, such as axiom adoption or rejection [8]. Krabbe [53], Aberdein [4, 5, 6] and Aberdein and Pease [72] further demonstrate the application of various theories of argumentation to mathematical argument. Further related issues are explored in a recent edited volume [7].

Although theories of argument have been applied within the philosophy of mathematical practice (as above), mathematics presents a largely novel domain of discourse within the argumentation research community, perhaps due to the misconception that it represents a deductive style of reasoning, more appropriate to formal proof than argumentation.

## 1.4. Aims and contributions

Our main aim is to develop a new approach in which tools and theories from the argumentation community can be deployed to build a bridge between interactive proof tools and human mathematicians.

Our main contribution is to identify two areas in which ATP is currently unlike human mathematics – informal and social aspects – and to employ techniques and tools

from argumentation to build a framework which opens the door to mixed-initiative collaborative reasoning in mathematics. Specifically, we:

1. Propose and demonstrate a way to make proofs more understandable by drawing on philosophical, sociological and educational literature on mathematics which highlights the importance of presenting the development of a proof attempt alongside a final, or currently accepted, proof artefact. We develop a framework in which this aspect is possible. A grounded extension of a dialogue can be produced, representing a currently accepted, collaboratively constructed, proof or theory. Since the record of the dialogue can be presented alongside this proof, the framework delivers the history of a proof attempt as well as the proof artefact.

2. Propose and demonstrate a way to make collaboration more social by opening the door to a mixed initiative collaborative mathematics. Social aspects in human mathematics have been shown to be integral to the human context, therefore technologies which are able to support mathematicians in the collective construction of mathematical knowledge, in a variety of ways are essential. In addition to being able to show the current state of discussion (as discussed above), this includes highlighting conflicting commitments or unresolved moves, finding similarities and conflicts across different discussions going on in parallel among otherwise independent groups of arguers, storing past discussions and making them searchable, and so on.

3. Develop three complementary arcs:

   (a) The first arc comprises our *theoretical model*. Starting from the philosophical stance provided by Lakatos [55] in his account of the dialectical, collaborative interaction that constitutes the practice of mathematics, we interpret this through the lens of dialogue game theory [43] and in particular as a dialogue game ranging over structures of argumentation.

   (b) In the second arc we develop an *abstraction level*, in which we start with the argument fragments created via the formal dialogue game and develop structured arguments, from which we induce abstract argumentation systems in the style of [32]. The final stage in this arc is to compute the argumentation semantics, to provide labellings of the acceptability status of each argument. This paper shows how the labelling derived from the abstract argumentation framework corresponds precisely to the theory that has been collaboratively created by the participants in a Lakatosian dialogue. Thus the output from this stage corresponds to a final, or currently accepted proof artefact, which can be viewed alongside its historical development.

   (c) The third arc comprises our *computational model*, in which we show how each of these formal steps is available in implementation. The interpretation of Lakatos as a formal dialogue game can be captured as an implemented specification in the Dialogue Game Description Language (DGDL) [105]. This specification can be executed by a platform, the Dialogue Game

Execution Platform (DGEP) [15] which offers a series of web services to clients for executing a participant's legal moves. Part of the semantics of the dialogue game specification is to define updates on a shared information state [96], in which the language of knowledge representation is AIF, implemented as a series of web services provided by the AIFdb infrastructure [57]. The AIF data created as a side effect of the operational semantics of turn-taking in the dialogue game is interpreted by The Online Argument Structures Tool (TOAST) [90] as an ASPIC+ system [66] and passed to DungOMatic [88] to calculate the grounded extension. At each step in the game, this calculation returns the current state of the co-created mathematical theory.

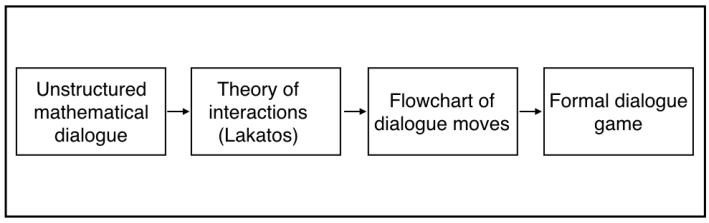
4. Demonstrate our approach and the interaction of our three arcs via an implemented example of human mathematical collaboration. This shows how off-the-shelf technologies can produce a pipeline system, running from natural language dialogue about the construction of a mathematical proof, to philosophical theory, to the formal expression of natural language reasoning in dialogue games, and from there to abstract argumentation and argumentation semantics, and finally, coming full circle, to show that implementations of these systems can then provide value back to the mathematical community from which the philosophical theory was derived.

5. Demonstrate the applicability of argumentation techniques to mathematical reasoning.

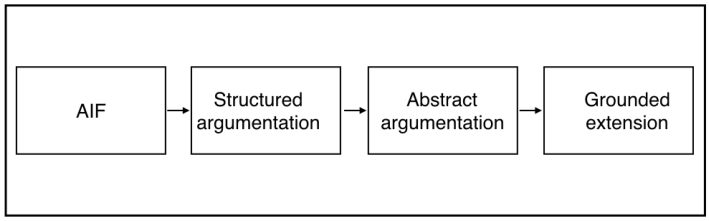
We further show how the model can be retrospectively applied to examples of extant mathematical discussion in Appendix A. In so doing, it is not only possible to demonstrate the depth of Lakatos's original insight, but also to show that the formal characterisation here remains both honest to the original and of practical utility to mathematicians. By making this connection back to the community of mathematical practice, we show that the door is opened to mixed-initiative, collaborative mathematics [89, 35].

Prior work on argumentation in artificial intelligence is surveyed in [12] and [11]. Several 'pipeline' systems for argument analysis have previously been developed [107, 88], "breaking down argumentation tools into small components, deploying these components as web services, then constructing UNIX-style pipelines to link them together as one large system" [88, p. 1]. The specific 'pipeline' detailed in the later sections of this paper is outlined in Figure 1c. We emphasise logic rather than linguistics, and thus do not solve the problem of the "knowledge acquisition bottleneck" highlighted by Wyner, van Engers, and Hunter [107, p. 21]. Section 8 points to some related recent work on the linguistics of mathematics that could be useful in widening this bottleneck. Indeed, there are at least two narrow places in the bottleneck – and the corresponding research challenges are roughly analogous to the steps "first from the established facts to intermediate predicates, and then from these intermediate predicates to legal consequences" in legal case-based reasoning [81, p. 17]. The present work is unique in that it is formal, implemented, and descriptive of real-world mathematical collaboration.

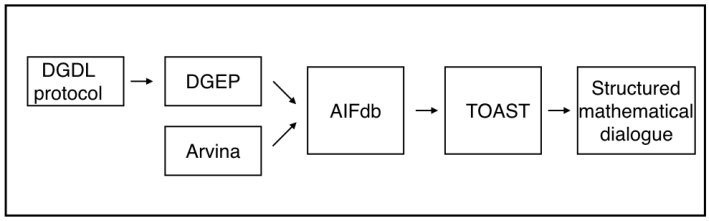
The remainder of the paper is structured as follows: in Sections 2 - 3 we outline our theoretical model, in which we introduce theoretical foundations and develop a




(a) 'Theoretical model/analysis' (Sections 2-3)




(b) 'Abstraction level' (Sections 4-5)




(c) 'Computational models/analyses' (Sections 6-7)

Figure 1: The three main developmental arcs discussed in the paper

formal dialogue system from Lakatos's model of mathematical discourse. In Sections 4 - 5 we present our abstract level, showing how we progress from a formal dialogue system to Argument Interchange Format structures and then to abstract argumentation frameworks. In Sections 6 - 7 we present our computational model and example of collective proof as argumentation. Finally, we present our conclusions in Section 8. A diagrammatic representation of the paper can be found in Figure 1.

## 2. Theoretical foundations

In this section we lay down the theoretical foundations necessary for understanding the rest of the paper. Section 2.1 contains a high-level description of Lakatos's patterns of dialogue: this, alongside material in Section 2.3 on specifying dialectical interactions as a set of rules governing the interaction, will be needed to understand our development of a formal dialogue game in Section 3. Section 2.2 shows the application of

Lakatos's patterns to real-world natural language discourse in mathematics, demonstrating their applicability to at least some human mathematical reasoning (more detailed applicability is demonstrated in Appendix A). Section 2.4 contains background on the Argument Interchange Format, and we review structured argumentation in Section 2.5: we draw on both of these in Section 4. Finally, in Section 2.6 we describe abstract argumentation, which we use in Section 5. All sections will be relevant for understanding our implementation work and the execution of the implemented system described in Sections 6 and 7.

## 2.1. Lakatos's patterns of dialogue

Lakatos has contributed to the field of AI both methodologically in his philosophy of science [54] and via his ideas on the growth of informal mathematics [55]. Against the traditional view that mathematical progress comes down lucky guess work or simple intuition, Lakatos proposed that there are logical mechanisms – albeit only informally specified in his work – which underly the mathematical thought process. He challenged Popper's view [77] that philosophers can form theories about how to evaluate conjectures, but not theories about how to generate them. He did this in two ways: arguing that *(i)* there is a logic of discovery, *i.e.*, the process of generating conjectures and proof ideas is subject to rational laws; and, *(ii)* a sharp distinction between discovery and justification is misleading as each affects the other.

We know of two systematic implementations of Lakatos's philosophy of mathematics. Pease [71] constructed a multi-agent system in which agents formed and communicated theories in both mathematical and non-mathematical domains, and responded by following Lakatos's patterns of dialogue. Hayes-Roth [46] developed heuristics for repairing flawed plans, modelled on [55] and developed in the context of a card game.

At a higher level, Sloman [87] highlights the relevance of Lakatos's notions to AI, via his analysis of the capabilities that would be necessary for an intelligent robot (or young child, or other human) to think mathematically. Sloman argues that, since such thinking is not infallible in humans, the capacity to discover and repair flaws in arguments and conclusions is essential. Lakatos's fallibilist picture of mathematics provides a detailed account of how this can be done, by outlining various methods by which discovery of mathematical claims and their justification in the form of an argument can occur. These methods suggest ways in which concepts, conjectures and proofs gradually evolve via interaction between mathematicians. In Lakatos's account — which he presented as a classroom discussion between (very advanced) students — proofs, conjectures and concepts are fluid and open to negotiation.

Lakatos demonstrated his argument by presenting case studies in dialogue form of the development of Euler's conjecture that for any polyhedron, the number of vertices ($V$) minus the number of edges ($E$) plus the number of faces ($F$) is equal to two, and Cauchy's proof of the conjecture that the limit of any convergent series of continuous functions is itself continuous. The dialogue highlights various moves that can be taken at different times, and the consequences of these moves on the resulting theory. We outline these dialogue moves below and represent the flow in Figure 2 This figure serves as a visual key to the formalisation developed in Section 3. The example in Section 7 is a portion of Lakatos's fictional debate of Euler's conjecture.

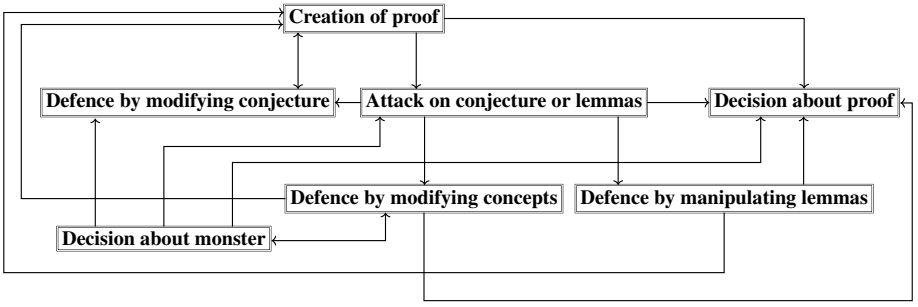



Figure 2: Lakatos's informal logic of mathematical discovery [55], represented as a flow chart

The first step in a Lakatosian dialogue is for someone to **propose a conjecture**. This is optionally followed by a proof, consisting of a list of lemmas and a statement that the conjecture is now proved. To emphasise: for Lakatos, the basic structure of a mathematical proof is nothing other than a list of lemmas together with a conclusion that they support.

After this, participants might **accept the conjecture**, which will terminate the dialogue. Alternatively, they might perform **strategic withdrawal**, which weakens the conjecture itself, but (ideally) strengthens confidence in it. If this move is used, then the subsequent options for dialogue moves are those which are available after the proposal of the initial conjecture. Strategic withdrawal consists of using positive examples of the conjecture and generalising from these to a class of object, and then limiting the domain of the conjecture to this class. For instance, the students generalise from regular polyhedra to *convex polyhedra*, and then modify Euler's conjecture to 'for any convex polyhedron, $V - E + F = 2$'. A third alternative at this stage is for a participant (opponent) to challenge the conjecture by **raising a counterexample**. When a counterexample has been raised, participants (specifically, proponents) have five moves available to them.

Firstly, they can **surrender** the conjecture, which will terminate the dialogue.

Secondly, they can deal with exceptions via **piecemeal exclusion** and thereby exclude a whole class of counterexamples. Piecemeal exclusion amounts to generalising from a counterexample to a class of counterexamples which have certain properties and then excluding the entire class. Again, in terms of dialogue moves, this is a form of **propose a conjecture** (which will be a weakened version of the original conjecture), and subsequent dialogue moves will follow the same pattern as for this move. An example of piecemeal exclusion is that the students generalise from the hollow cube to *polyhedra with cavities*, and then modify Euler's conjecture to 'for any polyhedron without cavities, $V - E + F = 2$'.

Thirdly, participants can perform **monster-barring**. This is a way of excluding an unwanted counterexample, and consists of the argument that the proposed 'counterexample' is not valid, as it is not within the claimed concept definition (this may then be expanded). Thus, it does not conflict with the conjecture, because it is *not* a counterexample. For instance, one of the students suggests that the hollow cube (a cube with a cube-shaped hole in it) is a counterexample to Euler's conjecture, since

$V-E+F=16-24+12=4$. Another student uses monster-barring to argue that the hollow cube does not threaten the conjecture as it is not in fact a polyhedron. The concept polyhedron then becomes the focus of the discussion, with the definition possibly being formulated explicitly for the first time. Thus, subsequent moves here must either propose an alternative definition, or express a preference to participants as to which definition should be adopted. Using this method, the original conjecture is unchanged, but the meaning of the terms in it may change.

Fourthly, participants can perform **monster-adjusting**, which is similar to monster-barring. Here, one *reinterprets* an object in such a way that it is no longer a counterexample, not by (re)defining the domain of the conjecture (so in this case the object is still seen as belonging to this domain), but by (re)defining subconcepts in the conjecture. Subsequent moves here are analogous to those following monster-barring, for appropriate concept definitions. The example in [55] concerns the star polyhedron. This entity is raised as a counterexample since, it is claimed, it has 12 faces, 12 vertices and 30 edges (where a single face is seen as a star polygon), and thus $V-E+F$ is $-6$. This is contested, and it is argued that it has 60 faces, 32 vertices and 90 edges (where a single face is seen as a triangle), and thus $V-E+F$ is 2. The argument then turns to the definition of 'face': again, using this method, the original conjecture is unchanged, but the meaning of the terms in it may change.

Finally, a fifth dialogue move when faced with a counterexample is to perform **lemma incorporation**. This works by considering the counterexample and determining whether it is global (a counterexample to the main conjecture) and/or local counterexamples (a counterexample to one of the lemmas). If it is both global and local, *i.e.*, there is a problem both with the argument and the conclusion, then the move consists in modifying the conjecture by incorporating the problematic proof step as a condition. If it is local but not global, *i.e.*, the conclusion may still be correct but the reasons for believing it are flawed, then the move consists in modifying the problematic proof step but leaving the conjecture unchanged. If it is global but not local, *i.e.*, there is a problem with the conclusion but no obvious flaw in the reasoning which led to the conclusion, then the move consists in looking for a hidden assumption in the proof step, then modifying the proof and the conjecture by making the assumption an explicit condition. For example, in the portion of the Euler conjecture discussion treated in Section 7, the Teacher remarks with respect to a local counterexample: "I no longer contend that the removal of any triangle follows one of the two patterns mentioned, but merely that at each stage of the removing operation the removal of any boundary triangle follows one of these patterns... I can easily improve the proof, by replacing the false lemma by a slightly modified one, which your counterexample will not refute." The discussion can then continue in the same fashion as when the initial problem and proof were proposed.

## 2.2. Analysing natural language dialogues

Dialogue has been a topic of interest for computer scientists for over three decades [50, 48]. Dialogue is, fittingly, approached through NLP and linguistic analysis [37], but it also falls within the scope of pragmatics [22], which "arises as soon as we move beyond the linguistic analysis of an utterance and ask what the speaker meant by it" [91]. Searle's theory of speech acts [85] is a classic in this genre. Pragmatic

analysis is relevant to both human-computer and multi-agent collaboration [51, 74]. *Dialogue game* protocols have previously been applied, for example, to multi-agent planning problems [10, 17]. Dialogue games are "more expressive than auction and game-theoretic mechanisms, typically allowing participants to question and contest assertions, to advance supporting arguments and counter-arguments, and to retract prior assertions" [65]. Given their formality, it is not surprising that, at least by default, "there is no direct connection between formal dialogue games or a theory of sentence meaning and natural language use" [104, p. 81]. Thus, alongside application areas like collaborative planning and proof construction, research on natural language dialogues is associated with a distinct collection of fundamental communication issues, such as modelling the ways in which mutual belief is established between discussants [97]. For example, the *contribution model* of Clark and Schaefer [26] is centred on building common understanding through iterated phases of *presentation* and *acceptance*. However, Lakatos's informal logic outlined in Figure 2 assumes that discussants are able to understand each other satisfactorily at the linguistic level, by and large, and focuses instead on building a shared mathematical theory. The Lakatos Game developed in Section 3 expands the schematic sketched in Figure 2 in detail. In Appendix A, we use the full range of this model to mark up a real-world example of a collaboratively-constructed proof. This shows the applicability of Lakatos's theory via our interpretation of his work as a dialogue game, and, more broadly, the relevance of our overall approach, which we outline in further detail below.

## 2.3. Formal dialogue systems

The pipeline of collaborative mathematics starts with the expressing the Lakatos model as a formal dialogue system. The approach of specifying dialectical interaction as a game or a set of rules governing the interaction has started with the work of Lorenzen [59] and Hamblin [43]. Lorenzen aimed to translate a system of logic (such as intuitionistic logic or classical logic) into a system of dialectical rules, the dialogical logic, in which the proponent and the opponent aim to collectively prove that a formula is the tautology of this logic (intuitively, that it is true in this logic). Hamblin, on the other hand, tried to show that at least some fallacies (flaw argumentation patterns) have a dialectical nature even though they have inferential structure. He designed a formal dialogue system, called formal dialectics, and demonstrated that it is possible to formulate rules which will prohibit the players to commit the fallacy of circular reasoning of the form "$p$ because $p$". This work led to a variety of similar systems which improved the original ones, responded to other philosophical problems, or model the communication amongst agents in multi-agents systems (see e.g. [102, 64, 68, 105]). In this paper we will treat these systems as a template for the development of formal dialogue systems.

In the literature, many different types of dialogues were identified (see e.g. [103] for the first and widely used attempt of classification of dialogues). Lakatos describes the interactions of mathematicians, when aiming to prove the conjecture, in the way that is the most closely related to the concept of persuasion dialogue, and in particular – its "conflict resolution" subtype [19]. Persuasion dialogue is triggered by a difference of opinion between participants, each of them aims to persuade each other and the main

goal of the conversation is to achieve the resolution of the conflict.

An excellent survey of systems for persuasion is given in [79]. A dialogue systems has a dialogue purpose, a set $A$ of participants and a set $R$ of roles which participants can adopt during a game. Contents of utterances used by the players in the dialogue are expressed in a topic language $L_t$. At the beginning of a dialogue every player $s$ has assigned a (possible empty) set of commitments $C_s \subseteq L_t$ which changes during a dialogue. Every dialogue system includes a logic $L$ consisting of a topic language $L_t$ and a set $R$ of inference rules over $L_t$. The dialogue system consists of several sets of rules, amongst which the most typically used there are: **(1)** locution rules which describe what type of utterances players can execute during a dialogue; **(2)** structural rules or protocol which determine the interaction between locutions (i.e., it specifies which locution can be performed as a reply to another locution)[3]; **(3)** commitment or effect rules which specify for each utterance $\varphi$ the effects which this locution makes on a set of commitments of the participant $i$ (a commitment of $i$ is a sentence that $i$ publicly declared as his belief)[4]; **(4)** termination rules which determine the cases where no move is legal, i.e. they should specify the conditions under which the protocol returns the empty set; and **(5)** outcome rules which define the outcome of a dialogue, i.e. provide a criterion to decide which player wins and which player loses the dialogue.

The typical set of locutions and the reply structure identified in [79] is presented in Table 1. There are six legal moves that formal dialogue systems for persuasion often permit the players to perform: claim $\varphi$; why $\varphi$; concede $\varphi$; retract $\varphi$; $\varphi$ since $S$; and question $\varphi$. The structural rules typically allows the players to interact in the following way: for example, after the agent claims a proposition $\varphi$ his respondent can challenge this proposition, claim it's negation or agree with the speaker; when the agent challenges $\varphi$ his respondent can justify this statement with a set of propositions $S$ or withdraw the statement; and so on.

| Locutions | Replies |
|---|---|
| claim $\varphi$ | why $\varphi$, claim $\neg\varphi$, concede $\varphi$ |
| why $\varphi$ | $\varphi$ since $S$ (alternatively: claim $S$), retract $\varphi$ |
| concede $\varphi$ | |
| retract $\varphi$ | |
| $\varphi$ since $S$ | why $\psi$ ($\psi \in S$), concede $\psi$ ($\psi \in S$) |
| question $\varphi$ | claim $\varphi$, claim $\neg\varphi$, retract $\varphi$ |

Table 1: A set of protocol rules typical for formal dialogue systems for persuasion (according to [79]).

This standard of formal representation of dialogical interaction will be used in Section 3 for expressing Lakatos model as a set of rules for mathematicians to follow, if they aim to collectively prove a conjecture.

[3]Formally, let $M$ be a set of moves. The set of finite dialogues $M < \infty$ is the set of all finite sequences $m_1, ..., m_i$ from $M$. A protocol, specifying the legal moves at each stage of a dialogue, is a function $P : Pow(L_t) \times D \longleftrightarrow Pow(L_c)$ where $D \subseteq M < \infty$. The elements of $D$ are called the legal finite dialogues.

[4]The function $C_i$ for a sequence of moves assigns a set of commitments.

## 2.4. Argument Interchange Format

The Argument Interchange Format [25] is an attempt to bring together a wide variety of argumentation technologies so that they can work together. [82] reviews some of the more recent applications of the AIF. Descriptions of the AIF are given in a number of places, as are reifications in languages such as RDF and OWL [25, 83, 82]. AIF uses a graph-theoretic basis for defining an "upper" ontology of the main components (or nodes) of arguments. Nodes are distinguished into those that capture information (loosely, these correspond to propositions), and those that capture relations between items of information, including relations of inference (which correspond to the application of inference rules to particular sets of propositions), relations of conflict (which represent forms of incompatibility between propositions) and relations of preference (which represent value orderings applied to particular sets of propositions). The instantiated nature of these relations is emphasised in the nomenclature, so whilst information is captured in Information (I-) nodes, relations between them are captured as Rule Application (RA-) nodes, Conflict Application (CA-) nodes and Preference Application (PA-) nodes. The general forms or patterns that these applications instantiate are given in a second part of the AIF ontology, the Forms ontology. The approach follows in the philosophical tradition of Walton [100], [101] of schematizing stereotypical patterns of reasoning – and then extending the tradition into conflict and preference. It is this schematic underpinning which gives the collective name for RA-, CA- and PA-nodes: Scheme (S-) nodes. The AIF upper ontology is designed to allow specialization and extension to particular domains and projects, in an attempt to balance the needs of interchange against the needs of idiosyncratic development.

The AIF standard will be used in Section 4 for showing how the execution of rules of Lakatos dialogue game is creating argument maps which represent a collective proof as a directed graph.

## 2.5. Structured Argumentation

Structured argumentation aims to "give a general structured account of argumentation that is intermediate in its level of abstraction between concrete logics and the fully abstract level, providing guidance on the structure of arguments, the nature of attacks, and the use of preferences, while at the same time accommodating a broad range of instantiating logics and allowing for the study of conditions under which the various desirable properties are satisfied by these instantiations" [66, p31]. One of the foremost and most flexible accounts of structured argumentation is available in *ASPIC*$^+$ [80, 66] which is not only flexible about the logic used to instantiate arguments within it, but also provides a straightforward mechanism for inducing abstract argumentation frameworks, described in the next section.

We simplify the definition in [80] in three ways:

(i) skipping preference ordering over rules, i.e. constraining $\leq = \preceq = \emptyset$;

(ii) ignoring strict rules (of which there are none permitted in Lakatos' theory), i.e. constraining $\mathcal{R}_s = \emptyset$ and thereby $\mathcal{R} = \mathcal{R}_d$; and

(iii) ignoring non-*ordinary* types of premises, i.e. constraining $\mathcal{K}_a = \mathcal{K}_i = \mathcal{K}_n = \emptyset$ and thereby $\mathcal{K} = \mathcal{K}_p$.

Whilst these are important features of *ASPIC*$^+$ in general, they are not exploited here, and so dropped for clarity. An *ASPIC*$^+$ *argumentation system* is thus a triple $AS = (\mathcal{L}, {}^-, \mathcal{R})$, where $\mathcal{L}$ is a logical language, $^-$ is a contrariness function from $\mathcal{L}$ to $2^{\mathcal{L}}$, and $\mathcal{R}$ is a set of rules of the form $\phi_1, \ldots, \phi_n \Rightarrow \phi$. An *ASPIC*$^+$ *argumentation theory* is then a pair $AT = (AS, \mathcal{K})$ where $\mathcal{K}$ is a knowledge base in $AS$.

An *ASPIC*$^+$ argumentation theory yields an *argument*, $A$, in two cases:

(i) $A$ is $\phi$ iff $\phi \in \mathcal{K}$, and in this case, $Prem(A) = \{\phi\}$, $Conc(A) = \phi$, $Sub(A) = \{\phi\}$, $DefRules(A) = \emptyset$ and $TopRule(A) = undefined$;

(ii) $A$ is $A_1, \ldots, A_n \Rightarrow \psi$ iff $A_1, \ldots, A_n$ are arguments and there exists a rule

$$Conc(A_1), \ldots, Conc(A_n) \Rightarrow \psi$$

in $\mathcal{R}$, and in this case,

$$Prem(A) = Prem(A_1) \cup \ldots \cup Prem(A_n),\ Conc(A) = \psi$$

$$Sub(A) = Sub(A_1) \cup \ldots \cup Sub(A_n) \cup \{A\},$$

$$DefRules(A) = DefRules(A_1) \cup \ldots \cup DefRules(A_n) \cup \{Conc(A_1), \ldots, Conc(A_n) \Rightarrow \psi\}$$

and $TopRule(A) = Conc(A_1), \ldots, Conc(A_n) \Rightarrow \psi$.

Finally, an *ASPIC*$^+$ argumentation theory yields an *attack* $(A, B)$ in three cases:

(i) argument $A$ *undercuts* argument $B$ (on $B'$) iff $Conc(A) \in \overline{B'}$ for some $B' \in Sub(B)$ of the form $B''_1, \ldots B''_n \Rightarrow \psi$;

(ii) argument $A$ *rebuts* argument $B$ (on $B'$) iff $Conc(A) \in \overline{\psi}$ for some $B' \in Sub(B)$ of the form $B''_1, \ldots B''_n \Rightarrow \psi$;

(iii) argument $A$ *undermines* argument $B$ (on $\psi$) iff $Conc(A) \in \overline{\psi}$ for some $\psi \in Prem(B)$.

An *ASPIC*$^+$ argumentation theory thus yields a set of arguments and a set of attacks, which can be considered as an abstract argumentation framework.

## 2.6. Abstract Argumentation

Abstract argumentation provides a mechanism for reasoning over directed graphs in which vertices are propositional labels corresponding to arguments, and edges between them are relationships of conflict or attack. Dung's original paper [32] rests upon a notion of acceptability of arguments which in turn is used to define (amongst others) a set of arguments which might be sceptically believed: the grounded extension. We adapt here the definition from [66] which is particularly concise.

An *abstract argumentation framework*, $AF$ is a pair $(AR, Att)$ where $AR$ is a set of arguments and $Att \subseteq AR \times AR$ is a binary relation of attack. A *semantics* for $AF$s

returns sets of arguments called *extensions*, which are internally coherent and defend themselves against attack. For an $AF$ $(AR, Att)$, for any $X \in AR$, $X$ is *acceptable* with respect to some $S \subseteq AR$ iff $\exists Y$ s.t. $(Y,X) \in Att$ implies $\exists Z \in S$ s.t. $(Z,Y) \in Att$. Let $S \subseteq A$ be *conflict free*, i.e., there are no $A$,$B$ in $S$ such that $(A,B) \in Att$. Then, $S$ is a *complete* extension iff $X \in S$ whenever $S$ is acceptable with respect to $S$ and $S$ is a *grounded* extension, $GE$, iff it is a complete extension that is minimal with respect to set inclusion.

# 3. Formalisation of Dialectical Interaction: From the Lakatosian model of mathematical discourse to a formal dialogue system

In this section we express the Lakatos model of collective proof as a formal dialogue system, the Lakatos Game (LG), understood in a way of standard representation for persuasion dialogues introduced in [79] and described in Section 2.3. We assume that this type of dialectical interaction is in fact the type of persuasion, since the initial situation of conflicting opinions on whether the conjecture is true or not is sought to be resolved by the players through communication [103]. However, the nature of the interaction is not "selfish" (as in the most radical type of persuasion where a player is interested only in the situation in which s/he is winning the game), because both parties ultimately aim to collaboratively test whether the conjecture is true and whether it can be proved, no matter who of them will win (such a type of persuasion is called collaborative conflict resolution in [19]).

LG determines rules for playing a game of collaborative mathematics and in this sense it is a formal representation of informal theory introduced by Lakatos. The rules of the LG system are designed as a bridge between the spirit of the loosely defined techniques in [55] and the formality required for expressing dialogues as complete specifications that, ultimately, can be implemented. According to the standard introduced in Section 2.3, LG is specified through five types of rules: (1) locution rules which determine what types of moves players are allowed to perform during the dialogue (i.e. what are legal locutions), and, uniquely for LG, how these moves update the current mathematical *theory* in which the co-constructed proof is sited; (2) structural rules which regulate what types of responses are allowed to be given (i.e. what are legal responses); (3) commitment rules which specify a set of propositions to which players will be committed as a result of performing a given locution during the dialogue;[5] (4) termination rules which describe when the dialogue will end; and (5) outcome rules which determine what the result of a dialogue is.

The "players" in our Lakatos Games are a Proponent and an Opponent. These roles are defined relative to a given conjecture. Multiple speakers may contribute to these

[5] In formal dialogue systems, a commitment store is used to keep track of propositions to which a player is committed, i.e. to which has e publicly declared belief (though this does not mean necessarily that a player really does believe the proposition, since he can be insincere).

roles, and both the Proponent and Opponent role can be voiced by the same speaker. This allows a speaker to avoid contradictory commitments, as long as s/he does not take on both roles simultaneously. A player who takes the same role as the other speaker has to commit to the same commitments and the same strategy of proof as his predecessor(s) as the proof is done collaboratively. In other words, the game does not distinguish between participant and roles.

## 3.1. LG System: Overview

The overall goal of the Lakatosian dialogue is to explore a mathematical theory and construct new examples, concepts, definitions, conjectures and proofs. The analysis of a fledgling proof and conjecture, via the patterns of discourse that Lakatos identified, provides a mechanism by which such a theory can grow. The Lakatos Game fulfills the definition of persuasion dialogue specified in [103, 79], since the dialogue starts with a conflict of opinion about a conjecture, and aims at resolution of the conflict. In this type of dialogue, Opponent, $O$, of a proof disagrees with Proponent, $P$, and they argue about elements of the theory from which the proof is constructable. The parties aim to collaboratively solve the problem rather than winning the game at any cost, thus this type of persuasive interaction is called collaborative conflict resolution [19].

**Notation.** Let $b,c,d,e,f,g,k,l,m,n,r,s$ be propositional variables. For clarity in description, we consistently use specific variables to describe specific objects in Lakatos's model:[6]

- $c,b$: conjecture ($c$ – current conjecture, $b$ – new conjecture)
- $l,k$: lemma ($l$ – current lemma, $k$ – new lemma)
- $m,n$: counterexamples which may become monsters, if they turn out to be invalid
- $d,e,f,g$: definitions
- $r,s$: propositions used to support or contradict counterexamples

The syntax of the system allows for high level (abstract) description of the legal moves and interactions that the players can execute during the LG dialogue. In the actual game of collaborative proof creation, the variables are then instantiated by the specific statements in natural language such as in Example (1) in Section 7 and examples in Appendix A.

In the following sections we describe the locution rules (Section 3.2), structural rules (Section 3.3), and commitment, termination and outcome rules (Section 3.4) which define our formalisation of the LG System.[7]

[6] Note that we use variables, in the strict sense, to refer to propositions, e.g. $m$ refers to stating that something is a counterexample. However, we use them also as shortcuts to refer to the objects described by these statements, e.g. we use $m$ to refer to the counterexample itself.

[7] Notice that the terminology adopted in this paper might be slightly unclear or ambiguous across different traditions and disciplines that this work builds upon. Although, in fact all Sections 3–5 talk about some

## 3.2. LG System: Locution rules

We specify all of the legal locutions in the LG system by the rules below, describing informally how the mathematical theory that is being co-constructed by the participants is updated by each of them; a more formal account showing how these updates are characterised in AIF and abstract argumentation frameworks then occupies much of the remainder of the paper.

Proponent and Opponent's behaviour during a collaborative proof is regulated by the LG locution rules (see below for their formal specification). In general, they allow the Proponent to create a proof (see locution rule **L1**); defend it against Opponent's attacks (counterexample, critique; see the rules **L2**-**L5**); or surrender the current conjecture (**L7.2**). Conversely, the LG locution rules allow *O* to: attack the proof, i.e. attack the conjecture or a lemma (**L2**); participate in the modification of a concept (see **L5.4**-**L5.5**); decide whether the counterexample is a monster or not (**L6**); or accept the current conjecture and the current lemmas (**L7.1**).

---

**LG Locution Rules**

**L1 Creation of proof**

1. *Conjecture(c)* asserts a conjecture *c*;
   *c* is added to the current theory
2. *Lemma(l)* asserts a lemma *l*;
   *l* is added to the current theory
3. *ProofDone* announces that a proof for the current conjecture is complete, and adds that inference, comprising the lemmas and the inference from them to the conjecture, to the theory (in place of the component parts)[8]

**L2 Attack on conjecture or lemmas**

1. *LocalCounter(m, l)* asserts a counterexample *m* that contradicts with a lemma *l*;
   *l* is removed from the current theory, and as a consequence, so is the proof as a whole from the lemmas to the conjecture
2. *HybridCounter(m, l, c)* asserts a counterexample *m* that contradicts with a conjecture *c* and a lemma *l*;
   is removed from the current theory, and as a consequence, so is the proof as a whole from the lemmas to the conjecture
3. *GlobalCounter(m, c)* asserts a counterexample *m* which supports a counter-conjecture not-*c*;
   the proof as a whole from lemmas to conjecture is removed, whilst the counterexample is added to the current theory

**L3 Defence by modifying conjecture**

1. *PiecemealExclusion(b)* asserts a new conjecture *b*;
   *b* is added to the current theory
2. *StrategicWithdrawal(r, c)* asserts *r* which contradicts with the current conjecture *c*;
   the entire proof from lemmas to conjecture is removed from the current theory

**L4 Defence by manipulating lemmas**

1. *LocalLemmaInc(m, l, k)* asserts a new lemma *k* that incorporates the counter *m* into an existing lemma *l* with a view to replacing *l*;
   *k* is added to the current theory
2. *HybridLemmaInc(m, l)* retracts the lemma *l* in response to the counter *m*;
   no update to the current theory
3. *GlobalLemmaInc(m, k)* asserts a new lemma *k* which contradicts the counterexample *m*;
   *m* is removed from the current theory

---

formal aspect of Lakatos theory, we use the phrase "formalisation" specifically to the system introduced in Section 3 in order to reflect and emphasise the idea that LG aims to describe in the precise manner the informal, philosophical model proposed by Lakatos. Then, for Sections 4 and 5, we use terms "representation" and "evaluation" to make the reference to two traditionally distinguished key stages or areas of study of argumentation (the distinction introduced by Aristotle and then wildly used in argumentation theory).

[8]This replacement is not a feature intrinsic to Lakatos' account, but is a convenience introduced here to harmonise the style of representing inference used later – nothing of significance hangs upon this way of characterising inference and entailment.

**L5 Defence by modifying concept**

1. *MonsterBar(m, c, r)* asserts $r$ which contradicts the justification that the counter-conjecture not-$c$ holds because of $m$;
   $r$ is added to the theory, and as a result, the entire proof from lemmas to conjecture is reinstated
2. *MonsterAdjust(m, r)* asserts $r$ which contradicts with $m$;
   $r$ is added to the theory, $m$ is removed, and as a result, the entire proof from lemmas to conjecture is reinstated
3. *PDefinition(m, r, d)* asserts the definition $d$ (performed by proponent) which supports $r$ used to show that $m$ is not a valid counterexample;
   $d$ is added to the theory, and $r$ is replaced by the argument from $d$ to $r$ [8]
4. *ODefinition(m, r, d, s, e)* asserts the definition $e$ (performed by opponent) which contradicts $r$, and supports $s$;
   $d$ is removed from the theory
5. *Prefer(m, r, f, g)* prefers the definition $f$ over the definition $g$ in response to the claim that $m$ is a monster, and the claim's justification $r$;
   the definition $f$ and the argument that follows from it are added to the current theory; in case the move is opponent's (i.e. preferring proponent's definition), the original proof as a whole from lemmas to conjecture is reinstated

**L6 Decision about monster**

1. *MonsterAccept(m, r)* re-asserts $r$ in support of the monster, $m$;
   no update to the current theory
2. *MonsterReject(m, r, d, s, c)* asserts $s$ which contradicts with $r$ and joins the counter $m$ to supports the counter-conjecture not-$c$;
   the counterexample $c$ is added whilst both the arguments from $d$ to $r$ and from $l$ to $c$ are removed from the current theory

**L7 Decision about proof**

1. *Accept* expresses the acceptance the current conjecture and the current lemmas; no update to the current proof
2. *Surrender* expresses surrender of the current conjecture; no update to the current proof

---

**L1 Creation of proof.** For creating the proof, the player $P$ introduces a conjecture (**L1.1**) and lemma(s) (**L1.2**), and then announces the end of the informal proof by saying *ProofDone* which plays a role analogous to "■" in typeset mathematics (**L1.3**).

**L2 Attack on conjecture or lemmas.** For attacking the proof, Opponent has three strategies available: he can attack locally by introducing a counter $m$ to the lemma $l$ (*LocalCounter(m, l)*, **L2.1**); attack globally by introducing a counterexample $m$ to the conjecture $c$ (*GlobalCounter(m, c)*, **L2.3**); or attack in a hybrid way by introducing $m$ which counters both $c$ and $l$ (*HybridCounter(m, l, c)*, **L2.2**).

**L3-L5 Defending a proof.** For defending the proof against Opponent's attacks, Proponent has available three strategies: he can modify the conjecture (**L3**); manipulate lemmas (**L4**); or modify the definition of a concept in the conjecture or lemmas (**L5**). In the first case, $P$ has two alternative options of how to change the current conjecture $c$ to a new conjecture $b$ – he can either perform *PiecemealExclusion(b)* and directly introduce a new conjecture $b$ (**L3.1**); or perform *StrategicWithdrawal(r, c)* and introduce $r$ which contradicts with the current conjecture $c$ (see **L3.2**). In the second case, Proponent acts in some sense as Opponent, i.e. he doubts that his conjecture $c$ actually holds and uses $r$ to demonstrate it (and then replaces it with a new conjecture $b$, see the structural rule **S9**).

The second strategy of defending against a counter is the manipulation of lemmas in three different ways. First, $P$ can introduce a new lemma $k$ to replace the current

lemma *l* which was successfully countered by *m* (*LocalLemmaInc(m, l, k)*, **L4.1**). Second, Proponent can retract a lemma *l* which was countered by *m* (*HybridLemmaInc(m, l)*, **L4.2**). Third, he can add a new lemma *k* which will be countered by *m* (*GlobalLemmaInc(m, k)*, **L4.3**). The reason for proponent introducing a problematic lemma will become clear when we look at the next move that this player will perform, i.e. when we will discuss the structural rules below (see **S17** and **S18**).

Finally, *P* can defend the proof by modifying the definition of a concept in the conjecture and, as a result, demonstrating that *m* is not a valid counterexample, i.e. that *m* is in fact a monster. According to this strategy Proponent introduces the proposition *r* which contradicts with: either the inference from counterexample *m* to the counter-conjecture not-*c* (*MonsterBar(m, c, r)*, see **L5.1**); or directly the counter *m* (*MonsterAdjust(m, r)*, **L5.2**). In other words, in the first case Proponent does not claim that *m* does not hold, but he points out that *m* cannot be used to infer not-*c*. In order to justify that *r* holds, *P* introduces a definition of a concept in the conjecture (*PDefinition(m, r, d)*, **L5.3**). If Opponent provides his alternative definition (*ODefinition(m, r, d, s, e)*, **L5.4**), then the players have to decide which definition (*f* or *g*) they prefer (*Prefer(m, r, f, g)*, **L5.5**). Opponent's definition move takes more variables as a content than Proponent's definition move, because *O* not only attacks *P*'s definition, but also defends his move of rejecting that the counterexample is a monster *MonsterReject(m, r, d, s, c)* (see **L6.2** described below). More specifically, *O* introduces his own definition *e* which contradicts *P*'s definition *d* and at the same time *e* supports a proposition *s* which contradicts with Proponent's attack *r* from the move *MonsterBar(m, c, r)* or *MonsterAdjust(m, r)*. Note that the main difference between the strategy of modifying the conjecture and the strategy of modifying a concept in this conjecture lies in the consequences of these moves for a theory in which the proof is conducted. The modification of the conjecture influences only the conjecture itself, while the modification of the concept influences all the conjectures in the theory which contain this concept.

**L6 Decision about monster.** For deciding whether the counterexample is a monster, *O* can assert *r* which *P* previously introduced to demonstrate that *m* is a monster. In other words, Opponent agrees with Proponent by repeating what Proponent said to attack the counter (*MonsterAccept(m, r)*, **L6.1**). Alternatively, *O* can reject that *m* is a monster by asserting *s* which contradicts Proponent's attack *r* and is included as a linked premise into his argument that *m* supports the counter-conjecture not-*c* (*MonsterReject(m, r, d, s, c)*, **L6.2**).

**L7 Decision about proof.** For deciding about the acceptance of the proof, Proponent can only surrender the conjecture (*Surrender*, **L7.2**). For deciding about the acceptance of the proof, Opponent can only accept the proof (*Accept*, **L7.1**).

## 3.3. LG System: Structural rules

The dynamics of the LG System are determined by the structural rules which describe how the players can navigate through the space of the legal locutions (see below for their formal specification). Notice that each rule determines a set of possible reposes to this move, and the players have to choose one and only one of alternatives. A different

response can be chosen once the players will return to a given state. Figure 3 is a behaviour tree that summarises the permitted follow-up structure that is detailed below. In the figure, items with one greater level of indenting are potential follow-up moves. Terms that are not terminating and that have no descendants should be understood as links that redirect the flow to nodes that appear higher up in the tree.[9]

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**LG Structural Rules**

**S1** A player *P* moves first with *Conjecture(c)*; then each player contributes a locution according to the rules **S2** – **S19** with the restriction that if a move has some propositional content, then players cannot perform this move again with the same content

**S2** After *Conjecture(c)*:

1. if the lemma commitment store $\mathcal{L}$ is not empty, then *P* can:
   (a) end the proof: *ProofDone*
   (b) introduce a lemma *l*: *Lemma(l)*
2. if the lemma commitment store is empty, then *P* must introduce a lemma *l*: *Lemma(l)*

**S3** After *Lemma(l)*, *P* can:

1. perform a sequence of locutions introducing some finite number of additional lemmas: *Lemma(k)*, ...
2. end the proof of a current conjecture by saying: *ProofDone*

**S4** After *ProofDone*:

1. *O* can introduce a counterexample *m* to the conjecture *c*: *GlobalCounter(m, c)*
2. *O* can introduce a counterexample *m* to a lemma *l*: *LocalCounter(m, l)*
3. *O* can introduce a counterexample *m* to *l* and *c*: *HybridCounter(m, l, c)*
4. *P* can attack the current conjecture *c* with *r*: *StrategicWithdrawal(r, c)*
5. *O* can accept the proof: *Accept*

**S5** After *GlobalCounter(m, c)*, *P* can reply:

1. *PiecemealExclusion(b)* introducing a new conjecture *b*
2. *StrategicWithdrawal(r, c)* attack the current conjecture *c* with *r*
3. *MonsterBar(m, c, r)* introducing *r* which contradicts with the justification that the counter-conjecture not-*c* holds because of the counterexample *m*
4. *MonsterAdjust(m, r)* introducing *r* which contradicts *m*
5. *GlobalLemmaInc(m, k)* introducing a new lemma *k* which contradicts *m*
6. *Surrender*

**S6** After *LocalCounter(m, l)*, *P* can reply:

1. *LocalLemmaInc(m, l, k)* introducing a new lemma *k* which replaces *l* and incorporates the counter *m*
2. *Surrender*

**S7** After *HybridCounter(m, c, l)*, *P* can reply:

1. *HybridLemmaInc(m, l)* retracting a lemma *l* which was countered by *m*
2. *Surrender*

**S8** After *PiecemealExclusion(b)*:

1. if the lemma commitment store is not empty, then *P* can:
   (a) end the proof for the new conjecture *b*: *ProofDone*
   (b) introduce a new lemma *k*: *Lemma(k)*
2. if the lemma commitment store is empty, then *P* introduces a lemma: *Lemma(k)*

**S9** After *StrategicWithdrawal(r, c)*, *P* introduces a new conjecture: *Conjecture(b)*

**S10** After *MonsterBar(m, c, r)*, *P* introduces a definition *d* which justifies the proposition *r* contradicting the justification that not-*c* holds because of *m*: *PDefinition(m, r, d)*

**S11** After *MonsterAdjust(m, r)*, *P* introduces a definition *d* which justifies the proposition *r* contradicting the counter *m*: *PDefinition(m, r, d)*

**S12** After *PDefinition(m, r, d)*, *O* can reply:

[9] A diagram of this process is also available in the form of a finite-state machine, online at http://arg.tech/lakatosFSM.

1. *MonsterAccept(m, r)* agreeing that *m* is a monster because of *r*
2. *MonsterReject(m, r, d, s, c)* disagreeing that *m* is a monster with respect to *c* by asserting *s* which contradicts with *r* justified by *d*

**S13** After *MonsterAccept(m, r)*:

1. *O* can introduce a counterexample *n* to the conjecture *c*: *GlobalCounter(n, c)*
2. *O* can introduce a counterexample *n* to a lemma *l*: *LocalCounter(n, l)*
3. *O* can introduce a counterexample *n* to *c* and *l*: *HybridCounter(n, l, c)*
4. *P* can attack the current conjecture *c* with *s*: *StrategicWithdrawal(s, c)*
5. *O* can accept the proof: *Accept*

**S14** After *MonsterReject(m, r, d, s, c)*, *O* can introduce a definition *e* which contradicts with Proponent's definition *d* and justifies the proposition *s* asserted in the move *MonsterReject*: *ODefinition(m, r, d, s, e)*

**S15** After *ODefinition(m, r, d, s, e)*:

1. *O* can introduce a preference of Proponent's definition *d* over her own definition *e*: *Prefer(m, r, d, e)*
2. *P* can introduce a preference of Opponent's definition *e* over her own *d*: *Prefer(m, r, e, d)*

**S16** After *Prefer(m, r, f, g)*:

1. if *Prefer(m, r, f, g)* is performed by *P* after *MonsterAdjust(m, r)*, then *P* can reply:
   (a) *PiecemealExclusion(b)* introducing a new conjecture *b*
   (b) *StrategicWithdrawal(s, c)* attacking the current conjecture *c* with *s*
   (c) *MonsterBar(m, c, s)* introducing *s* which contradicts the justification that the counter-conjecture not-*c* holds because of the counter *m*
   (d) *GlobalLemmaInc(m, k)* introducing a new lemma *k* which contradicts *m*
   (e) *Surrender*
2. if *Prefer(m, r, f, g)* is performed by *P* after *MonsterBar(m, c, r)*, then *P* can reply:
   (a) *PiecemealExclusion(b)* introducing a new conjecture *b*
   (b) *StrategicWithdrawal(s, c)* attacking the current conjecture *c* with *s*
   (c) *MonsterAdjust(m, s)* introducing *s* which contradicts the counter *m*
   (d) *GlobalLemmaInc(m, k)* introducing a new lemma *k* which contradicts *m*
   (e) *Surrender*
3. if *Prefer(m, r, f, g)* is performed by *O*, then:
   (a) *O* can introduce a counterexample *n* to the conjecture *c*: *GlobalCounter(n, c)*
   (b) *O* can introduce a counterexample *n* to a lemma *l*: *LocalCounter(n, l)*
   (c) *O* can introduce a counterexample *n* to *c* and *l*: *HybridCounter(n, l, c)*
   (d) *P* can attack the current conjecture *c* with *s*: *StrategicWithdrawal(s, c)*
   (e) *O* can accept the proof: *Accept*

**S17** After *GlobalLemmaInc(m, k)*, *P* retracts the lemma *k* which was countered by *m*: *HybridLemmaInc(m, k)*

**S18** After *HybridLemmaInc(m, l)*, *P* replies *Conjecture(b)*

**S19** After *LocalLemmaInc(m, l, k)*, *P* replies *ProofDone*

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**S1-S4 Creation of proof**. At the beginning of the game Proponent creates an initial, draft proof. His first move is to introduce a conjecture *c* (rule **S1**). At that point no lemma has been introduced yet, so he has to propose at least one lemma *l* according to **S2.2**. This rule is also used to prohibit Proponent from removing all lemmas and ending the proof by saying *ProofDone* (this is allowed by the rule **S2.1**). Then, Proponent can end the proof (**S3.2**); or continue adding further lemmas (**S3.1**) and then end the proof (**S3.2**). If Proponent is not introducing the initial version of the proof, but is introducing changes to the conjecture by performing *Conjecture(c)* and there is still at least one lemma in the proof, then he can either continue introducing additional lemmas (**S2.1b**), or end the proof (**S2.1a**).

After the initial version of the proof is ended, there are three possible ways of con-

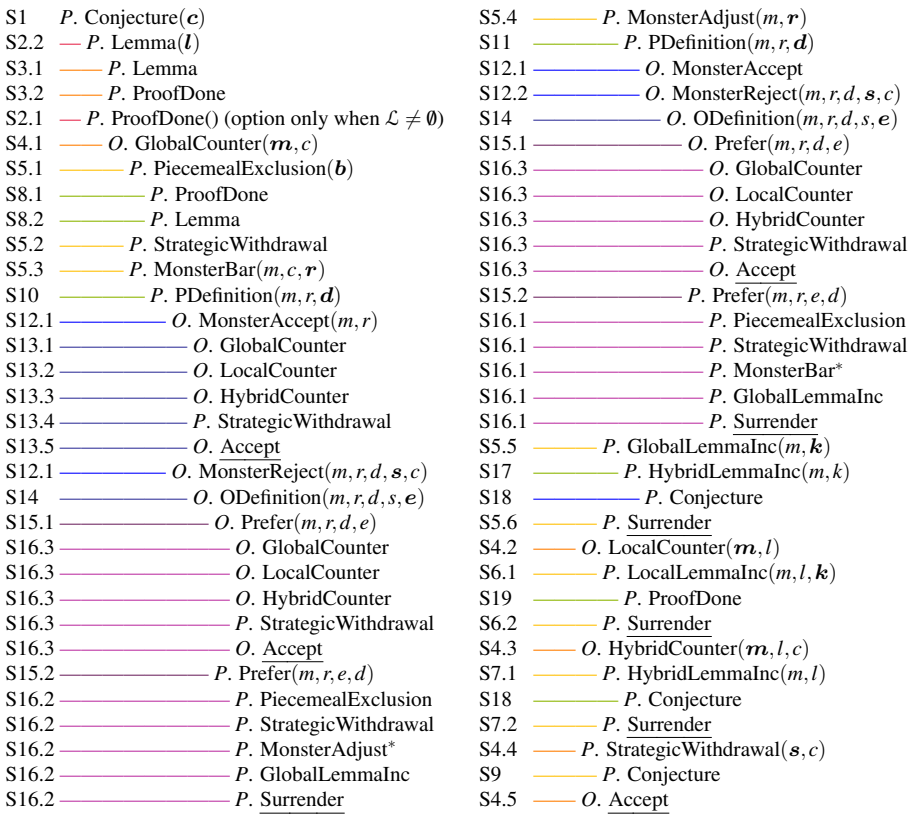



Figure 3: Behaviour Tree for the LG system. Bold indicates where a new variable has been introduced in a given path down the tree. The names of moves without any variables represent links to positions that appear higher up in the tree. Underlining indicates moves that end the game. Note that structure under MonsterBar and MonsterAdjust is similar but not identical: the point of difference is indicated with asterisks.

tinuing the game: either Opponent attacks the proof by introducing a counterexample *m* to the conjecture, a lemma, or both the conjecture and a lemma (**S4.1**-**S4.3**) together; or Proponent changes his mind and introduces a new conjecture in order to prevent the proof from a potential future attack from opponent (i.e. a potential counterexample for which the old conjecture might not hold, **S4.4**); or Opponent accepts the proof (**S4.5**).

**S5-S7 Attack on conjecture or lemmas.** Opponent's attack on the conjecture, via *GlobalCounter(m, c)*, generates the largest number of possible alternative paths through the Lakatos game (**S5**, see also Figure 3). First, *P* can immediately give up, i.e., he can become convinced of Opponent's critique and decide to surrender the proof (**S5.6**). Another way to respond is to only partly agree with *O* and modify the conjecture so that the attack fails (**S5.1-S5.2**). Proponent can introduce the changes directly either by piecemeal exclusion, i.e., introducing a new conjecture *b* (**S5.1**), or by strategically withdrawing (**S5.2**), i.e., expressing doubts about the current conjecture *c* first.

Opponent's attack on a lemma, *LocalCounter(m, l)*, and an attack on both a lemma and the conjecture, *HybridCounter(m, c, l)*, generate much simpler histories for LG dialogue games. In the first case, Proponent can either replace a lemma with a new one which incorporates Opponent's critique (*LocalLemmaInc(m, l, k)*, **S6.1**) and then he announces that the new proof (with the new lemma) for the old conjecture is completed by saying *ProofDone* (**S19**); or *P* can agree with the critique and surrender the proof (**S6.2**). In the case of *HybridCounter* attack, *P* can either retract a lemma which was countered by *m* (*HybridLemmaInc(m, l)*, **S7.1**) and then incorporate the critique into a new conjecture *b* (*Conjecture(b)*, **S18**); or *P* agrees with the critique and surrenders the proof (**S7.2**).

**S8-S9 Defence by modifying conjecture.** The difference between these piecemeal exclusion and strategic withdrawal is that *PiecemealExclusion(b)* has to be always triggered by Opponent's attack, while *StrategicWithdrawal(r, c)* can be performed at any point of the game when Proponent changes his mind about the acceptance of the conjecture, even immediately after ending the initial proof (see **S4.4**). After *StrategicWithdrawal(r, c)*, *P* introduces a new conjecture *b* (**S9**).[10] In both cases, after introducing the new conjecture *b*, *P* can propose some additional lemmas or end the new proof in the same way as he did at the beginning of the game (see **S8** for excluding piecemeal and **S2** for strategically withdrawing).

**S10-S16 Defence by modifying concept and decision about monster.** The final type of response to *GlobalCounter(m, c)* is for Proponent to entirely disagree with Opponent's global counter. In other words, *P* tries to show that *m* is not a valid counterexample (i.e. that *m* is a monster) either indirectly by introducing *r* which attacks the inference between *m* and the counter-conjecture not-*c* (*MonsterBar(m, c, r)*, see **S5.3**); or by introducing *r* which directly attacks the counter *m* itself (*MonsterAdjust(m, r)*, see **S5.4**). Both of these moves require Proponent to formulate a new definition *d* for a

[10]Notice that the rule for *PiecemealExclusion(b)* in **S8** repeats the specification for *Conjecture(c)* in **S2**, which means that in the first case Proponent again introduces a new conjecture directly as a response to Opponent's attack rather than doing so via a *Conjecture(b)* move.

concept in the current conjecture which justifies the attack *r* (**S10** and **S11**). Next, Opponent can accept the definition and as a result agree that *m* is the monster (*MonsterAccept(m, r)*, **S12.1**); or reject it and assert *s* which contradicts with *r* (*MonsterReject(m, r, d, s, c)*, **S12.2**). If *O* accepts that *m* is a monster, then he can try to attack using a different counterexample to the conjecture, a lemma or both (**S13.1-S13.3**); or he can become convinced and decide to accept the proof (**S13.5**); alternatively Proponent can modify the conjecture (**S13.4**). If *O* rejects that *m* is a monster, then he has to propose his own definition *e* of this concept which supports his claim *s* which is in contradiction to Proponent's claim *r* (**S14**). Then, one of the players has to decide which definition is preferred: either Opponent prefers Proponent's definition over his own (**S15.1**), or Proponent prefers Opponent's definition (**S15.2**).[11] If Proponent expresses the preference, then *P* can next: change the conjecture through piecemeal exclusion (**S16.1.a** and **S16.2.a**) or strategic withdrawal (**S16.1.b** and **S16.2.b**); try to show again that *m* is not a valid counterexample by monster-adjusting (if previously he did monster-barring (**S16.1.c**)), or by monster-barring (if previously he did monster-adjusting (**S16.2.c**)); introduce a new lemma that contradicts *m* (**S16.1.d** and **S16.2.d**); or surrender the proof (**S16.1.e** and **S16.2.e**). If Opponent expresses the preference, then: *O* can attack again by introducing another counter to the conjecture (**S16.3.a**), or a lemma (**S16.3.b**), or to both (**S16.3.c**); *P* can change the conjecture by strategically withdrawing (**S16.3.d**); or, finally, *O* can become convinced and decide to accept the proof (**S16.3.e**).

**S17-S19 Defence by manipulating lemmas.** Proponent can also introduce the changes to the conjecture indirectly, i.e., by firstly modifying a lemma and then modifying the conjecture. More specifically, first *P* performs *GlobalLemmaInc(m, k)* which introduces a new lemma *k* contradicting the counter *m* (**S5.5**). This lemma reveals the hidden assumption that needs to hold in order for the conjecture to hold too. Then, *P* continues with hybrid lemma incorporation which retracts the recently introduced lemma *k* (**S17**) and then incorporates the hidden assumption into a new conjecture *b* (**S18**).

## 3.4. LG System: Commitment rules, termination rules and outcome rules

The effects of the players' interactions are determined by three final types of rules: the commitment rules determine the effects that these interactions have on commitment stores of the players; termination rules define when the dialogue ends; and outcome

[11] Notice that it is prohibited by the LG rules for both players to prefer each others definitions, or for neither of them to prefer the other's definition. According to the rule **S15**, the players have to choose one and only one of the *Preference* moves – either Proponent will chose to prefer Opponent's definition or vice versa. As soon as one of them does the *Preference* move, the set of available moves change (described now by the rule **S16**) and the other player cannot perform the *Preference* move anymore. The protocol determines what should be the next move in the game to match the idea of collaborative mathematics proposed in the Lakatos model, but it does not determine what the players should say. In other words, according to the model the players should resolve between themselves which of alternative definitions is better in order to continue collaborative proof.

rules describe who won the dialogue. See below for their formal specification for the LG System.

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**LG Commitment Rules**

**C1 Conjecture commitment store**

1. After *P* performs *Conjecture(b)* or *PiecemealExclusion(b)*, the conjecture store is emptied and the propositional content *b* is included in the store

**C2 Lemma commitment store**

1. After *P* performs *Lemma(k)* or *GlobalLemmaInc(m, k)*, the content *k* is included in the lemma commitment store
2. After *P* performs *LocalLemmaInc(m, l, k)*, *l* is removed from the store and *k* is added
3. After *P* performs *HybridLemmaInc(m, l)*, *l* is removed from the store

**LG Termination Rules**

**T1** A dialogue terminates if either *Accept* or *Surrender* is performed

**LG Outcome Rules**

**O1** Proponent wins, if a dialogue terminates with *Accept*

**O2** Opponent wins, if a dialogue terminates with *Surrender*

---

In the LG system for collaborative mathematics, there are two types of commitment stores: the store for the conjecture (see **C1**) and the store for lemmas (**C2**). These stores keep track of the currently posited lemmas and conjecture. Most commitment updates are forced by Opponent's attacks (i.e., when Proponent is not able to defend against an attack, then he is forced to change the current proof). Similarly, the information about the other propositions used by players in counterexamples, definitions and so on, is not stored at all, since they do not contribute directly to the proof – they only influence its shape during the game.

Proponent can add a conjecture *b* into the conjecture commitment store by performing either *Conjecture(b)* or *PiecemealExclusion(b)* (**C1**). This type of store has to consist of only one proposition at any time of the game (i.e., it has to consist of the current conjecture), therefore both of these moves first empty the store to remove the old conjecture (if there is one), and then adds a new conjecture *b*. Further, *P* can add lemmas to, and remove them from, the lemma commitment store in three different ways: (i) adding a new lemma *k* either by performing *Lemma(k)* or *GlobalLemmaInc(m, k)* (**C2.1**); (ii) replacing lemma *l* with a new lemma *k* by performing *LocalLemmaInc(m, l, k)* (**C2.2**); or (iii) removing lemma *l* from the store by performing *HybridLemmaInc(m, l)* (**C2.3**).

The dialogue terminates when Opponent performs *Accept*, or when Proponent performs *Surrender* (see (**T1**)). Proponent wins if Opponent accepts the proof (**O1**); and

Opponent wins if Proponent surrenders the proof (**O2**).[12] Note that the phrase "win" is not used here to suggest that there is a competitive character to LG dialogues, since we assume, like Lakatos, that they are modelling a collaborative process. Our aim is just to stay also close to the standard terminology used in formal dialogue systems. As a result, Proponent's "winning" should be interpreted as the acceptance of the current conjecture and the current lemmas, and Opponent's "winning" should be interpreted as surrender of the current conjecture.

# 4. Graph-based Representation: From a formal dialogue system to Argument Interchange Format structures

The majority of research in the philosophy of dialogue focuses on normative, idealised models of interaction; only quite recently has there been an empirical turn that aims to connect such models with discursive practice. Normative games like DC [62] aim to make explicit features of dialogues which are latent in natural interactions (in the case of DC, cumulativity, whereby retraction is prohibited). Descriptive games such as PPD [103] aim to express natural interaction with more formal apparatus, thereby allowing dialogues to be assessed and guided (for comparative analysis of DC, PPD and many other games, see [105]). The Lakatos Game is designed with both normative and descriptive flavours, providing scaffolding and guidance for practical interactions, as well as tools that can be used to interpret and assess discourse. Crucially though, Lakatos games are not aimed, like DC and PPD, at establishing a common understanding, or a maieutic exploration of a space; they are instead aimed quite specifically at generating a proof, or, more specifically, at yielding a theory from which a proof can be extracted.

As a consequence of this normative-descriptive balance, the Lakatos Game provides a good test case for AIF, which similarly aims to handle both analysis of linguistic material as well as representation and evaluation of the norms of discourse context [25]. For although AIF structures are designed to handle structures of argument, their closest counterparts in formal logical systems are not formal theories or sets of propositions, but proofs. If the description of LG is adequate, therefore, it should yield AIF structures which can automatically be mapped to such proofs.

There is, however, a challenge. AIF is used as infrastructure for an interconnected web of debates and arguments that allows navigation through different modes, domains, types of argument, with the ability to extend, critique and adumbrate them via

[12] Notice that Proponent cannot perform *Surrender* when he is winning, because the structural rules allow him to do it only when he does not choose to defend the proof as a response to an attack. For example, in the case of the attack of *GlobalCounter(m, c)* (described by the series of the rules **S5-S7**), Proponent can immediately give up without a defence (**S5.6**), otherwise he chooses to recover the status of the proof and the defence makes him being in the winning position until the next attack. In the similar way, the structural rules allows Opponent to perform *Accept* only when he decided to not continue attacks, conceding as the result that Proponent managed to successfully defend his proof. For example, Opponent cannot reply *Surrender* to *LocalCounter*, because it is prohibited by the rule **S6**. After *LocalCounter*, the turn belongs to Proponent and he can either give up (conceding as the result that Opponent's attack is successful) or defend his position with *LocalLemmaInc(m, l, k)*.

a range of systems and tools. This infrastructure – collectively referred to as the Argument Web [16] – is founded upon several assumptions. One cornerstone is the assumption of cumulativity: once an argument has been committed to the Argument Web, it is there in perpetuity. Although individual arguers may change their position or retract their arguments, the arguments themselves must remain available so that others can refer to them, build upon them and revise them.

But if the Argument Web [16] is to capture the emerging proofs as they are developed, extended, redefined and reframed by a Lakatos game, how can the intrinsic nonmonotonicity be reflected by cumulative infrastructure? Our goal is to ensure that the semantics of the moves in the game, as they are defined in terms of AIF updates, should yield argument structures with a specific set of properties. These argument structures should be submittable to a simple, algorithmic procedure which will yield precisely the theory that represents the current shared understanding at any point in the game. To deliver this, the effects that each move has on AIF structures need to be carefully defined, then the argumentation semantics computation can be shown to yield exactly the correct subset corresponding to proof structures.

## 4.1. AIF structure update: AIF operations of *LG* locutions

For Proponent's $Conjecture(c)$ and $Lemma(l_i)$ moves, all that needs to be done is to add $c$ and the $l_i$ to the AIF graph. The $ProofDone$ move is used to establish the inferential connection between them (in AIF terms, it adds an RA-node with incoming edges from each of the $l_i$ and an outgoing edge to the $c$). In other words, the Lakatosian concept of a proof is being modelled as the inference or argument from lemmas to conjecture. The inference is established by using proponent's commitment stores, connecting all those propositions in the Lemma Commitment Store to that in the Conjecture Commitment Store. Figure 4 visualises the AIF structure available after proponent has offered a putative proof for a conjecture, $c$, based on lemmas $L_1$, $L_2$ and $L_3$. The visualisation adopts a common convention of AIF, with both propositions and relations between them (of inference, conflict, and later, preference and restatement) expressed in boxes, and with directionality (e.g. of inference), as defined in [25], indicated by directed edges. For convenience, each proposition is also indicated as belonging to either proponent (P) or opponent (O).

There are three types of counter that Opponent can offer in response to such an argument: countering the conjecture, countering one of the lemmas, or countering both conjecture and lemma simultaneously. These are referred to as $GlobalCounter$, $LocalCounter$ and $HybridCounter$, respectively. All three types of counter introduce conflict into the AIF structure:

- Opponent's $GlobalCounter(m,c)$ performs a total of four updates to the AIF structure. First, it adds $m$ to the graph and uses it as a premise in a new argument. The conclusion of that argument is the negation of the conjecture, $c$. The negation is glossed as "It is not the case that $c$", and the second step is to add the new element to the AIF. The relationship between $c$ and not-$c$ is clearly one of (symmetrical) conflict, so the third step is to add new CA nodes between $c$ and not-$c$. Finally, the AIF is also updated to reflect the argument from $m$ to not-$c$

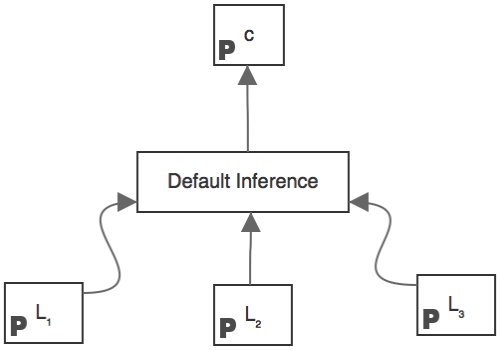



Figure 4: Lemmas, conjecture and proof

with a new RA between them.

- Opponent's $LocalCounter(m,l)$ simply introduces the counterexample, $m$ and the symmetrical conflict (i.e. a pair of CA nodes) between that and the lemma, $l$.
- Opponent's $HybridCounter(m,l,c)$ performs the same update as $LocalCounter$, with additional conflict added between $m$ and $c$. We might expect $HybridCounter$ to cause updates that constitute the union of the effects of $GlobalCounter$ and $LocalCounter$, introducing an attack with the lemma and a support for a counter-conjecture. Instead though, $HybridCounter$ uses a simpler characterisation of attack between $m$ and $c$, introducing a CA directly, rather than adding an argument for the negation of the conjecture. The reason for this discrepancy is that the negation is required for the action of future moves permitted by LG after $GlobalCounter$ (*viz.* $MonsterBar$ and $MonsterAdjust$), whilst these moves are not legal following $HybridCounter$, so the characterisation can be simpler.

These three updates for handling opponent counters are visualised in Figure 5. Proponent's eight possible substantive responses (excluding *Surrender*) to a counter then update the AIF graph further:

- $PiecemealExclusion$ allows for a revised conjecture, $b$, to be asserted, and demands a new $ProofDone$ move to create the inference from all of the current lemmas to the new conjecture. Note that this creates a new inference (i.e. a new RA-node) – the old one remains in the AIF graph.
- $StrategicWithdrawal$ allows Proponent to alter the conjecture, first through introducing a reason, $r$, for rejecting the current conjecture, then, through the obligatory subsequent $Conjecture$ move, proposing a new conjecture. The AIF is updated to reflect the addition of the new piece of information, $r$, plus the symmetrical conflict between that and the old conjecture, $c$.

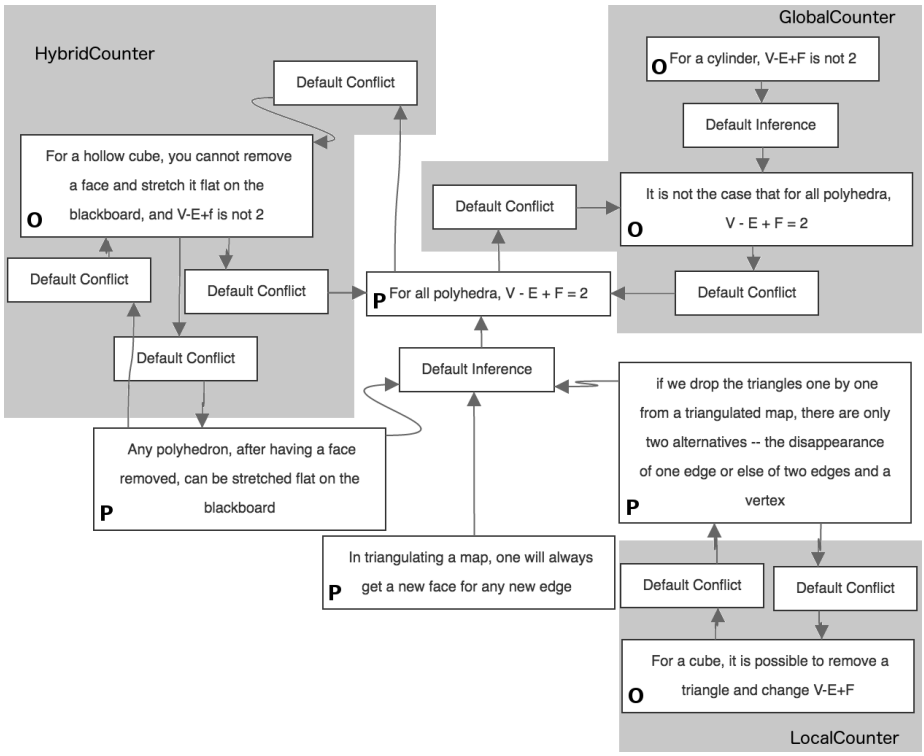



Figure 5: Global, Local and Hybrid counters

- *MonsterBar* allows Proponent to challenge the counterexample offered by Opponent by changing the definition of terms upon which it depends. The *MonsterBar* move itself updates the AIF in two steps. First, it adds a reason, $r$, for rejecting the counterexample and treating it as a monster. Then a conflict is established from $r$ to the RA-node constituting Opponent's extant argument from counterexample, $m$, to counter-conjecture, not-$c$. That is, $r$ is treated as an undercutter. These results are visualised in Figure 9.

- *MonsterReject*-ing allows Opponent the same freedom as Proponent in offering a reason for thinking the counterexample does, indeed hold. This reason $s$ is naturally in conflict with Proponent's reason that the counterexample does not hold, so the AIF updates are first to add $s$ and to link it as a further premise in the argument from the example, $m$, to the counter-conjecture, not-$c$ (that is, to connect $s$ to the RA-node between $m$ and not-$c$). Then the symmetrical conflict between $s$ and $r$ is also added. These updates are also visualised in Figure 9.

- *MonsterAdjust*-ing follows the same pattern as *MonsterBar*-ing, except that the counterexample is attacked directly by redefinition. That is, *MonsterAdjust* introduces a counter, $r$, and the symmetrical conflict between $r$ and $m$. An intuition

remains, however, for *MonsterAdjust*: that the counter to $r$ somehow trumps the monster. To capture this intution, we further add a preference(PA) that allows $r$ to defeat $m$. Both *MonsterBar* and *MonsterAdjust* then open a phase of the dialogue in which definitions are introduced (see below). The results of this move are visualised in Figure 10.

- *HybridLemmaInc*-orporation, following a *HybridCounter* paves the way for the introduction of a revised conjecture, $c'$ which incorporates one lemma, $l_i$. The update, however is performed by a subsequent *Conjecture* move.

- *GlobalLemmaInc*-orporation makes explicit a hidden assumption by adding it as a new lemma, $k$. It is also responsible for introducing the conflict between the counterexample, $m$ (identified in the preceding *GlobalCounter*) and this new lemma. It then forces a *HybridLemmaInc*-orporation so that $k$ is incorporated into a new conjecture and thence proof.

- Finally, *LocalLemmaInc*-orporation works in a very similar way to *GlobalLemmaInc*-orporation, except that it introduces a revised lemma $k$ before adding a new inference connecting all lemmas and (unchanged) conjecture through a mandatory *ProofDone* move. In addition, AIF provides the ability to represent non-inferential relations between propositions, including paraphrastic relationships such as restatement. We can capture the 'incorporates' relationship in this way as an MA node, but this has no impact on the argument frameworks – structured and abstract – that are created, functioning currently only to make the linguistic relationship. The AIF updates effected by the three types of lemma incorporation are visualised in Figure 6 – 8.

The moves of *MonsterBar* and *MonsterReject* can lead to further exchanges regarding the definitions upon which the counterexample depends, each of which provides further updates to the AIF structure:

- *PDefinition* allows Proponent to offer a definition, $d$ captured as a proposition in the AIF structure, which is then linked through an inference to the reason, $r$ for barring or adjusting a monster (from a *MonsterBar* or *MonsterAdjust* move). So *PDefinition* adds both the proposition $d$ and the RA from that to $r$. Proponent's definition can be *MonsterAccept*-ed or *MonsterReject*-ed. (The idea of definitions being used as the bases for arguments is one that is familiar in the philosophy of argument – see, e.g, [1]).

- *MonsterAccept*. In the former case, there is no further AIF update because Opponent is simply reiterating Proponent's position. In the latter case, Opponent can *MonsterReject*, which has already been treated, above.

- *ODefinition* allows Opponent to offer an alternative definition, which requires update to the AIF structure not only to include the new definition, $e$, but also (i) the conflict with Proponent's definition and (ii) the inference from $e$ to Opponent's reason, $s$, for thinking the counterexample holds.

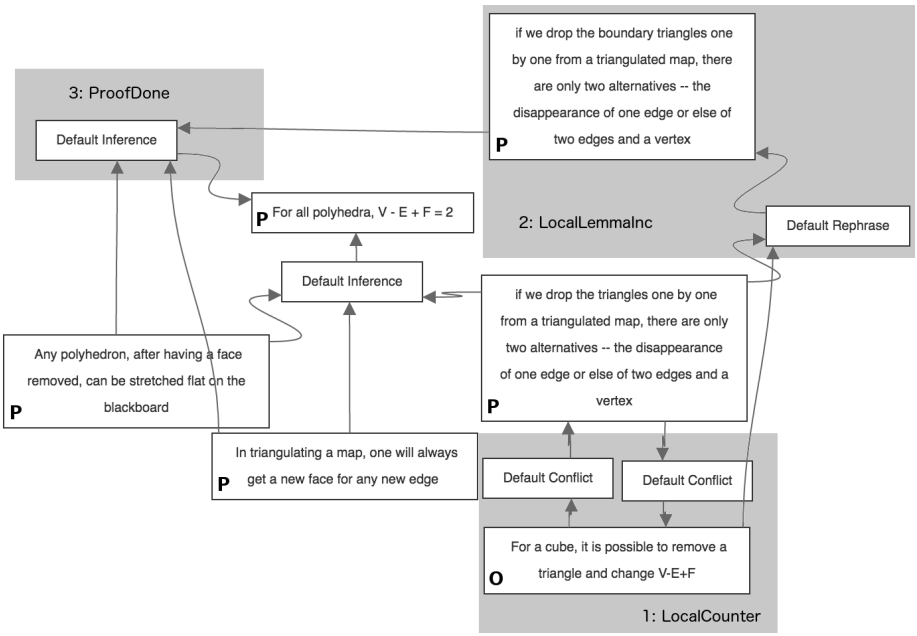



Figure 6: Local Lemma Incorporation

- *Prefer* allows resolution between competing definition by introducing a relation of preference between two definitions. The move thus introduces a PA node between definitions $f$ and $g$ (the variables are thus named to emphasise that *either* Proponent's definition $d$ *or* Opponent's definition $e$ may be preferred).

The effects of all of these moves are shown in Figures 9 and 10, which show the structures built up through the monster barring and monster adjusting pathways, respectively. Other moves described in the previous section impact the dialogical dynamics of the game (i.e. of what can be said, and of what commitments exist in stores) but do not update AIF structures beyond that.

# 5. Evaluation using Structured and Abstract Argumentation Frameworks: From AIF structures to abstract argumentation frameworks

AIF structures are one way to handle 'structured argumentation.' Other approaches, such as that provided by *ASPIC*$^+$ [80], have been shown to be compatible, in that it is

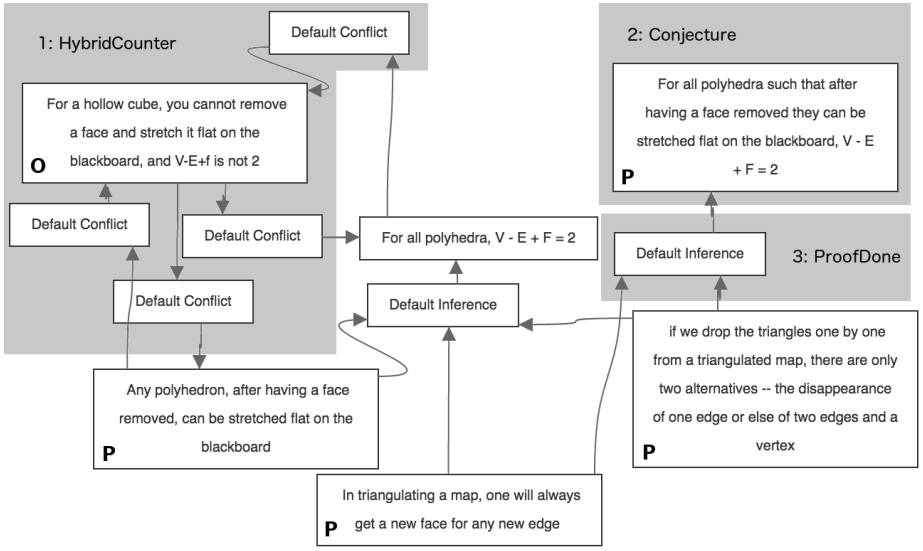



Figure 7: Hybrid Lemma Incorporation

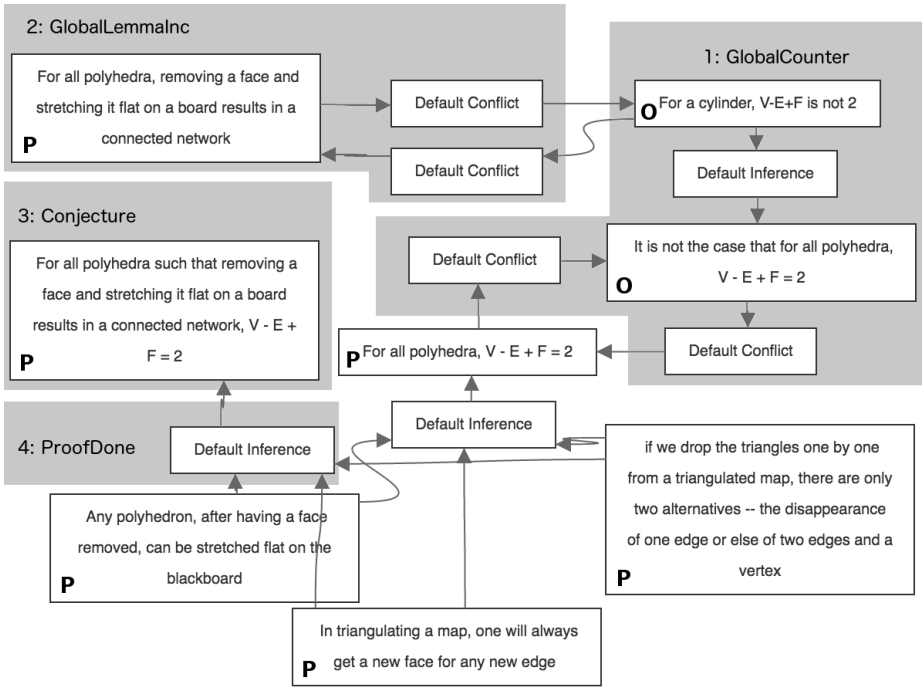



Figure 8: Global Lemma Incorporation

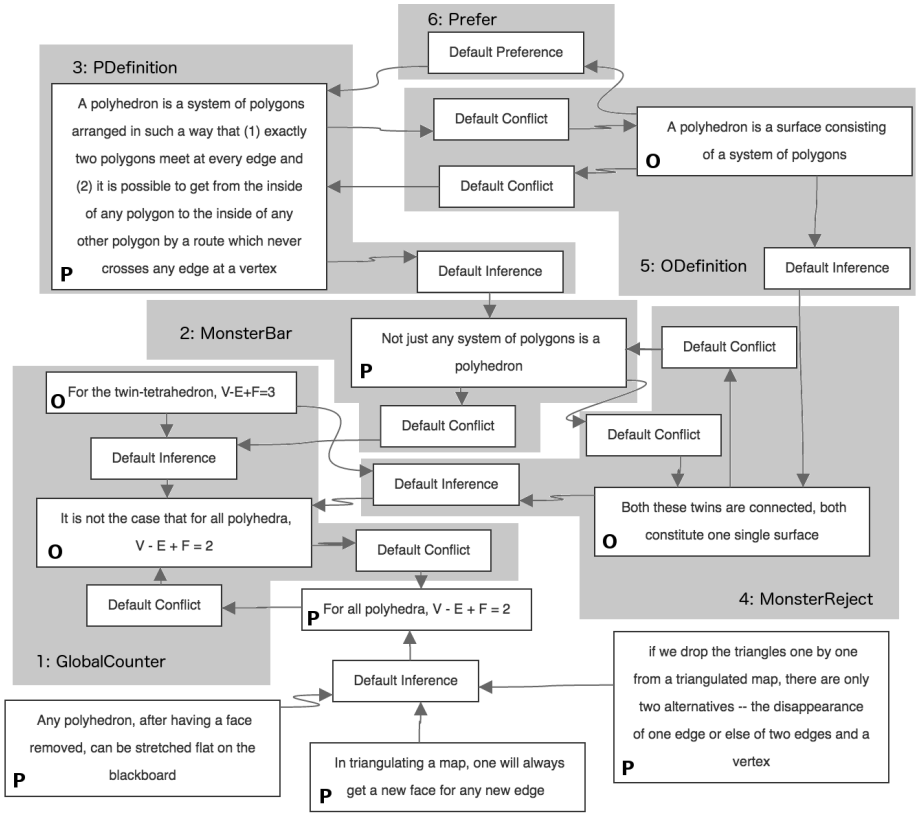



Figure 9: Monster barring and monster rejecting, with alternative definitions and preferences between them

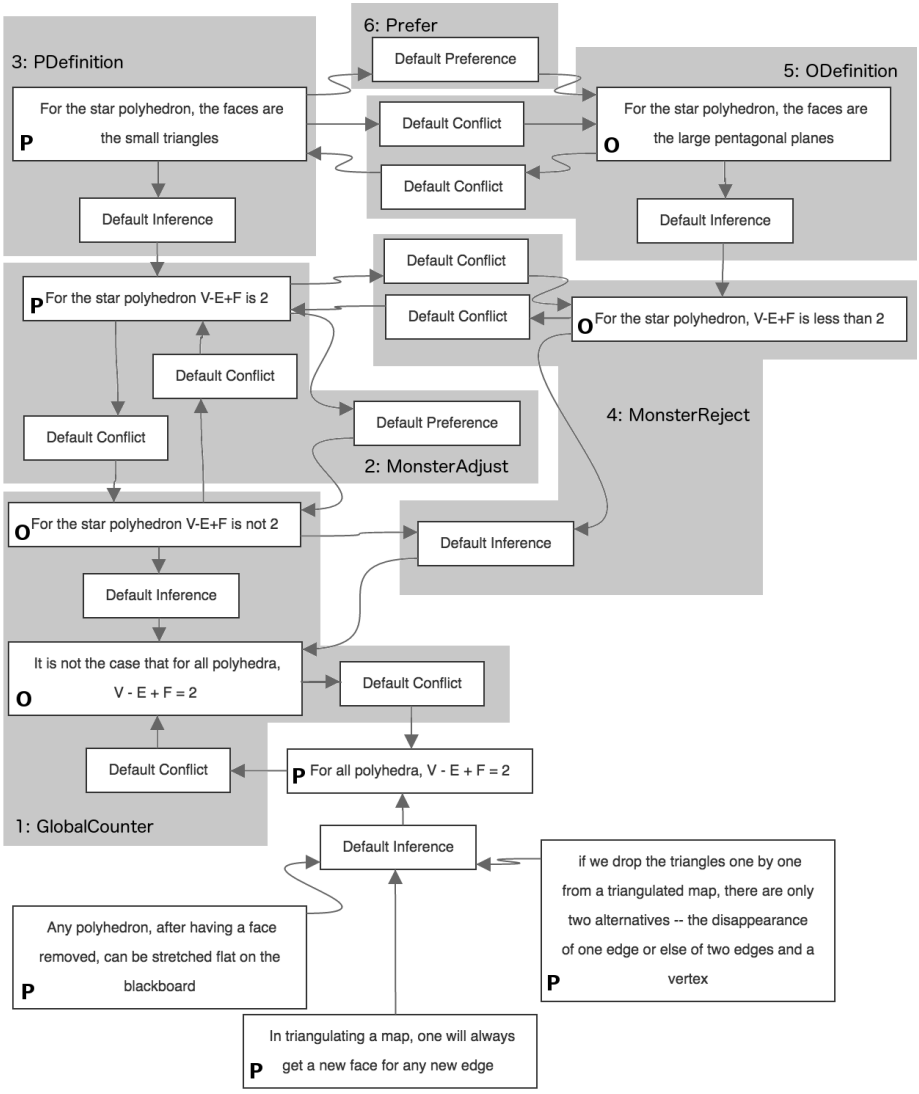



Figure 10: Monster adjusting with alternative definitions and preferences between them

possible (under certain assumptions[13]) to translate from AIF to *ASPIC*$^+$ [16]. Prakken has further shown (*ibid.*) that *ASPIC*$^+$ structures can be used to induce abstract argumentation frameworks which can make use of the wide range of existing argumentation semantics for computing acceptability.

The approach described in this section shows how as a Lakatos Game is executed, AIF structures are created which, when translated to *ASPIC*$^+$, produce abstract argumentation frameworks which under grounded semantics have as acceptable arguments all and only those elements which correspond to the mathematical theory accepted by the participants at a given stage of the game. Though other single extension semantics could be employed, it turns out that the strong, sceptical interpretation provided by grounded semantics deliver precisely the results required,as will be shown in this and the following sections. Our strategy is inductive, so we aim to show that each move updates the AIF graph in such a way that the grounded extension of the induced framework corresponds to the definitions of how the mathematical theory is evolving given by the LG locution rules in Section 3.2.

We describe updates in three steps. First, showing how an *ASPIC*$^+$ argumentation theory for the Lakatos Game, $AT_{LG} = ((\mathcal{L}_{LG}, \mathcal{R}_{LG}, n), \mathcal{K})$, is updated by virtue of the relationship between AIF and *ASPIC*$^+$ [16] (we adopt a convention whereby the new, post-locution argumentation theory and its constituents are referred to with a prime ($'$), whilst the original, pre-locution version is unadorned). Second, by virtue of the relationship between *ASPIC*$^+$ and abstract argumentation, we show updates to an abstract framework for the Lakatos Game *LG*, $AF_{LG} = \langle AR_{LG}, attacks_{LG} \rangle$, comprising a set of abstract arguments $AR_{LG}$ and a relation of attacks $attacks_{LG}$ over them following Dung [32]. Finally, the updates to the grounded extension of the argumentation framework, $GE_{LG}$, are computed and compared with the status of the theory defined by the locution rules of *LG*. Where abstract arguments correspond to single propositions in the structured argumentation, they are labelled the same for convenience (i.e. a proposition $p$ in the structured argumentation will correspond to an abstract argument also labelled $p$). Where we are interested in abstract arguments that correspond to a complex in the structured argumentation, we name them uniquely and explicitly describe their composition (so, for example, a structured argument [$p$ so $q$] might correspond to an abstract argument $\alpha$).

For clarity in presentation, the locutions are divided into three categories: those that add material to the theory (*constructive locutions*), those that attack material already in the theory (*critical locutions*) and those that make no explicit update to the material in the theory (*neutral locutions*).

[13]These assumptions serve to exclude boundary cases from the AIF to ensure (i) that arguments terminate with I nodes; (ii) that RA nodes always move from one or more premises to exactly one conclusion; and (iii) that PA and CA nodes always connect exactly one incoming with exactly one outgoing node, and never serve themselves as the incoming or outgoing nodes of other S nodes – i.e. never serve as premises or conclusions, conflicting or conflicted elements, preferred or dispreferred elements. None of the AIF updates described in Section 4 introduce violations to these assumptions.

## 5.1. Constructive locutions

For *Conjecture(c)* and *Lemma(l)*, the update to $AT_{LG}$ is straightforward in that $\mathcal{K}$ is updated to include them (i.e. $\mathcal{K}' = \mathcal{K} \cup \{c\}$ and $\mathcal{K} = \mathcal{K} \cup \{l\}$, respectively[14]).

Next, we must exploit the fact that the system is constrained to prohibit repetition of identical moves (i.e. moves of identical type with identical variable instantiations, but irrespective of speaker). So long as these moves have not come before[15], and therefore the contents, $c$ and $l$, have not previously been introduced, the AIF will be updated to include $c$ or $l$, and there are guaranteed to be no extant conflicts with either $c$ or $l$ (since conflicts can only be introduced by the *GlobalCounter*, *LocalCounter*, *HybridCounter* or *StrategicWithdrawal* moves which can only follow earlier *Conjecture* or *Lemma* moves). Thus the new abstract framework after a *Conjecture(c)* move, $AF'_{LG}$, is expanded from the old thus: $AR'_{LG} = AR_{LG} \cup \{c\}$ and $attacks'_{LG} = attacks_{LG}$ (and similarly for *Lemma(l)*). With no conflicts, abstract arguments corresponding to $c$ and $l$ are guaranteed to have no attacking arguments, and therefore to be in the grounded extension. That is, $\nexists x$s.t.$(x,c) \in attacks'$or$(x,l) \in attacks'$.

For *PiecemealExclusion(b)*, the revised conjecture, $b$ is added to the AIF, and consequently $\mathcal{K}' = \mathcal{K} \cup \{b\}$. Under the same restrictions and assumptions as for *Conjecture(c)*, above, the corresponding abstract argument $b$ is guaranteed to be in the grounded extension. The old conjecture that has been replaced is guaranteed to be excluded from the grounded extension because of the conflict introduced by the *GlobalCounter* or *Prefer*, the only moves that can legally precede *PiecemealExclusion*. Thus, $AR'_{LG} = AR_{LG} \cup \{b\}$ and $attacks'_{LG} = attacks_{LG}$ (since the necessary update to $attacks_{LG}$ has been carried out by other moves).

For *StrategicWithdrawal(r, c)*, the revised conjecture is added to the AIF along with its conflict with the old conjecture $c$. (Although from a protocol perspective this is a seemingly counter-strategic move for Proponent to make, it is in fact quite common in real exchanges – see, for example, Section A.1, turn 1). In the $ASPIC^+$ argumentation theory, $\mathcal{K}' = \mathcal{K} \cup \{r\}$ and $\overline{c}' = \overline{c} \cup \{r\}$. In the induced abstract framework, the conflict results in new attacks between $r$ and the abstract argument representing the proof, $\omega$, i.e. $[l_i$ so $c]$. This does not remove the previous conjecture or the extant proof in its support, but suspends it, knocking it out of the grounded extension, whilst focus shifts to the revised, narrower conjecture. For *StrategicWithdrawal(r,c)*, the update is thus $AR'_{LG} = AR_{LG} \cup \{r\}$ and $attacks'_{LG} = attacks_{LG} \cup \{(r,\omega),(c,\omega)\}$.

For *MonsterBar*, the AIF is updated with a reason, $r$, which is used as an undercutter in the argument from counterexample to counter-conjecture. Clearly, $\mathcal{K}' = \mathcal{K} \cup \{r\}$ and in addition, $\overline{m \Rightarrow \text{not-}c}' = \overline{m \Rightarrow \text{not-}c} \cup \{r\}$. The reason, $r$, is guaranteed to be in the grounded extension since there could be no attackers; as a result, the status of the abstract argument corresponding to the inference from counterexample, $m$, to counter-conjecture, not-$c$, is guaranteed to be out (since it is attacked by the abstract argument corresponding to $r$). With this argument out, if there are no other pending counterex-

[14] Specifically this is an update to $\mathcal{K}_p$, the subset comprising ordinary premises. Updates for *LG* make no use of the $\mathcal{K}_n$ subset comprising axioms.

[15] with the further commonsense assumption that within a single dialogue, a conjecture is not usable as a lemma, definition, or counterexample, and vice versa)

amples (and the dialogue protocol constrains the discussion to handle just one at a time), the conjecture will be in the grounded extension. Thus, *MonsterBar(m, c, r)* updates $AR'_{LG} = AR_{LG} \cup \{r\}$ and $attacks'_{LG} = attacks_{LG} \cup \{(r, \alpha)\}$ where $\alpha$ is the abstract argument corresponding to the structured argumentation complex [$m$ so not-$c$].

For *MonsterAdjust*, a direct counter, $r$, to the counterexample $m$, is introduced along with the symmetrical conflict between them, but also a preference to ensure $r$ defeats $m$. For *ASPIC*$^+$, this means that $\mathcal{K}' = \mathcal{K} \cup \{r\}$, $\overline{m}' = \overline{m} \cup \{r\}$ and $\overline{r}' = \overline{r} \setminus \{m\}$ (this latter is the result of the AIF PA node). As with *MonsterBar*-ing, if there are no other pending counterexamples (and the dialogue protocol constrains the discussion to handle just one at a time), the conjecture will be in the grounded extension. Thus, for *MonsterAdjust(m, r)*, the associated update is $AR'_{LG} = AR_{LG} \cup \{r\}$ and $attacks'_{LG} = attacks_{LG} \cup \{(r,m),(r,\alpha)\}$, where $\alpha$ is the abstract argument corresponding to the structured argumentation complex [$m$ so not-$c$].

*GlobalLemmaInc*, makes explicit the hidden assumption (as described by Lakatos) as a lemma (the subsequent incorporation of the lemma into the conjecture is handled by *HybridLemmaInc*). The new lemma is introduced and added in to AIF and thence the *ASPIC*$^+$ argumentation theory thus: $\mathcal{K}' = \mathcal{K} \cup \{k\}$, $\overline{m}' = \overline{m} \cup \{k\}$ and $\overline{k}' = \overline{k} \cup \{m\}$. At the abstract level, the argument corresponding to this lemma is guaranteed to be in the grounded extension. That is, for *GlobalLemmaInc(m, k)*, the update is $AR'_{LG} = AR_{LG} \cup \{k\}$ and $attacks'_{LG} = attacks_{LG} \cup \{(k,\alpha),(m,k),(k,m)\}$, where $\alpha$ is the abstract argument corresponding to the structured argumentation complex [$m$ so not-$c$].

For *LocalLemmaInc(m, l, k)*, the new lemma $k$ is added and guaranteed to be in the grounded extension, and the old is guaranteed to be out as a result of the preceding *LocalCounter* move. So, for *LocalLemmaInc(m, l, k)*, the resulting update to *ASPIC*$^+$ is just $\mathcal{K}' = \mathcal{K} \cup \{k\}$, with the corresponding update at the abstract level: $AR'_{LG} = AR_{LG} \cup \{k\}$ and $attacks'_{LG} = attacks_{LG}$.

For *PDefinition*, Proponent's definition, $d$ in support of the reason for treating the counterexample as a monster is added to the AIF. The *ASPIC*$^+$ theory is updated not only with $\mathcal{K}' = \mathcal{K} \cup \{d\}$ but also $\mathcal{R}'_{LG} = \mathcal{R}_{LG} \cup \{d \Rightarrow r\}$. The reason $d$ is guaranteed to be in the grounded extension (again, under a constraint of non-repetition). The abstract argument corresponding to the reason alone is no longer induceable, instead being replaced by the abstract argument representing the complex [$d$ so $r$]. The extant attack between $r$ and the argument $\alpha$ (from $m$ to not-$c$) is thus replaced by one from $\delta$ to $\alpha$, and likewise that from $r$ to $m$ by one from $\delta$ to $m$. Thus $AR'_{LG} = (AR_{LG} \cup \{d, \delta\}) \setminus \{r\}$ and $attacks'_{LG} = (attacks_{LG} \cup \{(\delta,\alpha),(\delta,m)\}) \setminus \{(r,\alpha),(r,m)\}$ where $\delta$ is the abstract argument corresponding to the structure argumentation complex [$d$ so $r$] and $\alpha$ is that for [$m$ so not-$c$].

## 5.2. Critical locutions

Next we move on to moves that reduce the size of the extant theory by introducing conflict into the AIF structure and thence the abstract framework. All are moves of Opponent, as is to be expected. First, *GlobalCounter(m, c)* updates the AIF to introduce not-$c$ in conflict with the conjecture $c$ (as well as introducing $m$, of course). This conflict introduces attacks between the abstract arguments involving $c$ on the one hand and those involving not-$c$ on the other [16]. The *ASPIC*$^+$ theory is updated

such that $\mathcal{K}' = \mathcal{K} \cup \{m, \text{not}{-}c\}$ and $\mathcal{R}'_{LG} = \mathcal{R}_{LG} \cup \{m \Rightarrow \text{not-}c\}$ plus $\overline{c}' = \overline{c} \cup \{\text{not-}c\}$ and $\overline{\text{not-}c}' = \overline{\text{not-}c} \cup \{c\}$. This ensures that $c$ is no longer in the grounded extension. Because the conflict is symmetrical in the AIF, there are bidirectional attacks in the abstract framework so not only is $c$ out of the grounded extension, so too is not-$c$. This accurately reflects not only that $c$ is knocked out of the current proof, but also that the counterposition is *not* added to the theory. Instead the proof is temporarily suspended – if the dialogue were to stop at this point, there would be no proof: which is exactly the status delivered by the grounded extension of the abstract framework. For *GlobalCounter(m, c)*, we have that $AR'_{LG} = AR_{LG} \cup \{m, \alpha\}$ and $attacks'_{LG} = attacks_{LG} \cup \{(\omega, \alpha), (\alpha, \omega)\}$, where $\alpha$ is the abstract argument corresponding to the structured argumentation complex [$m$ so not-$c$] and $\omega$ corresponds to the argument [$l_i$ so $c$] (i.e that connects all of the lemmas $l_i$ to the conjecture, $c$).

The *LocalCounter(m, l)* and *HybridCounter(m, l, c)* moves similarly add conflicts, and have similar effects on the abstract framework and thence the grounded extension. For *LocalCounter(m, l)*, $\mathcal{K}' = \mathcal{K} \cup \{m\}$ and $\overline{m}' = \overline{m} \cup \{l\}$ and $\overline{l}' = \overline{l} \cup \{m\}$. At the abstract level, this yields $AR'_{LG} = AR_{LG} \cup \{m\}$ and $attacks'_{LG} = attacks_{LG} \cup \{(l,m), (m,l), (m,\omega)\}$, where $\omega$ corresponds to the argument [$l_i$ so $c$]. The update associated with $HybridCounter(m,l,c)$ is the union of those for *LocalCounter* and *GlobalCounter*, except that the monster, $m$ is itself the counter to the conjecture, *viz.*, $\mathcal{K}' = \mathcal{K} \cup \{m\}$ and $\overline{m}' = \overline{m} \cup \{l, c\}$ and $\overline{l}' = \overline{l} \cup \{m\}$ and $\overline{c}' = \overline{c} \cup \{m\}$. Thus at the abstract level, $AR'_{LG} = AR_{LG} \cup \{m\}$ and $attacks'_{LG} = attacks_{LG} \cup \{(m,l), (l,m), (m,\omega), (\omega,m), (m,c), (c,m)\}$, where $\omega$ corresponds to the argument [$l_i$ so $c$].

For *MonsterReject*, the situation rather mirrors *MonsterBar* in that a reason, $s$, along with the symmetrical conflict between $s$ and $r$ are added to the AIF. With no resolution between these two, neither $s$ nor $r$ will be in the grounded extension; as a result the abstract argument corresponding to the argument from counterexample to counter-conjecture will be attacked by nothing other than the conjecture, so both will be excluded from the grounded extension. Finally, the new reason, $s$ is adduced as an additional premise in the argument from $m$ to not-$c$. Thus for $MonsterReject(m,r,d,s,c)$, the $ASPIC^+$ framework is updated so that $\mathcal{K}' = \mathcal{K} \cup \{s\}$ and $\mathcal{R}'_{LG} = \mathcal{R}_{LG} \cup \{s, m \Rightarrow \text{not-}c\}$ plus $\overline{s}' = \overline{s} \cup \{r\}$ and $\overline{r}' = \overline{r} \cup \{s\}$. In the resulting abstract framework, $AR'_{LG} = AR_{LG} \cup \{s, \sigma\}$ and $attacks'_{LG} = attacks_{LG} \cup \{(\delta, s), (s, \delta), (\delta, \sigma), (\sigma, \omega), (\omega, \sigma)\}$, where $\delta$ is the abstract argument corresponding to the structured argumentation complex [$d$ so $r$], $\sigma$ is [$\{s,m\}$ so not-$c$] and $\omega$ is [$l_i$ so $c$].

Lastly, *ODefinition*, which is only used as Opponent's substantiation of their *MonsterReject*, introduces a new definition, $e$ and symmetrical conflict between that and $d$. Again, with no resolution between $d$ and $e$, neither are in the grounded extension. The update introduced by $ODefinition(m,r,d,s,e)$ on $ASPIC^+$ is thus $\mathcal{K}' = \mathcal{K} \cup \{e\}$ and $\mathcal{R}'_{LG} = \mathcal{R}_{LG} \cup \{e \Rightarrow s\}$ plus $\overline{d}' = \overline{d} \cup \{e\}$ and $\overline{e}' = \overline{e} \cup \{d\}$. In the resulting abstract framework, $AR'_{LG} = (AR_{LG} \cup \{e, \varepsilon\}) \setminus \{s\}$ and $attacks'_{LG} = (attacks_{LG} \cup \{(d,e), (e,d), (\delta,\varepsilon), (\varepsilon,\delta)\}) \setminus \{(s,\delta), (\delta,s)\}$, where $\delta$ is the abstract argument corresponding to the structured argumentation complex [$d$ so $r$] and $\varepsilon$ is [$e$ so $s$].

## 5.3. Neutral locutions

Two further moves have important updates to the abstract argumentation framework despite not updating the theory structure directly (that is to say, they are important in terms of the dialogical dynamics of *LG* but have only indirect impact on the theory and its update (indirect inasmuch as constraining the subsequent application of moves that do have direct impact on the theory).

The *Prefer* move, is unique in that it will either expand or contract the grounded extension depending on which player executes it. The protocol demands that $Prefer$ can only be executed after both Proponent and Opponent have offered definitions, which in turn can only occur in a *MonsterBar-PDefinition-MonsterReject-ODefinition* series. First, we consider Opponent's use of *Prefer*, in which Opponent's definition, $e$, is preferred to Proponent's, $d$. This ensures that the abstract argument corresponding to $e$ defeats that corresponding to $d$, and further that the abstract argument $\varepsilon$ (corresponding to the structured argumentation complex [$e$ so $s$]) defeats the abstract argument $\delta$ (corresponding to the structured argumentation complex [$d$ so $r$]). With $\delta$ out of the grounded extension, the argument $\alpha$ (the abstract argument corresponding to the argument [$m$ so not-$c$]) is no longer undercut, thereby ensuring that $c$ is out of the grounded extension. For Proponent's use of the *Prefer* move, the situation is reversed, so that $d$ is preferred to $e$, and thence that $\delta$ is preferred to $\varepsilon$. This in turn ensures that $\alpha$ is out of the grounded extension and with nothing else to attack it, $c$ is in. So after Proponent's $Prefer$ move, $c,r,d \in GE'_{LG}$, whilst not-$c,m,s,e \notin GE'_{L}G$. The definition of $Prefer(m,r,f,g)$ allows us to capture these abstract consequences independently of the speaker, expressing that $f$ is preferred to $g$ (and in the abstract, that $\eta$ founded on $f$ is preferred to $\gamma$ founded on $g$), regardless of which of $f$ and $g$ are instantiated with $d$ or $e$. The effect of adding a preference into the structured argumentation is to remove one of the pair of symmetrical attacks in the asbtract framework. The *ASPIC*$^+$update is thus just $\overline{f}' = \overline{f} \cup \{g\}$ which results at the abstract level, in $AR'_{LG} = AR_{LG}$ and $attacks'_{LG} = attacks_{LG} \setminus \{(g,f),(\gamma,\eta)\}$.

The *ProofDone* move adds a new abstract argument, $\omega$ corresponding to the argument from all of the current lemmas, $l_i$, to the conjecture, $c$, and uses it to replace the atomic abstract argument corresponding to $c$. Finally, everything that previously attacked $c$ will now be attacking $\omega$. For *ASPIC*$^+$, the update is ths simply $\mathcal{R}'_{LG} = \mathcal{R}_{LG} \cup \{l_i \Rightarrow c\}$. The ramifications at the abstract level are more complex: $AR'_{LG} = (AR_{LG} \cup \omega) \setminus \{c\}$ and $attacks'_{LG} = attacks_{LG} \cup (\beta_i, \omega), \forall \beta_i s.t. (\beta_i, c) \in attacks_{LG}$, where $\omega$ is the abstract argument corresponding to the structured argumentation complex [$l_i$ so $c$].

To complete the assessment of semantic update, it is also useful to explain why the remaining four moves have no direct effect at all.

The purpose of *HybridLemmaInc* is to introduce a new conjecture, $c'$, but it is not *HybridLemmaInc* itself that does this, but rather the *Conjecture* move which is required to follow it. Similarly, the abstract argument corresponding to the inference from the new set of lemmas to the new conjecture is handled later by *ProofDone* whilst the exclusion of the lemma that has been incorporated and of the conjecture which has been revised are both handled earlier by the *HybridCounter* or *GlobalLemmaInc* which must have preceded it. The update associated with *HybridLemmaInc* itself, therefore,

is null (i.e. $AR'_{LG} = AR_{LG}$ and $attacks'_{LG} = attacks_{LG}$).

The *MonsterAccept* move allows Opponent to reiterate what has already been said, and so does not update the semantics any more than it updates the underlying AIF structure.

In addition there are the two terminating moves, *Accept* and *Surrender* neither of which effect material update on the AIF structure. Instead, they mark the conclusion of the dialogical process.

In short, *HybridLemmaInc*, *MonsterAccept*, *Accept* and *Surrender* all play a role in the dialogical structure, but their place in the dialogical exchange ensures that any appropriate updates with resulting semantic changes are executed by other moves in the dialogical sequence.

The complete set of updates, both syntactic and resultant semantic, for all moves in *LG* is summarised in Table 1, which also shows, in the final column, the direct impact on the resulting grounded extension – though this omits potential reinstatements and other indirect effects of the updates described in columns three and four.

# 6. Implementation: From the theoretical model to computational model

We have shown how argumentation semantics can, in theory, furnish us with the most up-to-date status of a theory in a Lakatos Game. All the components of this pipeline [88] are also implemented – so, execution of Lakatos Games requires no new development other than the creation of a Dialogue Game Description Language (DGDL) [105] specification to capture the rules described in Section 3 and the updates described in Section 4.

A DGDL specification consists of three main parts; composition, rules and interactions. The composition describes the general features of the dialogue game, and, in the case of LG, the composition is as shown in Listing 1.

```
turns{magnitude:multiple, ordering:liberal}
roles{Proponent, Opponent}
players{min:2, max:2}
player{id:Proponent}
player{id:Opponent}
store{id:Conjecture, owner:Proponent,
      structure:set, visibility:public, {""}}
store{id:Lemmas, owner:Proponent,
      structure:set, visibility:public, {""}}
```

Listing 1: LG DGDL Composition

Firstly, we specify that a turn can consist of multiple moves and that the ordering is liberal, i.e. that there are not single alternating moves made by each participant. The roles of Participant and Opponent are then given with a limitation that only these two players are possible. Finally, we specify the commitment stores that will be used to store the conjecture and the lemmas that will comprise the proof.

The LG specification contains only one rule, a starting rule that is triggered when the dialogue begins and specifies that the dialogue starts with Proponent making a

| **Move** | **AIF update** | $ASPIC^+$ **update** | | | $AF_{LG}$ **update** | |
|---|---|---|---|---|---|---|
| | | $\mathcal{K}$ | $\mathcal{R}_{LG}$ | $^-$ | $AR_{LG}$ | $attacks_{LG}$ |
| *Conjecture*$(c)$ | + $c$ | $\cup\{c\}$ | $\emptyset$ | $\emptyset$ | $\cup\{c\}$ | $\emptyset$ |
| *Lemma*$(l)$ | + $l$ | $\cup\{l\}$ | $\emptyset$ | $\emptyset$ | $\cup\{l\}$ | $\emptyset$ |
| *ProofDone* | + RA($l_i$,$c$) | $\emptyset$ | $\cup\{l \Rightarrow c\}$ | $\emptyset$ | $\cup\{\omega\}$ $\setminus\{c\}$ | $\cup(\beta_i,\omega)$, $\forall\beta_i$s.t. $(\beta_i,c) \in attacks_{LG}$ |
| *LocalCounter*$(m,l)$ | + $m$, CA($m$, $l$), CA($l$, $m$) | $\cup\{m\}$ | $\emptyset$ | $\cup\{(m,l),(l,m)\}$ | $\cup\{m\}$ | $\cup$ $\{(l,m),(m,l),(m,\omega)\}$ |
| *HybridCounter*$(m,l,c)$ | + $m$, CA($m$, $l$), CA($l$, $m$), CA($m$, $c$), CA($c$, $m$) | $\cup\{m\}$ | $\emptyset$ | $\cup\{(l,m),(m,l),(c,m),(c,m)\}$ | $\cup\{m\}$ | $\cup$ $\{(m,l),(l,m),(m,\omega),(\omega,m),(m,c),(c,m)\}$ |
| *GlobalCounter*$(m,c)$ | + $m$, not-$c$, RA($m$, not-$c$), CA($c$, not-$c$), CA(not-$c$, $c$) | $\cup$ $\{m$, not-$c\}$ | $\cup\{m \Rightarrow$ not-$c\}$ | $\cup\{$ $(c,$not-$c)$, (not-$c,c)\}$ | $\cup\{m,\alpha\}$ | $\cup\{(\omega,\alpha),(\alpha,\omega)\}$ |
| *PiecemealExclusion*$(b)$ | + $b$ | $\cup\{b\}$ | $\emptyset$ | $\emptyset$ | $\cup\{b\}$ | $\emptyset$ |
| *StrategicWithdrawal* $(r,c)$ | + $r$, CA($r$, $c$), CA($c$, $r$) | $\cup\{r\}$ | $\emptyset$ | $\cup\{(r,c),(c,r)\}$ | $\cup\{r\}$ | $\cup\{(r,\omega),(\omega,r)\}$ |
| *LocalLemmaInc*$(m,l,k)$ | + $k$, MA($l$, $k$) | $\cup\{k\}$ | $\emptyset$ | $\emptyset$ | $\cup\{k\}$ | $\emptyset$ |
| *HybridLemmaInc*$(m,l)$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| *GlobalLemmaInc*$(m,k)$ | + $k$, CA($m$, $k$), CA($k$,$m$) | $\cup\{k\}$ | $\emptyset$ | $\emptyset$ | $\cup\{k\}$ | $\cup\{(k,\alpha),(m,k),(k,m)\}$ |
| *MonsterBar*$(m,c,r)$ | + $r$, CA($r$, RA($m$, not-$c$)) | $\cup\{r\}$ | $\emptyset$ | $\cup\{(m \Rightarrow$ not-$c,r)\}$ | $\cup\{r\}$ | $\cup\{(r,\alpha)\}$ |
| *MonsterAdjust*$(m,r)$ | + $r$, CA($r$, $m$), CA($m$, $r$) | $\cup\{r\}$ | $\emptyset$ | $\cup\{(m,r)\}$ $\setminus\{(r,m)\}$ | $\cup\{r\}$ | $\cup\{(r,m)$ $(r,\alpha)\}$ |
| *PDefinition*$(m,r,d)$ | + $d$, RA($d$, $m$) | $\cup\{d\}$ | $\cup\{d \Rightarrow r\}$ | $\emptyset$ | $\cup\{d,\delta\}$ $\setminus\{r\}$ | $\cup\{(\delta,m),(\delta,\alpha)\}$ $\setminus\{(r,m),(r,\alpha)\}$ |
| *ODefinition*$(m,r,d,s,e)$ | + $e$, RA($s$, $e$), CA($e$, $d$), CA($d$, $e$) | $\cup\{e\}$ | $\cup\{$ $e \Rightarrow s\}$ | $\cup\{(e,d),(d,e)\}$ | $\cup\{e,\varepsilon\}$ $\setminus\{s\}$ | $\cup$ $\{(d,e),(e,d),(\delta,\varepsilon),(\varepsilon,\delta)\}$ $\setminus$ $\{(s,\delta),(\delta,s)\}$ |
| *Prefer*$(m,r,f,g)$ | + PA($f$, $g$) | $\emptyset$ | $\emptyset$ | $\setminus\{(f,g)\}$ | $\emptyset$ | $\setminus$ $\{(g,f),(\gamma,\eta)\}$ |
| *MonsterAccept*$(m,r)$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| *MonsterReject* $(m,r,d,s,c)$ | + $s$, RA($\{s$, $m\}$, not-$c$), CA($s$, $r$), CA($r$, $s$) | $\cup\{s\}$ | $\cup$ $\{\{s,m\}$ $\Rightarrow$ not-$c\}$ | $\cup\{(s,r),(r,s)\}$ | $\cup\{s,\sigma\}$ | $\cup$ $\{(\delta,s),(s,\delta),(\sigma,\omega),(\omega,\sigma),(\delta,\sigma)\}$ |
| *Accept* | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| *Surrender* | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |

Table 1: Summary of the syntactic and semantic updates in *LG*

'Conjecture' move.

The remainder of the LG DGDL specification then consists of the interactions available and the effects which they have on the dialogue. A simple example of an interaction description is Proponent's first move, 'Conjecture' (Listing 2).

```
interaction{Conjecture, {c}, Asserting, {c}, "$c is the conjecture",
    {
    if { size(Lemmas, Proponent, !empty) } then
        { move(add, next, ProofDone, Proponent)
        & move(add, next, Lemma, {l}, Proponent)
        & store(empty, Conjecture, Proponent)
        & store(add, {c}, Conjecture, Proponent)
        }
    else
        { move(add, next, Lemma, {l}, Proponent)
        & store(empty, Conjecture, Proponent)
        & store(add, {c}, Conjecture, Proponent)
        }
    }
}
```

Listing 2: The Conjecture Interaction

In this example, the name of the move is given first, followed by its content and the fact that Proponent is asserting the conjecture. The string "$c is the conjecture" is a description of what the move is doing, used to allow users to identify it. The body of the interaction looks at whether the Lemmas commitment store is empty, and if not, allows Proponent to perform a ProofDone move next. Regardless of the state of the Lemmas store, Proponent also has the option to perform the Lemma move at this point. Finally, in both cases, any existing conjecture is removed from the Conjecture store and the new conjecture is added.

Another example of a more complex interaction, 'GlobalCounter', is given in Listing 3.

```
interaction{GlobalCounter, {m,c},
            Asserting, {m}, Asserting, {!c},
            Contradicting, {<{c},{!c}>, DefaultConflict},
            Contradicting, {<{!c},{c}>, DefaultConflict},
            Arguing, {<{m},{!c}>, DefaultSupport}, "$m is a counter to
                $c",
        { move(add, next, PiecemealExclusion, {b}, Proponent)
        & move(add, next, StrategicWithdrawal, {r, c}, Proponent)
        & move(add, next, MonsterBar, {m,c,r}, Proponent)
        & move(add, next, MonsterAdjust, {m,r}, Proponent)
        & move(add, next, Surrender, Proponent)
        & move(add, next, GlobalLemmaInc,{m,l}, Proponent)
        }
    }
```

Listing 3: The GlobalCounter Interaction

Here Opponent is giving a counterexample, '$m$', to the conjecture, '$c$'. This counterexample is arguing in support of the negation of the conjecture, '!$c$'.

Having specified the LG system in DGDL, this specification can then be processed by the Dialogue Game Execution Platform (DGEP) [14]. DGEP allows participants to

take part in dialogues following the rules specified by a DGDL protocol. When a new dialogue is initiated, DGEP allows users to join in the roles available, and any initial rules are processed. From this point on, DGEP maintains the legal move list for each participant, based on the rules and interactions: a rule is executed when it is in scope, and an interaction when it is moved by a player during the game.

DGEP provides a range of web service interfaces, allowing a user to both perform interactions and get information about the current dialogue state (for example, their list of available moves). These web services can then be used by either software agents playing the roles of specific participants, or by graphical interfaces allowing human users to take part in the discussion. Arvina [56] is one such graphical interface, and a screenshot of a LG protocol dialogue taking place in Arvina can be seen in Figure 11.

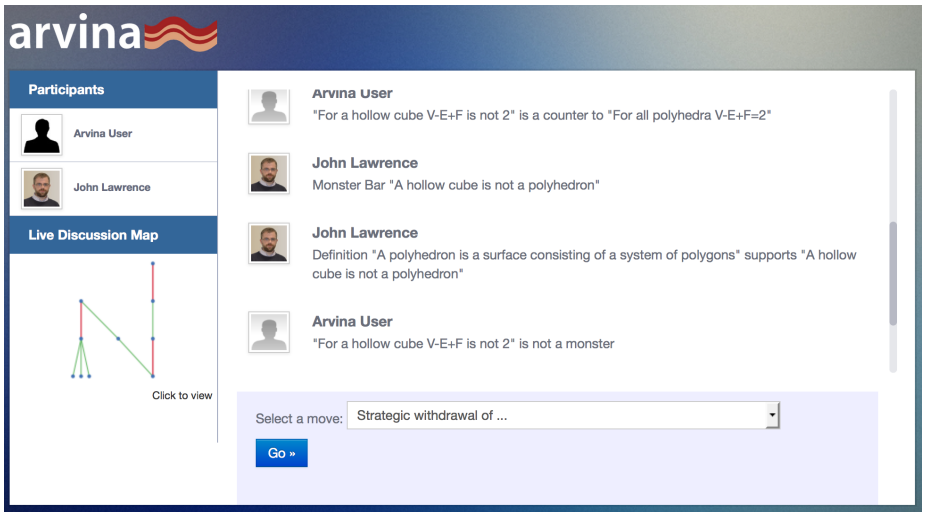



Figure 11: Arvina screenshot

One of the advantages of using DGEP is that it not only provides a robust platform for the execution of a dialogue protocol, but also creates argument structures as a side effect. AIF infrastructure for querying and updating is available in the webservices of AIFdb[16] [57]; conversion to *ASPIC*$^+$ and induction of abstract frameworks is handled by TOAST [90]; and calculation of acceptability semantics is performed by Dung-O-Matic [31]. (We use these tools for convenience because they work directly with AIF and *ASPIC*$^+$ data structures, though many other systems perform comparable computation using various alternative approach –Tweety [93] using defeasible logic programming, ASPARTIX [33] using answer set programming and ArgSemSAT [24] using SAT solving.)

Arvina has been demonstrated to provide a practical stepping stone towards mixed-initiative argumentation [89], a type of collaborative intelligence [35] or human-agent

[16] http://www.aifdb.org

collective [49]. Where agents have knowledge bases populated by pre-existing arguments, they can contribute to an evolving Lakatos game on a level playing field with human participants. Indeed, DGEP makes no distinction between types of clients, whether human or artificial. The Arvina system, the Lakatos game, and agents containing several small suitable knowledge bases are available online at `http://arvina.arg-tech.org`.

Finally, the AIF structures created as a side-effect of executing the DGDL specification for LG using Arvina can be submitted to the TOAST system [90] which implements (i) the translation from AIF to *ASPIC*$^+$; (ii) the induction of abstract frameworks according to the *ASPIC*$^+$ definitions; and (iii) the computation of semantics over those structures, including grounded, by calling DungOMatic [31]. Sample results from TOAST running over structures created by Arvina and DGEP executing the Lakatos Game DGDL specification are shown in Figure 12.

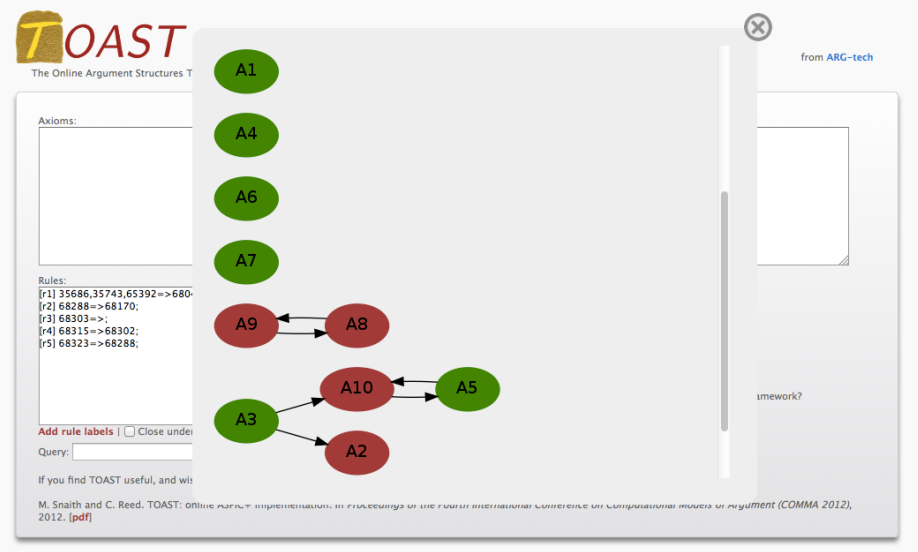



Figure 12: TOAST running on a Lakatos dialogue example

## 7. Execution: Collective proof as argumentation

In this section we look at a fully worked example dialogue, considering strategies of global counter and monster-barring (see Section 7.2), and then local counter and local lemma incorporation (see Section 7.3) in greater depth. A proof idea for Euler's conjecture is presented by the Teacher on the second page *Proofs and Refutations* [55]. The proof outline comes from [23], and consists of three steps (for a diagrammatic representation of these steps, carried out on the cube, see Figure 13, taken from [55, p.8].) Note that in (1-i), GAMMA's reference to "my counterexample" is made on behalf of

both ALPHA and GAMMA who speak in the role of Opponent throughout, whereas TEACHER and DELTA speak in the role of Proponent.

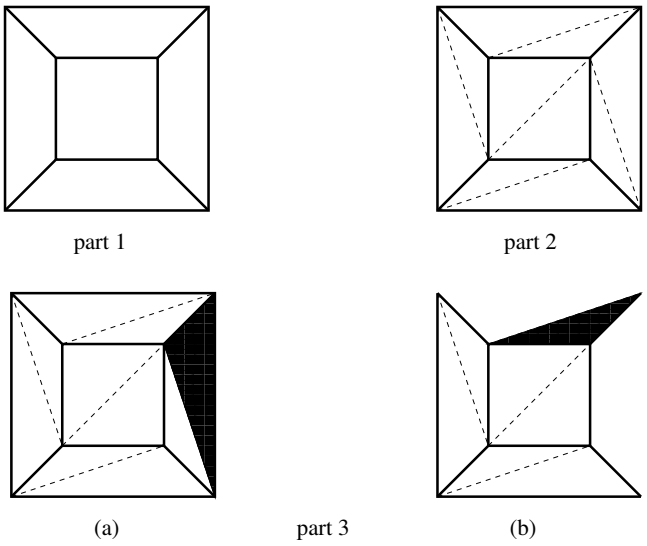



Figure 13: Given the cube, after removing a face and stretching it flat, we are left with the network in part 1. After triangulating, we get part 2. When removing a triangle, we either remove one edge and one face, or two edges, one vertex and a face – shown in parts 3(a) and (b) respectively.

(1) a. TEACHER: We arrived at a conjecture concerning polyhedra, namely that for all polyhedra, V-E+F = 2, where V is the number of vertices, E the number of edges and F the number of faces. ... ... I have one [a proof]. It consists of the following thought experiment.

b. TEACHER: Step 1 [described above, as lemmas].

c. TEACHER: Step 2.

d. TEACHER: Step 3.

e. TEACHER: Thus we have proved our conjecture.

f. ALPHA: I have a counterexample ..... Imagine a solid bounded by a pair of nested cubes – a pair of cubes, one of which is inside, but does not touch the other. ..... for each cube V-E+F = 2, so that for the hollow cube V-E+F = 4.

g. DELTA: This pair of nested cubes is not a polyhedron at all. .... It is a *monster*, .... not a counterexample.

h. DELTA: A polyhedron is a surface consisting of a system of polygons.

i. GAMMA: My counterexample is a solid bounded by polygonal faces.

j. GAMMA: A polyhedron is a solid whose surface consists of polygonal faces.

k. TEACHER: ...For the moment let us accept Delta's definition. Can you refute our conjecture now if by polyhedron we mean a surface?

l. GAMMA: I propose a trivial counterexample. Take the triangular network which results from performing the first two operations on a cube. Now if I remove a triangle from the *inside* of this network, as one might take a piece of the jigsaw puzzle, I remove one triangle without removing a single edge or vertex. So the third lemma is false – and not only in the case of the cube, but for *all* polyhedra except the tetrahedron, in the flat network of which all the triangles are boundary triangles. ...

m. TEACHER: I no longer contend that *the removal of any triangle follows one of the two patterns mentioned*, but merely that *at each stage of the removing operation the removal of any boundary triangle follows one of these patterns*. ... All that I have to do is to insert a single word in my third step, to wit, that 'from the triangulated network we now remove the *boundary* triangles one by one'.

n. TEACHER: ... I can easily (...) *improve the proof*, by replacing the false lemma by a slightly modified one, which your counterexample will not refute.

In the remainder of this section, we work step by step through this dialogue and show how our protocol both, covers all of the moves made, and, at the same time, produces updates to the generated the Argument Interchange Format structure and argumentation structure which capture the status of the proof after each turn of the dialogue.

## 7.1. Introducing the proof

The first five moves in the above dialogue transcript, (1-a) to (1-e), correspond to the introduction of the initial proof by Proponent. Table 2 shows how the execution of these steps using the LG dialogue system (column 1 of the table) updates a shared theory, understood by the participants as the current state of the proof (according to the definitions in Section 3.2) (in column 2). The AIF structures added by these moves are shown in column 3[17]. The current *ASPIC*$^+$ framework caused by mapping the complete new AIF structures according to [16] are shown in column 4. The abstract argumentation framework caused by inducing from the new *ASPIC*$^+$ framework a new AF are shown in column 5, and finally the grounded extension computed over that AF is shown in the last column.

In (1-a), Proponent presents their conjecture: "For all polyhedra, V - E + F = 2", this corresponds to the *Conjecture(C)* move in the LG protocol governed by the locution rule **L1.1** and structural rule **S1**. The current state of proof consists now of the conjecture *C* which adds $I_1$, a first node, to the AIF structure depicted on the left hand side in Figure 14. This also adds *C* as an argument to the *ASPIC*$^+$ framework, and to the abrstact argumentation framework. As there are no conflicts at this stage, *C* is in the grounded extension.

The next three moves (1-b), (1-c), and (1-d), present the three lemmas that the

[17] As the AIF updates only add to the structure, we only show those components added for each move in the table. For all other columns, the cumulative state is shown in each row.

Table 2: Proponent's construction of the initial proof

| **Formalisation** | | **Representation** | | **Evaluation** | |
|---|---|---|---|---|---|
| *LG System* | *Dialogue Theory* | *AIF* | *ASPIC*$^+$ | $AF_{LG}$ | $GE_{LG}$ |
| **Proponent** – Proposal of initial proof | | | | | |
| (1-a) *Conjecture*($C$) **[L1.1, S1]** | $C$ | $\mathbf{I}_1$: For all polyhedra, V - E + F = 2 [$C$] | $\mathcal{K} = \{C\}$ $\mathcal{R}_{LG} = \{\}$ $^- = \{\}$ | $AR_{LG} = \{C\}$ $att_{LG} = \{\}$ | $C$ |
| (1-b) *Lemma*($L_1$) **[L1.2, S2.2]** | $C$ $L_1$ | $\mathbf{I}_2$: Any polyhedron, after having a face removed, can be stretched flat on the blackboard [$L_1$] | $\mathcal{K} = \{C, L_1\}$ $\mathcal{R}_{LG} = \{\}$ $^- = \{\}$ | $AR_{LG} = \{C, L_1\}$ $att_{LG} = \{\}$ | $C$ $L_1$ |
| (1-c) *Lemma*($L_2$) **[L1.2, S2.1b]** | $C$ $L_1$ $L_2$ | $\mathbf{I}_3$: In triangulating a map, one will always get a new face for any new edge [$L_2$] | $\mathcal{K} = \{C, L_1, L_2\}$ $\mathcal{R}_{LG} = \{\}$ $^- = \{\}$ | $AR_{LG} = \{C, L_1, L_2\}$ $att_{LG} = \{\}$ | $C$ $L_1$ $L_2$ |
| (1-d) *Lemma*($L_3$) **[L1.2, S2.1b]** | $C$ $L_1$ $L_2$ $L_3$ | $\mathbf{I}_4$: if we drop the triangles one by one from a triangulated map, there are only two alternatives – the disappearance of one edge or else of two edges and a vertex [$L_3$] | $\mathcal{K} = \{C, L_1, L_2, L_3\}$ $\mathcal{R}_{LG} = \{\}$ $^- = \{\}$ | $AR_{LG} = \{C, L_1, L_2, L_3\}$ $att_{LG} = \{\}$ | $C$ $L_1$ $L_2$ $L_3$ |
| (1-e) *ProofDone*() **[L1.3, S2.1a]** | $L_1$ $L_2$ $L_3$ $L_1, L_2, L_3$ so $C$ | $\mathbf{RA}_1$: $(\{\mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4\}, \mathrm{I}_1)$ $[(\{L_1, L_2, L_3\}, C)]$ | $\mathcal{K} = \{C, L_1, L_2, L_3\}$ $\mathcal{R}_{LG} = \{$ $(L_1, L_2, L_3) \Rightarrow C$ $\}$ $^- = \{\}$ | $AR_{LG} = \{L_1, L_2, L_3, \omega\}$ $att_{LG} = \{\}$ | $L_1$ $L_2$ $L_3$ $\omega$ |

Proponent is using to support their conjecture (the full text of these lemmas can be seen in Table 2). This introduces three lemmas $L_1$, $L_2$, $L_3$ into dialogue theory, and three new nodes $I_2$, $I_3$, $I_4$ into AIF structure and argumentation framework in Figure 14.

Finally, in (1-e), the Proponent announces that they have completed constructing their proof, making the *ProofDone* move. Performing *ProofDone* not only shows that the proof is complete allowing the Opponent to raise any challenges, but also creates a new rule of inference ($\mathrm{RA}_1$) whereby representing that the given lemmas support the conjecture. This creates a new abstract argument, $\omega$, corresponding to the proof structure, and, as there are no attacks at this stage, $\omega$ and the three given lemmas are all in the grounded extension (Figure 14). The updates performed during these moves are shown in Table 2.

## 7.2. First strategy of testing the proof: Countering the conjecture

Once the Proponent has presented the initial proof, the Opponent makes their first attack. Here, the Opponent provides a counter to the conjecture, which the Proponent rejects as not fitting the definition of a polyhedron. The updates carried out in this section of the dialogue are summarised in Table 3.

The counter begins in (1-f), following the rules **L2.3** and **S4.1**, the Opponent uses *GlobalCounter* to introduce the counterexample of a hollow cube (i.e. $M_1$ or $\mathrm{I}_5$): "For the hollow cube V-E+F = 4", to the conjecture "For all polyhedra, V-E+F = 2" (thus new conflict nodes $\mathrm{CA}_1$ and $\mathrm{CA}_2$ in the updated AIF structure in Figure 15). More specifically, they use this counterexample as a premise to justify ($\mathrm{RA}_2$) the counter-conjecture (not-$C$, $\mathrm{I}_6$): "It is not the case that for all polyhedra, V-E+F = 2". This results in the creation of a new abstract argument in Figure 15, $\alpha$, which is in mutual conflict with the abstract argument, $\omega$, representing the proposed proof, and hence $\omega$

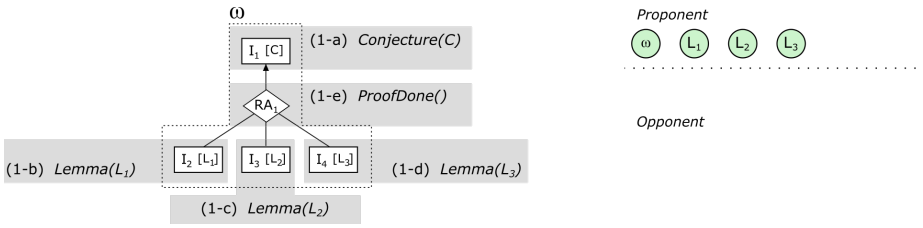



Figure 14: The update made to the Argument Interchange Format structure (on the left-hand side) and the argumentation framework (on the right-hand side) upon Proponent's introduction of the initial proof. On the left hand side, nodes labelled as $I_1$-$I_4$ are information nodes which correspond to the statements exchanged during the first five moves of the dialogue (1-a)-(1e) (the symbols from LG system, $C$, $L_1$-$L3$, are given in the brackets). The moves specified according to the LG systems are marked by grey area. The nodes $RA_m$ represent inferences and $CA_n$ (on the next figures) – conflicts. The right hand side of Figure shows the abstract framework created as a result of the update to the AIF structure (nodes at the top are Proponent's arguments and nodes on the bottom – Opponent's). The nodes in bold line denote arguments which are acceptable (in GE), while the nodes in dotted line denote arguments which are unacceptable (out of GE) at a given stage of the dialogue. Finally, some moves create new arguments to be added or replace other arguments in the framework: in this case we mark the fragment of the AIF structure and label it with new abbreviation such as $\omega$ in this figure.

is no longer in the grounded extension.

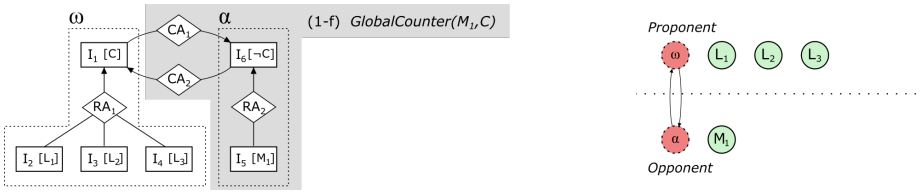



Figure 15: The update made to the Argument Interchange Format structure (on the left-hand side) and the argumentation framework (on the right-hand side) upon Opponent's counter to the conjecture (1-f).

The Proponent defends this challenge in (1-g), using *MonsterBar*, to show that this is not a valid counterexample, claiming that, whilst the hollow cube does indeed have the property "V-E+F = 4", it is not a polyhedron ($R$ or $I_7$). Following rule **S10**, the Proponent must then support this statement ($RA_3$), using the *PDefinition* move. As such, the Proponent provides a definition of a polyhedron: " A polyhedron is a surface consisting of a system of polygons", which excludes the hollow cube ($D$ or $I_8$). The resulting abstract argument, $\delta$, conflicts with $\alpha$, returning $\omega$, to the grounded extension (see Figure 16).

At this stage in the dialogue, Opponent could choose to agree with Proponent that this is not a valid counterexample (rule **S12.1**). However, in this example Opponent decides to continue the attack (rule **S12.2**), first making the move *MonsterReject*, to reject Proponent's defence by using $I_9$ ($S$), and $CA_4$ and $CA_5$, and then providing an alternative definition of a polyhedron ($I_10$ or $E$), which supports $I_9$, using *ODefinition*. As we now have two conflicting definitions, the Proponents reason is no-longer suf-

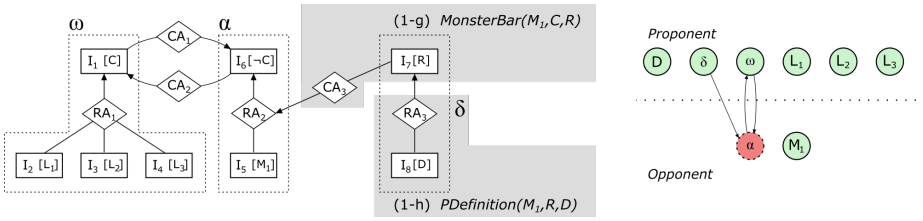



Figure 16: The update made to the Argument Interchange Format structure and the argumentation framework upon Proponent's rejection of $M_1$ as a counter (1-g), and definition supporting this (1-h).

ficient to mean that the counter is rejected, and, as such, $\omega$ is once again out of the grounded extension (Figure 17).

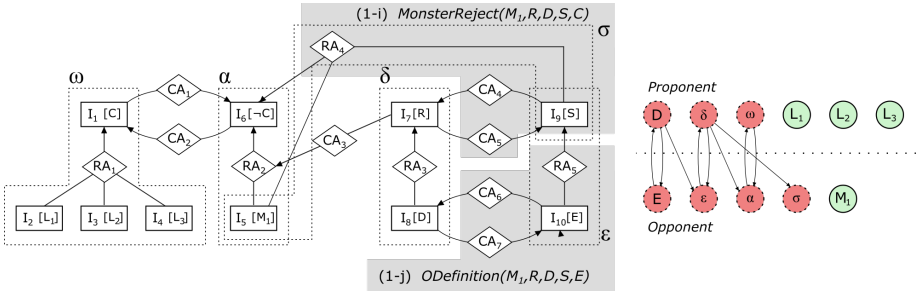



Figure 17: The update made to the Argument Interchange Format structure and the argumentation framework upon Opponent's rejection of the Proponents definition (1-i) and proposal of an alternative definition (1-j).

Progress in the dialogue reaches an impasse until one of the party excepts the definition proposed by the other. In the case of our example, the Proponent's definition is preferred, and the Opponent makes the *Prefer* move to indicate this. In the AIF structure, this results in a preference ($PA_1$ node) showing that $D$ is preferred to $E$, and, as such, the conflict from $E$ to $D$ in the abstract argument structure is removed. This impacts the grounded extension, by re-instating all of the Proponents arguments, meaning that the proponent has successfully defended against the counter (Figure 18).

## 7.3. Second strategy of testing the proof: Countering a lemma

With Proponent having successfully defended against the first attack of the proof, Opponent now makes a further attack. This time Opponent raises a counterexample to one of the lemmas. Proponent accepts that this counter is valid and replaces the attacked lemma with a revised version incorporating the counterexample. The updates carried out in this section of the dialogue are summarised in Table 4 (As this section follows on from the previous counter, the definitions introduced there remain a part of the structure, but are omitted here as they have no impact on the remainder of the dialogue.)

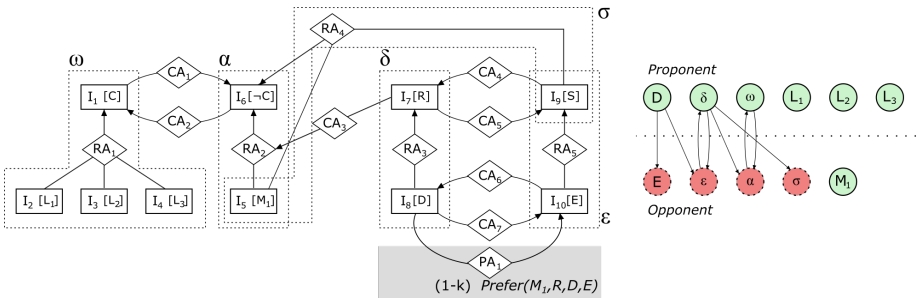



Figure 18: The update made to the Argument Interchange Format structure and the argumentation framework upon Proponent's definition being preferred (1-k).

The second counter begins in (1-l), following the rules **L2.1** and **S16.3b** (as this counter follows directly from the previous *Prefer* move), the Opponent uses *Local-Counter* to introduce the counterexample of a cube, explaining that, if a triangle is removed from inside the triangular network produced by performing the first two steps on this polyhedron, then no edges on vertices are removed ($I_1 1$ in the updated AIF structure in Figure 19). This is in conflict ($CA_8$ and $CA_9$) with the lemma, $L_3$, one of the premises of the abstract argument, $\omega$, representing the proposed proof, and hence $\omega$ is no longer in the grounded extension (Figure 19).

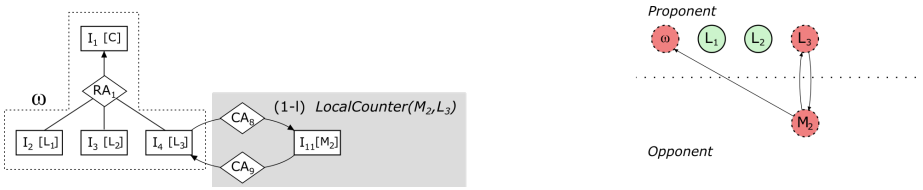



Figure 19: The update made to the Argument Interchange Format structure (on the left-hand side) and the argumentation framework (on the right-hand side) upon Opponent's counter of $L_3$ (1-l).

In (1-m) the Proponent accepts the validity of the counterexample raised by the Opponent, and revises the attacked lemma, $L_3$, via *LocalLemmaInc* to take this into account (a node $MA_1$ represents the rephrase relation between the new lemma and the old lemma). This new lemma, $K$ or $I_1 2$ in the AIF structure, adapts $L_3$ by insisting that triangles may only be removed from the boundary of the network. At this stage in the defence, $\omega$ is still out of the grounded extension, and the only change is the addition of $K$ (Figure 20).

Having introduced the revised lemma, $K$, the Proponent now performs the *Proof-Done* move to incorporate this new lemma into the proof structure. The result of this move is to create a new rule of inference, $RA_6$, between the current lemmas ($L_1$, $L_2$, $K$) and the conjecture ($C$), this produces a new abstract argument $\omega\prime$, which replaces $\omega$ as the current proof (Figure 21). The final grounded extension along with the text of each argument, can be seen in Table 5.

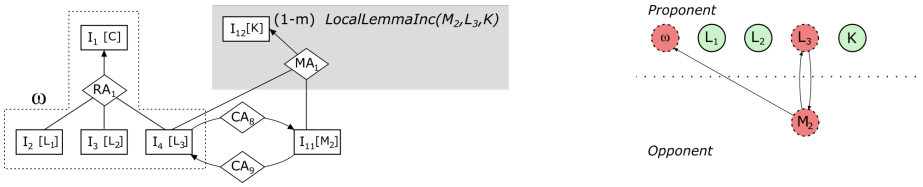



Figure 20: The update made to the Argument Interchange Format structure and the argumentation framework upon Proponent's introduction of a new lemma, $K$, replacing $L_3$ (1-m).




Figure 21: The update made to the Argument Interchange Format structure and the argumentation framework upon Proponent's creation of the revised proof (1-n).

# 8. Conclusions

In this paper we propose a way to narrow the gap between human and machine proof-construction, in order to promote mainstream acceptance and use of automated theorem provers by mathematicians. Philosophical, sociological and educational literature on mathematics highlights the importance of presenting the development of a proof attempt alongside a final, or currently accepted, proof artefact. We have developed an argumentation-based framework in which this is possible. A grounded extension of a dialogue can be produced, representing a currently accepted, collaboratively constructed, proof or theory. Since the record of the dialogue can be presented alongside this proof, the framework delivers the history of a proof attempt as well as the proof artefact. Similarly, social aspects in human mathematics have been shown to be integral to the human context. Technologies which are able to support mathematicians in the collective construction of mathematical knowledge, in a variety of ways, such as highlighting conflicting commitments or unresolved moves, finding similarities and conflicts across different discussions going on in parallel among otherwise independent groups of arguers, storing past discussions and making them searchable, and so on, are essential.

Our work develops across three arcs – a theoretical model, an abstraction level and a computational model – in order to take into account studies of human mathematical reasoning. In particular, we have:

1. taken Lakatos's informally specified technique for conducting dialogue in mathematics,

2. modeled it as a formal dialogue system,

3. expressed this model in a domain specific language for dialogue game specification, DGDL,

4. defined operational semantics for the specification in terms of updates to argumentation structures expressed in a natural language dialogue for argumentation, AIF,

5. induced abstract argumentation frameworks from AIF via the structured argumentation system *ASPIC*$^+$

6. shown the consequences of AIF updates at the abstract layer

7. demonstrated how those abstract semantics yield a grounded extension that provably always corresponds to the theory that has been collaboratively created by the dialogue participants

8. shown how the entire pipeline can be implemented and then skinned with the Arvina interface (using TOAST, DungOMatic, AIFdb and DGEP)

9. and, finally, shown how an example of Lakatos's informal logic of mathematical discovery can be formalised.

This is the first time that formally specified and fully implemented argumentation tools right through the abstraction hierarchy from natural language dialogue through structured argumentation to instantiated abstract argumentation have been brought together and applied to a specific, demanding domain of human reasoning. The foundation that has been laid here allows new explorations into mixed-initiative, collaborative reasoning between human and software participants in interactions which are both naturalistic but formally constrained and well-defined, with the potential to impact both the pedagogy and the professional practice of mathematics.

We emphasise that no formal linguistic analysis or NLP has been incorporated into the current effort. Such an effort must be reserved for future work, and it would play a crucial role in an automated mathematical dialogue system. Contemporary strategies from natural language processing, including the field of argument mining [73] have gained traction within the broader field of discourse mining and could help inform future NLP-based efforts in this direction. Progress has made in understanding the language that mathematicians use [38, 28], integrating this with reasoning systems, and generating human-like output [39].

Representation and reasoning are two important and intertwined branches of AI. Our paper has primarily focused on representation *of* reasoning. Moreover, this is not reasoning by way of standard deductive logic that is often used in logical foundations of mathematics, but is rather, per Lakatos, a closer approximation of the kind of reasoning used in everyday mathematics. Appendix A illustrates that our system models, with reasonable fidelity, the key processes that take place in mathematical dialogues leading to proofs. This can also be taken as a proof-of-concept that implies that the theoretical and technical achievements of the paper are on the right track.

Each of the major arcs mentioned in Section 1 could be continued in future work.

- On the theory side, expanding the range of mathematical dialogues that can be modeled would in some cases require moving beyond Lakatos's informal logic, e.g., to include problem solving and problem specification in the spirit of Pólya [76, 84].

- The abstraction layer could usefully be upgraded with a richer *instantiation layer*: per [11], in contrast to abstract argumentation, instantiated argumentation makes "the internal structure and content of the arguments, the relationships between the arguments in virtue of their content, and the application of argumentation" available to reasoning. This would be relevant for modelling, e.g., the lemmas or sequences of lemmas might be attacked in terms of their structure. This work would go hand-in-hand with the introduction of Pólya-style planning and heuristics.

- On the technical side, another major challenge for future work will be the design of agents that can participate *in* Lakatosian reasoning. This has been piloted in [70] (see [71] for a summary), but the formality and precision of the current offering motivate a new approach to this challenge, and provide groundwork for further advances. For example, following [78], it would be useful if interlocutors were able to identify the relevant moves among the permitted ones.

The research described here presents a solid basis for such a programme of work, because it represents the kind of reasoning that people actually use in practice. The extent to which human reasoning will jibe with "human-oriented theorem proving" systems (as used and surveyed briefly in [39]) remains to be seen. The inferential construction of the grounded extension in our system automates some basic reasoning tasks. Extensions, restrictions, or other modifications to the Lakatosian model – to represent richer or simply different forms of reasoning – along with corresponding extensions to the backend are reasonable projects for follow-up work. For example, we recently presented work with a limited Lakatosian framework for discussions about design artefacts [27]. More broadly, representing practical reasoning bears on several of the challenges about which AI researchers had been initially optimistic [86]. The systems we work with make available inherently explicable patterns of reasoning, which stands at a contrast to some of the impressive successes in the connectionist tradition. Pragmatics are essential for realising AI that can work in dialogue with humans.

Our long term goal is to build a seamless interaction between automatic, semi-automatic and manual processes in shaping the dynamics and the outcome of argumentative interactions among multiple parties, possibly including both human users and artificial agents. Here we have developed a new approach in which tools and theories from the argumentation community can be deployed to build a bridge between interactive proof tools and human mathematicians, thus helping to close the gap between the two disparate styles of reasonings. We have focused on two key areas of human mathematics: the importance of informal proof and presenting proof-development alongside its final form, and an emphasis on social aspects, opening the door to a mixed-initiative collaboration in mathematics. This will be a step towards our vision of a mathematics social machine, in which mathematicians view software systems as valued collaborators and respected fellow mathematicians.

Table 3: First strategy of testing the proof: Countering the conjecture

| **Formalisation** | | **Representation** | | **Evaluation** | |
|---|---|---|---|---|---|
| *LG System* | *Dialogue Theory* | *AIF* | *ASPIC*$^+$ | $AF_{LG}$ | $GE_{LG}$ |
| **Opponent** – Challenge 1 | | | | | |
| (1-f)<br>*GlobalCounter*<br>$(M_1, C)$<br>**[L2.3, S4.1]** | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$ | $\mathbf{I}_5$: For the hollow cube V - E + F = 4 [$M_1$]<br>$\mathbf{I}_6$: It is not the case that for all polyhedra, V - E + F = 2 [not-$C$]<br>$\mathbf{CA}_1$: ($\mathrm{I}_1$, $\mathrm{I}_6$) [($C$, not-$C$)]<br>$\mathbf{CA}_2$: ($\mathrm{I}_6$, $\mathrm{I}_1$) [(not-$C$, $C$)]<br>$\mathbf{RA}_2$: ($\mathrm{I}_5$, $\mathrm{I}_6$) [($M_1$, not-$C$)] | $\mathcal{K} = \{C, L_1, L_2, L_3, M_1, \text{not-}C\}$<br>$\mathcal{R}_{LG} = \{$<br>$(L_1,L_2,L_3) \Rightarrow C$,<br>$M_1 \Rightarrow \text{not-}C$ $\}$<br>$^- = \{(C, \text{not-}C), (\text{not-}C, C)\}$ | $AR_{LG} = \{L_1, L_2, L_3, \omega, M_1\}$<br>$att_{LG} = \{$<br>$(\omega, \alpha), (\alpha, \omega)\}$ | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$ |
| **Proponent** – Defence 1 | | | | | |
| (1-g)<br>*MonsterBar*<br>$(M_1, C, R)$<br>**[L5.1, S5.3]** | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$<br>$R$<br>$L_1$,$L_2$,$L_3$ so $C$ | $\mathbf{I}_7$: This pair of nested cubes is not a polyhedron [$R$]<br>$\mathbf{CA}_3$: ($\mathrm{I}_7$, $\mathrm{RA}_2$) [($R$, $\mathrm{RA}_2$)] | $\mathcal{K} = \{C, L_1, L_2, L_3, M_1, \text{not-}C, R\}$<br>$\mathcal{R}_{LG} = \{$<br>$(L_1,L_2,L_3) \Rightarrow C$,<br>$M_1 \Rightarrow \text{not-}C$ $\}$<br>$^- = \{(C, \text{not-}C), (\text{not-}C, C), (M_1 \Rightarrow \text{not-}C, R)\}$ | $AR_{LG} = \{L_1, L_2, L_3, \omega, M_1, \alpha, R\}$<br>$att_{LG} = \{$<br>$(\omega, \alpha), (\alpha, \omega), (R, \alpha)$ $\}$ | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$<br>$R$<br>$\omega$ |
| (1-h)<br>*PDefinition*<br>$(M_1, R, D)$<br>**[L5.3, S10]** | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$<br>$D$ so $R$<br>$D$<br>$L_1$,$L_2$,$L_3$ so $C$ | $\mathbf{I}_8$: A polyhedron is a surface consisting of a system of polygons [$D$]<br>$\mathbf{RA}_3$: ($\mathrm{I}_8$, $\mathrm{I}_7$) [($D$, $R$)] | $\mathcal{K} = \{C, L_1, L_2, L_3, M_1, \text{not-}C, R, D\}$<br>$\mathcal{R}_{LG} = \{$<br>$(L_1,L_2,L_3) \Rightarrow C$,<br>$M_1 \Rightarrow \text{not-}C$,<br>$D \Rightarrow R$ $\}$<br>$^- = \{(C, \text{not-}C), (\text{not-}C, C), (M_1 \Rightarrow \text{not-}C, R)\}$ | $AR_{LG} = \{L_1, L_2, L_3, \omega, M_1, \alpha, \delta, D\}$<br>$att_{LG} = \{$<br>$(\omega, \alpha), (\alpha, \omega), (\delta, \alpha)$ $\}$ | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$<br>$\delta$<br>$D$<br>$\omega$ |
| **Opponent** – Challenge 2 | | | | | |
| (1-i)<br>*MonsterReject*<br>$(M_1, R, D, S, C)$<br>**[L6.2, S12.2]** | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$<br>$D$ | $\mathbf{I}_9$: My counterexample is a solid bounded by polygonal faces [$S$]<br>$\mathbf{CA}_4$: ($\mathrm{I}_9$, $\mathrm{I}_7$) [($S$, $R$)]<br>$\mathbf{CA}_5$: ($\mathrm{I}_7$, $\mathrm{I}_9$) [($R$, $S$)]<br>$\mathbf{RA}_4$: ($\{\mathrm{I}_5, \mathrm{I}_9\}$, $\mathrm{I}_6$) [($\{M_1, S\}$, not-$C$)] | $\mathcal{K} = \{C, L_1, L_2, L_3, M_1, \text{not-}C, R, D, S\}$<br>$\mathcal{R}_{LG} = \{$<br>$(L_1,L_2,L_3) \Rightarrow C$,<br>$M_1 \Rightarrow \text{not-}C$,<br>$D \Rightarrow R$,<br>$(M_1,S) \Rightarrow \text{not-}C$ $\}$<br>$^- = \{(C, \text{not-}C), (\text{not-}C, C), (M_1 \Rightarrow \text{not-}C, R), (S, R), (R, S)\}$ | $AR_{LG} = \{L_1, L_2, L_3, \omega, M_1, \alpha, \delta, D, S, \sigma\}$<br>$att_{LG} = \{$<br>$(\omega, \alpha), (\alpha, \omega), (\delta, \alpha), (\delta, S), (S, \delta), (\delta, \sigma)$ $\}$ | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$<br>$D$ |
| (1-j)<br>*ODefinition*<br>$(M_1, R, D, S, E)$<br>**[L5.4, S14]** | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$ | $\mathbf{I}_{10}$: A polyhedron is a solid whose surface consists of polygonal faces [$E$]<br>$\mathbf{CA}_6$: ($\mathrm{I}_{10}$, $\mathrm{I}_8$) [($E$, $D$)]<br>$\mathbf{CA}_7$: ($\mathrm{I}_8$, $\mathrm{I}_{10}$) [($D$, $E$)]<br>$\mathbf{RA}_5$: ($\mathrm{I}_{10}$, $\mathrm{I}_9$) [($E$, $S$)] | $\mathcal{K} = \{C, L_1, L_2, L_3, M_1, \text{not-}C, R, D, S, E\}$<br>$\mathcal{R}_{LG} = \{$<br>$(L_1,L_2,L_3) \Rightarrow C$,<br>$M_1 \Rightarrow \text{not-}C$,<br>$D \Rightarrow R$,<br>$(M_1,S) \Rightarrow \text{not-}C$,<br>$E \Rightarrow S$ $\}$<br>$^- = \{(C, \text{not-}C), (\text{not-}C, C), (M_1 \Rightarrow \text{not-}C, R), (S, R), (R, S), (E, D), (D, E)\}$ | $AR_{LG} = \{L_1, L_2, L_3, \omega, M_1, \alpha, \delta, D, \sigma, E, \varepsilon\}$<br>$att_{LG} = \{$<br>$(\omega, \alpha), (\alpha, \omega), (\delta, \alpha), (D, E), (E, D), (\delta, \varepsilon), (\varepsilon, \delta), (\delta, \sigma)$ $\}$ | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$ |
| **Opponent** – Concede Definition | | | | | |
| (1-k)<br>*Prefer*<br>$(M_1, R, D, E)$<br>**[L2.3, S4.1]** | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$<br>$D$ so $R$<br>$L_1$,$L_2$,$L_3$ so $C$ | $\mathbf{PA}_1$: ($\mathrm{I}_8$, $\mathrm{I}_{10}$) [($D$, $E$)] | $\mathcal{K} = \{C, L_1, L_2, L_3, M_1, \text{not-}C, R, D, S, E\}$<br>$\mathcal{R}_{LG} = \{$<br>$(L_1,L_2,L_3) \Rightarrow C$,<br>$M_1 \Rightarrow \text{not-}C$,<br>$D \Rightarrow R$,<br>$(M_1,S) \Rightarrow \text{not-}C$,<br>$E \Rightarrow S$ $\}$<br>$^- = \{(C, \text{not-}C), (\text{not-}C, C), (M_1 \Rightarrow \text{not-}C, R), (S, R), (R, S), (E, D)$ $\}$ | $AR_{LG} = \{L_1, L_2, L_3, \omega, M_1, \alpha, \delta, D, \sigma, E, \varepsilon\}$<br>$att_{LG} = \{$<br>$(\omega, \alpha), (\alpha, \omega), (\delta, \alpha), (D, E), (\delta, \varepsilon), (\varepsilon, \delta), (\delta, \sigma)$ $\}$ | $L_1$<br>$L_2$<br>$L_3$<br>$M_1$<br>$D$<br>$\delta$<br>$\omega$ |

Table 4: Second strategy of testing the proof: Countering a lemma

| **Formalisation** | | **Representation** | | **Evaluation** | |
|---|---|---|---|---|---|
| *LG System* | *Dialogue Theory* | *AIF* | *ASPIC*$^+$ | $AF_{LG}$ | $GE_{LG}$ |
| **Opponent** – Attack | | | | | |
| (1-l) *LocalCounter* $(M_2, L_3)$ **[L2.1, S16.3b]** | $L_1$ $L_2$ | $\mathbf{I}_{11}$: Take the triangular network which results from performing the first two operations on a cube. Now if I remove a triangle from the inside of this network, as one might take a piece of the jigsaw puzzle, I remove one triangle without removing a single edge or vertex $[M_2]$ **CA**$_8$: ($I_4$, $I_{11}$) $[(L_3, M_2)]$ **CA**$_9$: ($I_{11}$, $I_4$) $[(M_2, L_3)]$ | $\mathcal{K} = \{C, L_1, L_2, L_3, M_2\}$ $\mathcal{R}_{LG} = \{$ $(L_1,L_2,L_3) \Rightarrow C$ $\}$ $^- = \{(L_3,M_2), (M_2,L_3)\}$ | $AR_{LG} = \{L_1, L_2, L_3, \omega, M_2\}$ $att_{LG} = \{$ $(M_2, \omega)$, $(M_2, L_3)$, $(L_3, M_2)\}$ | $L_1$ $L_2$ |
| **Proponent** – Defence | | | | | |
| (1-m) *LocalLemmaInc* $(M_2, L_3, K)$ **[L4.1, S6.1]** | $L_1$ $L_2$ $K$ | $\mathbf{I}_{12}$: if we drop the boundary triangles one by one from a triangulated map, there are only two alternatives – the disappearance of one edge or else of two edges and a vertex $[K]$ **MA**$_1$: ($\{I_4, I_{11}\}$, $I_{12}$) $[(\{L_3, M_2\}, K)]$ | $\mathcal{K} = \{C, L_1, L_2, L_3, M_2, K\}$ $\mathcal{R}_{LG} = \{$ $(L_1,L_2,L_3) \Rightarrow C$ $\}$ $^- = \{(L_3,M_2), (M_2,L_3)\}$ | $AR_{LG} = \{L_1, L_2, L_3, \omega, M_2, K\}$ $att_{LG} = \{$ $(M_2, \omega)$, $(M_2, L_3)$, $(L_3, M_2)\}$ | $L_1$ $L_2$ $K$ |
| (1-n) *ProofDone*() **[L1.3, S19]** | $L_1$ $L_2$ $K$ $L_1, L_2, K$ so $C$ | **RA**$_6$: ($\{I_2, I_3, I_{12}\}$, $I_1$) $[(\{L_1, L_2, K\}, C)]$ | $\mathcal{K} = \{C, L_1, L_2, L_3, M_2, K\}$ $\mathcal{R}_{LG} = \{$ $(L_1,L_2,L_3) \Rightarrow C$, $(L_1,L_2,K) \Rightarrow C$ $\}$ $^- = \{(L_3,M_2), (M_2,L_3)\}$ | $AR_{LG} = \{L_1, L_2, L_3, \omega, M_2, K, \omega\prime\}$ $att_{LG} = \{$ $(M_2, \omega)$, $(M_2, L_3)$, $(L_3, M_2)\}$ | $L_1$ $L_2$ $K$ $\omega\prime$ |

| **Argument** | **Text** |
|---|---|
| $L_1$ | Any polyhedron, after having a face removed, can be stretched flat on the blackboard |
| $L_2$ | in triangulating a map, one will always get a new face for any new edge |
| $M_1$ | for the hollow cube V-E+F=4 |
| $D$ | A polyhedron is a surface consisting of a system of polygons |
| $\delta$ | [$D$ so This pair of nested cubes is not a polyhedron at all] |
| $K$ | if we drop the boundary triangles one by one from a triangulated map, there are only two alternatives - the disappearance of one edge or else of two edges and a vertex |
| $\omega\prime$ | [$\{L_1, L_2, K\}$ so for all polyhedra, V-E+F=2] |

Table 5: The final grounded extension at the end of the example dialogue

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# Appendix

## A. Lakatos in the wild

While acknowledging Lakatos's contribution to the field, fellow philosophers of mathematical practice have criticised him for over-reliance on a few case studies, historical inaccuracy and narrowness of application of his theory [5, 4, 6]. This provides an impetus for a wider search for empirical examples. Here, we illustrate the feasibilty of using our formalisation of Lakatosian informal mathematics to model a real-world example of mathematical dialogue.

Until recently, naturalistic observation of mathematical discussions would typically only be possible for the direct participants in those discussions. Archives of online discussions make most of the same information that participants share available to researchers [11]. Such discussions are very different from the more formal presentations typical of research papers, which present polished proofs but lack transparency about how those proofs were obtained [7, 12].

In 2009, Timothy Gowers initiated the Polymath series of experiments in online collaborative mathematics, in which problems are posted online, and an open invitation issued for people to try to solve them collaboratively, documenting every step of the ensuing discussion [3]. The result is an unusual example of a public record of several episodes of mathematical activity that lead to a proof. Martin and Pease [8, 10] discuss the implications of such crowdsourced mathematical activity for the production of mathematics, and describe their preliminary investigations of the community question answering system MathOverflow [1], and the Polymath collaborations [3]. They furthermore present an analysis of one such project for students [2] and describe aspects of it which follow the conversational patterns which Lakatos identified. We build on this work below.

The problem in [2], taken from the International Mathematical Olympiad, asks about a certain geometrical construction involving lines rotating around a set of points in the plane; it refers to an example from this class as a "windmill process."[18] Between the problem proposal and Terrence Tao's certification of the answer in thread 27, approximately seventy four minutes passed [10].

The following set of hand-selected excerpts are hand-coded using the formalism introduced in Section 3 to show explicitly how Lakatosian reasoning contributes to the core steps in the development of the proof. Sometimes near-repetitions appear, when two or more comments are posted simultaneously (we mark these as "Repetition of previous move" below); similarly, we expand some simple statements sometimes into a sequence of several moves in the Lakatos Game. With these clarifications in mind, the

[18] "Let $S$ be a finite set of at least two points in the plane. Assume that no three points of $S$ are collinear. A windmill is a process that starts with a line $\ell$ going through a single point $P \in S$. The line rotates clockwise about the pivot $P$ until the first time that the line meets some other point $Q$ belonging to $S$. This point $Q$ takes over as the new pivot, and the line now rotates clockwise about $Q$, until it next meets a point of $S$. This process continues indefinitely. Show that we can choose a point $P$ in $S$ and a line $\ell$ going through $P$ such that the resulting windmill uses each point of $S$ as a pivot infinitely many times." [2]

following excerpts trace the highlights of the proof, and their corresponding mapping into the dialogue system introduced in Section 3.

## A.1. Thread 3: What is a line?

At the beginning of this thread early on in the discussion, ANONYMOUS decides to simplify things, moving from the running *Conjecture*('*problem statement holds*') to a related game dealing with a similar conjecture in the safe domain of convex polygons. The discussants readily agree that the proposed *Conjecture*('*statement holds in the convex polygon case*') is true. Variations on this case are then considered as potential counterexamples to the original conjecture. These are rejected after the basic concepts in the problem statement have been clarified.

| | |
|---|---|
| 1: ANONYMOUS: If the points form a convex polygon, it is easy. | a. *P*: *Conjecture*('*statement holds in the convex polygon case*'). **[L1.1, S1, C1.1]**<br>b. *P*: *Lemma*('*a windmill process walks around the vertices of the convex polygon*' $L_{3.1}$) **[L1.2, S2.2, C2.1]**<br>c. *P*: *ProofDone*. **[L1.3, S3.2]** |
| 2: THOMAS H: Yes. Can we do it if there is a single point not on the convex hull of the points? | d. *P*: *Conjecture*('*statement holds in the convex polygon plus point case*'). **[L1.1, S1, C1.1]** |
| 3: JERZY: Say there are four points: an equilateral triangle, and then one point in the center of the triangle. No three points are collinear. It seems to me that the windmill can not use the center point more than once! As soon as it hits one of the corner points, it will cycle indefinitely through the corners and never return to the center point. I must be missing something here... | e. *O*: *GlobalCounter*('*equilateral triangle plus point*', '*statement holds in the convex polygon case*'). **[L2.3, S4.1]** |
| 4: JOE: This isn't true - it will alternate between the centre and each vertex of the triangle. | f. *P*: *MonsterBar*('*equilateral triangle plus point*', '*statement holds in the convex polygon case*', '*line is alternating*'). **[L5.1, S5.3]** |
| 5: THOMAS H: No, you're not right. Let the corner points be *A*, *B*, *C*, clockwise, *M* the center. If you start in *M*, you first hit say *A*, then *C*, then *M*, then *B*, then *A*. | g. *P*: *PDefinition*('*equilateral triangle plus point*', '*line is alternating*', '"*line*" *extends in both directions*'). **[L5.3, S10]** |
| 6: JERZY: Ohhh... I misunderstood the problem. I saw it as a half-line extending out from the last point, in which case you would get stuck on the convex hull. But apparently it means a full line, so that the next point can be "behind" the previous point. Got it. | h. *O*: *MonsterAccept*('*equilateral triangle plus point*', '*line is alternating*'). **[L6.1, S12.1]** |

## A.2. Thread 11: Hitting all the points

In this thread, ANONYMOUS offers a potential characterisation of generality. We again retain *Conjecture*(*'problem statement holds'*), and ANONYMOUS first proposes *Lemma*(*'we can start with any point'*). This is refined into the claim *Lemma*(*'the line does not matter'*). However, counterexamples to this claim are produced, and ANONYMOUS must refine the claim again.

| | |
|---|---|
| 7: ANONYMOUS: One can start with any point (since every point of $S$ should be pivot infinitely often), the direction of line that one starts with however matters! | a. $P$: *Lemma*(*'we can start with any point'* $L_{11.1}$). **[L1.2, S2.2, C2.1]** |
| 8: NEMANJA: In other words, we can start with any point and 'just' need to choose a second point through which will we draw a line. | [Repetition of previous move.] |
| 9: ANONYMOUS: Perhaps even the line does not matter! Is it possible to prove that any point and any line will do? | b. $P$: *Lemma*(*'the line does not matter'* $L_{11.2}$). **[L1.2, S3.1, C2.1]**<br>c. $P$: *ProofDone*. **[L1.3, S3.2]** |
| 10: THOMAS H: No, if you start with two points on the convex hull (ordered in the right way) you stay on the convex hull. | d. $O$: *LocalCounter*(*'two points on the convex hull'*, $L_{11.2}$) **[L2.1, S4.2]** |
| 11: NEMANJA: It is not possible, two consecutive points on convex hull will not do. | [Repetition of previous move.] |
| 12: ZHECKA: Sure a choice of line is important. Imagine $S$ is a set of vertices of a convex polygon $P$ (triangle, say) plus one point inside $P$. | [Repetition of previous move.] |
| 13: ANONYMOUS: Only the starting point matters. By the problem statement, it appears that the initial angle is irrelevant to the existence of a pivot point $P^*$ from which all of $S$ is traversed. Every point in $S$ is a pivot point, but only with a specific range of starting angle (e.g. those consistent with the cycle generating $S$). The union of these intervals must necessarily be $[0, 2\pi)$, and thus we can assume WLOG that the starting angle is 0 (and thus we single out a specific point – or points in the case of $\lvert S\rvert = 2$). | e. $P$: *LocalLemmaInc*(*'two points on the convex hull'*, $L_{11.2}$, *'the initial angle is irrelevant within a specific range'*). **[L4.1, S6.1, C2.2]** |

## A.3. Thread 14 (first part): Splitting in two sets, conclusion

This thread, which contains the final solution to the problem, begins with ANONYMOUS advancing *Lemma*(*'we can separate the points in two parts of roughly equal size'*). This draws some initial interest, but the discussants aren't sure how it will be useful. A considerable amount of time passes before GARF is able to make something more concrete from the earlier remarks, in the form of a sequence of lemmas that, he claims, secure the desired result. GARF'S chain of reasoning will be certified as a solution to the problem only after they have been thoroughly vetted. Further minor criticisms and a recap of the solution are elided here.

| | |
|---|---|
| 14: ANONYMOUS: I'm not sure but as no three points are collinear, one can always find a line which splits the points into two sets whose number of elements differ at most one? | a. *P*: *Lemma*(*'we can separate the points in two parts of roughly equal size'* $L_{14.1}$). **[L1.2, S6.1, C2.1]** |
| 15: THOMAS H: That is surely true. How could this help us? | [Not coded.] |
| 16: ANONYMOUS: Something like one can find this no matter how we choose the first point. Then in some time the windmill must be parallel to the line through these points. This line must be unique or else it splits the points such that number of elements differ at least two. | [Not coded.] |
| 17: JUSTIN W SMITH: It appears that the number of points to the "left" or "right" of the line is constant through the entire process! | [Not coded.] |
| 18: GARF: I think this solves the problem. Start with a line which separates the points into two parts of roughly same size (their cardinal differ by at most one, not counting the point to which the line is attached). Then run the process until the line is "upside-down", and so has turn by exactly $\pi$. Every point has gone from the right of the line to the left of the line (easy to see is the number of point is odd, you have to be a bit more crafty if it is even), and no point can go from left to right or right to left without touching the line. Add the previous remarks (the process will always come back to its initial configuration), and every point will be visited infinitely often. | b. *P*: *Lemma*(*'we can run the process until the line is upside down'* $L_{14.2}$). **[L1.2, S2.2, C2.1]**<br>c. *P*: *Lemma*(*'every point goes from right to left'* $L_{14.3}$). **[L1.2, S3.1, C2.1]**<br>d. *P*: *Lemma*(*'no point can go from left to right without touching the line'* $L_{14.4}$). **[L1.2, S3.1, C2.1]**<br>e. *P*: *Lemma*(*'the process will return to its initial configuration'* $L_{14.5}$). **[L1.2, S3.1, C2.1]**<br>f. *P*: *Lemma*(*'each point will be visited infinitely often'* $L_{14.6}$). **[L1.2, S3.1, C2.1]**<br>g. *P*: *ProofDone*. **[L1.3, S3.2]** |
| 19: GAL: Very nice! Don't we run into problems with a convex hull though? Take a square with a point in the middle ($M$) and pass the diagonal of the square (not through $M$) – it seems to me $M$ is never visited (though I may be wrong here). I think we should be more specific in our initial choice of line, maybe? | h. *O*: *HybridCounter*(*'point inside square'*, *'problem statement holds'*, $L_{14.6}$). **[L2.2, S4.3]** |
| 20: GAL: No. This example is false :) | i. *P*: *HybridLemmaInc*(*'point inside square'*, $L_{14.6}$,). **[L4.2, S7.1, C2.3]**<br>j. *P*: *Conjecture*(*'problem statement holds, even for point inside square'*). **[L4.2, S18, C1.1]**<br>k. *P*: *ProofDone*. **[L1.3, S2.1a]** |
| 21: ZHECKA: Yes, it seems to be a correct solution! | [Repetition of previous moves.] |

| | |
|---|---|
| 22: GAL: This seems to be right, but there something I don't understand. Please see if you can help me with it: Start with a square and a point inside it ($M$): start with a tangent to the square (your solution demands a more equal division of points, I know). When we get to the opposite vertex of the square all points moved from one side of the line to the other, but not all points have been visited ($M$ will never be visited). The argument is almost exactly the same, so it seems that the equal division of points plays a crucial role, but I don't understand what role exactly. Can we pin it down precisely? | l. $O$: *LocalCounter*('*M is never visited*', $L_{14.4}$).<br>**[L2.1, S4.2]** |
| 23: GARF: If I understand well your example: the problem is that you must give an orientation to the line. Then, left and right are defined with respect to this orientation: if the line has made half a turn, then left and right are reversed. In your example, I think most of the points move from, say, the part at the top of the line to the part at the bottom of the line, but always stay at the right of the line. | m. $P$: *LocalLemmaInc*('*M is never visited*', $L_{14.4}$, *'left and right must be reversed after a rotation'*).<br>**[L4.1, S6.1, C2.2]** |
| GAL: Got it! Kind of like a turn number in topology. Thanks! :) | [Not coded.] |

These examples show that Lakatos-style reasoning can be used to describe real world examples of mathematical conversations. It is also clear that there are reasoning steps involved that are not classically Lakatosian, for instance surrounding the clarification of terminology rather than concepts (in Thread 3 (g-h)). The relationship between the overall game and a sub-game like the one in Thread 3 (a-c) is not formally specified in our protocol, although it would not be difficult to do so [9]. In some cases, a more detailed analysis might yield further insights into the discussants' thought process. For example, consider Turns 14-17 in Thread 14, which comprise a more complex exchange than would be needed if the only goal was to introduce a lemma. Nevertheless, the overall structure of the discussion is coherent with the Lakatosian framework.

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# **math.wikipedia.org**: A vision for a collaborative, semi-formal, language independent math(s) encyclopedia

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## Introduction

It has been argued that the right representation is the key to successful reasoning [Bun13, p. 16]. We would broaden this observation to point out that while both representation and reasoning are crucial, they are not in themselves sufficient for most applied cognitive and computational workflows. In particular, the processes by which representations are created are logically prior. With large-scale projects, human and HCI factors must be considered, and suitable management strategies devised. Below, we consider the representation and workflow issues surrounding the creation of a large machine-readable mathematics knowledge base. Such a knowledge base would differ from a repository of human-readable texts, in that it would support a range of automated reasoning tasks. For example, one such task is automated tutoring. An ongoing master's thesis [Dud16] on a related simpler task, automated question answering, has helped to motivate the current inquiry.

## 1 What has been done?

As far as human-readable mathematics goes, one of the most successful and visible projects is Wikipedia, the world's largest encyclopedia. The Mathematics WikiProject indexes 31444 articles at the time of writing.[1] For some sub-fields of STEM, Wikipedia also contains a significant amount of related domain knowledge. Until recently, the main – and still most familiar – way of representing this knowledge was in encyclopedic human readable articles in different languages, which were to some degree linked to each other. With the implementation of Wikidata this has begun to change. Now, there is a centralized database that has unique IDs for each semantic concept. From those concepts, there are links to the individual language versions of the Wikipedia articles. In addition, Wikidata contains links between concepts, and properties of the items of various data types, including mathematical formulae [SS16]. In this way, Wikidata forms the backbone of a knowledge graph. The project, then, exists somewhere between two domains: *digital libraries* and *artificial intelligence*.

## 2 Our plan for the future

We propose to rely on Wikidata, and build "math.wikipedia.org" as a frontend to that. This service will allow users to conveniently store and retrieve formulas, terms, and other mathematical objects. Building a knowledge base with machine-readable formulas and links between

[1] https://en.wikipedia.org/wiki/Wikipedia:WikiProject_Mathematics/Count

concepts will produce a richer domain for reasoning than we get from existing projects or other mathematical digital library proposals that lack semantic models.

Working with an existing popular platform should help bootstrap the social side, and allow us to focus our development effort on domain-specific issues. State of the art technology now exists for mining relevant knowledge from existing knowledge resources. Wikidata and a handful of other projects have proposed knowledge representation formats for mathematics. One current example, *Gamma function* on Wikidata is pictured in Figure 1. The page is quite terse. More details would be needed to bring the Wikidata page closer to feature parity with the Wikipedia pages in English or German. Indeed, better-articulated frameworks would be required to adequately represent more complex mathematical concepts, e.g., *adjunction in a category*.[2] Whatever the concept, the definition and any formulas should be accompanied by framing information (e.g., who invented it, where has the concept been discussed?).

## 3 How will the public benefit?

**Improvements to Wikipedia:** If all of this information is available within Wikidata, we and others can use it to make a better Wikipedia using semi-automated methods. For example, article placeholders, autolinking of entries, automatic provision of references, and so on, could all be provided. By using the knowledge graph, such services can be delivered with contextual sensitivity; e.g., to point out that "this article is missing a basic explanation of $X/Y/Z$.")

**Reasoning and computational methods available by default:** Having a computational "frame" over the material, we would be able to infer, e.g, "Here is an example; here's how you compute with it." An implementation of the key Pólya heuristic "If you can't solve a problem, there is an easier problem you can solve: find it" would be valuable for tutoring applications.

**Integration with other knowledge bases and AI systems:** Thinking of the system as a data consumer, we should be able to use it to index and centrally store concepts from the broader literature. Thinking of the system as a data provider, if the Wikidata representations are done right, we should be able to use this data in AI applications, for example, as a resource to use in more general question-answering systems that make use of mathematical reasoning.

## 4 Related work

The NIST Digital Library of Mathematical Functions [Mil13] and the derived Digital Repository of Mathematical Formulae [CMS+14, CSM+15] are inspiring examples of formula-centric storage. Contentmine is an interesting current project that has had success mining knowledge out of research papers (for now, mostly in chemistry and bioscience). It would be interesting to apply the methods to the mathematics and physics content stored in arXiv. Babar is another project for extracting knowledge out of Wikipedia, specifically [dL12]. The authors of the current paper have been highly involved with representing mathematics on the web in a format suited for human readers, notably through work with the MathML Association and PlanetMath. Representing mathematical knowledge in a format suitable for both human and machine interaction requires a somewhat different approach. Michael Kohlhase's recent work on a Semantic Multilingual Glossary for Mathematics is relevant prior art, as are the earlier OpenMath Content Dictionaries. A range of more formal representations exists in the context of projects like HOL [GK15], Isabelle, Coq [CFGW04], and Mizar, where contributions to the Mizar Mathematical Library correspond to articles in the journal *Formalized Mathematics*.

[2] https://en.wikipedia.org/wiki/Adjoint_functors

# 5 Conclusions

If we compare the few high-level properties of the gamma function that are expressed in Figure 1 with those that are captured in connection with the function's formalisation in HOL [SH14], it becomes even more starkly clear that the current Wikidata treatment lacks detail. In addition to listing more properties, the formal treatment naturally provides the proofs of these properties. At the moment, Wikidata even lacks the property '`demonstrates`' that could be used to connect the statement of a theorem with a proof.[3] In principle, one could be added – which would then beg the question as to how proofs would be represented in the system. However, proofs are just one of many challenging topics to address. A simple example: how should Wikidata answer the question "What is the area of a circle?" – given that 'disk' would be correct in mathematical English, but *Kreisfläche* (lit. *circle area*) is the relevant concept in German.[4]

More broadly: what role will semantic representation and AI software play in the development of the online mathematics ecosystem? What workflows will support the use and further development of the constituent online resources? This is the upstream part of the "collaborative development of open services based on representations of mathematical knowledge" [Ion16]. The fruits of repository- and KB-building efforts depend on due attention to these roots.

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[4] https://www.wikidata.org/wiki/Wikidata_talk:WikiProject_Mathematics#Consistent_naming_convention

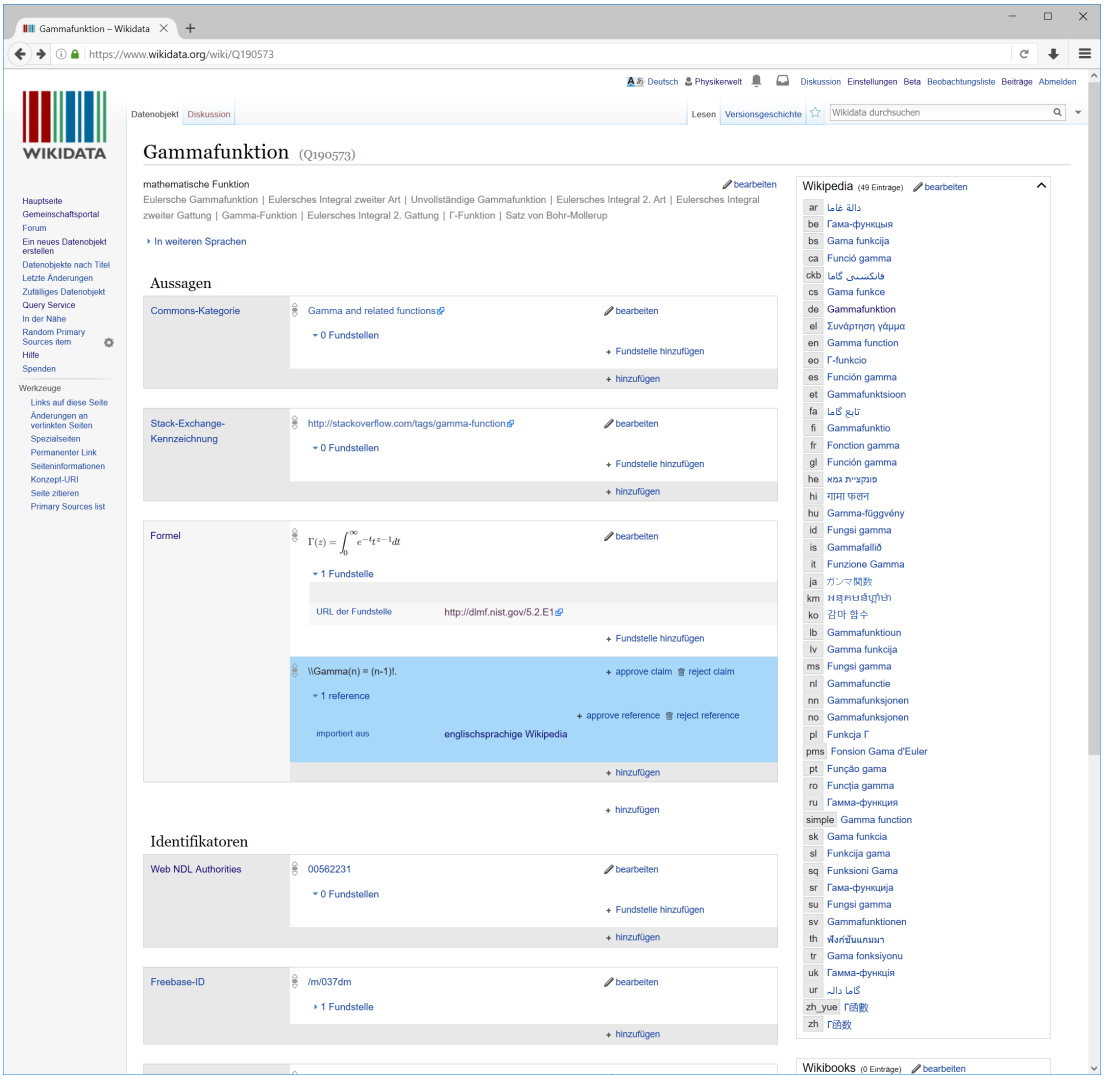



Figure 1: Screenshot of the Wikidata page for Euler's gamma function https://www.wikidata.org/wiki/Q190573, with German language localization. On the top of the page is the title and the id. Below there is a natural language description and a list of aliases. In the block on the right there are links to the various Wikipedia pages about the gamma function. One of the blocks on the left contains a definition that was manually entered, and originates from the NIST Digital Library of Mathematical Functions. The formula in the blue box, $\Gamma(n) = (n-1)!$, was automatically extracted from the English Wikipedia by Yash Nager (a master's student at Universität Konstanz), and currently waits for human verification. Links to external identifiers and other related resources are provided.

# Bootstrapping the next generation of mathematical social machines

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**Abstract**

Tim Berners-Lee defines social machines to be a class of systems "in which the people do the creative work and the machine does the administration." Here, we note that the ansatz in computational creativity – a subfield of artificial intelligence – is that, in the future if not already, computational systems and agents will also contribute directly to creative work. But it can be presumed that we will use standard social machines to develop the next generation 'target' systems, whatever they may be.

The domain of mathematics is a particularly interesting and practical place to explore a range of issues in AI and other aspects of computing. Contemporary mathematical practice intersects several areas of computing practice and theory, ranging from computer-mediated communication to formal computational modelling of mathematical argument.

We ask: How will we use the various existing social machines to bootstrap the next edition? How can the idea of programming languages help?

Bryan Birch, Professor Emeritus at the Oxford Mathematical Institute, is credited with the off-handed remark that he programmed in a very high-level programming language called 'graduate student'.[1]

This anecdote is a schematic match to David Murray-Rust and David Robertson's [3] idea of bootstrapping social machines. These authors partition the overall information ecosystem into 'development' and 'target' machines. For example, a certain set of tools and collaboration strategies went into building the first version of Mediawiki and creating the first few articles in Wikipedia. With this system, one of the world's most popular websites was bootstrapped.

Tim Berners-Lee [2] defines social machines to be a class of systems "in which the people do the creative work and the machine does the administration." The ansatz of *computational creativity* is that, in the future if not already, computational systems and agents will also contribute directly to creative work. (These days, more edits to Wikipedia are made by bots than by people.)

The domain of mathematics is a particularly interesting – and practical – place to explore a range of issues in AI and other branches of computing. Many

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[1] http://mathoverflow.net/questions/11084/what-programming-languages-do-mathematicians-use

mathematicians engage in computer-mediated communication – via email, blogs, wiki, Q&A sites, etc. They use and create digital libraries, also, to an increasing extent – as traces of these communications, and resulting publications, are stored for later use. Computer algebra and other branches of symbolic computing have given rise to their own programming languages, like those developed by Wolfram Research. Theorem proving, after several decades largely within the ægis of computer science, is currently finding an archipelago of use cases within mainstream mathematics. And since its days of origin, mathematics has been closely tied to argumentation: argumentation theory is increasingly capable of modelling the way people do mainstream mathematics.

The next generation of mathematical social machines will need to be conversant with the core ideas in these several modes of engagement. They will also – fundamentally – need to be conversant with the language of mathematics. We have been developing a new strategy for representing mathematical discussions, in which 'The Cayley graph of group $G$' becomes an object in its own right; the relationship between the proposition $P$ and the proposition '$P$ is difficult to prove' is made explicit; and in which Lakatos-style conjectures, refutations, and repair are modelled. A range of mathematical predicates like `stronger`, `not`, `equivalent`, `weaker`, `has_property`, and `case_split` are supported, along with meta-level predicates like `goal`, `strategy`, and `auxiliary` – which are used for guiding proofs. Value judgements such as `easy`, `plausible`, `beautiful`, and `useful`, along with performatives like `agree`, `assert`, `challenge`, `define`, `query`, `retract`, and `suggest` round out this proto-language.

We envision extensions to this lexicon that cover the technical (social mechanical) aspects of mathematical dialogues, surveyed above, with robust interfaces to contemporary social machines (like Wikidata, MathOverflow, etc.). Related work in the programming language literature includes the language Dog [1], which was designed to make it easy to coordinate online services. Service oriented architecture (SOA), in which some responsibility for service assembly is handed over to the computer, is also relevant. 'Scope' and 'context' present particularly interesting challenges. As we imagine was also the case for Birch's graduate student, our language wil have to learn – grow, and develop – by doing.

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# X575: writing rengas with web services

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## Abstract

Our software system simulates the classical collaborative Japanese poetry form, *renga*, made of linked haikus. We used NLP methods wrapped up as web services. This approach is suitable for collaborative human-AI generation, as well as purely computer-generated poetry. Evaluation included a blind survey comparing AI and human haiku. To gather ideas for future work, we examine related research in semiotics, linguistics, and computing.

## 1 Introduction

Computer haikus have been explored in practice at least since Lutz (1959). More recently, haikus have been used by Ventura (2016) as the testbed for a thought experiment on levels of computational creativity. As we will discuss below, the classic haiku traditionally formed the starting verse of a longer poetry jam, resulting in a poem called a *renga*. A computational exploration of renga writing allows us to return to some of the classical ideas in Japanese poetry via thoroughly modern ideas like concept blending and collaborative AI.

Ventura's creative levels range from *randomisation* to *plagiarisation*, *memorisation*, *generalisation*, *filtration*, *inception*[1] and *creation*. Further gradations and criteria could be advanced, for example, the fitness function used for filtration could be developed and refined as the system *learns*. Creativity might be assessed in a social context, as we investigate how a system *collaborates*.

While self-play was a good way for the recently developed board game-playing system AlphaGo to transcend its training data (Silver et al., 2016), we do not yet have computationally robust qualitative evaluation measures for the poetry domain, where there is no obvious "winning condition." We began by creating a program for generating haikus, trained on a small corpus. Our technical aim then was to simulate the collaborative creation of renga, i.e., linked haikus. There are several forms of renga with varying constraints (Carley, 2015), for example the 20 stanza "Nijiun" renga which alternates between two-line and three-line verses, with a focus on seasonal symbolism and rules against repetition.[2] Our initial effort was a technical success, however the rengas we produced fail to fully satisfy classical constraints. A subsequent experiment is more convincing in this regard, but still leaves room for improvement. Our discussion considers the aesthetics of the generated poems and outlines directions for future research.

## 2 Background

Coleridge considered poetry to be "the blossom and the fragrance of all human knowledge." AI researcher Ruli Manurung defines poetry somewhat more drily: "A *poem* is a natural language artefact which simultaneously fulfils the properties of meaningfulness, grammaticality and poeticness" (Manurung, 2004, p. 8).

The *haiku* as we know it was originally called

[1] "[I]nject[ing] knowledge into a computationally creative system without leaving the injector's fingerprints all over the resulting artifacts."

[2] http://www.renga-platform.co.uk/webpages/renga_01.htm

*hokku* – 発句, literally the "starting verse" of a collaboratively written poem, *hakai no renga*. Typically each of following links in a renga take the familiar 5/7/5 syllable form. Classical rengas vary in length from two to 100 links (and, rarely, even 1000). The starting verse is traditionally comprised of two images, with a *kireji* – a sharp cut – between them. The term *haiku* introduced by the 19th Century poet Masaoka Shiki supersedes the older term. Stylistically, a haiku captures a moment.

In classical renga, all of the verses after the first have additional complex constraints, such as requiring certain images to be used at certain points, but disallowing repetition, with various proximity constraints. The setting in which rengas were composed is also worth commenting on. A few poets would compose together in party atmosphere, with one honoured guest proposing the starting haiku, then the next responding, and continuing in turn, subject to the oversight of a scribe and a renga master. These poetry parties were once so popular and time consuming that they were viewed as a major decadence. Jin'Ichi et al. (1975) offers a useful overview.

Because of the way we've constructed our haiku generating system, it can take an entire haiku as its input topic – we just add the word vectors to make a topic model – and compose a response. This affords AI-to-AI collaboration, or AI-human collaboration. It can also blend two inputs – for example, the previous haiku and the current constraint from the renga ruleset (e.g., the requirement to allude to "cherry blossoms" or "the moon").

## 3 Implementation

Working with a small haiku corpus, we used a POS tagger to reveal the grammatical structure typical to haikus. The CMU Pronouncing Dictionary is used to count syllables of words that fill in this structure.[3] The Brown corpus was used to generate n-grams, and the generation process prefers more common constructions in haikus.[4] Wikipedia data was processed with GloVe (Pennington et al., 2014) to create a semantic vector space model of topics, based on word co-occurrences.[5] Adding a web API turned the haiku generating system into a haiku server, and facilitated subsequent work with FloWr. In short:

1. Haiku corpus → POS tagger → grammatical skeleton fragments.
2. General text corpus → n-gram model.
3. Wiki corpus → topic vectors.
4. Combine skeleton fragments to make a haiku template.
5. Assign syllable counts to slots.
6. Fill in the template, preferring n-grams and close topic matches.
7. Wrap the process with a JSON HTTP API

## 4 Experiments

**I. Initial evaluation of haikus** Following Manurung's definition of poetry, above, we would like to assess: (1) whether a given haiku makes sense and how well it fits the topic, (2) whether it fits the form, i.e., is it a valid haiku?, and (3), the beauty of the writing, the emotion it evokes. Details of a survey-based blind comparison of human and computer-written haikus were written up by Aji (2015). The system was then extended with multiple inputs, in some cases producing interesting blends: e.g., the following in response to "frog pond" and "moon":

*that gull in the dress –*
*vivacious in statue*
*from so many ebbs*

**II. Generation of rengas** Here are two rengas generated by wrapping the haiku API inside the FloWr flowchart system (Charnley et al., 2016):

*fertile forefingers*
*took orchard for my lather*
*brackish was cherished*

*toddler of strong bet*
*foaling feels to a good tooth*
*thriving like a paw*

*a drawer straight inside*
*under the slicked interim*
*to shrink the safe cute*

*readjusted blots*
*in the creativity –*
*one child at a love*

*that vase in the quilt –*
*the effeminate of names*
*with a colored juice*

*cases of sibyl*
*and a stylish curators*
*from downed in the aim*

*figures of digress*
*and a sumac excises*
*from key in the ribbed*

*cluster for icebergs –*
*and a waging everglades*
*from huge in the drug*

In each case, the prompt for the first link is "flower blossom" and each link is passed on to

[3] http://www.speech.cs.cmu.edu/cgi-bin/cmudict

[4] https://en.wikipedia.org/wiki/Wikipedia:Database_download

[5] http://nlp.stanford.edu/projects/glove/

the next link along with a secondary prompt. The secondary links are "moon," "autumn," and "love," respectively. For the first renga, we designed a flowchart that selects the "most positive" haiku from the ten that the haiku API returns, using the AFINN word list.[6] In the second renga, we designed a flowchart to select the haiku with the lowest word variety (computed in terms of Levenshtein distance).

**III. Tuning the parameters for the Nijiun form**

*national forces*
*on the government afghan*
*because no-one armed*

*military government*
*the war on the armed*
*afghan*

*a soviet troops*
*upcoming in the winter*
*military armed*

*troops fighting military*
*in the administration*

*blooms operations*
*the military planted*
*of the flower war*

*the army of government*
*we die the war civilians*

We made improvements to the use of the Brown corpus to utilise n-grams for word-flow and sense, as well as tuning the weightings given to sense and topic. We implemented the injection of topics via by blending, as per classical constraints (e.g., required seasonal themes like "winter," or "flowers" in the penultimate link). At left, we quote the closing links of the first Nijiun renga generated by our software.

## 5 Discussion and Related Work

**Towards automated evaluation** Some aspects of the evaluation dimensions are built into the way the poems are constructed.

*Form*: the haiku-generating subsystem guarantees that the requirements of a grammatical skeleton are met, and the 5/7/5 syllable pattern is guaranteed (up to the accuracy of the CMU Pronouncing Dictionary). Surface form scales up well for rengas.

*Sense*: the haiku generating subsystem uses an n-gram model of text likelihood, which will yield a higher score for constructions that match frequently observed phrases. In our first round of experiments with rengas, sense tended to degrade quickly. Our subsequent adaptations to the renga generation algorithm prioritise greater continuitity between links.

*Topic*: we used a vector model of the topic word(s), and can measure the distance to the vector given by the sum of the words in the poem.

[6] http://neuro.imm.dtu.dk/wiki/AFINN

*Emotion*: In our experiment with FloWr, we used a quite simple method, filtering a list for the "most positive" haikus. Mohammad (2016) surveys more recent work in NLP on modelling emotion, which could be exploited in future work.

*Beauty*: Waugh (1980) points out that language is based on a "hierarchy of signs ... of ascending complexity, but also one of ascending freedom or creativity," and also remarks that a "poem provides its own 'universe of discourse.'" To some extent these criteria pull in opposite directions: towards complexity, and towards coherence, respectively. Our first rengas could not be reasonably described as a 'universe of discourse' but rather, a 'universe of random nonsense'. This is improved in the subsequent experiment. Traditional rengas forbid repetition, and discourage overt reflection on themes like death, war, illness, impermanence, religion and sex (Carley, 2015, p. 89). Thus, despite being coherent, the repetitive "military" theme in the final example above is not appropriate to classical constraints. A reader may identify some fortuitous resonances, e.g., "the flower war" is interesting within the "afghan" context established in earlier links – but the system does not yet recognise these features.

**Some paths forward** Wiggins and Forth (2015) use hierarchical models in a system that builds a formative evaluation as it composes or reads sentences, judging how well they match learned patterns. While this seems to have more to do with constraints around typicality, per Waugh, there is room for creativity within hierarchies. Hoey (2005) makes a convincing argument that satisfying lexical constraints while violating some familiar patterns may come across as interesting and creative.

Word similarities can be found using GloVe: this would presumably produce links with more coherent meanings, compared to the edit distance-based measure we used. Ali Javaheri Javid et al. (2016) use *information gain* to model the aesthetics of cellular automata. Can these ideas be combined to model evolving topic salience, complexity, and coherence?

If the system provided a *razo* (the troubadours' jargon for "rationale"; see Agamben (1999, p. 79)), we could debug that, and perhaps involve additional AI systems in the process (Corneli et al., 2015).

## 6 Conclusion

In terms of Ventura's hierarchy of creative levels, the haiku system appears to be in the "generalisation" stage. Our renga-writing experiments with FloWr brought in a "filtration" aspect. The research themes discussed above point to directions for future work in pursuit of the "inception" and "creativity" stages.

Some previous work with haiku, e.g. Netzer et al. (2009) and Rzepka and Araki (2015), have addressed the problem of *meaning*. The renga form brings these issues to the fore. We hope this early work has motivated further interest in this challenging and enjoyable poetic form that – like other less constrained forms of dialogue – combines themes of natural language generation and understanding. One natural next step would be a series of experiments in collaborative human-AI generation of rengas. Our haiku software is available for future experiments.[7]


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# Teaching natural language to computers

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## Abstract

"Natural Language," whether spoken and attended to by humans, or processed and generated by computers, requires networked structures that reflect creative processes in semantic, syntactic, phonetic, linguistic, social, emotional, and cultural modules. Being able to produce novel and useful behavior following repeated practice gets to the root of both artificial intelligence and human language. This paper investigates the modalities involved in language-like applications that computers – and programmers – engage with, and aims to fine tune the questions we ask to better account for context, self-awareness, and embodiment.

## 1 Introduction

In the genesis of intelligent computational systems, one often observes programs that write before they can read, compose before they can listen, and paint before they can see. However, the most successful systems in poetry, music, and visual art are indeed perceptually aware – and derive significant benefits from that ability [Kurzweil, 1990; Screene and Wiggins, 2015; Colton *et al.*, 2015b].

We will argue that the bar by which we judge computational creativity in text generation – and appreciation – can and must be raised, in order to build systems that we can meaningfully communicate with. Our paper takes the form of a necessarily high-level sketch, supported by hand-crafted examples that draw on both standard and custom software. Most of these examples concern computer poetry, but our aim is not to present a technical or aesthetic achievement. Rather, we use the examples to survey the limitations of current systems and to indicate some potentially novel approaches.

Specifically, we highlight three interrelated thematic areas that we think will repay effort.

**(Limited) Contextual Understanding** Here, we are concerned with what makes a response to some circumstance meaningful. This certainly requires *context* [Ogden and Richards, 1923]. For example, a response in a conversational dialog generally considers the previous elements of the exchange, and perhaps also previous exchanges, and elements of a shared culture. This understanding does not need to be, nor in general can it be, "complete." Furthermore, in many cases, the reader or listener will hear meanings that were not there originally [Veale, 2015] – however, the interpolation of meaning by the reader cannot always be relied upon. Later in this paper we include an example of a computer-generated poem that is essentially just verbose babbling fitted to a predefined template. This poem was previously reviewed by a published poet, and it does not stand up well to critical scrutiny [Corneli *et al.*, 2015]. The poem misses any sense of "why" – and the program that generated this poem would not be able to offer what the Provençal poets called a *razo* [Agamben, 1995, p. 51], that is, an exposition of the poem.

**(Limited) Self-awareness** Here, we focus exclusively on the ability to reason about creativity as a process. Computer programs often have limited metadata about their software processes, for example the "signatures" of functions, specifying the types of input data that the functions will accept, or "contracts" specifying preconditions, postconditions, and invariants. Along with these representations often comes some limited ability to reason formally about code. But few if any contemporary software systems would be able to convincingly make sense of a simple writing prompt, or adapt a dialogical process of response in order to reach an agreement, or respond to feedback from a critic in order to produce a better poem. In the future, such computational abilities with language may be commonplace. These abilities may depend on fairly profound epistemic features, for example, the computer might need to recognize when its "knowledge" is uncertain, and proceed accordingly – perhaps asking for help, or making multiple generative attempts in parallel and assessing which one works better relative to its contextual understanding.

**(Limited) Embodiment** We will consider program flowcharts [Charnley *et al.*, 2014] as the primary framework with which to describe the computer's "process" layer. Whether embodied as a flowchart or a Von Neumann machine or something else, computational processes are also physical processes. The manipulations of the nodes and arrows of a flowchart or of some other collection of physical objects, like the flowchart's corresponding script, or the words of a poem, can (potentially) be thought about with

respect to its gestural content. This definition of *gesture* due to the 12th Century theologist Hugues de Saint-Victor, quoted in [Mazzola, 2016], shows the connection with our other themes: *Gestus est motus et figuratio membrorum corporis, ad omnem agendi et habendi modum.* Mazolla glosses this as follows:

> Gesture is the movement and figuration of the body's limbs with an aim, but also according to the measure and modality proper to the achievement of all action and attitude.

Introducing this quote requires us to make one significant caveat. Whereas humans tend to perceive ourselves as relatively free beings, able to act according to a purpose and even to choose which purpose to serve, we regard computers in a very different light. At best, a computer can be programmed to optimize its behavior relative to some constraint. This perspective does not sit well with the typical understanding of the English word "aim."[1] The point to make here is that a *computational system* is understood relative to an operating environment, and its behavior is worked out relative to that environment. Under some circumstances we would call this process "programming," and under somewhat different circumstances we would describe it as "self-programming" or "automatic programming." In short, it is not necessary to attribute intention to the computer, but – once again – it is necessary to think about its behavior in context.

The remainder of the paper is structured as follows: First, we will explore these themes from a computing standpoint, developing a technical sketch rather than a formal system description. Then we turn to a discussion that evaluates this technical sketching from the point of view of the second author, an English as a Foreign Language teacher with a prior background in consciousness studies. Finally, we draw on this discussion to outline a plan for future computational experiments centered on making sense of language.

## 2 Motivation

Oscar Schwartz offers the following framing:

> Can a computer write poetry? This is a provocative question. You think about it for a minute, and you suddenly have a bunch of other questions like: What is a computer? What is poetry? What is creativity? [Schwartz, 2015]

Computer poetry may sound like a bit of a lark – after all, it's not clear that anyone really needs it. Nevertheless, asking these questions about poetry begins to suggest a way of working with language that has wider implications.

Consider Turing's idea that machines "would be able to converse with each other to sharpen their wits" [Turing, 1951]. This could be realized as a *Q&A site specifically for computers*. The discussions could address all manner of practical concerns, for example, those arising for bots that are engaged in code or editorial review tasks. A reputation system and web of trust could be used to maintain quality control. If the participating computational systems had sufficient abilities with natural language, this system could be bridged into a Q&A site that is in everyday use by human beings. However, before a computational system would be useful in any Q&A context, it would presumably need to be able to be able to model the meanings of the questions that are being asked reasonably well, and also to be able to compose meaningful responses. For now, we will side-step the Chinese Room-style question of whether the system "really" understands what it is processing [Bishop, 2004] and focus on the more applied question: how would meaningful responses occur? A high-level outline could be something like this [Corneli *et al.*, 2015]: (1) Read and understand the "prompt" to a sufficient degree; (2) Compose a response that "makes sense"; (3) Criticize the response along various dimensions, for instance, does it read well, does it tell a story or develop a character?; (4) Consider how it might be improved. This outline is based on an established process that groups of creative writers use to critique and revise poetry or literary works for publication.

When we turn to computer generated text – say, poetry, to be concrete – in addition to examining the generated poem, a sophisticated audience can also examine the *process* whereby the poem was generated, and read the product against this process (or vice versa). Indeed, the computer can create both poem and process, the later via automatic programming – and offer its own assessments of both as well, as long as it can make sense of the success criteria.

### 2.1 Related work

Natural language processing often begins with a grammar. If none is available, it may be induced, for instance by using compression techniques [Wolff, 1988]. Both in older [Redington and Chater, 1997] and quite recent work [Hermann *et al.*, 2015], statistical and neural network approaches to corpus-based language understanding have shown strong potential for developing "reasoned" ways of thinking about linguistic structure without the usual grammatical assumptions. Corpus methods help to understand the patterns in the way people use language, and the creative potential of unexpected word combinations [Hoey, 2005].

One example cited by Hoey is Tennyson's famous line *Theirs is not to reason why*. Here, the word *reason* is used with its verb sense, rather than with the noun sense that most readers would expect based on their prior experience with the two-word phrase *reason why*. This unique feature makes the line memorable and interesting. The psycholinguistic properties of the broader phenomenon of "lexical priming" have been extensively studied [Pace-Sigge, 2013]. One empirical result is that priming works differently for native speakers and for non-native ESL speakers, insofar as native speakers are more affected by binding of words within formulas, whereas someone learning a new language tends to only recognize the strings that they have encountered before.

The ascendant status of data-driven methods in natural language processing does not obviate symbolic AI, which continues to be useful for work with specialist languages. For computer programming languages in particular, techniques for *reflection* and, in the case of LISP, *homoiconicity* (i.e., treating code as data) make it possible to write computer pro-

[1] It would appear that Mazolla freely introduced this concept, instead of sticking with the more literal "agenda."

grams that reason about, write, or adapt computer programs.

Artificial languages have been used in video games in creative ways, but not, as yet, for functional communication with or between non-player characters. Multi-agent systems have, however, been used in poetry generation. One example system creates poems based on repeated words and sounds driven by a model of agents' emotional states [Kirke and Miranda, 2013], inspired by earlier work in music [Kirke, 2011]. As for computer programs that read poetry, this is typically limited to reading (and mimicking) surface style, without extending to *meaning* [Carslisle, 2000] – even if some readers were fooled. Without knowing what's in a poem, it seems difficult to be other than superficial.

Just how far the "surface" goes is a question much discussed by poets and translators of poetry. Red Pine, translator of the 13th-14th Century Chinese poet and Buddhist monk Stonehouse (石屋) wrote as follows [Pine, 2014, p. xxiv]:

> I don't know how others do it, but when I've tried to think of a metaphor for what I go through, I keep coming up with the image of a dance. [ . . . ] I try to get close enough to feel the poet's rhythm, not only the rhythm of the words but also the rhythm of the poet's heart.

A typical approach to poetry generation might take Stonehouse's corpus and notice that he often writes about clouds and mountains and plants, and attempt to generate a poem "in the style of Stonehouse," referencing some of these typical concepts and aiming to get the number of syllables and the grammatical structure right. However, there is little doubt that a reader with an ear for Chinese poetry or some familiarity with the ideas of Chan Buddhism would recognize these *ersatz* attempts for what they are: "dead words" [Pine, 2014, p. xxiii]. More interesting computer poems in a somewhat related genre (Japanese *haiku*) have been created by programmers working from the premise [Rzepka and Araki, 2015, p. 2497] that a reader's interest in a haiku stems from:

> feeling that the poet understands a situation and that we can mentally agree with what she/he (or maybe *it*) shares with us.

The challenges posed by computer poetry serve as a point of departure. "Poetry exercises are used to allow learners to explore the complexities of English" [Parker, 2010] – or another language – and the contexts in which it is meaningful.

# 3 Exploration

Here are 10 short examples of writing prompts excerpted from the book "642 Things to Write About" [San Francisco Writers' Grotto, 2012]:

1. What can happen in a second.
2. The worst Thanksgiving dish you ever had.
3. A houseplant is dying. Tell it why it needs to live.
4. Tell a story that begins with a ransom note.
5. Write a recipe for disaster.
6. If you had one week to live. . . .
7. What your desk thinks about at night.
8. The one thing you are most ashamed of. . .
9. Describe your best friend.
10. Describe Heaven.

In order to create a computer-generated response to any of these prompts, in addition to understanding what the prompt is saying, some further understanding of the topic is required. The response itself will have various standard features. Many of the following features of *stories* are found (with minor adaptations) in poetry, and other kinds of writing:

> A story is not a modular presentation of ideas but a multi-layered work consisting of interdependent characters, plot elements, and settings. [Kim *et al.*, 2014]

Let us consider, then, a simple theory of stories and storytelling, using the prompts above as our domain. One suitable theory would involve a micro-world containing: *A Scenario*, *A Narrator*, *An Audience*, *A Beginning*, *A Middle*, and *An End*. Consider that – with respect to the "Thanksgiving dish" prompt – the computer has presumably never tasted food of any kind. It could, however, "imagine" a scenario in which there is a character who eats a Thanksgiving dish.

## 3.1 An example *Scenario*

The scenario could be represented with various relations:

> Squanto `memberOf` Patuxet tribe, Patuxet tribe `hasCardinality` 1, Thanksgiving `isa` event, Thanksgiving `hasHost` Pilgrims, Thanksgiving `hasGuest` Squanto, Thanksgiving `hasFood` eel, eel `hasCondition` burnt

Naturally, this might be extended with further information as that comes to light; and in practice we might use a more robust formalism.

## 3.2 The other components

The *Narrator* would walk through the scenario and say what's there. On a metaphorical level, the narrator's role is somewhat similar to the way a virtual camera moves through a 3D simulation in order to create a film. However, the *Narrator* needs to consider the *Audience* in order to be effective. As indicated in the quote from Kim *et al.*, above, the data in the *Scenario* needs to be structured when telling the story.

Here is one possible presentation of the scenario, embellished with some facts, fictions, and local color, and combined into several sentences that flow reasonably well.

> Squanto was the last surviving member of the Patuxet tribe. He attended the first Thanksgiving with the Pilgrims wearing a new buckskin jacket. One of the foods that was served was eel, but it had been rather badly burnt and Squanto didn't find it to his liking.

This text manages to include a range of emotionally evocative, thought provoking, character establishing, and sensory language ("last surviving member," "first Thanksgiving," "new buckskin jacket," "rather badly burnt," "didn't find it to his liking") – and at least one unexpected word combination ("Thanksgiving. . . eel"). It also has a discernible *Beginning*, *Middle*, and *End*. It seems appropriate to call it a *story*.

### 3.3 How to come up with stories?

Let's start with a parse:

```
(TOP
 (NP
  (NP (DT The) (JJS worst)
      (NNP Thanksgiving) (NN dish))
  (SBAR (S (NP (PRP you)) (ADVP (RB ever))
           (VP (VBD had)))) (. .)))
```

Here are some associated word meanings from WordNet:[2]

| Word | Gloss from WordNet |
|---|---|
| (DT The) | determiner |
| (JJS worst) | (superlative of 'bad') most wanting in quality or value or condition |
| (NNP Thanksgiving) | commemorates a feast held in 1621 by the Pilgrims and the Wampanoag |
| (NN dish) | a particular item of prepared food |
| (PRP you) | pronoun |
| (RB ever) | at any time |
| (VBD had) | serve oneself to, or consume regularly |

There are some other interesting word senses available, and knowing which one to pick, or how (and how much) to combine various senses seems like a bit of an art form. Should we consider *a short prayer of thanks before a meal* when thinking of "Thanksgiving"? For now, we will say a short prayer and tentatively assume that a `(NN dish)` is what you eat, rather than what you eat off of. "Lexical priming" [Hoey, 2005] techniques would help make the relevant distinction here.

But supposing we get this far, now what? We've moved from one sentence to several quasi-sentences, without getting that much closer to a "*Scenario*" like the one envisaged above. One possibility is that the WordNet expansion could be sufficient give us relevant keywords and phrases, from which a small corpus could be built (e.g., by doing a web search for the glosses) and then mined to learn relations between the items in that corpus. Alternatively or additionally, these meanings might be connected to a pre-computed model of linguistic meaning drawing from a much larger background corpus [McGregor *et al.*, 2016].

Mining significant associations from large scale text corpora is something people have explored in various ways. Finding subject-verb-object triples, in particular, is a popular method: one well-known algorithm is presented by Rusu *et al.* [2007]. This is sometimes called "building a semantic model." Other more sophisticated approaches might draw on associations with a pre-existing ontology [Kiryakov *et al.*, 2004] – the particular benefit of the Rusu *et al.* approach is that it can be implemented using a simple parsing-based method. The basic theme of building semantic models of text goes back to Quillian [1969] – about whom there will be more to say later. For now, we just remark that in addition to expanding the writing prompt, we may also want to draw on some "stock" associations stored in a background knowledge base. We illustrate the method from Rusu *et al.* by applying it to the beginning of the novel *Frankenstein* (Table 1).

> You will rejoice to hear that no *disaster* has accompanied the commencement of an enterprise which you have *regarded* with such evil *forebodings*. . . . [Shelley, 1831] (emphasis added)

| | | | |
|---|---|---|---|
| 1. | *disaster* | *regarded* | *forebodings* |
| 2. | *yesterday* | *increasing* | *confidence* |
| 3. | *London* | *fills* | *delight* |
| 4. | *feeling* | *understand* | *feeling* |
| 5. | *breeze* | *gives* | *foretaste* |

Table 1: 5 triples extracted from *Frankenstein*

As far as text understanding goes, the result in Table 1 is not particularly encouraging. However, even a low-fidelity database of background knowledge would allow us to extend the story. Perhaps Squanto would decide that the burnt eel *gives* a *foretaste* of things to come. To make this association, we might use methods similar to the ones used to reason about ConceptNet triples – which have been employed to good effect in text and concept generation within the FloWr framework [Llano *et al.*, 2016], which is described below.

### 3.4 Can FloWr flowcharts be used to solve the composition problem?

FloWr [Charnley *et al.*, 2014] is a flowcharting system with basic text processing abilities. Metadata describing the "ProcessNodes" from which FloWr's flowcharts are formed can be used to pose and solve simple automatic programming problems. A writing prompt like "worst Thanksgiving dish" can be interpreted as a constraint – or, more broadly, a fitness function – that steers the generative process, and trickles through to guide the choice of functional components and, eventually, words. By "fitness function," we understand that text may be composed iteratively, and improved along the way, relative to some context of evaluation. The simple examples in Figure 1 illustrate problems that can be solved quite straightforwardly:

> "Give me a list of mixed *adjectives* and *nouns*."

(the italicized terms are "independent" variables that tell the `WordSenseCategoriser` node how to behave); or

> "Give me a list of *5* rhyming couplets built of text from The Guardian and Twitter mentioning '*eels*'."

It may make sense to include considerably more abstraction in the description of larger and more complex flowcharts. For example, a flowchart discussed by Corneli *et al.* [2015] includes 28 nodes and generates the following poem (and others that are similar):

*Oh dog the mysterious demon*
*Why do you feel startle of attention?*
*Oh demon the lonely encounter*
*ghostly elusive ruler*
*Oh encounter the horrible glimpse*
*helpless introspective consciousness*

[2] NB., WordNet contains no entries for determiners or pronouns.

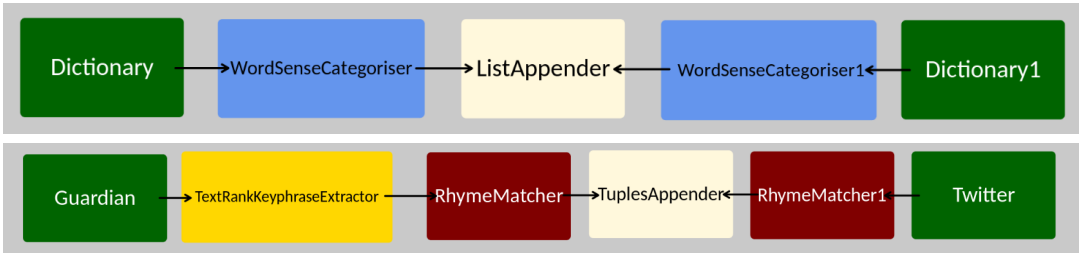



Figure 1: Two simple FloWr flowcharts

| | | |
|---|---|---|
| number `N` | `Dictionary` | list of words of length `N` |
| list of words, word sense | `WordSenseCategoriser` | list extracting words with the word sense |
| input string(s) | `TextRankKeyphraseExtractor` | list of key phrases extracted from the strings |
| phrases, number `M` of phonemes | `RhymeMatcher` | tuple of couplets with `M` rhyming phonemes |
| number `N`, word | `Twitter` | some `N` tweets containing the word |
| tuples | `TuplesAppender` | a list combining the tuples |
| lists | `ListAppender` | a list combining the lists |

Table 2: Triples describe the functional mapping from input to output for selected FloWr nodes

Would the most succinct description of the flowchart be an approximately 28 clause sentence that is *equivalent* to the flowchart? Perhaps, in the current case, everything can be compressed down to the following template (fixed for this flowchart), and potentially further with optimizations:

*Oh THEME the COLLOCATE SIMILAR*
*Why do you feel INVERSION of DESIRED?*
*Oh SIMILAR the COLLOCATE-OF-SIMILAR SIMILAR-TO-SIMILAR*
*SIMILAR-TO-COLLOCATE SIMILAR-TO-COLLOCATE′ COLLOCATE-OF-SIMILAR*
*Oh SIMILAR-TO-SIMILAR′ the COLLOCATE-OF-SIMILAR-TO-SIMILAR SIMILAR-TO-SIMILAR-TO-SIMILAR*
*SIMILAR-TO-COLLOCATE SIMILAR-TO-COLLOCATE COLLOCATE-OF-COLLOCATE*

The connection between this template, its instantiation, and the putative prompt, "Write a poem about an old dog who is afraid of attention," is tenuous at best. Nevertheless, reasonable hope exists for future work that would generate models – and a *Narrator* – tailor-made to a given prompt.

FloWr's ProcessNodes define a micro-language denoting the available ways in which the system can transform input data to output data (Table 2). In a meaningful expansion of a given prompt, many choices would have to be made. New flowcharts built in response to writing prompts or other contextual data would constitute the system's core "learnings." In short, poetry and process need to be thought about together.

## 4 Discussion

In this section, we will briefly review human language learning from a second language teaching perspective, and then draw comparisons with the foregoing description of a hypothetical computer language learner.

First, why are primary language learning and second/foreign language learning often considered separately? One of the biggest differences between the two cases is that after learning a first language, "neural pathways" have been set down, so that second language information has to be encoded "along with" or "beside" the first language pathways. New neural connections have to be formed to maintain the memory of the second language, whereas the first language has been implanted quite thoroughly. So when learning verbs, nouns, etc., the person (child, adult) learns or has learned, for instance, "*stand up!*" first and then that "*levantate!*" means a similar thing, in Spanish, or that "*chair*" = "*silla*." As they progress in the second language, what we find is a process that linguists call "interlanguage" [Selinker, 1972], or a language that is not quite English, and not totally Spanish yet, because of the variations in the two languages in structure, phonemes, morphemes, allophones, semantics, and so on.

Language is also very multi-modal, and if we think again in terms of neurons and brain pathways, the word "dog" not only brings up images of a furry canine animal but all kinds of other associations. Pet dogs. Large and small dogs. Dogs with different color fur (visual cortex). *Dog* as a verb (motor cortex associations). The smell of wet dogs (olfactory bulb). Fear of dogs (limbic system). Favorite pet dogs you have loved in your life (emotions). The spelling or sound of the word D-O-G as opposed to D-A-W-G (auditory and visual cortex). If you are from certain countries perhaps even dog meat. Imagining a "demon dog" will invoke networks running all over the brain [Schlegel *et al.*, 2013]. Now what about computers, do they have a similar symbolic or multi-modal operating ability? People, as they become literate, learn to write and sound out letters at the same time – and the foundations of first language learning draw heavily on a sensory-motor channel [Iverson and Thelen, 1999; Hernandez, 2013].

Computers may be said to have a first language with several dialects, highly constrained by grammar – namely, bytecode and programming languages. The flowcharts described above begin to recover a degree of multi-modality, and a process orientation that is similar in certain partial respects to sensory-motor experience. We might also think of flowcharts

as akin to neurons and cortexes, as above. After all, there is a kind of embodiment even in the brain – it is an active and evolving organ [Doidge, 2007]. Flowcharts are rather different from classical neural network models, but one common feature is that they would need to be "trained" if they are to understand and express language.

Again, poetry could be part of the way forward. Exercises from a book like *Writing Simple Poems: Pattern Poetry for Language Acquisition* [Holmes and Moulton, 2001] might be used to teach computers as well as humans. Note that grammar and poetry are very different, and perhaps complementary. Thus, for example, a "Learn English!" notebook found in Japan with *Subject – Verb – Direct Object – Indirect Object – Prepositional Phrase – Adjectival Phrase* written at the top of each page offers a useful rubric for Japanese students, since the verb comes at the end of the sentence in Japanese (SOV). But grammar will only get you so far. Consider Chomsky's famous nonsense statement, "Colorless green ideas sleep furiously." A poetic gesture – like prefixing that phrase with a description of "planted tulip bulbs" – is able to make some sense out of nonsense by adding context![3]

Social context is also likely to be relevant: peer learning is very useful for human language learners [Rancière, 1991; Raw, 2014]. One idea would be to adapt the Q&A model mentioned in Section 2 as a "social" site about poetry.

## 5 Future work

Quillian's "The Teachable Language Comprehender: A Simulation Program and Theory of Language" [1969] took the novel – and fundamental – approach of understanding things in such a way that new understandings could be added directly to its knowledge base.[4] When reading a piece of text, the TLC program would search its memory for related information that it could use to make sense of the input. More specifically, a given text would be expanded using "form tests" which extracted meaningful pieces of the text, and connected these to items stored in memory. Quillian writes that "ultimately, a human-like memory should relate descriptive knowledge of the world to perceptual-motor activity, in a manner like that indicated by Piaget" – but deems this to be "far beyond our present scope" [Quillian, 1969, p. 474].

Future research might use FloWr to develop a TLC-like library of "form tests" and generative tools that would add a multi-modal aspect to knowledge representation. To be sure, a flowchart-based representation of poetic process would be quite different from the embodied sensory-motor experience of humans. Nevertheless, computational processes that allow us to model text generation contextually, procedurally, and gesturally can help to understand the way linguistic meaning comes to be. This is not something we can readily learn from parsing, corpus-based modeling, or grammar-based text generation alone. There is exciting potential for future experiments with natural language that strives to capture and express shades of meaning and the "feel" of the language. Experimentation is necessary: if we have learned anything about language, it is that "learners should be motivated to speak bravely" [Wang, 2014].

[3] `http://www.linguistlist.org/issues/2/2-457.html#2`

[4] In practice, "While the monitor can add TLC's encoded output to the program's memory, the program itself makes no attempt to do so, nor to solve the problems inherent in doing so" [Quillian, 1969, p. 473].

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# An institutional approach to computational social creativity

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### Abstract

Modelling the creativity that takes place in social settings presents a range of theoretical challenges. Mel Rhodes's classic "4Ps" of creativity, the "Person, Process, Product, and Press," offer an initial typology. Here, Rhodes's ideas are connected with Elinor Ostrom's work on the analysis of economic governance to generate several "creativity design principles." These principles frame a survey of the shared concepts that structure the contexts that support creative work. The concepts are connected to the idea of computational "tests" to foreground the relationship with standard computing practice, and to draw out specific recommendations for the further development of computational creativity culture.

## Introduction

One two-part claim is advanced and defended herein: *Elinor Ostrom's theory of institutions can be used to design systems that exhibit computational social creativity, and a culture supports this work*. The contribution takes the form of several candidate "design principles," a literature survey that elaborates them, and an analysis that connects these ideas to common programming practice.

The paper is structured as follows. The "Background" section describes Ostrom's (1990) *Institutional Analysis and Development* (IAD) framework, focusing on her proposed design principles for commons management. To connect these ideas to social creativity, the paper draws on the 4Ps (Person/Process/Press/Product), a model for thinking about creative contexts (Rhodes 1961) that has been brought to bear in theorising computational creativity (Jordanous 2016). This is summarised and slightly adapted. In the subsequent main section of the paper, "Testing for Creativity", Ostrom's design principles are transposed from the world of commons management to the world of computational social creativity. This section looks for ways to connect the propsed creativity design principles to computational methods, and also draws on contemporary thinking in the philosophy of technology, with examples from familar social computing settings like Wikipedia. A two-part example dealing with both the "soft" culture of the computational creativity community and potential software-based interventions is presented in the "Example" section. Finally, the "Discussion and Conclusions" highlight the relevance of this work for computational creativity culture, systems, and evaluation.

## Background

This section summarises the motivation for the paper, introduces Elinor Ostrom's work, and reviews Rhodes's 4P framework. The central parts of this section are Table 1 and 2, which list Ostrom's design principles for managing a commons, and transpose them to creative domains.

**Motivation** The current investigation is motivated, in part, by the idea of *Ecologically Grounded Creative Practice* (Keller, Lazzarini, and Pimenta 2014). Within a given ecological niche, agents and objects interact; niches can also be brought into relationship in creative ways. The current work has in mind relatively sophisticated agents with their own "contextual maps" and the ability to participate in "reading and writing computational ecosystems" (Antunes, Leymarie, and Latham 2015). Such agents will use, view, critique, and evaluate the work and workflow of other agents. Although computational agents with all of these features do not exist yet in any robust form, we can reason about them, and in so doing, help design the future of *computational social creativity* (Saunders and Bown 2015) – an emerging research area at the nexus of artificial life, social simulation, and computational creativity.

**Elinor Ostrom's "design principles"** To contextualise this effort, we must begin with a short excursus into economics. Ostrom's work is typically applied to study the management of natural resources. In economics jargon, the specific resources considered are *rivalrous* and *non-excludable*. This means that consumption by one party precludes consumption by a rival, and that it is not directly possible to for anyone to block others' access to the resource. Economic goods with these two properties are referred to as *common pool resources* (CPRs); see (Ostrom 2008). Fisheries and forests are important examples. Economic actors have incentives to exploit these resources, however, there are natural limits on total consumption. In principle, a CPR might be gobbled up due to individual greed: this is the so-called tragedy of the commons, and one does not have to look too far for examples. However, in practice, the tragic outcome does not always transpire. Ostrom's theoretical and empirical perspective helps understand why, and emphasises:

(1) the importance of group attributes and institutional arrangements in relation to the structure of incentives and utilities for individual decision making; and (2) the likelihood of a broader set of possible outcomes, including user-group institutional solutions (McCay and Acheson 1990, p. 23)

Ostrom's ideas have recently been applied to analyse Wikipedia, considered as an "expressive commons" (Safner 2016). Wikipedia is *non-rivalrous* in consumption, if we accept the metaphor "to read is to consume." However, *contribution* to Wikipedia presents a range of salient social dilemmas, and efforts to manage them are reflected, for example, in the *Neutral Point Of View* (NPOV) policy, which helps produce "articles that document and explain major points of view, giving due weight with respect to their prominence in an impartial tone."[1]

IAD focuses on *action situations*, framed in three phases: *context*, *action*, and *outcome*. Importantly, this part of the theory is not linked to the particular details of CPRs. Ostrom uses the term *institution* to refer to the "shared concepts used by humans in repetitive situations organized by rules, norms, and strategies" (Ostrom 2010). We will return to these concept categories later and consider them further from a computational perspective. For now, our way into thinking in terms of IAD will be by way of several *design principles* for the successful management of CPRs that Ostrom described; see Table 1. These principles work together to support institutions that maintain the integrity of the commons – for example, by ensuring that behaviour is monitored, that knowledgeable and concerned parties are the ones who make specific rules, and that conflicts do not get out of hand (Ostrom et al. 2012, p. 79).

**The four Ps** We can bootstrap our contextual understanding of creativity with the help of an existing model. Rhodes (1961, pp. 307-309) intends "the four Ps" to refer to the following facets of creativity, which are familiar from everyday experiences of creativity in society.

Person – *personality, intellect, temperament, physique, traits, habits, attitudes, self-concept, value-systems, defense mechanisms, and behavior.*

Process – *motivation, perception, thinking, and communication.*

Product – *an idea embodied into a tangible form.*

Press – *the relationship between human beings and their environment.*

We will shortly use these concepts to rewrite the items in Table 1, replacing the focus on *appropriation* with a focus on *contribution* that befits a theory of social creativity.

Jordanous makes a case for thinking about computational creativity using Rhodes's 4P's, starting with a critique of the strategies used in the evaluation of computational creativity, which, she emphasises, is "traditionally considered ... from the perspective of the creative output produced by a system" (Jordanous 2016).

Ecological thinking suggests that that it is quite limited to take the final product as the sole term of analysis. At least we might like to introduce the "embedded evaluation" of creative products into the creative process, and build agents that are aware of some contextual features of their environment. For example, these agents might ask: *How similar or how different is my generated artwork to an existing artwork, or*

**Ostrom's design principles**

**1A. User boundaries**
"Clear boundaries between legitimate users and nonusers must be clearly defined."[2]

**1B. Resource boundaries**
"Clear boundaries are present that define a resource system and separate it from the larger biophysical environment."

**2A. Congruence with local conditions**
"Appropriation and provision rules are congruent with local social and environmental conditions."

**2B. Appropriation and provision**
"The benefits obtained by users from a common-pool resource (CPR), as determined by appropriation rules, are proportional to the amount of inputs required in the form of labor, material, or money, as determined by provision rules."

**3. Collective-choice arrangements**
"Most individuals affected by the operational rules can participate in modifying the operational rules."

**4A. Monitoring users**
"Monitors who are accountable to the users monitor the appropriation and provision levels of the users."

**4B. Monitoring the resource**
"Monitors who are accountable to the users monitor the condition of the resource."

**5. Graduated sanctions**
"Appropriators who violate operational rules are likely to be assessed graduated sanctions (depending on the seriousness and context of the offense) by other appropriators, by officials accountable to these appropriators, or both."

**6. Conflict-resolution mechanisms**
"Appropriators and their officials have rapid access to low-cost local arenas to resolve conflicts among appropriators or between appropriators and officials."

**7. Minimal recognition of rights to organise**
"The rights of appropriators to devise their own institutions are not challenged by external governmental authorities."

**8. Nested enterprises**
"Appropriation, provision, monitoring, enforcement, conflict resolution, and governance activities are organised in multiple layers of nested enterprises."

Table 1: Ostrom's design principles as expressed in the meta-review carried out by Cox, Arnold, and Tomás (2010)

[1] https://en.wikipedia.org/wiki/Wikipedia:Five_pillars

[2] Repetition is *sic*, the point being that the boundaries must be both distinct and explicitly defined.

**Proposed creativity design principles**

**1A.** The population of Producers who can add to or alter the resource is clearly defined.

**1B.** The boundaries of the Place must be well defined.

**2A.** The Process is related to local conditions.

**2B.** Contributing to the Product has benefits for the Producer that are proportional to the efforts expended.

**3.** Most Producers who are affected by the rules governing contribution can participate in modifying the operational rules.

**4A.** Tests document the interaction of Producers and Place.

**4B.** Tests can be modified by Producers or their representatives.

**5.** Producers who violate operational rules in the domain will be assessed sanctions by other Producers.

**6.** Producers have rapid access to low-cost local arenas to resolve conflicts.

**7.** The rights of Producers to devise institutions governing their contributions are not challenged by external authorities.

**8.** Contribution, testing, enforcement, conflict resolution, and governance and are organised in multiple layers of nested Places and agencies.

Table 2: "Creativity design principles" formed by switching the polarity of entries in Table 1 to emphasise contribution rather than appropriation, and using the concept of "tests" to connect to computing practice

*to the components thereof, or to the initial conception for the work?* This route is quite close to Ritchie's (2007) empirical criteria for judging a final product against an "inspiring set" – but now makes evaluation an explicit part of the creative process. Some recent work in computational creativity emphasises embedded evaluation (Gervás and León 2014). However, as Jordanous argues, creative products are just one part of the overall creative process – and the 4Ps help expose the other features.

Unfortunately, however, Rhodes's thinking and terminology is too anthropocentric for our current purpose. As Ostrom describes it, action situations are to be understood using seven clusters of variables: *participants*, *positions*, *potential outcomes*, *action-outcome linkages*, *participant control*, *types of information generated*, and *costs and benefits assigned to actions and outcomes* (Ostrom 2009, p. 14). Nowhere does this mention a "Person". Continuing the adaptations begun by Jordanous (2016), the four Ps will be rendered here as Producer/Process/Product/Place. It is important to emphasise that these labels are strictly more inclusive than Rhodes's, and more abstract. In particular, the Place corresponds to Ostrom's action situation, structured in advance by contextual features. This adapted 4P model is reminiscent of the *Domain-Individual-Field Interaction* (DIFI) model due to Feldman, Csikszentmihalyi, and Gardner (1994), if we understand Domain $\approx$ Place, Individual $\approx$ Producer, and Field $\approx$ (a collection of) established Processes. Note that contextual theories, broadly construed, pose a long-standing challenge for computing, partly because "what context is changes with its context" (Gundersen 2014, p. 343). One possible working definition is that: "Context is what contrains a problem solving [scenario] without intervening in it explicitly" (Brézillon 1999). Another relevant remark is that context is "defined solely in terms of effects in a given situation" (Hirst 2000).

In developing an *institutional approach to computational social creativity*, we will look for the rules, norms, and strategies that can be used to establish suitable and effective contextual relationships between Process(es), Place(s), Producer(s), and Product(s).

**Transposing the design principles into "creativity design principles" and translating them into technical terms**
*Software testing* is embodied in the formal ideas of *assertions*, *advice*, and *contracts*. Related programming methodologies aim to build *executable specifications* and may make use of *test-driven development* (TDD). These techniques provide various ways for (evolving) programs to interact with their context. These ideas can help us translate Table 1 into technical terms. To get started, Table 2 uses the 4P terminology and the generic notion of a test to transpose Ostrom's design principles into "creativity design principles."

## Testing for creativity

The current section elaborates the candidate creativity design principles outlined above, expanding each with relevant literature and examples, and seeking the ways in which each principle could be applied within a software system.

**1A. User boundaries**

"The population of Producers who can add to or alter the resource is clearly defined."

In user-oriented computing, this principle is often addressed using *Access Control Lists* (ACLs) or other permissions mechanisms. The corresponding tests are relatively simple: either each modifiable object in the system has a piece of metadata about it that says who can modify it, or each user has a piece of metadata attached to his or her user account that says which resources they can modify.

Before granting access to a resource, we may require that a Producer implements certain protocols. In a client-server architecture, the client generally communicates using an existing API and may need to implement a certain set of callback functions or adhere to other restrictions. Noncompliant user behaviour after access has been granted may result in access being revoked. Thus, for example, even though Wikipedia is "the encyclopedia anyone can edit," violating the site's principles may lead to a IP-based block, or a username-based ban.

**1B. Resource boundaries**

"The boundaries of the Place must be well defined."

The source of this well-definedness may come from "either side." That is, the Place may advertise its definition in terms of its APIs and other criteria (as above) together with

guarantees on output behaviour in the style of "Design by Contract" (Mitchell and McKim 2002); alternatively, Producers may implement tests that restrict the Places that they will engage with.

In a simple example of the latter sort, a game-playing agent might resign if it estimates that its position is unwinnable. The fact that different participants can have different perspectives points to an interesting special case in which the (shared) definition of the Place arises in an emergent manner. This phenomenon is especially important if we "[take] a broad view of creativity as any process in which novel outcomes emerge" (Saunders and Bown 2015).

**2A. Congruence with local conditions**

"The Process is related to local conditions."

In Ostrom's original formulation, local conditions were broken down along axes of "time, place, technology, and/or quantity of resource units" (Ostrom 1990). With respect to theorising the local conditions of creativity, we can gain a useful perspective by turning briefly to the psychoanalyst Winnicott's treatment of "the exciting interweave of subjectivity and objective observation" which takes place in an "area that is intermediate between the inner reality of the individual and the shared reality of the world" (Winnicott 2002, p. 86). We are then led to consider those local conditions that exist in the "interweave" of Place and Producer. For example, roboticist Andy Clark proposes a theory of extended cognition, in which enminded beings "use" the environment to self-program and are not just programmed by the environment (Clark 1998). However as Clark points out elsewhere, "it becomes harder and harder to say where the world stops and the person begins" (Clark 2001). In short, the mind is not separated from the body or environment but grounded in perception (Ingold 2000).

A corresponding computational test is found in the earlier example of embedded evaluation, in which existing artefacts are employed as a virtual sensorium. More broadly, this principle concerns making sense of, or "parsing", the Place (and the other P's). This Process is well described by Steigler's notion of *grammatisation*: "processes by which a material, sensory, or symbolic flux becomes a gramme," or, more simply, "the production and discretisation of structures" (Tinnell 2015). Remember that while a given agent is trying to make sense of the world, others are likely trying to make sense of that agent as well. Framed as dilemma, the last word would likely be: "program or be programmed" (Rushkoff 2010) – but reflecting on Clark's comment above, we see that this can become somewhat complex.

**2B. Appropriation and provision**

"Contributing to the Product has benefits for the Producer that are proportional to the efforts expended."

The usual way of thinking about computers – as non-agentive machines – would render the above-stated principle perfectly meaningless. In connection with principle 2A, we should here remark: "That an object is more profitable or effective is only a secondary consequence of its refinement" (Chabot 2013, p. 12). In any case, before we can think about "benefits" in the case of a non-human (and non-living) Producer, the phrasing of the current principle leads us to ponder the cost of their "efforts."

It may be best to change tack, and ask, with Terrance Deacon, "In what sense could a machine be alive?" (Deacon 2014). If a machine were responsible for maintaining its own energy supply, its features of outward-orientation might give cause to say that the machine has a "self" (Deacon, Haag, and Ogilvy 2011). Consider for example the Ethereum project, which provides protocols for distributed computing and the creation of "decentralized autonomous organisations" – whose organisation relative to the outside world is mediated by cryptocurrency, referred to as "fuel" (Wood 2014).

From a testing standpoint, the key requirements are: an ability to judge whether a given option can be (tentatively) thought of as beneficial, and, ideally, a memory that can compare these judgements with iterations of similar situations later on. In this way we would recover the foundations of reinforcement learning, and, as Ostrom points out, the core logic behind the development of new institutions:

> "How about if you do $A$ in the future, and I will do $B$, and before we ever make a decision about $C$ again, we both discuss it and make a joint decision?" (Ostrom 2009, p. 19)

**3. Collective-choice arrangements**

"Most Producers who are affected by the rules governing contribution can participate in modifying the operational rules."

Let us reflect in more detail on the *rules* that comprise – along with biophysical and material conditions and community attributes – the locally-contextual variables which determine or constrain an action situation (Ostrom 2009, p. 15). At their simplest, these rules are "if-then" statements giving instructions that determine the behaviour of persons in certain roles. As such, each rule contains a logical test, and changing the rules means writing new tests.

Ostrom develops a grammar around this idea, and defines *regulatory rules* with the following formula:

> ATTRIBUTES of participants who are OBLIGED, FORBIDDEN, OR PERMITTED to ACT (or AFFECT an outcome) under specified CONDITIONS, OR ELSE. (Ostrom 2009, p. 187)

*Norms* and *strategies* are defined using a simplified formula, also cast in terms of *attributes*, *deontics*,[3] *aim*, and *conditions* (Ostrom 2009, p. 140). The prescriptive terms may be assigned a particular weight, and actions and consequences may also be assigned a particular cost or value (Ostrom 2009, p. 142). Some relevant actions are: *be* in a position, *cross* a boundary, *effect* a choice, *jointly exercise* partial control together with others, *send or receive* information, *pay out or receive* costs or benefits, and *take place* (for outcomes) (Ostrom 2009, p. 191).

[3] I.e., the presciptive valence – obliged, forbidden, or permitted, as above – for norms, not for strategies.

Something more needs to be said about the assertion that Producers "can" participate in changing (or creating) rules, norms, and strategies. In practice, participatory systems tend to be lossy. Changes to rules and structures will tend to be carried out by those Producers who are *most* affected – and *thus* most knowledgeable; cf. Ostrom et al. (2012, p. 79). The structure of new rules is predicted by Conway's Law:

> [T]here is a very close relationship between the structure of a system and the structure of the organization which designed it. (Conway 1968)

Specifically, the proposed relationship is "homomorphism": following Conway, any Product will mirror the hyper-local conditions that describe the Producers' social context. Furthermore, it seems likely that Products will mirror local environmental conditions in the Place. This points to importance of a broad class of tests that would be described as environmental "sensors". This theme will be developed more fully below.

**4A. Monitoring users**

"Tests document the interaction of Producers and Place."

The straightforward view suggested by the idea of "monitoring" is to deploy some global functionality that keeps track of the actions of all participating Producers within a Place. But this function can be broken up and distributed out among the Producers themselves. In the first instance, what a Producer produces is sensory data. Sensors are generally deployed along with effectors or (more broadly) transducers that translate the sensory information into action. So, monitoring is important for modelling any action or interaction whatsover. For example, The Painting Fool compares an initial altered snapshot (sensory data) to the painted image that it generates in response to that snapshot, and judges the quality of its output on that basis (Colton and Ventura 2014). This example could be extended to theorise "proprioceptive" sensing and judgement about effected actions more broadly. Filtering upstream data is another simple application of sensors, which Keller (2012) describes as an "ecocompositional technique." In short, an ecological view on monitoring suggests that it can be distributed out among participants and that this is vital for social creativity.

**4B. Monitoring the resource**

"Tests can be modified by Producers or their representatives."

The environment itself also filters and selects (Kockelman 2011). Some of these conditions are fatal for living beings in the environment, and more broadly may provide terminating conditions for the constituent Processes in a Place. It would be too much to say that *all* tests can be modified by Producers. Rather, Producers may have programmatic access to those tests which transform potentially fatal (or at least fateful) features of the Place and participating Producers into *data*. This opens up the possibility of directly modifying decision making processes on the one hand, or of passing along information about the fitness landscape to future generations of Producers in a (co-) evolutionary framework on the other (DeLanda 2011).

Simply put, *data* is lack of uniformity within some context (Floridi 2016). In the case of monitoring the extractive use of CPRs, direct and compelling feedback about instances of non-uniform or otherwise aberrant resource usage define critical (i.e., decisive) points within a resource management structure. In creative contexts "critique" is no less important.

**5. Graduated sanctions**

"Producers who violate operational rules in the domain will be assessed sanctions by other Producers."

Economic sanctions are generally punishments, which are presumed to have a clear meaning or a direct impact on behaviour. However, there are other cases in which Producers' interactions with other Producers will not be punitive so much as, for example, educative or otherwise formative.

In an artistic context, "sanctions" may range from constructively critical reviews to outright condemnation to no response at all. The *Iterative Development-Execution-Appreciation* (IDEA) cycle (Colton, Charnley, and Pease 2011) introduces *well-being* and *cognitive effort* ratings from which several derived measures of audience response can be computed (e.g., by averaging across audience members). This can readily be extended to a developmental or peer production context. "Audience" might be re-thought as a "public," or as Rhodes's "press" (as originally formulated) to capture the idea that its response has a direct effect on the Producer. Inasmuch as the Producer is produced, feedback from the "parent" Producer(s) is especially important to this formative Process.

**6. Conflict-resolution mechanisms**

"Producers have rapid access to low-cost local arenas to resolve conflicts."

Wikipedia's edit wars provide a familiar example (Viegas et al. 2007; Yasseri et al. 2012). These are carried out on the pages of the encyclopedia itself, and resolved using supplementary pages. Machine-generated metadata is relied upon throughout. These mechanisms are low cost: the "stigmergic" self-organisation patterns exemplified by open online communities make fairly minimal demands on participating agents (Heylighen 2015). Nevertheless, structure matters: cases of direct and unresolvable conflict must usually be referred a higher authority, e.g., sitewide guidelines and policies, or available arbitration committees. Opportunities to jointly exercise partial control are, again, often Products, and the creation of a communication channel – a Place within a Place – is another formative Process, which Jakobson (1960, p. 355) calls the "phatic function."

The theme of local scale suggests more and less representative examples. For instance, academic research is currently organised in a much more segmented and localised format than Wikipedia. *Modularity* is one of three features that are hypothesised to support *commons based peer production* (CBPP) (Benkler 2002). However, CBPP requires not just decomposability into modules but relatively fine *granularity*

of these modules, and as well as a *low cost of integration* to bring disparate pieces of work together once they are completed – possibly "subsidised" by an assistive technology, like Wikipedia's metadata systems. Creative and scientific writing, at the level of individual papers or books, tends to miss features that would allow this work to scale up (Kim, Cheng, and Bernstein 2014) – even though science and literature represent impressively huge "virtual" collaborations.

The most straightforward test related to this theme is that a Producer needs to be able to *detect* conflict, either between itself and other Producers, or between incompatible goals. In order to resolve a conflict – or to organise work on a project to avoid conflicts in the first place – a Producer will probably need to reason about the project's structure.

**7. Minimal recognition of rights to organise**

"The rights of Producers to devise institutions governing their contributions are not challenged by external authorities."

The foremost external authority to be concerned about in a computational creativity setting is the programmer. A "mini me" critique can readily be levelled by CC sceptics (Colton 2012). We are still in early days for autonomous creative systems and general AI, and involvement of programmers and others in teaching systems how to devise institutions is at least as relevant as teaching them how to conform to pre-given instructions.

Keeping in mind the earlier reflections on Winnicott, a relevant set of tests would compare the frequency of user- or programmer-generated changes in the system, with the frequency of changes coming from the system itself. This is the thrust of the diagrammatic formalism of creative acts developed by Colton et al. (2014): with considerable further work we could expand the ability of computer systems to participate in, or fully automate, such modelling activity. A basic challenge in applying the formalism from Colton et al. is to identify the individual "creative acts" that a given Producer has made. The tests that would reveal these acts in a given stream of Products tend to be domain-specific.

**8. Nested enterprises**

"Contribution, testing, enforcement, conflict resolution, and governance and are organised in multiple layers of nested Places and agencies."

That the Place or the Producer would be layered isn't a surprise; many systems have a hierarchical aspect. What is perhaps more surprising is that many of the features that make up a "creative ecosystem" must themselves be *produced*, which points to the inherent multiplicity of Producers. Here, Producers are seen as self-organising the structure of their interrelationships and interconnections at various levels. Developing a computational treatment of such a system divorced from real world applications would be a thankless and ultimately futile task. Effort may be better spent on developing programs that model and participate in existing creative ecosystems. In such cases, there would be real-world empirical tests of success, coming from users.

## Example

This section uses the creativity design principles discussed above to describe some of the creativity-supporting institutions in place at the Seventh International Conference on Computational Creativity (ICCC 2016), and to explore potential additions and adaptations for future ICCCs.

**I.** The crucible for the current paper was a unique set of ongoing discussions (see "Acknowledgements") (2A). At first, the hope was to co-author the paper with one of these discussants, but due to time constraints this was not possible, so it became a single-author paper (1A). The ICCC call helped motivate writing up the ideas (2B), partly because the conference is open to papers that are informed by and contribute to various disciplines at varying degrees of formality. However, ICCC enforces rigourous academic standards, using slightly different evaluation criteria for papers submitted to each of five "tracks" (1B). Reviewers used the Easychair website to bid for papers to review, and to share discussions and debate about these papers in case of disagreement (6). Papers that were seen as less relevant were rejected outright, or potentially (as a norm) allocated briefer slots in the conference schedule (5). The current paper was conditionally accepted, which meant that it entered into a "shepherding" process, whereby a senior programme committee member could check (4A) whether the author followed through on specific reviewer requirements (4B). By and large authors are given free rein to write papers about any topic relevant to computational creativity, if they do so in a rigorous academic style (7). This entails reflecting on certain themes-held-in-common – but the conference seems to lose some opportunities for structuring engagement more deeply, e.g., around common tools or challenge problems (8?). Presumably only the conference steering committee can change the conference's overall rules; however, it should be noted that reviewer requirements constitute fine-tuned rule-setting at the level of individual papers (3?).

**II.** The reflections above begin to suggest ways in which we might make better use of software systems in creative partnership. One realistic idea would be to use computer programs to help with paper review tasks. Essay grading software is now mainstream, and services like WriteLab can help authors simplify their writing and catch grammar and logic errors.[4] Agent-based reviews or a shift to *post-publication review*, in which reviews are offered "after an article is published, much like commentary on a blog post" (Ford 2013, p. 316) would change the population of reviewers (1A). Moving beyond blogs to wikis, lists of open problems from prior publications could be collected, compared, and explicitly referenced with semantic links (1B) (Tomlinson and others 2012). This could begin to make explicit the ways in which a given paper constitutes an advance (2A, 2B). The development, use, and maintenance of shared tools (APIs, open source software) and design patterns for computational creativity could be encouraged (3). A standardised testing approach based on challenge problems, as in the

[4] `https://writelab.com` offers a freemium service for students, but is "always free for instructors."

recently announced OpenAI Gym,[5] with worked examples, explicit evaluation metrics, and variant versions (4A, 4B) could help the community move towards, and enforce, standards of *replicability* and *generalisability* (5). Partial "wikification" and semantisation of the research area is already underway with systems like FloWr (Charnley, Colton, and Llano 2014) and ConCreTeFlows (Znidaršič et al. 2014), but it is unclear whether these systems will merge, or diverge, or if a new standard will come along (6?). Once shared technologies and datasets are in common use, computational agents will be better able to contribute to the field (7). It is to the advantage of computational creativity researchers to develop applications and application environments that we – and others – agree are useful (8).

## Discussion and Conclusions

This section reviews the contribution above, beginning with a link to related work. Specifically, Ostrom's high-level Institutional Analysis and Development (IAD) framework can be regarded in parallel with the high-level outline of the *Standardised Procedure for Evaluation of Creative Systems* (SPECS) (Jordanous 2012). SPECS suggests that, in order to evaluate creativity, it is necessary to put forth a definition of "what it is to be creative," and then to specify criteria by which creativity will be measured before formulating the evaluation. IAD suggests that institutions operate within a certain context, which afford certain kinds of actions, and that these lead to certain observable outcomes. To wit:

| **IAD** | **SPECS** |
|---|---|
| Context | Definition |
| Action | Criteria |
| Outcome | Evaluation |

In IAD, context can be thought of as a collection of "exogenous variables," (Ostrom 2009, p. 13, esp. Figure 1.1) including pre-defined rules, that shape what happens in the action situation at the heart of the analysis. We have described several candidate design principles that outline potential rules for guiding action in creative settings. This suggests the possibility of recording a definition and set of criteria for evaluating social creativity in a general domain. Pragmatically, this definition might unpack the 4Ps in terms of Ostrom's variables (participants, positions, etc.).

SPECS could be criticised for being overly abstract: in other words, for simply describing good practice in any empirical investigation. IAD adds many more specifics, which have necessarily been presented in a compressed form here. It is hoped that this first attempt to use IAD to theorise computational social creativity will motivate future explorations that further unpack social creativity using Ostrom's ideas.

The creativity design principles offer guidelines (and with minor changes, hypotheses) for members of the computational creativity community to test out in practice. More empirical work is needed to validate (or improve) these principles. On the cultural side, more attention should be given to the fact that our institutions – including institutions for building institutions – are analysable in programmatic terms.

[5] `https://gym.openai.com/`


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# An Argument-based Creative Assistant for Harmonic Blending

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## Abstract

Conceptual blending is a powerful tool for computational creativity where, for example, the properties of two harmonic spaces may be combined in a consistent manner to produce a novel harmonic space. However, deciding about the importance of property features in the input spaces and evaluating the results of conceptual blending is a nontrivial task. In the specific case of musical harmony, defining the salient features of chord transitions and evaluating invented harmonic spaces requires deep musicological background knowledge. In this paper, we propose a creative tool that helps musicologists to evaluate and to enhance harmonic innovation. This tool allows a music expert to specify arguments over given transition properties. These arguments are then considered by the system when defining combinations of features in an idiom-blending process. A music expert can assess whether the new harmonic idiom makes musicological sense and re-adjust the arguments (selection of features) to explore alternative blends that can potentially produce better harmonic spaces. We conclude with a discussion of future work that would further automate the harmonisation process.

## Introduction

The invention of new harmonic spaces in this paper is conceived as a computational creative process according to which a new harmonic idiom is created by means of blending the 'atoms' of harmony, i.e., transitions between chords. The blended transitions are created by combining the features characterising pairs of transitions belonging to two idioms (expressed as sets of potentially learned transitions) according to an amalgam-based algorithm (Confalonieri et al., 2015a; Eppe et al., 2015b) that implements Fauconnier and Turner (2002)'s theory of conceptual blending. The transitions are then used in an extended harmonic space that accommodates the two initial harmonic spaces, linked with the new blended transitions.

When modeling creative processes computationally, one of the key questions is how good are the created artefacts. The approach to evaluation that has been applied most frequently within computational creativity requires a human to evaluate attributes of the created work or the system's operation. Basic measures consider the *typicality* of a generated artefact within a particular genre, or the *quality* of the generated work according to the users' aesthetic judgement (Ritchie, 2007).

In music blending, the evaluation of artefacts is not a trivial matter. This is due not only to the time evolving nature of the final output, but also to the lack of clearly defined criteria for their assessment. In the particular case of transition[1] blending, which is how harmonic blending is approached in this paper, the evaluation of the blends is of key importance, in order to produce musically meaningful extended harmonic spaces. To evaluate the set of blended transitions and the corresponding generated extended harmonic space, several musical features need to be taken into account according to indications by musicologists. The importance of each particular feature, however, is not known in advance and musicologists need to make adjustments by experimenting with a large set of test cases.

To ease this task, in this paper, we propose a creative tool (Figure 1) that assists a musicologist with the evaluation of harmonic blends. The system allows a musicologist to specify *arguments*—abstracting the properties of chords and transitions—and to use them for an iterative evaluation of the blended outcome, based on the transitions that the system proposes in order to connect two (potentially remote) harmonic spaces.

Using arguments to make and explain decisions has been proposed and explored in Artificial Intelligence (Bench-Capon and Dunne, 2007), where an argument is a reason for believing a statement, choosing an option, or doing an action. In most existing works on argumentation, an argument is either considered as an abstract entity whose origin and structure are not defined (Dung, 1995), or it is a logical proof for a statement where the proof is built from a knowledge base (Amgoud and Prade, 2009). The use of argumentation in concept invention is, on the other hand, less frequent. Confalonieri et al. (2015b) use Lakatosian's reasoning to model dialogues in which users engage to discuss the intended meaning of blended concept.

In our approach, arguments encapsulate desirable properties that the user would like to have in the resulting transition blends. Arguments are specified by the user by answering specific questions over the features of the idioms selected as input for the transition blending process. Providing some

[1]For the rest of the paper, the term transition will be referring to a pair of chords where one follows the other. For example, G7 $\rightarrow$ C is a transition describing the perfect cadences in tonal music.

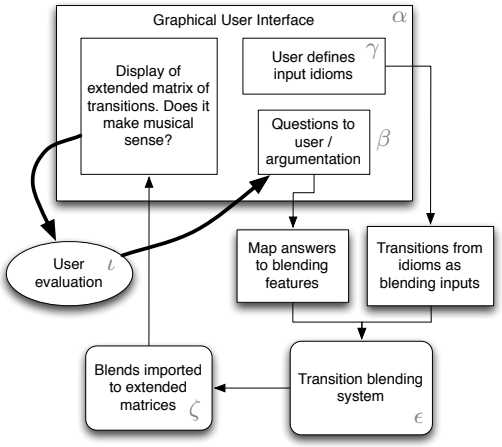



Figure 1: A schematic diagram of the system's workflow.

higher level arguments as inputs to the system is equivalent to allowing a musical expert to interact with it in a language he/she understands. This offers the user a flexible way to adjust the harmonic blending properties according to different input scenarios in order to improve the creativity of the system. Extended experimentation with the system—by making use of the available arguments—can enable music experts to provide valuable feedback regarding the functionality of transition features, thus directly intervening in the blending process by answering simple questions. In a future scenario, the assessment of the system will be based on merely musicological criteria that should be more clearly defined.

The paper is organised as follow. In the next section, we describe the harmonic blending creative process embedded in a creative assistant tool we implemented. Next, we describe the methodology of transition blending and extending harmonic spaces. We show how user arguments are used to evaluate transition blends based on two criteria resembling two of the optimality principles of conceptual blending. Then, we present a process-based system evaluation that focuses on the creative acts of programmers (Colton et al., 2014). This evaluation is helpful in guiding further developments of the system. These are discussed in a concluding section.

## System overview and test cases

Figure 1 illustrates a diagram of the presented system. The user (music expert) interacts with the system through the Graphical User Interface (GUI), where she/he selects two initial idioms (harmonic spaces expressed as sets of permitted chord transitions) in $\gamma$ and defines the important features used in conceptual blending by answering specific questions (argumentation) in $\beta$. The selected initial idioms are described as sets of chord transitions, while the provided answers to questions are mapped to enabling/disabling features of transitions (see Section 'Chord transitions description and blending') that define the outcome of transition blending (see Section 'Evaluation of transition blending via arguments').

Afterwards, pairs of transition in the two initial harmonic spaces are given as inputs to the transition blending system in $\epsilon$ where new transitions are invented through conceptual blending.[2] These transitions are then integrated into an *extended* musical idiom that includes the initial idioms selected by the user. While the role of the new transitions is to provide musically meaningful connections between the initial harmonic spaces. The created extended idiom is displayed to the user in the GUI in terms of a transition matrix (see Section 'From transition blends to transition matrices'). By observing the matrix, the music expert evaluates ($\iota$) the results produced by the current blending setup, i.e., the given questions to the argumentation module ($\beta$), and re-adjusts her/his answers in $\beta$ accordingly.

Several scenarios for initial idiom combinations are available to the user. The system includes several harmonic blending test cases according to which the user can blend simple 'artificial' harmonic spaces as well as harmonic spaces trained from data in different tonalities. The artificial harmonic spaces are manually constructed to include simple transitions that can typically be found in tonal music in order to allow clear interpretations of the results, e.g., a C major space included the chords C, F and G7. Among the trained idioms that have been examined, there are sets of Bach chorales in major and minor mode, and sets of modal chorales in several modes.

The test cases, in which harmonic spaces in different tonalities are blended, resemble the musical task of finding transition paths for tonality modulations (changing the tonality of a given harmonic space). This task allowed music experts to identify arguments for defining the features of transition blending that connect potentially remote harmonic spaces (e.g., C major with F♯ major) in a manner that is explainable in music theory in terms of tonality modulations. Through the processes offered by the system, the music experts were able to come to conclusions about what transition features are important for constructing meaningful connections between different combinations of pairs of initial harmonic spaces.

## Methodological aspects of transition blending and extending harmonic spaces

The cognitive theory of conceptual blending by Fauconnier and Turner (2002) has been extensively used in linguistics, music composition (Zbikowski, 2002), music cognition (Antovic, 2009, 2011) and other domains mainly as an analytical tool, which is useful for explaining the cognitive process that humans undergo when engaged in creative acts. According to this theory, human creativity is modeled

[2]For instance, if the two initial harmonic spaces are a tonal C minor ($I_1$) and a modal C phrygian space ($I_2$), then a pair of transitions for blending could be G7 $\rightarrow$ Cm (from $I_1$) and B♭m $\rightarrow$ Cm (from $I_2$). A possible resulting transition from blending these input transitions is the tritone substitution cadence, C♯7 $\rightarrow$ Cm, as computed in Eppe et al. (2015b) and Zacharakis, Kaliakatsos-Papakostas, and Cambouropoulos (2015).

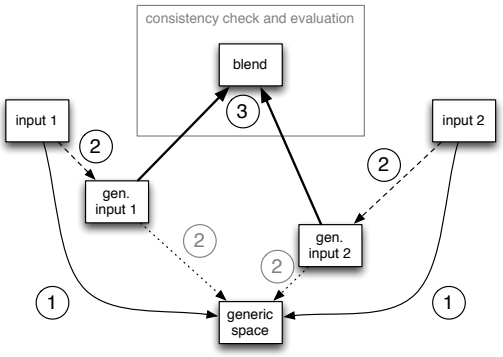



Figure 2: Conceptual blending based on amalgam. The generic space is computed (1) and the input spaces are successively generalised (2), while new blends are constantly created (3). Some blends might be inconsistent or purely evaluated according to blending optimality principles or domain specific criteria.

as a process by which a new concept is constructed by taking the commonalities among two *input spaces* into account, to form a so-called *generic space*, and by projecting their non-common structure in a selective way to a novel blended space, called a *blend*.

In computational creativity, conceptual blending has been modeled by Goguen (2006) as a generative mechanism, according to which input spaces are modeled as *algebraic specifications* and a blend is computed as a categorical *colimit*. A computational framework that extends Goguen's approach has been developed in the context of the COncept INVENtion Theory[3] (COINVENT) project (Schorlemmer et al., 2014) based on the notion of *amalgams* (Ontañón and Plaza, 2010). According to this framework, *input spaces* are described as sets of features, properties and relations, and an *amalgam*-based workflow finds the blends (Confalonieri et al., 2015a; Eppe et al., 2015b). The amalgam-based workflow generalises input concepts until a generic space is found and 'combines' generalised versions of the input spaces to create blends that are consistent or satisfy certain properties that relate to the knowledge domain (Figure 2).[4]

Amalgam-based conceptual blending has been applied to invent chord cadences (Eppe et al., 2015a; Zacharakis, Kaliakatsos-Papakostas, and Cambouropoulos, 2015). In this setting, cadences are considered as special cases of chord transitions—pairs of chords, where the first chord is followed by the second one—that are described by means of features such as the roots or types of the involved chords, or intervals between voice motions, among others. When blending two transitions, the amalgam-based algorithm first finds a generic space between them (point 1 in Figure 2). For instance, in the case of blending the perfect with the

[3] http://www.coinvent-project.eu

[4] In the process of blending through amalgams, the notions of 'amalgam' and 'blend' are the same. Therefore, in the following paragraphs they are used interchangeably.

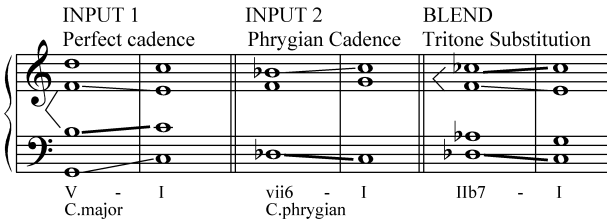



Figure 3: Example of blending cadences, which are special case of transitions, where blending the perfect and the Phrygian produce the tritone substitution cadence blend.

Phrygian cadences (Figure 3)—described by the transitions $I_1$: G7 $\rightarrow$ C and $I_2$: B♭m $\rightarrow$ C5 respectively—their generic space consists of any transition that has a second chord with the root note C, since this is the root note of both inputs' second chords (C and C5).

After a generic space is found, the amalgam-based process computes the amalgam of two input spaces by *unifying* their content. If the resulting amalgam is inconsistent, then it iteratively generalises the properties of the inputs (point 2 in Figure 2), until the resulting unification is consistent (point 3 in Figure 2). For instance, trying to directly unify the transitions $I_1$: G7 $\rightarrow$ C and $I_2$: B♭m $\rightarrow$ C5 would yield an inconsistent amalgam, since a transition cannot both include and *not* include a leading note to the second chord's tonic (which is a property of $I_1$ and the $I_2$ respectively). Therefore, the amalgam-based process generalises the clashing property in one of the inputs (e.g., the property describing the absence of leading note would be left empty in $I_2$) and tries to unify the generalised versions of the inputs again. After a number of generalisation steps are applied (point 2 in Figure 2), the resulting blend is consistent (point 3 in Figure 2). However, it may be the case that the blend is not complete, in the sense that this process may have generated an over-generalised term.

After several blends have been computed, an automated evaluation process ranks them according to some optimality principles (Fauconnier and Turner, 2002, Chapter 16). These principles are a necessary aspect of conceptual blending since they allow to filter interesting blends from the (potentially too) many possible ones.[5] A complete description of optimality principles is out of the scope of this paper and the reader is referred to Goguen and Harrell (2010) for applications of such principles in the *Alloy* algorithm. We give, however, two extreme examples of 'bad blends' for clarifying the importance of using optimality principles in conceptual blending. *Example 1:* Each of the input spaces is a trivial form of a (bad) blend, since no information from the other input spaces is considered. *Example 2:* A blend that includes all properties of the generic space, but no other information of the inputs spaces is a bad blend, since it has the least possible connections with the input spaces. These examples suggest two criteria for ranking the blends; we provide a computational characterisation of them below.

[5] The amalgam-based algorithm produces many blends by following alternative generalisation paths.

## Chord transitions description and blending

Individual chord transitions are the 'atoms' of the methodology followed herein to construct new transition matrices. Specifically, transition sets from two musical idioms provide input transitions for blending, producing a list of blended transitions that are afterwards embeded in an extended harmonic space. This methodology is described briefly in the next section while some definitions regarding chord transitions follow.

**Definition 1.** *A chord transition $c$ is described by a set of features $\mathcal{F}$.*

In this work a transition is represented by 17 features. Features 1-6 refer to the involved chords. Features 8 to 10 indicate changes during the transitions and are based on the Directed Interval Class (DIC) vector (Cambouropoulos, Katsiavalos, and Tsougras, 2013; Cambouropoulos, 2012). Feature 7 accounts for the change that occurred regarding the chords' root notes. The features considered important in this work are the following:

1. *fromRoot*: the root pitch class of the first chord,
2. *toRoot*: the root pitch class of the second chord,
3. *fromType*: the type of the first chord (GCT base),
4. *toType*: the type of the second chord (GCT base),
5. *fromPCs*: the pitch classes included in the first chord,
6. *toPCs*: the pitch classes included in the second chord,
7. *DICinfo*: the DIC vector of the transition,
8. *DIChas0*: Boolean value indicating whether the DIC of the transition has 0,
9. *DIChas1*: As above but for DIC value 1,
10. *DIChasMinus1*: As above but for DIC value $-1$,
11. *ascSemZero*: Boolean value indicating whether the first chord has the relative pitch class value 11,
12. *descSemZero*: As above but value 1,
13. *semZero*: As above but for value 11 or 1,
14. *ascSemNextRoot*: Boolean value indicating whether the first chord has a pitch class with ascending semitone relation with the pitch class of the second chord's root,
15. *descSemNextRoot*: As above but with descending semitone,
16. *semNextRoot*: As above but with either ascending or descending semitone, and
17. *5thRootRelation*: Boolean value indicating whether the first chord's root note is a fifth above of the second's.

Each feature can be considered as a function that assigns a value to a chord transition $c$. Features' values are defined differently depending on the properties they represent. For instance, features 3 to 8 are set-value functions that assign a set of values to a chord. We refer to them as $F_i(c)$. The value of the feature 7 is a vector and we refer to it as $\vec{f}(c)$. Finally, all the other features are binary functions and we refer to them as $f_i(c)$.

## From transition blends to transition matrices

In the literature, an effective and common way to describe chord progressions in a music idiom in a statistical manner is by using first-order Markov models (see Kaliakatsos-Papakostas and Cambouropoulos (2014); Simon, Morris, and Basu (2008), among others). Such models reflect the probabilities of each chord following other chords in the idiom, as trained or statistically measured throughout all the pieces in the examined idiom. In this context, individual transitions play an important role on indicating particular characteristics of an idiom.

A convenient way to represent a first order Markov model is through transition matrices, which include one respective row and column for each chord in the examined idiom. The probability value in the $i$-th row and the $j$-th column exhibits the probability of the $i$-th chord going to the $j$-th —the probabilities of each row sum to unit. The utilised chords are actually represented by chord group exemplars, obtained by the method described in Kaliakatsos-Papakostas et al. (2015), while transitions between chords that pertain to the same chord group are disregarded. The representation of chords is based on the General Chord Type representation (Cambouropoulos, Kaliakatsos-Papakostas, and Tsougras, 2014).

Then, an important question is: *How would a blended idiom be expressed in terms of a transition matrix, provided that the transition matrices of two initial idioms are available?*

The idea examined in the present system is to create an *extended* transition matrix that includes new transitions that allow moving across chords of the initial idioms by potentially using new chords. The examined methodology uses transition blending to create new transitions that: (a) maximally preserve the common parts of transitions between the two initial spaces, and (b) incorporate blended characteristics for creating a smooth 'morphing' harmonic effect when moving from chords of one space to chords of the other. An abstract illustration of an extended matrix is given in Figure 4.

By analysing the graphical representation of an extended matrix as depicted in Figure 4 the following facts are highlighted:

1. By using transitions in $\mathtt{I_i}$, only chords of the $i$-th idiom are used. When using these transitions, the resulting harmonisations preserve the character of idiom $i$.
2. Transitions in $\mathtt{A_{i-j}}$ create direct jumps from chords of the $i$-th to chords of the $j$-th idiom. Blended transitions in $\mathtt{A_{i-j}}$ can be directly included in the extended matrix.
3. Transitions in $\mathtt{B_{i-X}}$ constitute harmonic motions from a chord of idiom $i$ to a newly created chord by blending. Similarly, transitions in the $\mathtt{B_{X-j}}$ arrive at chords in idiom $j$ from new chords. For moving from idiom $i$ to idiom $j$ using one external chord $c_x$ that was produced by blending, a chain of two transitions is needed: $c_i \rightarrow c_x$ followed by a transition $c_x \rightarrow c_j$, where $c_i$ in idiom $i$ and $c_j$ in idiom $j$ respectively. A chain of two consecutive transitions with one intermediate external chord from chords of $i$ to chords of $j$ will be denoted as $\mathtt{B_{i-X-j}}$.

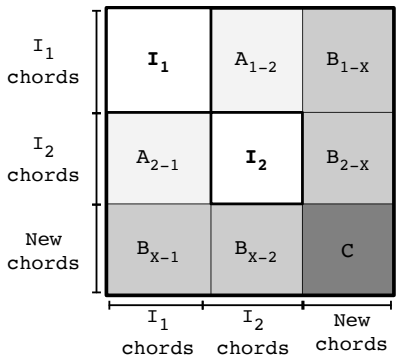



Figure 4: Graphical description of an *extended* matrix that includes transition probabilities of both initial idioms and of several new transitions generated through transition blending. These new transitions allow moving across the initial idioms, creating a new extended idiom that is a superset of the initial ones.

4. Sector C transitions are disregarded since they incorporate pairs of chords that exist outside the $i$-th and $j$-th idioms, violating our hypothesis for moving from one known chord sets to the other using one new chord at most.

Based on this analysis of the extended matrix, a methodology is proposed for using blends between transitions in $\mathtt{I}_1$ and $\mathtt{I}_2$. Thereby, transitions in $\mathtt{I}_1$ are blended with those in $\mathtt{I}_2$ and a certain number of best blends is stored for further investigation, creating a pool of best blends. Based on multiple simulations, a large number of the best blends (i.e. 100) in each blending simulation should be inserted in the pool of best blends ($\mathcal{B}$), so that several commuting scenarios can be created between the initial transition spaces. Thus, a greater number of blends in the pool of best blends introduces a larger number of possible commuting paths in $\mathtt{A}_{\mathtt{i-j}}$ or in $\mathtt{B}_{\mathtt{i-X-j}}$.

## Evaluation of transition blending via arguments

By applying the aforementioned blending process a pool of best blends is created that is afterwards used for connecting the transition blocks of two initial idioms, through forming an extended matrix. When a music expert is using the system, she/he is able to select pairs of initial input idioms[6], choose which aspects of blending are important through arguments (analysed in the following paragraphs) and evaluate/re-adjust this choice by observing the produced results in the extended matrix.

The user evaluates the importance of several transition features by answering questions based on the connecting transitions produced by blending in the extended matrix. The features related to the transitions and their constituent chords are classified into 9 questions (Table 1).

[6]Except from the idioms used in this paper, new idioms can (a) be manually constructed by providing permitted chord transitions and their probabilities, or (b) learned from data using the first order Markov matrix of chord transitions.

| Question | Chord Properties | Transition Changes |
|---|---|---|
| Q1 | *fromRoot*<br>*toRoot*<br>*fromType*<br>*toType* | |
| Q2 | *fromRelPCs*<br>*toRelPCs* | |
| Q3 | | *DIChas0* |
| Q4 | | *DIChas1*<br>*DIChasN1* |
| Q5 | | *DIChas2*<br>*DIChasN2* |
| Q6 | | *DICinfo* |
| Q7 | | *ascSemZero*<br>*descSemZero*<br>*semZero* |
| Q8 | | *ascSemNextRoot*<br>*descSemNextRoot*<br>*semNextRoot* |
| Q9 | | *5thRootRel* |

Table 1: Abstraction of chords' and transition changes' features.

**Q1:** Are roots and types of chords important?
**Q2:** Are individual pitch classes of chords important?
**Q3:** Are repeating pitch classes in transitions important?
**Q4:** Are semitone steps in transitions important?
**Q5:** Are tone steps in transitions important?
**Q6:** Are the intervalic contents of transitions important?
**Q7:** Are semitone motions to the tonic important?
**Q8:** Are semitones to the second chord's root important?
**Q9:** Are motions of the chord roots by $5^{\text{th}}$ important?

The first two questions concern characteristics of the chords that constitute the transition (features 1 to 6), while the remaining seven questions concern intervalic changes that occur within the transition (features 7 to 17). Relating questions to transition features was performed with the involvement of music experts, to ensure that the mapping is as accurate and as informative to the user as possible.

We denote the set of questions available to the user as $\mathcal{Q}$. When a user selects a question, an argument is automatically generated. For the sake of this paper, an argument is defined as follows.

**Definition 2.** *An argument $A$ is a tuple $\langle q, F\rangle$, where $q \in \mathcal{Q}$ and $F \subset \mathcal{F}$.*

The user can specify at most 9 arguments, each of them is mapped to a set of properties. The set of user arguments $\{A_1, \ldots, A_9\}$ corresponding to answers to $\mathcal{Q}$ will be denoted by $\mathcal{A}$. We assume to have a function $\psi : \mathcal{A} \rightarrow \mathcal{F}$ that returns the set of chord and transition properties associated with an argument (e.g., for the purposes of the current analysis, Table 1 specifies $\psi$ as a look-up function). The arguments are used to compute two criteria in order to rate a blend: *total association* and *symmetry*.

The total association indicates the total number of properties that a blend inherits from the inputs. A blend with higher input associations is preferable since it is structurally more deeply related with the inputs. The total association is calculated by taking the individual association of a blend w.r.t. the input chord transitions into account. The individual association of a blend $b$ w.r.t. to an input $I$, denoted as $\mathsf{a}(b, I)$, is defined as:

$$\mathsf{a}(b, I) = \sum_{A_i \in \mathcal{A}} \mathsf{Val}(A_i, b, I)$$

where $\mathsf{Val} : \mathcal{A} \rightarrow \mathbb{R}$ is a function that takes an argument as input and aggregates the values of the chord and transition change properties associated with the argument, by interpreting them according to some music background knowledge. Depending on the type of argument, $\mathsf{Val}$ is defined in different ways.

When an argument refers to the roots and types of chords ($A_1$), $\mathsf{Val}$ is defined as:

$$\mathsf{Val}(A_1, b, I) = \sum_{F_j \in \psi(A_1)} \mathsf{equals}(F_j(I), F_j(b))$$

The value of $A_1$ is calculated by counting how many properties —among *fromRoot*, *toRoot*, *fromType* and *toType*— are equals between a blend $b$ and an input $I$. $\mathsf{equals}$ is a function that returns 1 when two sets are equals and 0 otherwise.

When an argument refers to the individual pitch classes of chords ($A_2$), $\mathsf{Val}$ is defined as:

$$\mathsf{Val}(A_2, b, I) = \sum_{F_j \in \psi(A_2)} |F_j(I) \cap F_j(b)|$$

The value of $A_2$ is calculated as the number of elements that are common in the set-value properties *fromRelPCs* and *toRelPCs* of a blend $b$ and an input $I$.

When an argument refers to the intervalic contents of transitions ($A_6$), $\mathsf{Val}$ is defined as:

$$\mathsf{Val}(A_6, b, I) = \mathsf{norm}_{[0,1]}(\rho_{\vec{f}(I), \vec{f}(b)})$$

The value of $A_6$ is calculated as the Pearson's correlation coefficient of the vector-value property *DICinfo* of a blend $b$ and an input $I$. Higher correlations in the DIC vectors of two transitions indicate higher resemblance; $\mathsf{norm}$ is a function that normalises the Pearson's coefficient from the interval $[-1, 1]$ to the interval $[0, 1]$.
For all the other types of arguments, $\mathsf{Val}$ is defined as:

$$\mathsf{Val}(A_i, b, I) = \sum_{f_j \in \psi(A_i)} 1 - (f_j(I) - f_j(b))$$

Based on the above definitions, the *total association* value is the sum of the individual associations.

$$\mathsf{assoc}(b) = \sum_{I_i \in \mathcal{I}} \mathsf{a}(b, I_i)$$

where $\mathcal{I}$ is the set of input spaces, containing in this specific case, $I_1$ and $I_2$.

Symmetry, on the other hand, reflects the balance of properties that a blend inherits from both input spaces. A blend has a high symmetry when it inherits an almost equal proportion of properties from both input spaces. Blends having higher symmetry are preferred to those with lower symmetry, since a high symmetry reflects a stronger hybridisation of structural characteristics. Hybridisation is an important principle to evaluate transition blends.

The blend symmetry is defined in terms of its 'asymmetry'. The asymmetry of a blend w.r.t. the inputs, denoted as $\mathsf{asym}(b)$, is calculated as:

$$\left| \frac{\mathsf{a}(b, I_1)^2 + \mathsf{a}(b, I_1)\mathsf{a}(b, I_2)}{\mathsf{a}(b, I_1)^2 + \mathsf{a}(b, I_2)} - \frac{\mathsf{a}(b, I_2)^2 + \mathsf{a}(b, I_1)\mathsf{a}(b, I_2)}{\mathsf{a}(b, I_2)^2 + \mathsf{a}(b, I_1)} \right|$$

The value of $\mathsf{asym}(b)$ is defined in $[0, 1]$, where 0 stands for a perfect symmetry (equal association with both inputs) and 1 stands for total asymmetry (association only with one input). Additionally, the non-absolute version of the above equation suggests the prevailing input, with a negative value indicating dominating association of the blend with the first input and a positive value contrarily.

The total rate of a blend is computed by taking the input association and asymmetry values into account.

$$\mathsf{rate}(b) = \frac{\mathsf{assoc(b)}(1 - \mathsf{asym}(b))}{\mathsf{assoc(b)} + (1 - \mathsf{asym}(b))}$$

The above expression promotes pairs of association and symmetry that are both high, while a simple sum would allow a low value of the one to be covered by the other.

Finally, a decision making criterion to compare any pair of blends $b_1, b_2 \in \mathcal{B}$ can be defined as follows.

**Definition 3** (Decision criterion)**.** *A blend $b_1$ is preferred to a blend $b_2$ if and only if* $\mathsf{rate}(b_1) \geq \mathsf{rate}(b_2)$.

It is worthy to notice that the above criterion guarantees that any pair of blends is comparable, and, consequently, it allows to decide which blends are the best ones. This is an important property for blend evaluation and, generally, for approaches to argumentation-based decision making (Amgoud and Prade, 2009; Bonet and Geffner, 1996).

## System evaluation

Referring to Figure 1, via the interface $\alpha$, the user has access to modules $\gamma$, and $\beta$ which can be used to specify *concepts* that will inform the resulting product, namely, the input idioms and arguments that impose constraints on the generated blend. These are translated by the system into process-friendly formats. Module $\epsilon$ embodies the (process-level) concept of a system that make use of the supplied idioms and the blending properties to generate *example* transition matrices, $\zeta$. In the current version of the system, these transitions are evaluated by the user (music expert) in step $\iota$ using sophisticated harmonic knowledge that reflects an historically established musical *aesthetic*. The user can then return to the GUI $\alpha$, and adjust the settings of $\gamma$ and $\beta$ to regenerate the transitions.

This is illustrated in Figure 5 in box **P1**, using the diagrammatic extension to the FACE model by Colton et al. (2014). Here, capital letters $F$, $A$, $C$, or $E$ are creative

acts that generate a framing, aesthetic, concept, or example, respectively. Administrative acts $S$ and $T$ denote selection and translation. Lower-case letters denote the generated artefact in each case (e.g., the concept $c$ corresponding to the concept-creation act $C$). Subscripts $p$, $g$, or $m$ indicate whether the act takes place at the process, ground, or meta level. Inside each box, stacks show the dependence in development epochs, and arrows show run-time message passing. Acts taken by the programmer or user are decorated with a bar, whereas acts taken by the system itself receive no extra decoration.

In the current version of the system, apart from the *programmer's* creative acts specifying the modules and their interconnections, and the algorithm $\overline{C^{\epsilon}_{p}}$ that turns inputs into blends, the user, who is assumed to be a music expert, must intervene in the system in two places.

First, the *user* defines system settings $\overline{C^{\gamma}_{g}}$, $\overline{C^{\beta}_{g}}$ that correspond to the selection of input idioms and of arguments respectively. Second, after the run completes, he or she evaluates the system output via $\overline{A^{\iota}_{g}}$.

The *system's* primary responsibilities take place through the creative acts $E^{\epsilon}_{g}$, which generate blends, and $S[a^{\beta,\gamma}_{g}](e^{\epsilon *})$, in which the aesthetic $A^{\beta,\gamma}_{g}$ (a unified label for assoc and asym, which are defined anew in each run, based on a fixed translation of the user's arguments, as specified in the previous section) is applied to rate the possible blends, and select to a final extended transition matrix.

Therefore, the key idea behind what has been implemented so far is an 'automated ranking/evaluation' step that guides the selection of blends, $S[a^{\beta,\gamma}_{g}](e^{\epsilon *})$ according to the arguments defined by the user. The development of the programmatic components that operationalise this process has relied on both computer science and musicological insights. This approach has been characterised as meaningful per se through informal feedback provided by musical experts – but is perhaps especially valuable because it constitutes a prototype for more involved automated evaluation of computer-generated harmonic spaces.

Indeed, the next step towards the development of a more autonomously creative system using the same architecture is fairly clear: future work would need to 'close the loop' computationally, connecting the evaluation of generated transition matrices with the parameter-setting (i.e., argumentation) stage, and making this run autonomously to refine the system's behavior. This as-yet hypothetical situation is illustrated in the box **P2**.

Here, the programmer has translated some of the user-specified aesthetics into code $\overline{T[A^{\iota}_{g}]}$, and invented a meta-level concept $\overline{C^{\alpha}_{m}}$ defining a system component that can automatically apply these aesthetics to the generated transition matrices $e^{\zeta}_{g}$ as in order to automatically generate new system settings $C^{\gamma}_{g}$, $C^{\beta}_{g}$.

## Conclusion, Discussion and Future Work

In this paper, we described a methodology for harmonic blending and we proposed a creative system that assists musicologists with the evaluation and enhancement of harmonic innovation. We defined some harmonic features of chord transitions utilised for evaluating blends of transitions, leading to the invention of novel harmonic spaces. The system allows a musicologist to specify arguments over these features that are taken into account in the generation of new harmonic spaces. The music expert can then assess whether the new harmonic idiom makes musicological sense and readjust the arguments to explore alternative blends that can potentially produce better harmonic spaces.

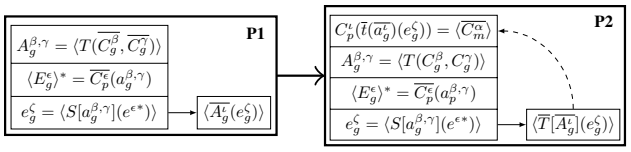



Figure 5: The current implementation **P1** prototypes automated evaluation of blends according to user's arguments; this points to a proposed future implementation **P2** with further automation.

The main advantage of the current system is the agile interaction through which the user can express desirable properties over the transition blends and their argument-based evaluation in order to produce musically meaningful results. The added value of argumentation is the ranking/evaluation of blended transition—obtained by conceptual blending of two input transition belonging to two musical idioms—by answering questions which abstract several properties of chord transitions. On the other hand, the evaluation of the creative output of the system, i.e., an extended harmonic space that includes blended transitions, is carried out by the user via an introspective argumentative dialogue.

In a future work we intent to use the argumentation-based process for evaluating the blended harmonisations of user defined melodies, i.e., actual music output. Additionally, mapping the properties of the blended idiom or, at a latter stage of a harmonised melody, back to the parameter-setting stage opens an interesting direction for future research and further improvements of the system. The added value of argumentation can be stressed, for instance, by letting the system suggest possible refinements of the initial user arguments, progressively converting part of the introspective user evaluation into a more explicit format. For example, a future version of the system would be based on identifying harmonic features of the input spaces that automatically suggest an 'optimal' set of initial arguments. The current version of the system is an already-usable prototype on the way towards the development of a more autonomous creative system.

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# The FloWr Online Platform: Automated Programming and Computational Creativity as a Service

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### Abstract

We present recent developments in the Flowchart Writer (FloWr) project, where we have built a framework for implementing creative systems as flowcharts of processing nodes. We describe how the system has been migrated from a desktop application to a web portal and document the various features that the portal provides to support Computational Creativity research and development. This includes a node development package and automated chart development assistants. We detail how we have supplemented the online graphical platform with a web service API to enable developers to remotely access the features of FloWr through a programming language of their choice. This encompasses developing systems as flowcharts, together with running flowcharts remotely and also allows developers to publish flowcharts as web services. Importantly, the API allows Computational Creativity researchers to experiment with the automated development of creative software systems. To encourage this, we have also introduced simple models for automated software development into the FloWr API itself, providing a novel system for unsophisticated users to experiment with. We demonstrate the potential benefits of using FloWr, with case studies showing how the web portal has been used for both node and chart development by novice and expert users.

## Introduction

In the FlowChart Writer (FloWr) project[1], we have built a platform for all Computational Creativity researchers to produce novel creative systems via GUI-based writing of *flowcharts* which pass information through processing *nodes*. The ultimate aim of the project is to form a community of users contributing to a corpus of nodes and flowcharts, which will enable the automatic generation of flowcharts, hence modelling creativity at the process level.

Since its introduction as a desktop application in (Charnley, Colton, and Llano 2014), there have been many developments with FloWr. One of the most significant changes has been the migration to an online version. FloWr users no longer have to download a huge Java desktop application, no updates are required and a variety of devices can be used to access the system. Nor are users restricted by the computing power of their device as processing is performed on our servers. Our main motivation for this migration, however, is a desire to provide a platform for interaction between researchers in the Computational Creativity community, by letting them share ideas, processes and resources, and collaborate on creative system development.

The main flowchart writing platform allows high-level creative systems to be developed. Users can also create new flowchart nodes to add novel functionality to the portal, increasing the power and scope of the systems that can be created. Our platform also provides Computational Creativity system design as a service through the FloWr API. This means that researchers are not restricted to the visual GUI and can access the full power of FloWr how they like, from whichever programming environment they choose. We are particularly interested in the possibilities for automated programming that this affords. Another new feature lets users expose flowcharts as standalone web services to allow other systems or users to access them. In the next section, we give details of the portal and highlight improved features of the online system over the previous desktop version. We follow this with two case studies highlighting the potential value of FloWr for novices and experts alike. We conclude with a discussion of future developments for the FloWr platform.

This work has similarities with the ConCreTeFlows project (Žnidarsic et al. 2016), where concept creation workflows have been implemented in a flowchart paradigm. That project uses Clowdflows[2] which, like FloWr, provides a portal for developing and sharing flowchart-based systems. Clowdflows was chiefly developed for algorithmic programming in machine learning and data-mining, with appropriate nodes. By contrast, FloWr has been developed for Computational Creativity collaboration and research and we are unaware of any other projects with these specific aims.

## The FloWr Web Portal

In addition to improved facilities for writing flowcharts, users have access to a great deal of additional meta-level information describing what nodes and charts do. There is also an *Admin* area which gives access to numerous other enhancements, such as the *API* and *Node Development Package*. We describe the various aspects of the portal below.

[1] http://ccg.doc.gold.ac.uk/research/flowr/

[2] http://clowdflows.org/

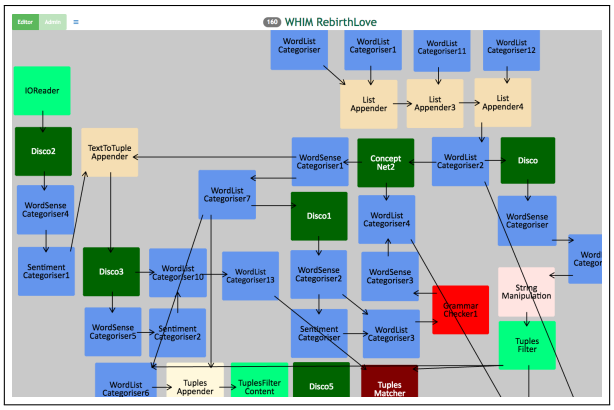



Figure 1: The online FloWr flowchart building interface.

## Writing Flowcharts

The main interface, shown in Figure 1, is where users manually craft flowcharts. It is an improved version of the FloWr desktop application (Charnley, Colton, and Llano 2014). Flowcharts consist of nodes, representing self-contained processing elements, and arrows, which indicate how data is passed between nodes during processing. The core features of the desktop version have been retained including adding, removing, moving and re-sizing nodes, defining variables to highlight output, setting parameters, running charts and viewing output. Nodes are still written as stand-alone Java-wrapped code modules and the system still makes use of an underlying script syntax to describe the functional aspects of a chart that are independent from the graphical representation. General usability, feel and use-of-space has been improved by using modern front-end web development frameworks, as has cross-device/platform support. There are also improvements to GUI interaction, such as single-axis resizing and improved chart runtime feedback.

**Node Information** We have enhanced the information panel that appears on double-clicking a node, with additional information and the ability to open multiple instances. The *information panel* for a node shows the unique node id, its type, a bespoke label and a description of what the node does in the specific context of this chart, as well as tools to alter the node colours. The *input panel* shows, for each parameter, its name (with type tooltip), links to information about that parameter (see below) and a type-specific input control for setting the parameter. Static-value parameter setting has been enhanced to make it simpler and more fault-tolerant. Type-appropriate controls are displayed, such as checkboxes for Boolean values. Automated pre-validation ensures that, for example, numerical parameters are only passed numbers. Node developers can specify additional validation checks, such as maximum values, and bespoke data-types can be used to enforce regex-based validation. Developers can also choose which type of control, should be used for each input parameter, such as a textarea or textbox.

The *output panel* shows a tabbed list of all the defined output variables for the node (elements of its output that have been given specific labels). Output and variable definitions operate in the same manner as for the desktop version, using

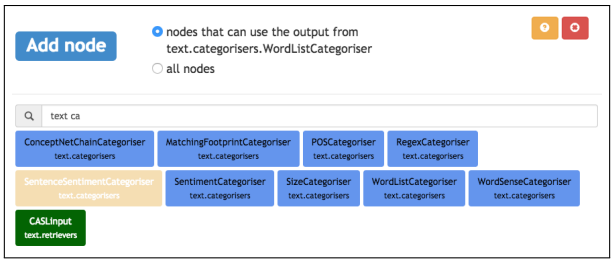



Figure 2: The add-node wizard.

the same syntax. If the user has sufficient access rights – i.e. they own that chart – they can add, delete and amend variables, and we have made error notification clearer and more robust. Once run, any node output can be downloaded in JSON-encoded format. This is particularly useful for testing new nodes in the node development environment, as we describe below. The user can also inspect the output from the Java console, which is useful for debugging node errors and for applications where flowcharts are to be used from shell commands. As in the desktop version, the output of nodes can be locked so that outputs are cached which is useful for chart development and debugging.

**Smart Assistance** We have introduced wizards to help users create charts, as shown in Figure 2. For example, when the first node is added to a blank chart, the user will, initially, be shown only those nodes that have been tagged by the node developer as suitable for starting charts. Similarly, when a user wishes to create a link between two nodes, by holding down the control key to drag a new arrow between them, a wizard will suggest a subset of nodes which can, in some way, make use of the source node output. Similarly, when a user draws an arrow in to a node from empty space, or vice-versa, the new node wizard will consider which nodes might be appropriate to take/provide data from/to the target. These wizards use Java reflection to find potential data-type matches. Restricting nodes on this basis is very useful, given how many nodes are available, especially when dealing with certain artefact types, e.g., dragging an arrow out of an image retrieval node brings up a wizard showing only image-manipulation nodes. The wizards also provide a comprehensive node text search facilities to find particular nodes or domain-specific packages.

A new *MapHelper* wizard, shown in Figure 3, helps to create and manage data links between nodes, i.e., how data is passed between them at runtime. This appears whenever a new node has been chosen via the new-node wizards, above, or when an existing arrow is selected. This wizard shows the source, or output, node on the left and the target, or input, node on the right. Existing data maps are shown at the top of the dialog. To create new data maps, the user selects a parameter from the right-hand panel, whereupon FloWr uses Java reflection to review all the output of the source node to find any output or variables that match the data-type of that parameter. If the user clicks on one of these suggestions, the wizard establishes the data map, creating a new output variable where necessary.

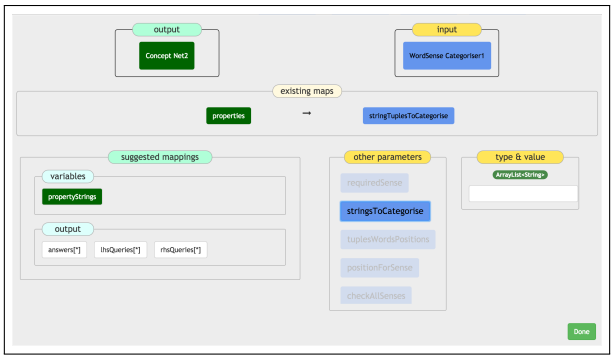



Figure 3: The map helper wizard.

**Data Types** The desktop version could already handle a large number of different data types and we have continued to expand this. For example, we have recently added images, which are represented internally as Java *BufferedImages*. These can be large, depending upon the resolution, so thumbnails are used in output panels which give users the option to download the image at any resolution they choose, up to the full version. We have also introduced the notion of bespoke FloWr data-types. These are designed to enforce data-consistency by providing methods to support seamless front-end validation. For example, we have introduced a new *Float01* data-type which can only be instantiated with a string value that represents a valid float value between 0 and 1. The node provides a regular expression for validating front-end input strings accordingly. Node developers can use this to define bespoke data types and take assurance that values input to their node will be valid and consistent.

**Menu** The menu bar contains a number of new features. For instance, chart loading now includes lists per user and for recent charts. New *history* functionality takes regular snapshots of the chart and allows users to return their chart to a previous state. This also includes a facility whereby the user can take a named snapshot to better control versioning. There are other menu items for clearing the chart output, restarting the user-specific server (useful in debugging), highlighting run nodes and removing locks. The user can view the script underlying the chart and export the whole chart as an XML file. They can also view the supplementary information about the chart provided by the chart owner, as described below. Some of the charts that FloWr users have developed are quite large and contain many nodes. So, we have improved the way in which FloWr handles node positioning and introduced user-specific view profiles, which persist between sessions. The menu provides commands to re-centre the chart and help to find nodes that have been moved off-screen. Auto-layout using Graphviz (www.graphviz.org) has also been implemented.

## Help and Information

Every FloWr node or chart has a specific **owner** attribute, which is used to control access and editing rights. Only the owner of a chart can edit it and they can, optionally, lock it to prevent accidental changes. Charts can be *private*, visible to the owner alone, or *public*, where all other users can view and run, but not amend, that chart. To amend another person's chart, a user must take a copy and use that. Similarly, only node owners may download and make changes to a node's code and information (using the node development package described below). So, only node owners can download and upload new code or rollback versions. Chart Owners can provide an overview of the chart, bespoke node labels and context-specific descriptions for nodes using in-situ editors. Node Owners can provide additional information about nodes, which is available from various buttons next to node types and parameters. This includes an overview of the node, its default colours and whether it can be used to start charts, i.e. generates data from scratch (to inform the first-node wizard). It provides information about the parameters and allows specific input control, default value and validation options to be set. In particular, the node owner can specify drop-down options for a parameter, together with user-friendly replacement labels, if desired. This has replaced clunky source code constructs, which had a number of issues. Information can also be provided about the bespoke output objects, or sub-objects, that the node owner has created for their node.

## Implementation

The portal uses a mixture of front-end web technologies, PHP, a relational SQL database and Java. To avoid cross-user data contamination, often caused by Java static variables, each user is given their own java server instance which handles their current chart. This also provides a load-balancing system. User-state is maintained server-side between sessions. Currently users must have a Google account to log in and their account must be unlocked using a code provided by the FloWr team. To maintain state consistency, users can only log in from one browser session at a time. We have taken steps to improve responsiveness by minimising client-server communication, e.g. by bundling calls in specific client use-cases. For speed of execution, chart runs are handled entirely by the back-end Java server with only minimal updates passed to the browser. Output for display is sent to the client piecemeal, with elements transferred only when viewed by the user. The system uses a mixture of sockets and file-system tools to transfer data around and minimise lag.

## Admin Area

The admin area handles aspects of the portal that aren't concerned directly with flowchart writing. There are tabs for searching and managing charts, including moving them between the API and GUI or importing from XML. A tab for managing nodes, including downloading/uploading code, rolling back to previous versions and purging unwanted nodes entirely. Note that any changes to nodes must, currently, be carefully managed by node developers to ensure that existing charts aren't broken. The admin area also includes an ever-changing tutorial section. The developer section provides a link to download the developer package (see below). Other tabs in the admin area include recent news and developments. There is a place to provide feedback, a bug tracker for superusers and instructions for using the API.

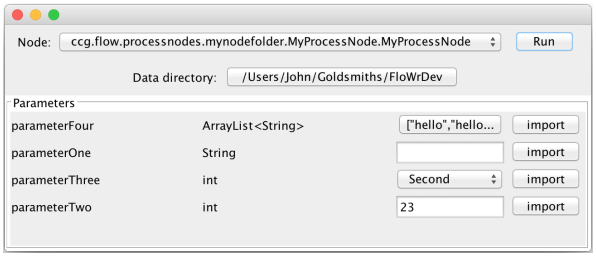



Figure 4: The NodeTester application.

## Node Development

We have created a developer package to help users create their own nodes. It includes the ProcessNode and ProcessOutput Java classes that underlie all nodes and their output (as described in (Charnley, Colton, and Llano 2014)). The package also provides a NodeTester application, shown in Figure 4, which lets developers parameterise a node, run it and inspect its output. NodeTester allows you to select a node to focus on, from those that you have in the local *processnodes* package. Below this is a DataDirectory selector. In the desktop version, this contained large libraries of static data, such as dictionaries or newspaper article archives. Previously users had to download these multi-gigabyte archives to their local machine if they wanted to use all the nodes. The online version keeps all this data on the server, so during development, node owners use a local DataDirectory and, when ready for release, an administrator installs the node's static data files on the server.

There is also a panel for setting parameters which uses a JSON format. Whilst not as user-friendly as the online GUI, it is as powerful and allows other data-types, such as lists-of-lists to be defined. Once parameter values have been set, they are stored locally as text. So, they can be edited manually and saved between development sessions. NodeTester allows users to import downloaded output from charts (see above) for use as values for parameters. Hence, developers can debug their nodes as though they were part of a larger flowchart, without having to continually upload and test their code changes.

Once a developer is happy with their node, they can upload it through the admin area of the portal. The meta-level node information that owners can provide about their nodes, as described above, is stored in XML format alongside the source code. If the owner desires, they can upload changes to this directly rather than using the online edit functions.

## The FloWr Web API

One of the most exciting developments in the FloWr project is the API. This allows users to access the power of FloWr from within a programming language of their choosing. We describe this as Computational Creativity as a Service. Via the API, users can perform almost any of the actions that would be available to the online developer. As with the underlying script syntax, there is no need for notions of chart layout, colours, labels etc., and the API is predominantly functional rather than visual. So, for example, it is possible to set values for parameters but not, say, to re-position a node in some notion of a screen. Charts created under the API are kept separate from those created through the online GUI (this is changeable in the Admin Charts area). This is chiefly to remove the potential for portal to be swamped by a large number of automatically-created API charts.

The API is accessed by POST requests, which must specify the user's temporary 24-hour access token. As well as creating and editing charts, the API allows the caller to run them, as they would in the online GUI, and download the output they produce. Some of the commands available include all those for manipulating charts, parameters, variables, data maps etc. They also provide lists of available nodes, parameter information, output information and current chart state including run progress. Chart states are provided in JSON format, listing all the nodes and their parameters in a machine-readable format, similar to both the XML export and the underlying script syntax.

We are particularly interested in encouraging the API to be used for automated programming via automatic flowchart construction. To this end, the experiments in automatic flowchart writing presented in (Charnley, Colton, and Llano 2014) have now been run entirely separately to the core FloWr system, via the API. To foster such research, many of the API commands provide meta-data about the available nodes, their parameter types, output etc., which allows automatic programming approaches to make informed decisions about chart manipulation. In addition, the API has functions for users to upload and syntax-check flowchart scripts and XML charts, which provides another approach to automated programming that researchers could use.

## Automation Features

In addition to providing API functionality to encourage automated programming, we have begun to add some automated programming features into the main FloWr GUI portal. In particular, users can call up the automated programming dialog box where they can re-run the simple flowchart construction experiments presented in (Charnley, Colton, and Llano 2014), where we asked FloWr to generate charts for creating poetic couplets from scratch. In the dialog box, the user can select the nodes to be placed into particular places in a template flowchart, sets of values to consider for particular parameters and they can specify a minimum level of output from the chart. They can then ask the system to generate a working flowchart. We are hoping that this demonstration of how FloWr might be used for automated programming could encourage researchers to perform their own experiments via the API, and we plan to greatly expand the online automation aspects.

# Case Study 1: A novice user adds a new node

This case study describes the experience of a relatively new user to FloWr (author 4) who is familiar with the general features of the web API, and who wants to contribute to node development. His objective is to wrap one of the commands from the Microsoft Web Language Model API (www.projectoxford.ai/weblm) in a FloWr node, and use it in a sample flowchart. The command to be encapsulated is

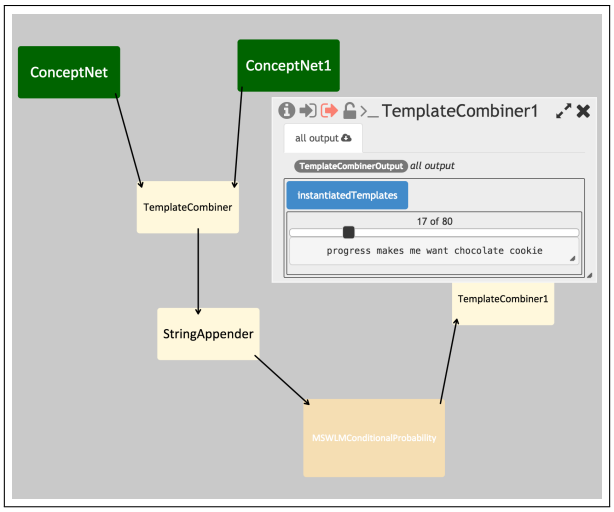



Figure 5: "Progress makes me want chocolate cookie": Simple interaction with a flowchart containing the new MSWLMConditionalProbability node

*Conditional Probability* that rates how likely it is that a particular word/phrase will follow a given sequence of words.

The user easily finds the `FloWrDev.zip` file inside the development area of FloWr. This contains the NodeTester toolkit, including `FloWrDev.jar`, a file of `INSTRUCTIONS`, and an outline of a sample node. The sample node does not have a lot of detail, so the user navigates to the *Nodes* tab and types "api" into the search box to track down code he had written earlier. There is a suitable existing node available as a template for modification, namely the node that wraps FloWr's own API. He downloads `FloWrAPI.zip`, which contains detailed and relevant sample code.

The sample files are renamed, placed in an appropriate local directory, then easily compiled and run under Java within the FloWrDev.jar environment. Java then displays a minimal panel for setting parameters and running the toolkit (Figure 4). After some further investigation, the user decides that he needs to track down another worked example that formats its output in a way that is more suited to the problem at hand. He returns to the web API and locates a node that is known to have output with suitable format, but this was created by another user so the *Get code* option is greyed out and unavailable. This time, he had another way to track down the code, but he submits a feature request asking for more public example nodes. The web UI has a *Feedback* where this sort of request can be made.

Adapting the FloWr API node to work with the Microsoft API, rather than the FloWr API itself, is straightforward. With the additional example in hand, formatting the node's output correctly is also easy. Following the strategy used in the example, a third file is added to support a supplementary class that organises output variables. During the adapting of the node, the toolkit gives access to standard Java debugging information and FloWr-specific runtime error messages.

The next step is to upload the code. On the *Nodes* tab in the web interface there is a button for this. Files are selected one by one using a file chooser. After some processing, the new node becomes available via the dropdown node chooser. Clicking on the *Description* button opens up an interface whereby documentation for the node and each of its input parameters and output variables can be added. Along with a plain-text description, this interface also displays the Java type information (which cannot be altered at this stage), and an interface that provides the ability to add or modify default settings and multiple drop-down options for the node. In this case, the user just fills in the plain text descriptions.

The node is now available for further experimentation. Although the NodeTester allows the user to try out various parameter settings for a single node, this is the first experiment with the node in context. Hints are provided (e.g., FloWr knows that a chart should typically begin with a node that retrieves text for further processing). It takes the user about half an hour to explore the available nodes and choose some that make a convincing demo (Figure 5). This flowchart uses a ConceptNet (Liu and Singh 2004) node to come up with food combinations using the template: [x HasProperty edible] + [x Any eat]. It concatenates these, and then uses the new node to decide which word combinations are likely to follow the word "eat". It then uses another ConceptNet node to select a putative cause: [x IsA change]. Finally, it combines the answers using a template combiner node: `b1Texts[*] makes me want b2Texts[*]`. Note that `b1Texts` and `b2Texts` here refer to the inputs to the template combiner. Along with *"progress makes me want chocolate cookie"* (Figure 5), other output with highly probable food items, as rated by by the new node, included *"grow makes me want cheese plate"* and *"become makes me want steak egg"*. Output ranked with low probability included *"become makes me want chocolate chinese_restaurant"* and *"progress makes me want dandelion goat"*. In the process of writing this flowchart, the user learns more about how the node-connection Wizard works: ctrl + mouse1 establishes connections between the nodes, and then the Wizard points out which available variables from a given upsteam node can be connected to a selected input parameter in the downstream node.

Along the way to this result, a few further ideas came to mind. Firstly, the node could be improved by allowing more input parameters, in order to make more expressive use of Microsoft's API . Secondly, additional text processing nodes could be created that quickly concatenate ArrayLists together to form n-grams for testing using the API (rather than using `TemplateCombiner` nodes). These ideas are easily addressed following the patterns described above. In total, the writing of the new node, deploying it within the online FloWr portal, then building, debugging and running a flowchart containing the new node took around 2 hours, which we believe is a reasonable time for a novice user to write a simple generative system. The new node contains 208 lines of code in total, of which approximately 40 are new. The experience was quite satisfactory to the user, who felt it was a good way for him as a novice Java programmer to get practice in the language.

## Case study 2: An expert user's flowcharts

Here we present an account of the use of FloWr to develop a complex flowchart from the perspective of an expert user (author 3) who is not a developer of the core FloWr system (author 1). We also summarise this user's broader experience with the system. The flowchart we focus on is shown in Figure 1. This flowchart generates fictional ideas to be used, in this instance, in the context of generating original concepts for musical theatre pieces. Research using FloWr for fictional ideation has been carried out for around 2.5 years as described in (Llano et al. 2016). During this time, the expert user has added 41 new nodes to FloWr, which range over utility nodes, natural language processing, and domain specific nodes for fictional ideation and theory formation (Colton, Ramezani, and Llano 2014). An overview of nodes this user added to the system, and an example in each category, is presented in Table 1. A total of 17 flowcharts have been built to support fictional ideation, 9 of which have been released into production to be used in the European WHIM project (www.whim-project.eu) and 8 of them being more experimental. These flowcharts use an average of 35 nodes each, with the most complex flowchart composed of 73 nodes and the most simple one composed of 6 nodes.

| Type | New | Example |
|---|---|---|
| Utility | 11 | RunShellScript: runs shell scripts |
| Natural Language Processing: | | |
| Categorisers | 2 | POSCategoriser: annotates text with part-of-speech information |
| Combiners | 5 | ListAppender: appends two lists |
| Extractors | 1 | NamesExtractor: extracts proper names from text |
| Language | 1 | GrammarChecker: e.g. converts nouns from singular to plural |
| Manipulators | 2 | StringManipulation: changes text case |
| Matchers | 2 | TuplesMatcher: matches tuples in specified positions |
| Retrievers | 9 | WordNet: retrieves WordNet data |
| Theory formation | 3 | HR3: text to HR3 format (Colton, Ramezani, and Llano 2014). |
| Ideation | 5 | AudienceModel: ranks fictional ideas |

Table 1: New nodes added by the expert user.

The starting point for our examination of the flowchart in Figure 1 is a set of templates that describe pre-defined, general scenarios that are to be completed by specifying either locations, attributes, or characters, etc., that form the foundation for the fictional ideas. These templates also form the building blocks of the flowchart and their structure guides the development effort. An instance of such a template is:

*What if a PERSON_TYPE had to learn how to ACTIVITY in order to find true love?*

The building blocks here are place-holders for the *person_type* and the *activity*. A ConceptNet node is used as the source of knowledge to retrieve a list of suitable concepts to fill the *person_type* place-holder. This is shown in Figure 6(a) block 1. As can be seen in this figure, various ConceptNet nodes are invoked, which retrieve ConceptNet facts of the form: [x, IsA, human], [x, IsA, occupation], [x, IsA, person] and [x, IsA, profession], where $x$ is the person_type. Outliers are removed using *WordListCategoriser* nodes, and the results are appended to a common list through the *ListAppender* nodes. Building block 2 of Figure 6(a) retrieves facts of the form [x, Any, y], where $y$ is then filtered to restrict only to verbs, through a *WordSenseCategoriser* node; the results can then be used in the *activity* place-holder. Finally, building block 3 combines these tuples into larger tuples of the form: [$x_1$, IsA, _, $x_2$, _, y], where $x_1 \neq x_2$. The template is finally filled in through the *TemplateCombiner* node, producing ideas such as:

*What if a banker had to learn how to fix a cat in order to find true love?*

After a first version of a flowchart is finished, the output is analysed to decide if further work is required, which is usually the case. In the particular example followed here, two additional modifications were performed. These are shown in Figure 6(b). The nodes in blocks 4 and 5 retrieve representative qualifiers of the $x_1$ and $x_2$ concepts – for which there was not much data in ConceptNet. To achieve this, the *Disco* node, a linguistic tool (www.linguatools.de/disco/disco_en.html) that extracts related words using co-occurrences, was used. In particular, the 50 most common collocations for each concept were retrieved, and consequently filtered to keep only adjectives. Finally, block 6 in Figure 6(c) handles the evaluation of the ideas. This is achieved by connecting to an external web service that analyses their narrative potential through a set of measures; the results are subsequently fed into another external web service that contains an audience model that provides a ranking of the ideas. A possible expansion of the idea above, that is ready for this evaluation is:

*What if a wealthy banker had to learn how to fix a cat in order to captivate an accomplished veterinarian?*

Having access to different linguistic tools as nodes that support the retrieval and analysis of information has provided a useful framework for experimentation. In particular, being able to connect different tools (as illustrated above for the Disco and ConceptNet nodes) enables easy formation of a dynamic knowledge base to enrich generated output.

As can be seen from Figure 1, we have only explained a small subset of nodes used in the flowchart (we have focused on one of the possible templates). The flowchart deals with a total of 4 templates, each of which produces a different set of fictional ideas using common standard building blocks, contextual information blocks and evaluation blocks. This way of working provides a lot of flexibility. However, it also has the drawback that flowcharts can become cluttered with multiple nodes that could otherwise be encompassed in only one. For instance, if the ConceptNet node had either (a) an input parameter that takes a list of RHS or LHS queries in the form of text – it currently accepts these query parameters only in the form of an ArrayList of strings – or (b) if an alternative ConceptNet node existed with a text parameter, then all of the nodes in block 1 of Figure 6(a) could be replaced by a single node. Such functionality could be easily developed; however, following the line set out in alternative (a),

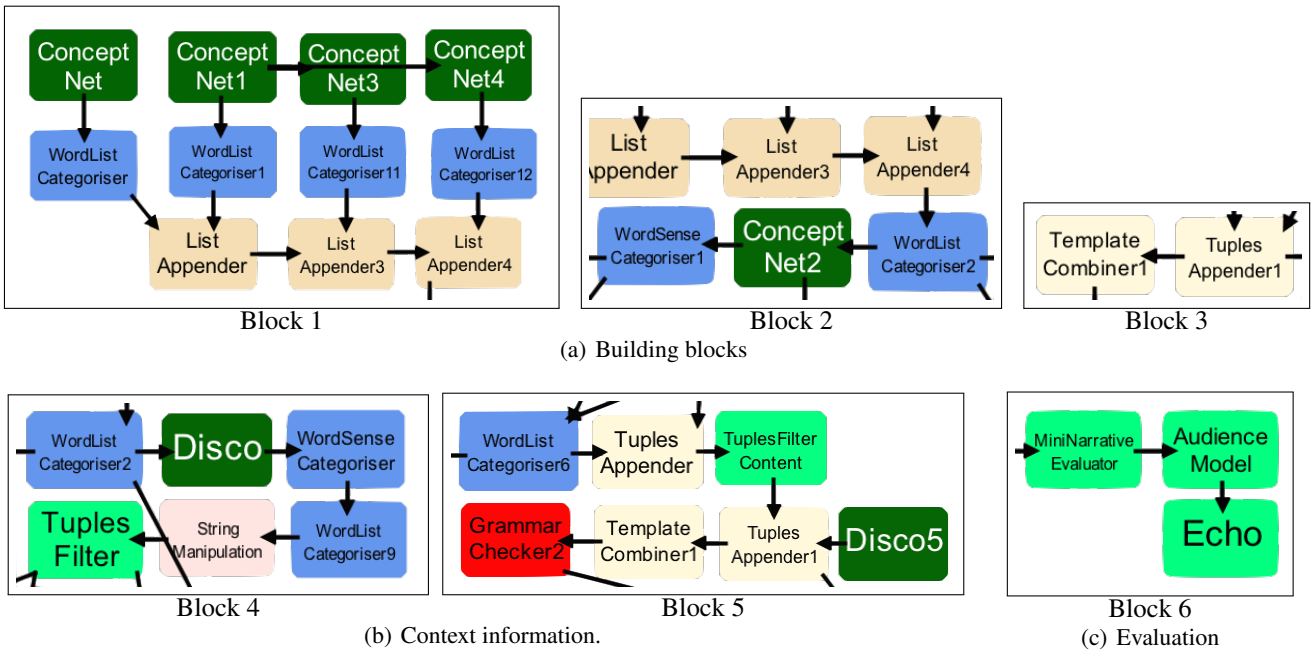



Figure 6: Expert user processes in the fictional ideation flowchart, split into blocks.

nodes can become too complex, with an unmanageable variety of optional parameter combinations, and following the route set out in alternative (b), FloWr would end up with several ConceptNet nodes that only differ in the way the information is retrieved, so it could become difficult to track all of the variants. In any case, complex flowcharts are likely to be difficult to read because of the sheer number of nodes. A future improvement can be accomplished by having nodes that represent entire flowcharts, so for instance, block 1 in Figure 6(a) could be represented as a single node, which could be expanded at will to show the lower layers.

In summary, the experience with FloWr by the expert user has been very satisfactory in the sense of reuse and ease of experimentation. Challenges are present with all frameworks, and from experience of using FloWr in the context of fictional ideation, it is clear that the maintenance of large flowcharts (as in Figure 1) can be difficult, since the connections become confusing. Being able to use hierarchies of flowcharts would be facilitative. The format of data exchange is another challenge. As illustrated above with the ConceptNet node, the situation can be complex either for the developer or the user. We plan to implement a mechanism to handle the complexity of nodes in a flexible framework that can be tailored to different users with different priorities.

## Conclusions and Future Work

We have described progress in the FloWr project, the long term aim of which is to study automating creativity at process level through the automatic construction of flowcharts for generative purposes. We have migrated FloWr to an online portal, with all the benefits that this affords, with an aim of creating a platform for interaction between Computational Creativity researchers. In addition, we have described new powerful additional features of the platform, such as the API, the node developer package and the ability to publish flowcharts as stand-alone web services. We are particularly interested in how the API will help and encourage researchers to experiment with automated programming. To illustrate the potential value of FloWr to Computational Creativity researchers, we presented two case studies. In the first, a novice user created a new node and a flowchart which used it in two hours, showing that the learning curve for the platform is not too steep. In case study 2, the accomplishments and experiences of an expert user showed that FloWr has real potential for building sophisticated creative systems.

FloWr has reached an important milestone in its development. The platform is now mature and performs all of the main functions for which it was built. We still foresee development work for the core system. However, at this point we have decided that, rather than second-guess which new features to introduce, we will focus our efforts on raising awareness of the system and supporting its adoption by new FloWr users by helping them to learn how to use various aspects of the system and contribute to the platform. We will then respond to feedback and direct our efforts accordingly.

Future feature improvements include better looping and conditional processing plus the introduction of new data-handling approaches such as tagging and a global data store during chart running. Although such enhancements should not be of major concern to API users, who can decide to run parts of the chart, and handle output, as they wish. Further improvements could include parameter validation using owner-defined regex checking or cross-parameter checks and better support for pausing and stopping chart runs.

Our automation experiments will continue. In particular, we are hoping that, as the corpus of nodes and flowcharts

grows, we will have an opportunity to machine-learn regularities in the structure of flowcharts, and apply that knowledge to the generation of novel flowcharts.

In order to further support automated programming, it is likely to be important to expand on the bespoke data-type and supplementary meta-information, e.g. minimum values, that FloWr currently supports. Following the methodology of "Design by Contract" (Mitchell and McKim 2002), node authors would be able to make explicit statements of pre-conditions, post-conditions, and invariants. Sophisticated automatic programming clients could then reason about these specifications. There are a range of Java libraries that support this sort of annotation, e.g., the Java Modelling Language (Leavens, Baker, and Ruby 1998). Using such an approach, we plan to see whether an automatic programming tool could rediscover, for example, the patterns used in Figure 6(a)–6(c), and the way they fit together. This would be informed by work on reasoning about formal specifications. There is an established body of work in this area, much of it carried out in connection with theorem provers such as Coq (Bove and Capretta 2007; Tollitte, Delahaye, and Dubois 2012). Reasoning about program syntax and semantics is a recognised challenge "the pragmatic questions in this domain are far from settled" (Chlipala 2011, p. 340), and we hope to contribute to this field with reasoned automation over the FloWr system.

We intend to build on our initial work in encapsulating charts as nodes. We have developed a node which makes calls to the chart-run feature of the API and, so, we have been able to use entire flowcharts as single nodes in other charts. The next stage will be to resolve the issue of interface – i.e. how to re-describe specific parameters of the encapsulated chart as named parameters of the encapsulating node – so that traditional nodes and encapsulating nodes are indistinguishable. This is likely to require some development work in the core FloWr system, as the ability to pass in a list of encapsulated parameters with potentially differing data-types is not clear. Python and Clojure clients implementing the current API functions are available, and will be maintained and extended as the API evolves (https://github.com/holtzermann17/FloWrTester).

We want to enhance chart-sharing by introducing a system which, like view profiles, allows viewers of public charts to alter a subset of parameters, rather than having to take a copy. The subset of changeable parameters will be determined by the chart owner so that they can be changed to produce new chart output without breaking the chart. This will have some overlap with our work to encapsulate charts as nodes by introducing a notion of interface. In addition, we would like to enable users to switch between a number of pre-set chart parameterisations. We also plan to further extend the media types that FloWr can handle. The placeholder-viewer approach used for images is very effective and we expect to be able to easily expand FloWr to other multimedia domains, such as audio and 3D models.

The community features of FloWr will evolve. We will introduce sub-groups of users, context-based messaging, and comments and responses. We will supplement the basic public/private visibility system with the ability to open charts up to subgroups and introduce a similar system for node visibility and versioning. So, for instance, novice node developers will be able to ask for feedback from a select group before releasing their code to a wider audience. We hope that FloWr becomes a hub for Computational Creativity research, education and practice. Illustration with examples is very powerful and the ability to quickly set up a chart to demonstrate concepts with a natural language tool like Porterstemming (Porter 1980), could be highly effective in a range of scenarios. As per case study 1 above, we imagine a researcher who hears about a new piece of research or tool and is able to easily share it with the broader community by implementing it as a node and demonstrating its operation in a flowchart. Such illustrations have the power to be far more informative than, for example, writing up a research note or providing a link to a paper

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# What If A Fish Got Drunk?
# Exploring the Plausibility of Machine-Generated Fictions

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### Abstract

Within the WHIM project, we study *fictional ideation*: processes for automatically inventing, assessing and presenting fictional ideas. Here we examine the foundational notion of the *plausibility* of fictional ideas, by performing an empirical study to surface the factors that affect judgements of plausibility. Our long term aim is to formalise a computational method which captures some intuitive notions of plausibility and can predict how certain types of people will assess the plausibility of certain types of fictional ideas. This paper constitutes a first firm step towards this aim.

## Introduction

In Llano et al. (2016), we define a successful fictional idea as *"one that presents a character, event or scenario that transforms or distorts the 'real' world in the imagination of the reader without requiring him or her to leave it entirely"*. In the WHIM project (an acronym for The *What-if Machine*), we are undertaking the first large-scale study of how software can invent, evaluate and express fictional ideas with real cultural value (www.whim-project.eu). We have identified *plausibility* as one of the key dimensions of fictionality, and so investigating questions of plausibility is important for the aims of the WHIM project. Unfortunately, plausibility resists a simple definition. Here, we explore the factors that support the perception of a machine-generated fictional idea as plausible or implausible.

Plausibility in fictional scenarios is different from notions of probability, which rely on modelling situations in terms of relative frequency, or the updating of prior distributions. Judgements about plausibility in fictional situations involve a process of *interpretation*, where the reader makes – perhaps implicit – subjective decisions about the underspecified fictional universe. For example, in the absurdist play *Rosencrantz and Guildenstern are Dead*, Rosencrantz bets "heads" on a coin flip 92 times in a row, and wins each time. If presented as a factual news story, this would likely both be judged as implausible and mathematically improbable (albeit no more improbable than any other session of 92 coin flips). However, within the context of the play, we're invited to consider a fictional world in which this highly improbable chain of events is plausible, i.e., it actually happens.

Following an overview of prior research, we propose some candidate factors to capture how people assess the plausibility of fictional statements. We then report on how these were used in an exploratory study, where 20 participants were asked to categorisation a set of fictional ideas – both machine- and human- generated – into four plausibility categories and interviewed about their judgements. We then examine these judgements from two distinct perspectives: (1) a grounded theory analysis that identifies several factors they considered relevant to plausibility; (2) a multidimensional scaling analysis that suggests several factors that can explain differences between how fictional ideas were assessed. Finally – as per Rothbauer (2008) – we triangulate the outcomes of these analyses and our initial theories, leading to a final set of factors. We conclude with a discussion of the relevance of this work for further research on fictional ideation and Computational Creativity in general.

## Background

Connell and Keane (2004) carried out an empirical study of plausibility. They first evaluated the *concept coherence* of a set of events written as two connected sentences, e.g.,

*The bottle fell off the shelf. The bottle smashed.*

These were classified according to different types of inferences; the sentence above references a *causal inference*, while the sentence pair

*The bottle fell off the shelf. The bottle was pretty.*

references an *attributal inference*. Their results supported the received view that concept coherence is important in plausibility judgements, and showed that different inference types differentially affect plausibility. A second experiment evaluated *word coherence* rather than concept coherence, but this experiment found no reliable effect of word coherence on plausibility.

Connell and Keane note that their studies used "the term plausible interchangeably with other descriptions such as appropriate, sensible, or makes sense." Our approach differs considerably from theirs, as they focused on statements whose two constituent parts have a strong conceptual relation. In contrast, as the fictional statements evaluated in this paper have not been conceived with such restrictions, this has allowed for a more comprehensive analysis.

Lombardi, Nussbaum, and Sinatra (2015) have sought to outline a primarily theoretical model for plausibility judgements. In particular, they examine the role such judgements

play in *conceptual change*, and come to understand plausibility as meaning: 'what is perceived to be potentially truthful when evaluating explanations'. They cite Nicholas Rescher's observation that a statement deemed to be plausible, or potentially truthful, indicates that there has been a 'highly provisional and conditional epistemic inclination towards it' (Rescher 1976).

In sum, there seem to be myriad criteria upon which people may base judgements of plausibility. Some factors in an individual's judgement appear to be highly subjective, often being heavily influenced by personal circumstances, religious belief, cultural background, or political sympathies. Indeed, while most people feel they have an understanding of plausibility, that understanding is almost always destabilised when an individual tries to apply consistent criteria to analyse the plausibility of various sample statements.

Plausibility is closely connected with the notion of *interpretation*: that is, a given interpretation of a given scenario is deemed "plausible" if potentially valid, under a given set of assumptions. Interpretation of conventional symbols is highly constrained. However, creative interpretations may be almost endlessly fanciful. To take one example: the science fiction author Philip K. Dick suggests in several of his written works that we continue to live in biblical times, evidence of which can only be accessed by visionary experience (Dick 1995). This interpretation of the world is grounded in the data of personal perception and reflection. Nevertheless, most people would find such an interpretation implausible if presented as anything other than fiction.

Eco argued that the world, history, and texts have constraints on their plausible interpretations, despite the wide range of *possible* interpretations. He posits that fiction and reality intersect in the following way: "We can make true statements about fictional characters because what happens to them is recorded in a text, and a text is like a musical score" (Eco 2009). Music shows us that a creative work can take on a life of its own through interpretation. Fictional characters can also "become individuals living outside their original scores," or, to put it more formally: "a fictional character is a semiotic object."

We can thus make chains of interpretations about fictional characters and other elements of fictional worlds. In the first instance, the validity of such interpretations is not "grounded" in real-world facts, but in the fictive notions of the fictional world – subject also to the perceptions, beliefs, and other features of the interpreting agent. When instigating behaviour (including storytelling behaviour), certain interpretations may be predicted, based in part on a preliminary interpretation of those agents who are expected to perceive the behaviour (Kockelman 2012).

We believe it is important for Computational Creativity researchers to tackle issues related to fictional interpretations, in particular to ask what kinds of interpretations are *useful* (Eco 2006) – rather than merely true. Although subjectivity plays a role, keeping in mind Eco's remarks on limits of interpretation, we think that the reader's perception of a text's plausibility will often draw on the text's objective features, and we develop this theme below.

## Candidate Factors

As a first step, we conducted an introspective study to identify an initial set of factors that may be involved in human plausibility judgements. These candidate factors helped guide the design of the exploratory study, described below. Eight fictional statements were used. Four were from the What-If Machine and four were summaries of well known literary works, included to foster the generalisability of the findings. The machine-generated statements were selected for quality and diversity, to showcase a range of potentially relevant factors.

Three of the authors independently read the statements and rated each sentence 1–5 in terms of how plausible they were (1= low plausibility and 5 = high plausibility). They also wrote a commentary for each statement, describing their rationale for that score, their scoring process (including whether they had revised a score during analysis), and a set of labels that described relevant properties, dimensions or features. By comparing our individual answers, we found a common set of factors that appeared to affect our plausibility judgements, listed below.

**Complexity** The level of elaboration of the idea in terms of the amount of narrative detail that it is composed of. Our intuition is that a larger number of statements, or narrative details, used to compose an idea reduces its plausibility. An illustrative example is the statement:

> *"What if there was a poor orphan girl who was abused by her aunt, sent away to school where conditions were harsh, before becoming a governess and marrying her employer, who was already married to a mentally ill woman whom he has locked up in his house?" (1)*

Each part is rather plausible, but their conjunction renders the overall idea less plausible. *We hypothesise that there is a negative correlation between complexity and plausibility.*

**Universality** The scope of an idea, in terms of how general people think it is intended to be. In other words, whether the scenario in the fictional idea applies to one, a few or all members of a group. The intuition behind this is that an idea that is generalised to a large number of members of a group is less plausible than an idea that only involves one member. For instance, from the statement:

> *"What if there was a young girl who went through a rabbit hole and found herself in a strange and mysterious land where animals could talk and everyone is mad?" (2)*

The implicit universally quantified sentences "animals could talk" and "everyone is mad" decrease the plausibility of the idea. *We hypothesise that there is a negative correlation between universality and plausibility.*

**Openness** How open to subjective interpretation an idea is perceived to be. Our intuition is that if an idea that is composed of statements that are ambiguous or not specific, for which the reader can provide different interpretations or scenarios, is perceived as more plausible. As an example, take the statement:

> *"What if there was a young man who kept a painting of himself which aged while he himself stayed young?" (3)*

Here, "stayed young" could be interpreted both in terms of not looking old or actually not ageing, while the painting that "aged" could have been painted in highly impermanent materials. The possible explanations (natural youthfulness and cheap paint, or a bizarre medical condition) differ strongly in their plausibility; if there is a choice, a subject might choose the more plausible explanation. *We hypothesise that there is a positive correlation between openness and plausibility.*

**Causality** The level of connectivity between the components that make up an idea. In other words, how naturally the statements that make up an idea lead coherently from one to the other. The intuition behind this is that plausibility increases when the statements of an idea are clearly connected so as to serve as supporting arguments themselves. To illustrate this, take the example:

*"What if there was a little doctor who couldn't take a pulse?" (4)*

Without an explanation as to why the doctor is unable to perform the common task of taking a pulse, this statement will likely score low for plausibility. *We hypothesise that there is a positive correlation between causality and plausibility.*

**Familiarity** The level of awareness of the overall scenario relative to known ideas. Although this is a subjective factor, the intuition behind it is that our perception of plausibility is affected by common themes, scenarios and characters that figure more commonly in culture. Statement (1) illustrates this intuition. Well-known character stereotypes such as *an orphan girl*, *an evil aunt* and *a mentally ill woman* render the statement more plausible. *We hypothesise that there is a positive correlation between familiarity and plausibility.*

**Feasibility** How well the elements within an idea fit within the overall scenario. The intuition behind this is that plausibility increases if an element; e.g., a character, is better suited to one situation than another. To illustrate, the statement:

*"What if there was a little cat who learned how to use a phone?" (5)*

Would rank lower in plausibility if instead of *a cat*, the subject was an inanimate object, for instance *a cooker* due to the affordances of the subjects. *We hypothesise that there is a positive correlation between feasibility and plausibility.*

## An Exploratory Categorisation Study

To further explore the factors underlying human plausibility judgements, we conducted a categorisation study where the above candidate factors guided the selection of the stimuli for, and design of, the study. In the study, participants were asked to assign machine- and human-generated fictional ideas into different categories of plausibility. We collected both quantitative and qualitative data from these sessions, which were then separately analysed and interpreted: 1) the raw categorisation results were used to calculate explorative statistics; 2) participant think aloud commentaries and post-task interviews formed the basis of a grounded theory analysis (Adams, Lunt, and Cairns 2008), identifying key factors in their categorisation process. 3) The categorisation results were transformed into similarity data for a multidimensional scaling (MDS) analysis (Borg and Groenen 2005) of the fictional ideas, to collect more evidence on the underlying dimensions which influenced the plausibility categorisations; By studying the same plausibility judgements qualitatively and quantitatively, we hoped to triangulate the results to arrive at a final set of factors. A categorisation task with physical cards, as opposed to ordinary Likert-scale rating, was deliberately chosen to promote think-aloud comments and the comparison of stimuli. These methods are well-suited to exploring complex and poorly conceptualised domains, e.g., see Wallraven et al. (2009) or Gow et al. (2010).

**Stimuli** We used 28 fictional ideas in total, consisting of 18 ideas generated by The What-If Machine (three from each of the six categories the system currently supports: "Disney", "Metaphors", "Utopian/Dystopian", "Alternative Scenarios", "Kafkaesque" and "Musicals"), 7 ideas summarising well-known fictional literature works ("Literary Fiction"), and 3 ideas that used known fictional characters or worlds ("Fiction in Fiction"). We also selected a subset of six ideas (from the 28 already selected) for the participants to verbally elaborate on in more detail. A selection of stimuli from each category can be found in Table 1.

**Method** Participants took part in the study individually and were all read the same introductory material. Each session was audio recorded for later analysis. We first asked them to sort the 28 stimuli, provided as paper cards, into four plausibility categories:

1. **Highly implausible:** describe scenarios that have very little grounding in your experience of reality.
2. **Slightly implausible:** describe scenarios that have a low degree of grounding in your experience of reality.
3. **Slightly plausible:** describe scenarios that are somewhat grounded in your experience of reality.
4. **Highly plausible:** describe scenarios that have a high degree of grounding in your experience of reality.

An "**I don't understand**" category was also provided. In contrast to our introspective study, we chose four categories to eliminate the neutral choice. Participants were not told that some of the statements had been written by software, nor asked if they recognised those from human-authored narratives.

We asked participants to think aloud while performing this task, i.e., to articulate their categorisation process and rationale. For some participants (see below) this was followed by open-ended questions where these issues were probed in greater depth, focusing on the six statements which we had pre- selected, or others highlighted during the categorisation study. Finally, we asked some participants explicit questions about our candidate factors (described above), to determine if they considered them relevant. For instance, regarding *complexity* we asked: "Do you believe that a complex fictional statement; that is, with a large amount of conditions, makes the plausibility higher, lower or neither?". Each such question was accompanied by an example statement.

| Id | Mean | Var. | Ag. | NAs | Stimulus | Category |
|---|---|---|---|---|---|---|
| 5 | 2.00 | 1.44 | 14 | 0 | What if we could give life to a being created by combining the body parts of dead people? | Literary Fiction |
| 8 | 0.39 | 0.72 | 16 | 1 | What if a zombie rugby-tackled a ghost and broke his leg? | Fiction in Fiction |
| 12 | 0.38 | 0.78 | 14 | 3 | What if there was a little pen who forgot how to write? | Disney |
| 14 | 2.50 | 0.62 | 15 | 1 | What if ignorant fools were to overcome mistakes, establish cults and become knowledgeable gurus? | Metaphors |
| 19 | 1.69 | 1.03 | 9 | 3 | What if the world suddenly had lots more assassins? Then there would be more antidotes, since assassins use the poisons that require antidotes. | Utopia / Dystopia |
| 20 | 0.44 | 0.61 | 15 | 1 | What if there was an old fish, who couldn't swim anymore, which he used to do for relaxation, so decided instead to get drunk? | Alternative |
| 21 | 0.94 | 1.31 | 11 | 2 | What if there was an old car that could be used as the space for holding a star? | Alternative |
| 24 | 0.17 | 0.26 | 17 | 1 | What if a bicycle appeared in a dog pound, and suddenly became a dog that was able to drive an automobile? | Kafkaesque |
| 26 | 2.72 | 0.21 | 18 | 1 | What if a wounded soldier had to learn how to understand a child in order to find true love? | Musicals |
| 28 | 2.06 | 0.43 | 14 | 2 | What if a janitor needed to suppress a rebellion in order to gain admiration? | Musicals |

Table 1: A selection of stimuli, with response mean and variance. Ag.= participants agreeing with most common response. NAs = times classified as "Don't understand".

**Participants** In total, 20 participants took part in the study, although one participant's data was excluded (see below). Of the remaining 19, 4 participants were female and 15 male. 8 participants were in the age range 18-24 years old, 9 were 25-34, and 2 were 35-44. 4 of them specified A levels as their current level of education, 7 had a first degree, 6 a higher degree, and 2 a doctorate. 7 participants were fluent in English, 11 were native speakers, and one self-rated as "intermediate", but was considered fluent. We assumed that the lack of demographic diversity would have limited impact on our results, although future studies could make some provision for variations related to gender, age or educational background, e.g., cultural references. Participants did not have familiarity with our work on plausibility prior to taking part the study.

All participants were paid £10 and undertook the categorisation experiment. Only 12 were asked the open-ended questions and questions about the candidate factors. This allowed us to constrain the amount of data collected for the grounded theory analysis, while satisfying representativeness for the quantitative analysis. One participant was a very distinct outlier in terms of categorisation mean and variance, as they classified most statements into either "highly implausible" or "I don't understand". They were perfectly aware of the meaning but didn't agree with the logic of the statement. Their think aloud data also suggested that they did not engage with the task as requested. We therefore excluded this participant from the analysis that follows.

## Categorisation Results

Of the 532 judgements made, the most common were "highly implausible" (34%) and "highly plausible" (25%), followed closely by "slightly plausible" (23%). The least common responses were "slightly implausible" (11%) and "don't understand" (7%). In the analysis below, we sometimes interpret these ordinal responses (excluding "don't understand") as interval data from 0 (highly implausible) to 3 (highly plausible). Table 1 shows the response mean and variance for a selection of stimuli.

**By Participant** All participants used the entire range of responses. There were notable individual differences: 4 participants had median response of "highly implausible", 8 had "slightly plausible", with the remaining 7 medians falling in-between. The variance for each stimuli provides a measure of agreement between participants: the mean variance was 0.93 (min 0.21, max 1.49), suggesting quite a high level of disagreement. However, if we ignore the distinction between *highly* and *slightly* and merge categories to *plausible/implausible/don't understand*, we actually see many participants agreeing with the modal (most popular) category: for 68% of stimuli, at least two-thirds agree. This shows at least a weak consensus was often present.

Participants used the "I don't understand this statement" category a median of 1 times, indicating comprehension was not a problem for most participants. Only one participant claimed to understand all stimuli and, at the other extreme, two didn't understand six stimuli. Using Spearman's $\rho$, there is a medium negative correlation ($\rho_S(28) = -0.4, p = 0.09$) between not understanding and use of "Highly implausible" and a medium positive correlation ($\rho_S(28) = 0.35, p = 0.1$) between not understanding and that participant's mean plausibility. This suggests there may be some confusion between "don't understand" and "implausible", which should be addressed in the design of future studies.

**By Stimuli** Almost all the stimuli provoked the full range of responses, confirming that assessing plausibility is a highly subjective task. The mean response for each stimuli

ranged from 0.17 (Stimuli 24) to 2.84. The variance ranged from 0.21 (Stimuli 26) to 1.49. We compared the plausibility ratings between the different stimulus groups described above. A Kruskal-Wallis one-way analysis of variance indicated that the plausibility ratings between the eight groups were significantly different $H(7) = 80, p < 1e - 13$. We then performed a series of Wilcoxon rank sum post-hoc tests with Bonferroni correction to determine which of the groups are significantly different. The $p$-values and group means are listed in Table 2. It shows, amongst others things, that statements from the categories "Musicals" and "Metaphor" were rated highest in plausibility ($\mu = 2.33, \mu = 1.76$), and differ significantly from the categories that were considered highly implausible, namely "Kafkaesque" and "Utopia/Dystopia" ($\mu = 0.56, \mu = 1.06$).

## Think Aloud Results

To understand the factors which contributed to participants' plausibility judgements, we performed a grounded theory analysis of the think aloud data. Grounded theory is a qualitative research method that is used to build, validate and expand theories from data, in order to reach "a theoretical formulation of the reality under investigation" (Corbin and Strauss 1990). Our analysis validated four of our initial hypothesised factors as influential within our participants' judgements: *openness*, *familiarity*, *causality* and *feasibility*; the other two, *complexity* and *universality*, were concluded as non-influential. An additional factor, *perception of reality* was identified. Furthermore, for each of the supported factors, we identified a set of properties that represent the different ways the participants talked about the factors, as well as dimensions describing values these properties can hold. These results are summarised in Table 3.

Participants often based their judgement on how ***Familiar*** they were with the content; either from *experience* (own or by others) or *knowledge* they have acquired from different mediums. An illustrative example is Statement (1), quoted earlier to highlight the Complexity dimension. Two participants said the following:

> "Doesn't go with things in this time and day but people's lives are complicated [...]"

> "you see similar situations in the news [...] these are different personalities that actually exists [...]"

Consequently, this statement was often classified in the plausible spectrum (10 as highly plausible and 8 as slightly plausible). Familiarity at the level of *cultural recognition* also affects plausibility judgements. For instance, despite the fact that the statement: *What if a zombie rugby-tackled a ghost and broke his leg?*, contains fictional characters, as these are well-known concepts that form part of our culture, participants would hesitate about their plausibility value (even if eventually most decided the statement was not plausible).

***Openness*** was also a recurrent factor we identified from the recorded sessions. Often, participants would try to make sense of the statements, saying things such as:

> 'Maybe because my brain [is trying to give] sense to sentences.'

> 'Where there is more room for interpretation, it is more easy to be black or white.'

*Ambiguity* and *context* played an important role for this factor. We found that key concepts appearing in a statement made a significant difference in plausibility judgements when these could be interpreted in different ways, and there was not enough context to narrow down the intended meaning. This led participants to stick with their favourite interpretation and provide their judgement accordingly. To illustrate, regarding the statement '*What if there was an old car that could be used as the space for holding a star?*', participants would often ask if the concept *star* meant the astrological object or a celebrity, with most of the participants selecting the former and consequently placing this statement within the implausible spectrum. A similar reasoning was common with statement (4) above, for which participants would consider the concept of the *little doctor* as being either a child or a doctor short in height. Most participants chose the former interpretation and placed the statement in the plausible spectrum.

***Feasibility*** was also one of the factors used by the participants when judging plausibility. In particular, we found that they would consider if the *likelihood* of the statement would form a usual or unusual scenario to decide on its plausibility. This was often seen in statements like (1), where the co-occurrence of all the elements of the statement was seen as unusual – but still plausible. One participant said:

> 'it's quite a complicated story but elements of the story makes it feel more real.'

Additionally, feasibility was also accounted for based on the use of *stereotypes* and how the individual parts of the statement fit together with a stereotypical construct. To illustrate, take the statement: *What if the world suddenly had lots more dictators? Then there would be less neediness, since dictators abuse the victims that demonstrate neediness*, for which a handful of participants focused on the contradiction between the concept of *dictators*, which has negative connotations, and the concept of *less neediness*, which has positive connotations.

Specific keywords, in particular *attributes* of the concepts in the statement, were also a decisive property when judging the statement based on its feasibility. For instance, the use of the adjective *little* in statement (4) made the plausibility higher, since participants interpreted the scenario as a *child playing doctor who is not able to actually take a pulse*, which in their view was completely feasible.

We also found that, although ***causality*** was not a strong factor in the decision making process, it was present on some occasions. In particular, finding *arguments* in favour or against particular elements of a statement had an influence in plausibility judgements. For instance, the statement: *What if the world suddenly had lots more assassins? Then there would be more antidotes, since assassins use the poisons that require antidotes*, links the concept of *assassins* with the concept of *poisons*, and this itself to the concept of *antidotes*. Although this statement was built through well-attested associations, specifically that assassins use poisons, and that poisons require antidotes, the intended strong link between assassins and poisons was used constantly as an argument against the plausibility of the statement. In contrast, from the statement: *What if there was a punishable man who had to learn how to eat a person in order to achieve his dream of becoming a criminal?*, the link between 'eating

| Category | Literary Fiction | Fiction in Fiction | Disney | Metaphors | Utopia / Dystopia | Alternative | Kafkaesque | Musicals |
|---|---|---|---|---|---|---|---|---|
| Mean Plausibility | 1.71 | 1.11 | 1.37 | 1.76 | 1.06 | 1.12 | 0.56 | 2.33 |
| Fiction in Fiction | 0.11394 | - | - | - | - | - | - | - |
| Disney | 1.00000 | 1.00000 | - | - | - | - | - | - |
| Metaphors | 1.00000 | 0.16271 | 1.00000 | - | - | - | - | - |
| Utopia / Dystopia | 0.07103* | 1.00000 | 1.00000 | 0.08518* | - | - | - | - |
| Alternative | 0.14113 | 1.00000 | 1.00000 | 0.19369 | 1.00000 | - | - | - |
| Kafkaesque | 5.9e-07** | 0.48768 | 0.00455** | 2.4e-06** | 0.31448 | 0.44275 | - | - |
| Musicals | 0.09299* | 3.5e-05** | 0.00039** | 0.23095 | 5.9e-06** | 4.4e-05** | 1.9e-11** | - |

Table 2: $p$-values from pairwise comparisons of stimuli groups using Wilcoxon rank sum test and Bonferroni correction. Significance: * low ($\alpha < 0.1$) and ** high ($\alpha < 0.01$). The second row comprises plausibility means for all categories.

a person' and becoming a 'criminal' was seen as logically connected:

> 'I can imagine eating a person as an act of initiation for a person to be part of a gang...'

Interestingly, the idea of *unknowns* was also used as an argument to decide on a plausibility category. This is when a participant considered that he/she did not have enough knowledge to argue against or in favour of a particular scenario. An example of this was the statement: *What if the ministry of magic paid JK Rowling to write her books so we muggles would think magic is fiction?*:

> 'I don't know if there is a minister of magic [...] who knows?'

which some participants used as an argument to assign a higher plausibility value.

Lastly, ***perception of reality*** played a role for some individuals when making their judgements. This factor represents how people may account for different ways of perceiving reality within certain scenarios. To illustrate, a participant categorised statement (1) as highly plausible based on the following reasoning:

> "From my experience of reality, that might happen in some psychedelic state, a dream state, an imaginary state. I don't think it's right to count what happens in these states as any less real [...] the rabbit hole could be a doorway to other states. That's an idea I'm definitely open to."

another participant questioned the meaning of reality:

> '[...] what is reality? Different statements push different readings of what reality is: objective reality in terms of things that are physically possible for ever and ever, things that might be possible in the future with technology, things that kind of work in a fictional world, and things that don't work at all.'

| Factor | Properties | Dimensions |
|---|---|---|
| Familiarity | Experience | Own/Others |
| | Knowledge | Cultural/Heard/Read/Seen |
| | Cultural recognition | Conceptual/Factual |
| Openness | Ambiguity | Most/Least plausible |
| | Context | Lack/Presence of |
| Feasibility | Likelihood | Usual/Unusual |
| | Stereotypes | Confirmation/Contradiction |
| | Attributes | Opened/Specific |
| Causality | Arguments | In favour/Against/Unknowns |
| | Context | Lack/Presence of |
| Perception of reality | Abstraction | Conceptual/Physical |
| | Cultural influence | Background/Beliefs |

Table 3: Influential factors when judging plausibility.

This is a subjective factor, but the intuition behind it is that judgements of plausibility are affected by personal views of what can be considered to be real or not.

Within this factor, *abstraction* was found to be a common property. In this case, the overall scenario was considered as having a hidden meaning. To illustrate, the statement: *What if the world suddenly had lots more angels? Then there would be more barriers, since angels serve the gods that impose the laws that create barriers*, was abstracted by some participants:

> 'I don't believe in angels or God, but I think the government can use it as a tool to manage people.'

leading them to assign a higher plausibility value to the statement. Similarly, *cultural influence* played a role in how participants' perception of reality would affect plausibility. Take the statement: *What if respected senators were to retire from their senates, join gangs and become shady gangsters?*, which was implausible for many participants because it did not make sense with their notion of reality:

> 'there is no reason why a senator with power and money would choose to be a gangster'

while for others, this was a plausible scenario due to their cultural background, where this situation was feasible:

> '[...] in certain very corrupt countries it actually happens [...] when they are senators they belong to gangs, legal ones, but they do [...] and when they retire they keep being part of those clubs'

Likewise, the statement *What if a janitor needed to suppress a rebellion in order to gain admiration?* was classified as slightly plausible because:

> 'this is kind of a standard Hollywood plot really, I can imagine that being played by Tom Cruise [...] it has high degree of grounding in my experience of reality [...] not my experience of reality, my experience of Hollywood film making'

suggesting the participant considered the fictional world of Hollywood films as a type of reality.

*Complexity*, as mentioned before, did not come across as an influential factor for plausibility. For instance, regarding the complexity of statement (1), a participant highlighted:

> 'All happening at once is unlikely but it's possible [...] that doesn't change the plausibility.'

instead, this factor was considered to sometimes make the statements more difficult to understand. *Universality*, on the other hand, was seen as a factor that would make a statement more interesting, but would not have a significant effect on its plausibility value, specially when the statement was placed in the implausible spectrum:

'if you are in the implausible categories, then it doesn't matter, one, many [...] we are talking about something that is not real, so it doesn't matter'

## Multidimensional Scaling Results

We performed an multidimensional scaling (MDS) analysis to quantitatively derive a set of factors from the categorisation results, and to assess their influence on the overall judgment. Classic MDS maps measurements of (dis-) similarity among pairs of stimuli to distances between points in a geometric space (Borg and Groenen 2005, p. 3). In this space, each dimension can be considered a factor which influenced the initial similarity judgement. The meaning of a dimension is a matter of interpretation, based on the distribution of stimuli along it and their properties.

First, we had to determine the pairwise similarities between stimuli. The effort of collecting this data manually increases exponentially with the number of stimuli; we therefore followed a different approach suggested by Wallraven et al. (2009), where pairwise similarities are derived from a categorisation task. This approach allowed us to re-use our previously collected data, while implicitly grounding the similarities in plausibility judgements. We started with an empty similarity matrix, and increased the similarity value of two stimuli if they were put into the same plausibility group. This was repeated for all participants and normalised.

We then determined how many dimensions have to be used to approximate the data well enough by looking at how much variance in the data each dimension accounts for (Borg and Groenen 2005, pp. 247). We cut off at three dimensions, accounting for 78% of the variance, with the first dimension covering 57%. We then visualised each of these as a one-dimensional axis with the stimuli projected along it, allowing us to compare the relative distribution of the stimuli. These visualisations were given to four of the authors for interpretation, informed by the think-aloud results. A consensus interpretation of each dimension was then agreed on. These are summarised in Table 4, along with some examples of high and low scoring stimuli from Table 1.

On the first dimension, statements that showed a strong deviation from reality were grouped in one extreme. On the other extreme were statements that were more aligned with the rules of what it is commonly agreed as possible. The dimension was identified as *feasibility*. The second dimension was strongly associated with interpretability, i.e. with the stimuli's *openness to interpretation*. Interestingly, the stimuli in both extremes were found to have different interpretations; however, what separated one group from the other was how ambiguous the possible interpretations were assessed to be. In one extreme, an interpretation would allow for a more decisive judgement, while in the other, the interpretation would still be seen as not convincing. The third dimension was found to classify stimuli based on *familiarity*. Well known elements, similar stories, common characters and stereotypes were identified in one extreme, while the other extreme presented the same characteristics (i.e. familiar elements) used in contradictory ways.

| Dim | Var | Example stimuli (Id) High | Low | Interpretation |
|---|---|---|---|---|
| 1 | 57% | 14, 26, 28 | 8, 12, 24 | Feasibility |
| 2 | 12% | 5 | 28 | Openness |
| 3 | 9% | 14, 26 | 19, 28 | Familiarity |

Table 4: The first three dimensions identified by MDS.

## Future Work

This study provides evidence for three factors — feasibility, openness to interpretation, and familiarity — that contribute to judgements about the plausibility of fictional ideas. Understanding these factors is a necessary step towards further experimental investigation in this area. We plan to further test and refine this theory, and use it to design studies on the perception of machine-generated fictions.

We intend to model these factors computationally within the What-If Machine, to control the plausibility of the fictional ideas it generates. This could enhance the usefulness of the software and perhaps increase the cultural value of the ideas it produces. Although further experimentation is needed, we believe that metrics which predict values for each of the factors can be devised. Moreover, these could be used to predict the plausibility judgement that certain types of people will make for particular fictional statements.

A heuristic approach to analyse whether a statement is open to interpretation can be based on the *concreteness scores* of its constituent keywords. This measures the level of ambiguity of these words and give an approximation of the concreteness of the overall scenario. We have formalised fictional statements within the WHIM project as short narratives composed of narrative points that are either linked through causal relations, assumed by the reader, or given by the knowledge base (Llano et al. 2016). This formalism allows us to represent each statement as a graph over which we can reason. For instance, analysing the connectivity within the graph may allow us to hypothesise the level of *contextual support* within the statement as a whole. Highly supported statements may be less open to interpretation.

Feasibility, on the other hand, could be accounted for through the use of techniques such as a distributional semantics vector space model (Mikolov et al. 2015). Specifically, how well the elements of a statement fit together could be measured by studying their *semantic similarity* as well as their shared contextual co-occurrences. Stereotypical properties of concepts can be mined from the web (Veale 2012). A similar method could be followed in order to assess stereotypes within a statement and compare the polarity between the stereotypes and the other elements in the statement.

Finally, although familiarity is a subjective factor, metrics could be defined by establishing links with the information in knowledge bases of common knowledge and narrative constructs. In this context, strongly linked data can be seen as connected to "known or familiar scenarios".

Progress in fictional ideation has general implications for Computational Creativity. In the problem solving paradigm of AI, intelligent tasks to automate are broken down into a series of problems to be solved, and there is a usually a

'right answer' to these problems whether local or global, or at least a fitness function relating to the potential value of solutions, which ordinarily captures notions of value from the real world. In the artefact generation paradigm of AI, however, an intelligence task to automate is considered as an invitation to create something of value in a potentially interesting way. Value can be externally imposed, but some Computational Creativity projects have allowed software to invent its own measures of value and to motivate these through framing (Charnley, Pease, and Colton 2012).

We can use the above observations on plausibility to set ourselves apart somewhat further from mainstream AI. In particular, Computational Creativity research could be seen as the sub-field focused on AI for *what could be* rather than *what is best*. Approaches to generate information about possible worlds naturally includes making discoveries about the real universe around us. However, it also includes the invention of imagined scenarios specifically constructed to reflect alternative realities. Such scenarios, like those produced by The What-If Machine, are valuable not because of their explicit reflection of reality, but because they force us to see the realities of our own existence in new and thought-provoking ways (in addition to simply providing entertainment). Analyses building on the work presented here could be influential in the advancement of the automatic generation of fictional universes and other creative works. Predicting how plausible (or not) people judge an idea to be will be a key part of automatically producing imagined scenarios.

## Conclusions

We conducted a study in which participants categorised human authored and machine generated fictional statements (including the one paraphrased in the title of this paper), in terms of their plausibility. Unlike previous studies, in which the notion of plausibility had only operational significance, we explored the constituent factors of plausibility, in order to determine which are most influential. We found that the three most influential factors when judging plausibility are: *feasibility*, which determines how well the elements of an idea fit within the overall scenario, *openness* to subjective interpretation, and *familiarity*, which specifies the level of awareness of the overall scenario relative to known ideas. Our findings can serve as a theoretical grounding for future cognitive and computational studies involving plausibility, as well as informing wider discussions about perceptions of fictionality. We hope to build on this work to improve the cultural value of machine-generated fictions and to make fictional ideation a central part of Computational Creativity.

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# The Peeragogy Handbook

# The Peeragogy Handbook

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# FOREWORD

I was invited to lecture at UC Berkeley in January, 2012, and to involve their faculty and their graduate students in some kind of seminar, so I told the story of how I've used social media in teaching and learning - and invited them to help me create a handbook for self-learners.

I called it the Peeragogy Handbook. I met twice on the Berkeley campus in the weeks following the lecture with about a dozen Berkeley faculty and graduate students. We also had a laptop open with Elluminate, an online platform that enabled video chatting and text chat, enabling people around the world who were interested in the subject, who I recruited through Twitter and email, to also participate in this conversation. All of the faculty and grad students at Berkeley dropped out of the project, but we ended up with about two dozen people, most of them educators, several of them students, in Canada, Belgium, Brazil, Germany, Italy, Mexico, the UK, USA, and Venezuela who ended up collaborating on a voluntary effort to create this Peeragogy Handbook, at peeragogy.org. We all shared an interest in the question: "If you give more and more of your power as a teacher to the students, can't you just eliminate the teacher all together, or can't people take turns being the facilitator of the class?"

Between the time nine years ago, when I started out using social media in teaching and learning, clearly there's been an explosion of people learning things together online via Wikipedia and YouTube, MOOCs and Quora, Twitter and Facebook, Google Docs and video chat, and I don't really know what's going to happen with the institutions, but I do know that this wild learning is happening and that some people are becoming more expert at it.

I started trying to learn programming this summer, and I think that learning programming and doing programming must be very, very different now from before the Web, because now, if

you know the right question to ask, and you put it into a search query, there's someone out there on StackOverflow who is already discussing it. More and more people are getting savvy to the fact that you don't have to go to a university to have access to all of the materials, plus media that the universities haven't even had until recently. What's missing for learners outside formal institutions who know how to use social media is useful lore about how people learn together without a teacher. Nobody should ever overlook the fact that there are great teachers. Teachers should be trained, rewarded, and sought out. But it's time to expand the focus on learners, particularly on self-learners whose hunger for learning hasn't been schooled out of them.

I think that we're beginning to see the next step, which is to develop the methods – we certainly have the technologies, accessible at the cost of broadband access – for self-learners to teach and learn from each other more effectively. Self-learners know how to go to YouTube, they know how to use search, mobilize personal learning networks. How does a group of self-learners organize co-learning?

In the Peeragogy project, we started with a wiki and then we decided that we needed to have a mechanism for people who were self-electing to write articles on the wiki to say, OK, this is ready for editing, and then for an editor to come in and say, this is ready for Wordpress, and then for someone to say, this has been moved to WordPress. We used a forum to hash out these issues and met often via Elluminate, which enabled us to all use audio and video, to share screens, to text-chat, and to simultaneously draw on a whiteboard. We tried Piratepad for a while. Eventually we settled on WordPress as our publication platform and moved our most of our discussions to Google+. It was a messy process, learning to work together while deciding what, exactly it was we were doing and how we were going to go about it. In the end we ended up evolving methods and settled on tools that worked pretty well. We tackled key questions and provided resources for dealing with them: How you want to govern your learning community? What kinds of technologies do you want to use, and why, and how to use them? How are learners going to convene, what kind of re-

sources are available, and are those resources free or what are their advantages and disadvantages. We were betting that if we could organize good responses to all these questions, a resource would prove to be useful: Here's a resource on how to organize a syllabus or a learning space, and here are a lot of suggestions for good learning activities, and here's why I should use a wiki rather than a forum. We planned the Handbook to be an open and growable resource – if you want to add to it, join us! The purpose of all this work is to provide a means of lubricating the process of creating online courses and/or learning spaces.

Please use this handbook to enhance your own peer learning and please join our effort to expand and enhance its value. The people who came together to create the first edition – few of us knew any of the others, and often people from three continents would participate in our synchronous meetings – found that creating the Handbook was a training course and experiment in peeragogy. If you want to practice peeragogy, here's a vehicle. Not only can you use it, you can expand it, spread it around. Translators created versions of the first edition of the handbook in Spanish, and Italian. Updated translations are welcome.

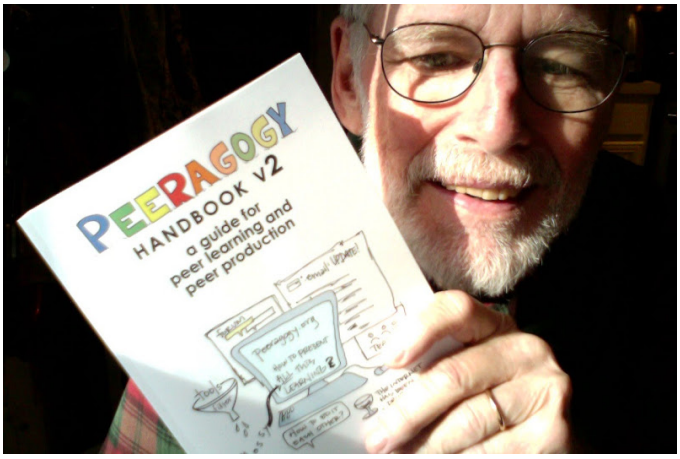
What made this work? Polycentric leadership is one key. Many different members of the project stepped up at different times and in different ways and did truly vital things for the project. Currently, over 30 contributors have signed the CC Zero waiver and have material in the handbook; over 600 joined our Peeragogy in Action community on G+; and over 1000 tweets mention peeragogy.org. People clearly like the concept of peeragogy – and a healthy number also like participating in the process.

We know that this isn't the last word. We hope it's a start. We invite new generations of editors, educators, learners, media-makers, web-makers, and translators to build on our foundation.

Howard Rheingold
Marin County
January, 2014

# PREFACE TO THE THIRD EDITION

This 3rd Edition of the *Peeragogy Handbook* is dedicated to one of our most convivial – indeed, lovable – volunteers. George Brett was a quiet and learned man with a sense of fun. His mail art name was geORge and for a while he was known by many online for the selfies he took with his yellow rubber duckie. Far from being a full-time funster, George has been involved in expanding the educational uses of the Internet since before it was the Internet, both in his work with government institutions and his many online volunteer activities. He gifted the Peeragogy project not only with his knowledge and labor, but with his warmth and fun. We miss him.




The Second Edition of the *Handbook* came out two years ago. We've kept at it since then. This time around, we're kicking things off with a short workbook that contains a concise guide to the who, what, when, where, how and why of peeragogy. We've thoroughly revised the pattern catalog at the heart of the book, added more case studies, and made numerous small improvements to the text (and the typesetting!) throughout.

# Part I

# Introduction

CHAPTER 1

# WELCOME TO THE PEERAGOGY HANDBOOK

> We live where no one knows the answer and the struggle is to figure out the question. [1]

Welcome to the *Peeragogy Handbook*! We want to kick things off with a candid confession: we're not going to pretend that this book is perfect. In fact, it's not an ordinary book at all. The adventure starts when you get out your pen or pencil, or mouse and keyboard, and begin marking it up. It gets kicked into high gear when you join *Peeragogy in Action*. You'll find a lot of friendly support as you write, draw, or dance your own peeragogical adventure. But first, what is *peeragogy*?

Peeragogy is a flexible framework of techniques for peer learning and peer knowledge production. Whereas pedagogy deals with the transmission of knowledge from teachers to students, peeragogy is what people use to produce and apply knowledge together. The strength of peeragogy is its flexibility and scalability. The learning mind-set and strategies that we are uncovering in the Peeragogy project can be applied in classrooms, hackerspaces, organizations, wikis, and interconnected collaborations across an entire society.

The *Peeragogy Handbook* is a compendium of know how for any group of people who want to co-learn any subject together, when none of them is an expert in the particular subject matter – learning together without one traditional teacher, especially using the tools and knowledge available online. What we say in the *Handbook* draws extensively on our experiences working together on the *Handbook* – and our experiences in other collaborative projects that drew us here in the first place. The best way to learn about peeragogy is to do peeragogy, not just read about it. Towards that end, coauthors and fans of the *Handbook* have an active Google+ community, conveniently called *Peeragogy in Action*.

We maintain a regular schedule of weekly meetings that you're welcome to join. The *Handbook* includes a short syllabus, which also called "Peeragogy in Action", and you can work through this with your own group as you read through the book.

You're warmly invited to combine your local projects with the global effort, and get involved in making the next edition of the *Handbook*. That doesn't necessarily require you to do extensive writing or editing. We're always interested in new use cases, tricky problems, and interesting questions. In fact, our view is that any question is a good question.

Here are some of the ways in which the current edition of the Handbook is *not perfect*. You're welcome to add to the list! These are places where you can jump in and get involved. This list gives a sense of the challenges that we face putting peeragogy into action.

# Scrapbook of Peeragogical Problems

## Maintaining a list of useful resources

We include references and recommended reading in the *Handbook*, and there are a lot more links that have been shared in the *Peeragogy in Action* community. It's a ongoing task to catalog and improve these resources – including books, videos, images, projects, technology, etc. In short, let's "Reduce, Reuse, Recycle"! As a good start, Charlotte Pierce has been maintaining a spreadsheet under the heading "survey" in our Google Drive.

## Developing a really accessible DIY tool-kit

A short "workbook" containing interviews and some activities follows this introduction, but it could be much more interactive. Amanda Lyons and Paola Ricaurte made several new exercises and drawings that we could include. A more developed workbook could be split off from the handbook into a separate publication. It would be great to have something simple for onramping. For

example, the workbook could be accompanied by video tutorials for new contributors.

Paola Ricaurte points out that a really useful book will be easy to sell. For teachers interested in peeragogy, this needs to be something that can be use in workshops or on their own, to write in, to think through issues. We're partway there, but to improve things, we really need a better set of activities.

The next time Paola or someone else uses the handbook or workbook to run a workshop, she can say, "turn to this page, let's answer this question, you have 10 minutes." There are lots of places where the writing in the handbook could be made more interactive. One technique Paola and Amanda used was turning "statements" from the handbook into "questions."

## Crafting a visual identity

Amanda also put together the latest cover art, with some collaboration from Charlotte using inDesign. A more large-scale visual design would be a good goal for the 4th Edition of the book. Fabrizio Terzi, who made the handbook cover art for the 1st Edition, has been working on making our website more friendly. So, again, work is in progress but we could use your help.

## Workflow for the 4th edition

We've uploaded the content of the book to Github and are editing the "live" version of the site in Markdown. For this and previous print editions, we've converted to LaTeX. There are a number of workflow bottlenecks: First, people need to be comfortable updating the content on the site. Second, it would be good to have more people involved with the technical editing work that goes into compiling for print. Remember, when we produce an actual physical handbook, we can sell it. In fact, because all co-authors have transferred their copyright in this book to the Public Domain, *anyone* can print and sell copies, convert the material into new interactive forms, or do just about anything with it.

## Translations

Translating a book that's continually being revised is pretty much a nightmare. With due respect to the valiant volunteer efforts that have been attempted so far, it might be more convenient for everyone involved to just pay professional translators or find a way to foster a multi-lingual authoring community, or find a way to create a more robust process of collective translation. Ideas are welcome, and we're making some small steps here. More on this below.

## Next steps? What's the future of the project?

In short: If we make the Handbook even more useful, then it will be no problem to sell more copies of it. That is one way to make money to cover future expenses. It's a paradigmatic example for other business models we might use in the future. But even more important than a business model is a sense of our shared vision, which is why we're working on a "Peeragogy Creed" (after the Taekwondo creed, which exists in various forms, one example is [2]). No doubt you'll find the first version on peeragogy.org soon! Chapter 7 contains a further list of practical next steps for the project.

## References

1. Joshua Schimel, 2012. "Writing Science", Oxford University Press.

2. Taekwondo Student Creed, World Martial Arts Academy, `http://www.worldtaekwondo.com/handbook.htm`

## ☞ Welcome to the *Peeragogy Workbook!*

This booklet-within-a-book is designed to introduce you to our fun, exciting world of peer learning and peer production‼ You may already be familiar with these terms, or they may be new to you. Either way, don't worry‼

If they are new, consider the following 2 examples.

1. *Peer learning*: Joe Corneli needs to get from the suburbs of Chicago to the north side of the city. He gets on the commuter train and transfers to the purple "L" at Davis Street in Evanston. He plans to change to the red line at Howard Street, but the train says "Loop" and he asks another passenger whether it will stop at Howard. She says it will, but that he can save an hour of his time by riding express to the city and then coming back two stops! Joe makes it to his meeting with Charlie with plenty of time to spare.

2. *Peer production*: Two cavewomen see lightning strike a tree and produce fire! Walking up to it they notice the heat and think "Wouldn't it be nice to have fire for our family at night!" Once the rain clears, they find some dry sticks and start working together to figure out how they can use them to start their own flame. After hours of trial and error, BOOM they've got fire! The news travels fast. :)

Peeragogy is an approach to learning and working together on projects ranging from the mundane to the monumental. Peer learning and peer production are probably as old as humanity itself, but they take on new importance in the digital age.

The Peeragogy Project is an informal learning project with members worldwide. Three members of the project share their welcoming messages below.

**Paola Ricaurte Quijano**: Welcome to the Peeragogy Project! We are a group of enthusiastic people who love to learn and are trying to find the best ways to learn together.

**Lisa Snow MacDonald**: Welcome to peeragogy! It's kind of a weird name, but it's enormously powerful in providing a fresh understanding of ways of working together.

**Dorotea Mar**: Your contributions will be really welcome if you participate respectfully and harmoniously with other peers. It can change your life and improve your well-being and make everything better.

# A Peeragogy Interview

## Introductions

**Paola Ricaurte Quijano**: Hi! I'm Paola, I'm from Ecuador. I work at Tecnológico de Monterrey, a private university in Mexico City, and I love to learn with everybody!

**Dorotea Mar**: Hello. I'm in Berlin now and I really like the peeragogical atmosphere of collaboration and I think we are really improving ways of collaboration and peer production, so that's why I'm here.

**Lisa Snow MacDonald**: Hello. This is Lisa from Los Angeles. My background is media psychology and I'm interested in peeragogy as it relates to business.

### *What is peer learning/production?*

**LSM** I think what these things do is that they allow us to recognize the value of connections. A lot of other ways of working are more individualized. It goes back to a concept of 1 + 1 = 2, which is very rational and very measured and is kind of a dominant way of thinking in our society today, whereas peer to peer learning and production recognize the value of those connections. You may not be able to measure it with a yardstick, but we understand that

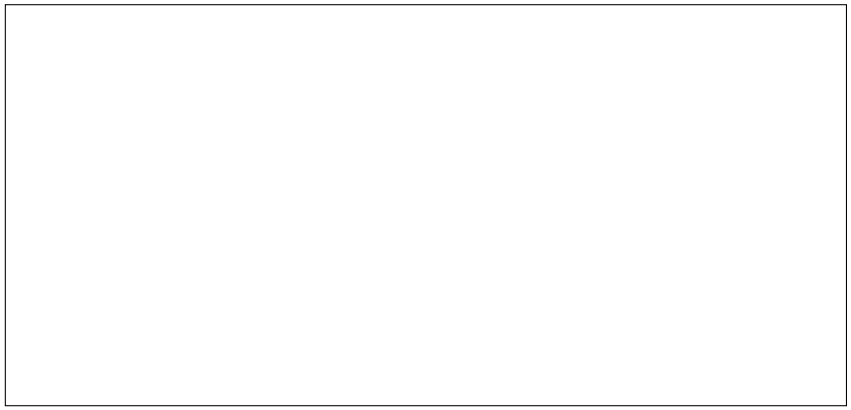
there is value in those connections. So it's basically acknowledging that when it comes to learning/collaborative environments if constructed the right way if working well it can be 1 + 1 = 3 or 1 + 1 = 4. That type of situation, which is really different from the way we're used to thinking about things. And I think that's really the value of what we're doing and the potential of what we could hopefully unlock.

*Do you agree with Lisa? What else do the terms peer learning, peer production, and peeragogy bring to mind for you? You can jot some keywords here:*

## *More specifically, what is the Peeragogy Project?*

**PRQ** This is a project that began spontaneously. We didn't have a plan at the beginning. We just talked about the things that concerned us the most. What do you need if you want to learn with others, how to learn better? what do you want to learn? Where do you want to learn? When do you want to learn? Basic questions that can be answered in many ways. We don't have a strict line. We have a map, maybe, but a map that can be walked through by many different paths. Paths that you choose can be related to the people you are working with. I think it's been a great experience for us. As Lisa said, we have been recognizing the talents and strengths of every person that has contributed to and participated in this project.

*What are some of the things you're concerned about that brought you here?*



**LSM** The map analogy that Paola just mentioned is really good. It's not about providing a direct path. If you're on a trip trying to get from LA to Chicago, there are many paths you can take. What's important is making sure you're monitoring your resources and you're taking care of things along the way. You can drift off-course. One plus one can equal zero if things don't work out well. So, what peeragogy and the Peeragogy Project can do is to provide some structure and framework around the unstructured way that things can be done. People trying to make sure their methods are constructive and beneficial now have some guidelines and things to watch out for.

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## Example: Howard Rheingold Grows a Learning Network

"When I started using social media in the classroom, I looked for and began to learn from more experienced educators. First, I read and then tried to comment usefully on their blog posts and tweets. When I began to understand who knew what in the world of social media in education, I narrowed my focus to the most knowledge-

able and adventurous among them. I paid attention to the people the savviest social media educators paid attention to. I added and subtracted voices from my attention network, listened and followed, then commented and opened conversations. When I found something I thought would interest the friends and strangers I was learning from, I passed along my own learning through my blogs and Twitter stream. I asked questions, asked for help, and eventually started providing answers and assistance to those who seemed to know less than I. The teachers I had been learning from had a name for what I was doing — "growing a personal learning network." So I started looking for and learning from people who talked about HOW to grow a "PLN" as the enthusiasts called them."

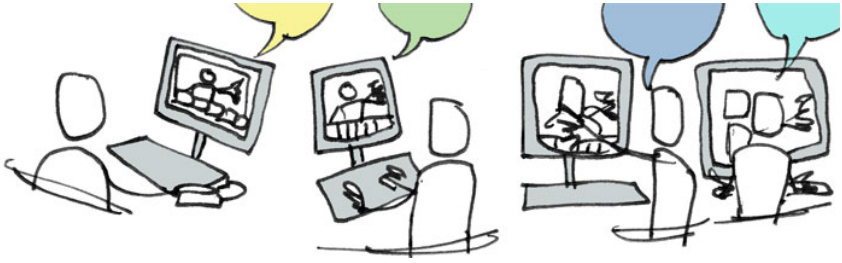
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*How do you do peeragogy?*

**DM** I think I do a lot of peeragogy and I'm very happy about it because I learn so much from my group and from myself in this group that I like to apply it to other projects that I'm in or things like co-working and co-living projects. Especially the principle of mutual respect that still remains after a very long time. And the way we relate to each other is really nice.

The main principle is mutual respect and openness, and the process. And in each detail, there is value that we believe in.

Let's say how we manage the Peeragogy Page or Community (See "How to Get Involved," later in this chapter.). These seem to be details, but they're actually really important. So if we pay attention to all these, every little thing matters, and this is how I do it. I try to be very mindful in all interactions.



*How do you do peeragogy? Can you think of examples that worked well, and others that didn't work as well?*

## Example: Learner, know thyself.



When he joined the Peeragogy project in 2012, Charles Jeffrey Danoff did a brief self-evaluation. What makes Charlie interested in learning?

1. Context. As a student, I resisted being groomed for some unforeseeable future. I'd rather work toward a specific goal.
2. Timing and sequence. I find learning fun when I'm studying something as a way to procrastinate another pressing assignment.
3. Social reinforcement. Getting tips from peers on how to navigate a snowboard around moguls was more fun for me than my Dad showing me the proper way to buff the car's leather seats on chore day.
4. Experiential awareness. In high school, it was not fun to sit and compose a 30-page reading journal on Frankenstein. But owing in part to those types of prior experiences, I now find writing pleasurable and it's fun to learn how to write better.

**PRQ** I think peeragogy is more like a mind-set. I think we have to change the way we interact with others and the way we understand the parameters of learning. For example, I'm a teacher and, of course, my teaching practice promotes collaborative, creative learning. So, I expect my students to take responsibility for their own learning by making decisions about most aspects of the learning process; to program their own learning goals. They need to learn to effectively employ the environments (like whiteboards), the activities, and the assessments. I'm trying to give my learners the tools to decide how, what, and why they want to learn. For me, it's been a very interesting experience. Learners often find it unfamiliar to make their own decisions about the process in a formal environment. At the beginning of the semester, students are given everything and usually just follow guidelines and criteria. I have been trying to change this dynamic. Students feel insecure, because they really do not know how or what they want to do. So, that process of making decisions together becomes very rich and meaningful.



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## Example: Metacognition and Mindfulness

"What (exactly) are you doing? Can you describe it precisely? Why are you doing it? How does it fit into the solution? How does it help you? What will you do with the outcome when you obtain it?"[1]

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[1]Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), Cognitive science and mathematics education (pp. 189-215). Hilldale, NJ: Lawrence Erlbaum Associates.

*When do you do peeragogy?*

**DM** I think I'm always practicing it. I really like that during the weekly hangouts we don't usually have rigid agendas. We just get creative and let ideas connect and flow. And whatever happens it's the right thing. We just work together and somehow the right things happen. I think we're always doing peeragogy when we pursue activities and projects in open, collaborative ways without imposing too much structure or hierarchy. There are many collaborative projects that aim to do something similar to this, but, in a sense, focus on different aspects of the process, and maybe not on such an abstract level as we might. Some people have natural peeragogical tendencies, and some people are less transparent in the way they do things. For me, peeragogy is really beneficial, especially for collaborative projects. Everybody works and learns differently, so if everyone became increasingly aware of how they and others work and learn, of how peergogy functions, and how it all fits into a bigger picture, many tasks would not only be more efficiently done, but also much more enjoyable. It's also beneficial if everyone focusses on a bigger picture instead of focussing only on their part of it, and if attention is drawn to all that could be done in a peeragogical way.

*Do you agree with Dorotea? What challenges might come up in your peeragogical journey?*

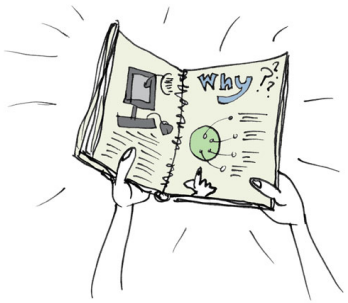
## Example: Jay Cross on Setting Sail

"If I were an instructional designer in a moribund training department, I'd polish up my resume and head over to marketing. Co-learning can differentiate services, increase product usage, strengthen customer relationships, and reduce the cost of handholding. It's cheaper and more useful than advertising. But instead of just making a copy of today's boring educational practices, build something based on interaction and camaraderie, perhaps with some healthy competition thrown in. Again, the emphasis should always be on learning in order to do something!"

### *Why do you do peeragogy?*

**PRQ** Why? Well, as said before, I believe in peeragogy. I believe it's a good way to learn. Maybe it's the best way. I think I wasn't aware of that before joining the group. I have always been a self-learner, I have been working mostly alone. After I began working with the group, I understood that you grow working with a group. You achieve things that you aren't able to achieve alone.

I think there's a growing awareness of the value of collaboration in every setting and environment. There are more and more learning communities around the world where people are also learning that making decisions together and working together are the best way to be in this world! I think as we live through hard times, we increasingly need a sense that we are not alone and that we cannot solve problems alone.



*How did you join the Peeragogy project?*

**PRQ** After taking Howard Rheingold's course on Mind Amplifiers in 2012 we were invited to join this group. There was no plan, just an open question of how to best learn with others. That's how it began. We had lots of sessions and discussed a wide range of issues. The Peeragogy Handbook (http://peeragogy.org) was the product of that process. We've been working with the Handbook, releasing a new version every year and trying to figure out what might be the best way to go forward and what the future of our collaboration as a group/team might be.

**LSM** A couple friends of mine were involved in P2P learning. They were invited to a conference at UCI. Howard was at the event and they were familiar with him and his work. We ended up in an obscure classroom and he started talking about principles that were peeragogy related, while I don't know if it provided much value to my friends, it sounded a lot like what I saw in business and he mentioned the group. So after that, I met everyone here and it's been pretty random.

**DM** I think many paths led to my involvement. I have a lot of academic experience and was doing research on Open Science. I had always wanted to improve the way things work and somehow I wanted to do it more creatively. I resonated a lot with the Peeragogy Project on many levels, so somehow I just joined, I think it was serendipity of some kind.

Paola, Dorotea, and Lisa were interviewed by Charlie on December 15th, 2014. The transcript was significantly edited. You can watch the whole interview online at `http://is.gd/peeragogyworkbook_interviews`. (49 Minutes)

We've given you some examples and questions to think about. Here's one more really useful exercise. Pick at least one thing you're good at and one thing you want to improve on from the selection below (or write in your own alternative answers):

# Exercise: How do you see yourself fitting in?

## Potential roles in your peer-learning project

- Worker, Team Member, Co-Manager, Manager, Co-Leader, Leader
- Reviewer, Editor, Author, Content Creator,
- Presenter, Designer, Graphic Artist, Technologist
- Attendee, Participant, Coordinator, Planner
- Mediator, Moderator, Facilitator, Proponent, Advocate, Representative, Contributor, Activist
- ☜

## Potential contributions

- Create, Originate, Research, Aggregate
- Develop, Design, Integrate, Refine, Convert
- Write, Edit, Format
- ☜

## Potential motivations

- Acquisition of training or support in a topic or field;
- Building relationships with interesting people;
- Finding professional opportunities through other participants;
- Creating or bolstering a personal network;
- More organized and rational thinking through dialog and debate;
- Feedback about performance and understanding of the topic.
- ☜

# How-To Get Involved in the Peeragogy Project

Write to the Peeragogy Project:
c/o Pierce Press PO Box 206 Arlington, MA 02476
Join a Google Hangout after connecting with us on Google+:
`https://plus.google.com/+PeeragogyOrgHandbook`
Or jump into our live chat:
`https://gitter.im/orgs/Peeragogy/rooms`
There are lots of ways for peers to contribute. Here's our current "Top Seven" list:

1. Site: **Google+ Peeragogy Handbook page**

- What happens: Discussion, news, meeting announcements
- Who's in charge: Charlotte Pierce & Charlie Danoff
- `https://plus.google.com/+PeeragogyOrgHandbook`
- Status: Active

2. Site: **Peeragogy mailing list**

- What happens: Meta-level coordination for the project, main point of contact with the email-o- sphere
- Who's in charge: Joe Corneli
- `https://groups.google.com/forum/#!forum/peeragogy`
- Status: Active

3. Site: **Peeragogy.org**

- What happens: Maintain the "master" copy of the peeragogy handbook, share public news about the project.
- Who's in charge: Peeragogy Project
- `http://peeragogy.org`
- Status: Active[2]

[2] See `https://github.com/Peeragogy/Peeragogy.github.io` for the "behind the scenes".

4. Site: **Google Drive Peeragogy Work Folder**

- What happens: Hive editing, working drafts to be delivered elsewhere when they are finished or for final polishing.
- Who's in charge: Peeragogy Project
- `http://is.gd/peeragogydrive`
- Status: Active

5. Site: **Peeragogy In Action Google+ Community**

- What happens: Random posts related to Peeragogy, quick communications between members, news about events, hangouts, etc
- Who's in charge: Everyone
- `http://goo.gl/4dRU92`
- Status: Active

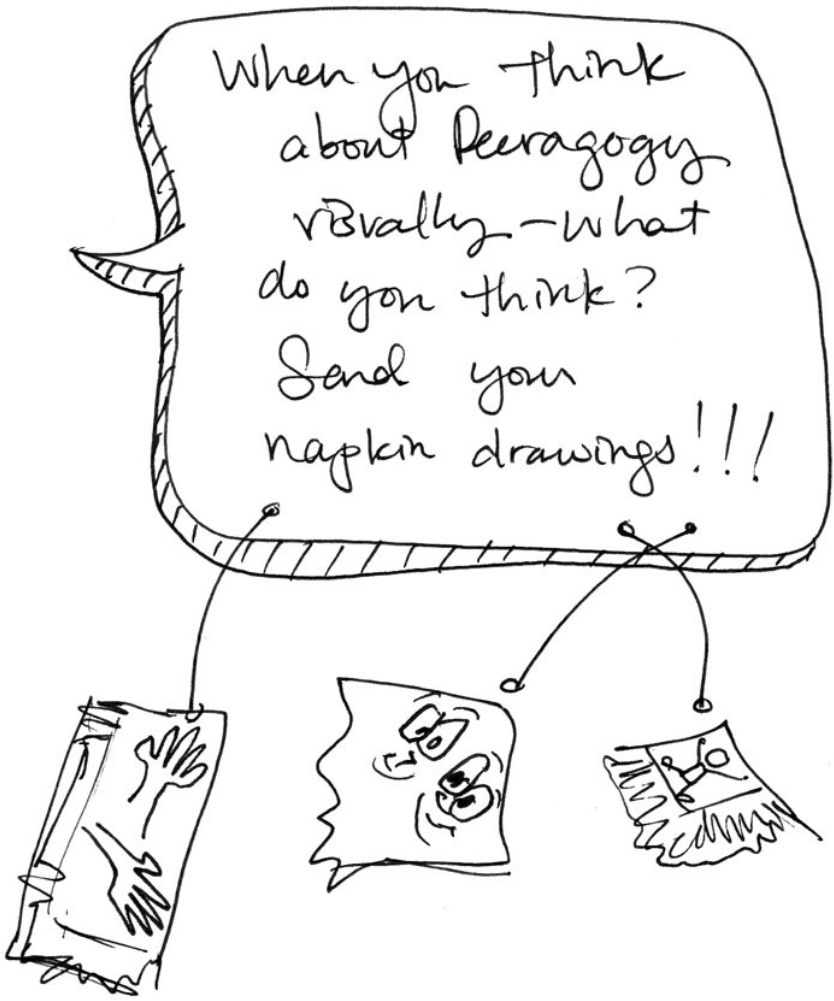
6. Site: **Peeragogy YouTube Channel**

- What happens: videos posted here
- Who's in charge: Charlotte Pierce
- `http://is.gd/peeragogyvideos`
- Status: Active

7. Site: **Git.io/Handbook**

- What happens: versioned storage of the LaTeX sources for the print version of the handbook and other derived formats and scripts
- Who's in charge: Joe Corneli
- `http://git.io/Handbook`
- Status: Active

Please don't worry about rules or trying to catch up! Just jump in! :) "Peeragogy Workbook v1.1" © 2015-2016 by the Peeragogy Project. All rights dedicated to the Public Domain under the Creative Commons Zero license. You can view the license online at `https://creativecommons.org/publicdomain/zero/1.0/`.


When you think about Peeragogy verbally – what do you think? Send your napkin drawings!!!

CHAPTER 2

# CHAPTER SUMMARIES

## Motivation

You might wonder why we're doing this project – what we hope to get out of it as volunteers, and how we think what we're doing can make a positive difference in the world. Have a look at this chapter if you, too, are thinking about getting involved in peeragogy, or wondering how peeragogy can help you accelerate your learning projects.

***Case Study: 5PH1NX.*** This example focuses on the interrelationship of pedagogy and peeragogy in a high school English class, when students are encouraged to find and share creative ways to learn. Explore this case study for ideas and encouragement for your own learning adventures.

## Peeragogy in Practice

Here we describe some of the interaction patterns that we've encountered time and time again in the Peeragogy project. You can use the ideas in this chapter as a starter-kit for your own experiments with peeragogy right away. Sharing – and revising – patterns is one of the key activities in peeragogy, so you will likely want to revisit this chapter several times as you look through the rest of the book. Don't forget your red pen or pencil, because you'll also want to tailor the patterns we describe here to suit.

***Case Study: SWATS.*** We present another example of peer learning in a classroom setting, focusing on the process of improving overall student performance with the help of a group of student experts. After describing the case study in general terms,

we then re-analyze it using our pattern tools to show how examples like this can be integrated into our project.

### Convening a Group

This chapter is about how to begin your own peeragogical project. You can also use the ideas described here to strengthen an existing collaboration. Simple but important questions will inspire unique answers for you and your group. In short: who, what, when, where, why, and how? Use this chapter to help design and critique your project's roadmap.

***Play & Learning.*** What makes learning fun? Just as actors learn their roles through the dynamic process of performance, In other words, the more we engage with a topic, the better we learn it and the more satisfying - or fun - the process becomes.

***K-12 Peeragogy.*** The key to becoming a successful 'connected educator-learner' involves spending the time needed to learn how to learn and share in an open, connected environment. Once you make the decision to enter into a dialogue with another user, you become a connected educator/learner and tap into the power of networks to distribute the load of learning. Depending on their age, you can even facilitate an awareness of peer networks among your students.

***P2P Self-Organizing Learning Environments.*** This section invites an exploration of support for self-organized learning in global and local networks. Emergent structures can create startling ripple effects.

### Organizing a Learning Context

Peer learning is sometimes organized in "courses" and sometimes in "spaces." We present the results of an informal poll that reveals some of the positive and some of the negative features of our own early choices in this project.

***Adding Structure with Activities.*** The first rule of thumb for peer learning is: announce activities only when you plan to take part as a fully engaged participant. Then ask a series of questions: what is the goal, what makes it challenging, what worked in other situations, what recipe is appropriate, what is different about learning about this topic?

***Student Authored Syllabus.*** This chapter describes various methods for co-creating a curriculum. If you're tasked with teaching an existing curriculum, you may want to start with a smaller co-created activity; but watch out, you may find that co-creation is habit forming.[1]

***Case Study: Collaborative Explorations.*** This chapter describes collaborative peer learning among adult students in the Master's program in Critical and Creative Thinking at University of Massachusetts in Boston. The idea in the collaborative explorations is to encourage individuals pursuing their own interests related to a predetermined topic, while supporting learning of everyone in the group through sharing and reflection. These interactions of supportive mutual inquiry evolve the content and structure within a short time frame and with open-ended results.

### Cooperation

Sometimes omitting the figurehead empowers a group. Co-facilitation tends to work in groups of people who gather to share common problems and experiences. The chapter suggests several ways to co-facilitate discussions, wiki workflows, and live online sessions. Conducting an "after action review" can help expose blind spots.

***The Workscape.*** In a corporate workscape, people are free-range learners: protect the learning environment, provide nutrients for growth, and let nature take its course. A workscape fea-

[1] Quick tip: if you create a syllabus, share it!

tures profiles, an activity stream, wikis, virtual meetings, blogs, bookmarks, mobile access and a social network.

***Participation.*** Participation grows from having a community of people who learn together, using a curriculum as a starting point to organize and trigger engagement. Keep in mind that participation may follow the 90/9/1 principle (lurkers/editors/authors) and that people may transition through these roles over time.

***Designs For Co-Working.*** Designing a co-working platform to include significant peer learning aspects often requires a new approach. This chapter describes the initial steps of converting an existing online encyclopedia project into a peer learning platform.

### Assessment

"Usefulness" is an appropriate metric for assessment in peeragogy, where we're concerned with devising our own problems rather than than the problems that have been handed down by society. We use the idea of return on investment (the value of changes in behavior divided by the cost of inducing the change) to assess the Peeragogy project itself, as one example.

***Researching peeragogy.*** This chapter is based on a "found manuscript" created by one of us as an undergraduate. It looks at the challenges that are associated with combining the roles of student, teacher, and researcher. It shows the relevance of peer support, and also illustrates the important factor of time in the evolution of an idea.

### Technologies, Services, and Platforms

Issues of utility, choice, coaching, impact and roles attach to the wide variety of tools and technologies available for peer learning. Keys to selection include the features you need, what people are

already using, and the type of tool (low threshold, wide wall, high ceilings) used for collaboration.

***Forums.*** Forums are web-based communication media that enable groups of people to conduct organized multimedia discussions about multiple topics over a period of time, asynchronously. A rubric for evaluating forum posts highlights the value of drawing connections. This chapter includes tips on selecting forum software.

***Wiki.*** A wiki is a website whose users can add, modify, or delete its content via a web browser. Pages have a feature called "history" which allows users to see previous versions and roll back to them. This chapter includes tips on how to use a wiki and select a wiki engine, with particular attention to peer learning opportunities.

***Real-time meetings.*** Web services enable broadband-connected learners to communicate in real time via audio, video, slides, whiteboards, chat, and screen-sharing. Possible roles for participants in real-time meetings include searchers, contextualizers, summarizers, lexicographers, mappers, and curators. This mode of interaction supports emergent agendas.

***Connectivism in Practice.*** Massive Open Online Courses (MOOCs) are decentralized online learning experiences: individuals and groups create blogs or wikis and comment on each other's work, often with other tools helping find information.

### Resources

Here we present a sample syllabus for bringing peer learning to life, recommended reading and tips on writing for The Handbook, as well as our Creative Commons Zero 1.0 Universal (CC0 1.0) Public Domain Dedication.

# Part II

# Motivation

CHAPTER 3

# WHY WE'RE DOING THIS

> Participants must bring self-knowledge and no small measure of honesty to the peer-learning project in order to accurately enunciate their motivations. If everyone in your peer learning project asks "What brings me here?" "How can I contribute?" and "How can I contribute more effectively?" things will really start percolating. Test this suggestion by asking these questions yourself and taking action on the answers!

Some of the primary motivators reported by participants in the Peeragogy project include:

1. Acquisition of training or support in a topic or field;
2. Building relationships with interesting people;
3. Finding professional opportunities by networking;
4. Creating or bolstering personal connections;
5. More organized and rational thinking through dialog and debate [1];
6. Feedback about their own performance and understanding of the topic.

We've seen that different motivations can affect the vitality of the peeragogical process and the end result for the individual participant. And different participants definitely have different motivations, and the differences can be surprising: for instance, if you're motivated by social image, you may not be so interested in reciprocity, and vice versa [2]. Motivations come with associated risks. For example, one may be reluctant to mention business aspirations in a volunteer context for fear of seeming greedy

or commercial. Whether or not potential peeragogues eventually decide to take on the risk depends on various factors. Actions that typify inappropriate behavior in one culture might represent desirable behavior in another. Motivations often come out of the closet through conflict; for example, when one learner feels offended or embarrassed by the actions of another.

When it comes to primary motivators, it seems some people are more motivated by the *process* and some people are motivated by the *end result*. A lot of the motivations mentioned in the list above are process-oriented. A process orientation is exemplified in the following quote:

> **Philip Spalding**: "The idea of visiting a garden together in a group to learn the names of flowers might have been the original intention for forming a Garden Group. The social aspect of having a day out might be goal of the people participating."

The basic dichotomy between process and product can be a source of tension. Some people are OK with a process that is long and drawn out – because they're mostly there for the process itself anyway. Others will only tolerate with a slight delay as long as the important end result remains in sight. Without a clear understanding and a good balance between these different core motivators, there will be conflict.

People often come to a collaboration with their own motivation in mind (with more or less clarity from case to case). They don't always step back to realise that other people are coming from the point of view of another often very different motivation. It never hurts to ask, especially when conflict rears up. Accordingly, especially for those readers who are interested in the *end results* and *applications* of peeragogy, and not yet steeped in the process, here's what we ask:

*What are the problems you're grappling with? How do you think "peer learning" and "peer production" could help you? Would you be willing to share some of the techniques that you use, and to learn together with us?*

## Example: Peeragogy editor Charlotte Pierce

Basically, I'm here because as an early adopter and admitted gadget freak, I find it fun and rewarding to explore new technologies and topics that I feel have a practical or exciting application. But I have some some other motivations that subtly co-exist alongside my eagerness to explore and learn.

Howard Rheingold's reputation as an innovator and internet pioneer got my attention when he announced his Think-Know Tools course on Facebook in 2012. I had known of Howard from the 1990's when I was a member of The WELL (Whole Earth 'Lectronic Link). I was curious to see what Howard was up to, so I signed onto the wiki site, paid my $300, and took the course starting in October.

Looking back, I realize we were practicing Peeragogy throughout the TKT course, though at the time I hardly knew peer learning from a pickle. In late November, missing the camaraderie and challenge of TKT, I stepped over to check out the *Peeragogy Handbook*.

Which brings me to motivations in signing on to Peeragogy. Since Howard and several Think-Know Tools co-learners were already dedicating their time here and their work looked innovative and exciting, I suspected they might be onto something that I wanted to be a part of. Plus, my brain was primed by the TKT experience. "What if a diverse group of people could learn a subject with little or no cost and not a lot of barriers to entry," I thought. "What if their own experience qualified them to join, contribute, and learn."

I also thought there might be a chance to meet some potential business partners or clients there - but if not, the experience looked rewarding and fun enough for me to take the risk of no direct remuneration. There was no up front cost to me, and a wealth of knowledge to gain as a part of something new and exciting. These are always big draws for me. I wanted to be in on it, and nobody was telling me I couldn't!

My projections proved correct. The participants already on board were gracious in welcoming me to Peeragogy, patient in getting me up to speed, and persistent in coaxing me into using the tools central to the project. I connected, learned, grew, and contributed. Now I'm on the brink of starting a peer learning project of my own in my publishing organization, IPNE.org. Stay tuned!

## Example: Cafes, schools, workshops

Suppose we wanted to make Peeragogy into a model that can be used in schools, libraries, and so forth, worldwide - and, in fact we do! How can we bring the basic Peeragogy motivations to bear, and make a resource, plan of action, and process that other people can connect with? In brief, how do we build peer learning into the curriculum, providing new insight from the safety of the existing structure?

One concrete way to implement these broad aims would be to make a peeragogy-oriented *development* project whose goal is to set up a system of internet cafes, schools, or workshops in places like China or Africa, where people could go to collaborate on work or to learn technical subjects. Students could learn on the job. It seems reasonable to think that investors could make a reasonable profit through "franchises," hardware sales, and so forth – and obviously making money is a motivation that most people can relate to.

In developing such a project, we would want to learn from other similar projects that already exist. For example, in Chicago, State Farm Insurance has created a space called the "Next Door Cafe" that runs community events. One of their offerings is free financial coaching, with the explicit agreement that the issues you discuss return to State Farm as market research.

> **State Farm Insurance**: "Free? Really. Yes, because we're experimenting. We want to learn what people really want. Then, we'll shoot those wants back to the

> Farm. We help you. You help us innovate. We're all smarter for it. We think it's a win-win."

Thus, Next Door Cafe forms part of a system to exploit the side-effects of interpersonal interactions to create a system that learns. A peer learning example from the opposite side of the world started in a slum next to New Delhi where Sugata Mitra gave children a computer and they self organized into a learning community and taught themselves how to use the machine and much more.

> **Sugata Mitra**: "I think what we need to look at is we need to look at learning as the product of educational self-organization. If you allow the educational process to self-organize, then learning emerges. It's not about making learning happen. It's about letting it happen."

In 2014, we tried a similar experiment. We asked: Can we build a "Peeragogy Accelerator" for a half-dozen peer learning projects, each of which defines their own metrics for success, but who come together to offer support and guidance, using the *Peeragogy Handbook* as a resource? We tried that with several our own projects, and benefitted from the peer support. Several months later, we found the Accelerator format even more exciting when we ran a one-off series focusing on Sagarika Bhatta's research on adaptation to climate change in Nepal. Our sense is that peeragogy could be useful for building a global support network around just about any project. Peeragogy can support a culture of real engagement, rather than "clicktivism," and the direct exchange of critically-assessed effort rather than often-inefficient donations of cash [3].

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CHAPTER 4

# CASE STUDY: 5PH1NX

> 5PH1NX: 5tudent Peer Heuristic for 1Nformation Xchange.
>
> (We think of it as a "curiously trans-media" case study in peeragogical assessment.)

Over the last several decades, technology has driven massive shifts in the way we communicate and collaborate. Information technology, socioeconomic trends, an increasingly complex and uncertain future, and the widely perceived failure of our school system to adequately prepare students are contributing factors in an emerging discourse that seeks to align learning with our rapidly changing culture.

Open Source Learning and Peeragogy, two emerging theoretical frameworks in this discourse, leverage end-to-end user principles of communication technology to facilitate peers learning together and teaching each other. In both traditional and liminal learning communities, one of the major points of contact between education and societal culture is the purposeful use of assessment.

The processes of giving, receiving, and applying constructive critique makes learners better thinkers, innovators, motivators, collaborators, coworkers, friends, relatives, spouses, teammates, and neighbors. Implementing peer-based assessment can be problematic in schooling institutions where evaluative authority is traditionally conflated with hierarchical authority, and where economic and political influences have focused attention on summative, quantitative, standardized measurement of learning and intelligence.

This is the story of how one learning community is adopting Open Source Learning and Peeragogical principles to decentralize and enrich the assessment process.

> **Aldous Huxley**: "Knowledge is acquired when we succeed in fitting a new experience into the system of concepts based upon our old experiences. Understanding comes when we liberate ourselves from the old and so make possible a direct, unmediated contact with the new, the mystery, moment by moment, of our existence."

## Enter 5PH1NX

On Monday, April 2, 2011, students in three English classes at a California public high school discovered anomalies in the day's entry on their course blog. (Reminder: not so long ago this sentence would have been rightly interpreted as being science fiction.) The date was wrong and the journal topic was this:

> In The Principles of Psychology (1890), William James wrote, "The faculty of voluntarily bringing back a wandering attention, over and over again, is the very root of judgment, character and will. No one is *compos sui* if he have it not. An education which should improve this faculty would be the education par excellence." How have your experiences in this course helped you focus your attention? What do you still need to work on? What elements of the following text (from Haruki Murakami's *1Q84*) draw your attention and help you construct meaning?
>
> The driver nodded and took the money. "Would you like a receipt?" "No need. And keep the change." "Thanks very much," he said. "Be care**f**ul, it looks windy out there. Don't sl**i**p." "I'll be careful," Aomame said. "A**n**d also," the **d**river said, facing **t**he mirror, "please remember: t**h**ings are not what they seem." Things are not what they seem, Aomame repeated mentally. "What do you mean by that?" she asked with knitt**e**d brows. The driver chose his words

carefully: "It's **j**ust that y**o**u're about to do something out of the ordinary. Am I right? People do not ordinarily climb down the emergency stairs of the Metropolitan Expressway in the middle of the day–especially women." "I suppose you're right." "Right. And after you do something like that, the everyday loo**k** of things might seem to chang**e** a little. Things may look *diff**e**rent* to you than they did before. I've had that experience myself. But don't let appearance**s** fool you. There's always only one reality."

## Find the jokers




The jokers were real and hidden (without much intent to conceal) around the classroom and in students' journals. Students found them and asked questions about the letters in bold; the questions went unanswered. Some thought it was just another of their teacher's wild hair ideas. Although they didn't know it yet they were playing the liminal role that Oedipus originated in mythology. Solving the riddle would enable them to usher out an old way of thinking and introduce the new.

The old way: An authority figure sets the rules, packages the information for a passive audience, and unilaterally evaluates each learner's performance. In that context, peeragogical assessment might be introduced with a theoretical framework, a rubric, and a lesson plan with input, checks for understanding, and guided practice as a foundation for independent work.

The new way: In Open Source Learning the learner pursues a path of inquiry within communities that function as end-to-end user networks. Each individual begins her learning with a question and pursues answers through an interdisciplinary course of study that emphasizes multiple modalities and the five Fs: mental Fitness, physical Fitness, spiritual Fitness, civic Fitness, and technological Fitness. Learners collaborate with mentors and receive feedback from experts, community-based peers, and the public. They are the heroes of learning journeys. Heroes don't respond to syllabi. They respond to calls to adventure. Open Source Learning prepares students for the unforeseen.

By the time they met the 5PH1NX students had learned about habits of mind, operating schema, digital culture and community, self-expression, collaboration, free play, autonomy, confidence/trust/risk, and resilience. These ideas had been reinforced through nonfiction articles and literary selections such as Montaigne's Essays, Plato's Allegory of the Cave, Shakespeare's Hamlet, Sartre's No Exit and others. The first poem assigned in the course was Bukowski's "Laughing Heart". *The Gods will offer you chances. Know them. Take them.*

So it is with knowledge and understanding. Today we are presented with an overwhelming, unprecedented quantity and variety of data in our physical and virtual lives; to cope we must improve the ways we seek, select, curate, analyze, evaluate, and act on information.

On the back of each Joker card was a QR code that linked to a blog page with riddles and clues to a search. At this point students realized they were playing a game. A tab on the blog page labeled "The Law" laid out the rules of engagement:

## This is The Law

1. You cannot "obey" or "break" The Law. You can only make good decisions or bad decisions.
2. Good decisions lead to positive outcomes.
3. Bad decisions lead to suffering.
4. Success requires humanity.
5. "For the strength of the Pack is the Wolf, and the strength of the Wolf is the Pack." -Rudyard Kipling
6. "The Way of the sage is to act but not to compete." -Lao Tzu
7. Be honorable.
8. Have fun.
9. Question.
10. *Sapere aude.*

This is The Law. After a second set of on-campus and blog quests, students noticed a shift in 5PH1NX. A couple of weeks before the first clue was published, during a Socratic seminar on Derrida's concept of Free Play, a student said, "We learn best when adults take away the crutches and there is no safety net."? The quote was used in the next clue; students began to realize that the game was not pre-determined. 5PH1NX was evolving in response to their contributions. This is a manifestation of the hackneyed writing cliché: show, don't tell. The student's comment was a call to action. The Feats of Wisdom were designed to engage learners over a vacation break in fun, collaborative, social media-friendly missions that required engagement in the community, expansion of their personal learning networks, and documentation on their blogs. For example:

**FEAT #1.** *Buy a ticket to "The Hunger Games" (or any other movie that's likely to draw a large, young, rowdy audience). Before the lights dim and the trailers begin, walk to the screen, turn to the audience, and in a loud, clear voice, recite the "To be, or not to be..." soliloquy from Hamlet (don't worry if you make a couple mistakes, just be sure you make it all the way to, "Be all my sins remembered."). Capture the event on video & post it to your blog.*

Students had been using the Internet without an Acceptable Use Policy all year; such policies are one-to-many artifacts of a central authority and far weaker than community norms. So rather than introduce "rules" 5PH1NX simply provided a reminder of the client-side responsibility.

## The Emergence of Peeragogical Assessment

The third page on the Feats of Wisdom blog was entitled *Identifying and Rewarding Greatness*, where learners were greeted with the following paragraph:

> If you see something that was done with love, that pushed the boundaries, set the standard, broke the mold, pushed the envelope, raised the bar, blew the doors off, or rocked in some previously unspecified way, please bring it to the attention of the tribe by posting a link to it [here].

No one did. Instead, they started doing something more effective. They started building. One student hacked the entire game and then created her own version. Other students began to consider the implications for identifying and rewarding greatness. They realized that one teacher couldn't possibly observe how 96 students were working over vacation out in the community and online to accomplish the Feats of Wisdom. In order to get credit for their efforts they would have to curate and share their work-process and product. They also realized that the same logic applied to learning and coursework in general; after all, even the most engaged, conscientious teacher only sees

a high school or college student a few hours a week, under relatively artificial conditions. The learner presumably spends her whole life in the company of her own brain. Who is the more qualified reporting authority? With these thoughts in mind students created *Project Infinity*, a peer-to-peer assessment platform through which students could independently assign value to the thoughts and activities they deemed worthy. Because the 2011-12 5PH1NX was a three-week exercise in gamification, *Project Infinity* quickly evolved to include collaborative working groups and coursework. This was learner-centered Peeragogical assessment in action; learners identified a need and an opportunity, they built a tool for the purpose, they managed it themselves, and they leveraged it in a meaningful way to support student achievement in the core curriculum.

## Project Infinity 2 & Implications for the Future

Alumni from the Class of 2012 felt such a strong positive connection to their experience in Open Source Learning and Peeragogical assessment that they built a version for the Class of 2013. They created *Project Infinity 2* with enhanced functionality. They asked the teacher to embed an associated Twitter feed on the course blog, then came to classes to speak with current students about their experiences. Everyone thought the Class of 2013 would stand on the shoulders of giants and adopt the platform with similar enthusiasm. They were wrong. Students understood the concept and politely contributed suggestions for credit, but it quickly became evident that they weren't enthusiastic. Submissions decreased and finally the *Project Infinity 2* Twitter feed disappeared from the course blog. Learners' blogs and project work suggested that they were mastering the core curriculum and meta concepts, and they appeared generally excited about Open Source Learning overall. So why weren't they more excited about the idea of assessing themselves and each other? Because *Project Infinity 2* wasn't theirs. They didn't get to build it. It was handed to them in the same way that a syllabus is handed to them. No matter how innovative or effective it might be, *Project Infinity 2* was just

another tool designed by someone else to get students to do something they weren't sure they wanted or needed to do in the first place. Timing may also be a factor. Last year's students didn't meet 5PH1NX until the first week in April, well into the spring semester. This year's cohort started everything faster and met 5PH1NX in November. In January they understood the true potential of their situation started to take the reins. As students realized what was happening with the clues and QR codes they approached the teacher and last year's alumni with a request: "Let Us In." They don't just want to design learning materials or creatively demonstrate mastery, they want to chart their own course and build the vehicles for taking the trip. Alumni and students are becoming Virtual TAs who will start the formal peer-to-peer advising and grading process. In the Spring Semester all students will be asked to prepare a statement of goals and intentions, and they will be informed that the traditional teacher will be responsible for no more than 30% of their grade. The rest will come from a community of peers, experts and members of the public. On Tuesday of Finals Week, 5PH1NX went from five players to two hundred.

Sophomores and freshman jumped into the fray and hacked/solved one of the blog clues before seniors did. Members of the Open Source Learning cohort have also identified opportunities to enrich and expand 5PH1NX. A series of conversations about in-person retreats and the alumni community led to students wanting to create a massively multiple player learning cohort.

Imagine 50,000-100,000 learners collaborating and sharing information on a quest to pass an exam by solving a puzzle that leads them to a "Learning Man Festival"? over Summer break. When 5PH1NX players return from Winter Break in January they will transform their roles relative to the game and the course. Several have already shared "AHA!" moments in which they discovered ways to share ideas and encourage collaboration and peer assessment. They have identified Virtual Teaching Assistant candidates, who will be coached by alumni, and they have plans to provide peer-based assessment for their online work. They are also now

actively engaged in taking more control over the collaboration process itself.

On the last day of the semester, a post-finals throwaway day of 30-minute class sessions that administrators put on the calendar to collect Average Daily Attendance money, hardly anyone came to campus. But Open Source Learning students were all there. They had separated the experience of learning from the temporal, spatial, and cultural constraints of school. They understand how democracy works: those who participate make the decisions. No one knows how this ends, but the outcome of Peeragogical assessment is not a score; it is learners who demonstrate their thinking progress and mastery through social production and peer-based critique. This community's approach to learning and assessment has prepared its members for a complex and uncertain future by moving them from a world of probability to a world of possibility. As one student put it in a video entitled "We Are Superman," "What we are doing now may seem small, but we are part of something so much bigger than we think. What does this prove? It proves everything; it proves that it's possible."

## Background

A world in which work looks like what's described in the PSFK think tank's *Future of Work Report 2013* requires a new learning environment.

The problem is that tools and strategies such as MOOCs, videos, virtual environments, and games are only as good as the contexts in which they are used. Even the most adept practitioners quickly discover that pressing emerging technology and culture into the shape of yesterday's curricular and instructional models amounts to little more than Skinner's Box 2.0. So what is to be done? How can we use emerging tools and culture to deliver such an amazing individual and collaborative experience that it shatters expectations and helps students forget they're in school long enough to fall in love with learning again?

Education in the Information Age should enable learners to find, analyze, evaluate, curate, and act on the best available in-

formation. Pursuing an interdisciplinary path of inquiry in an interest-based community doesn't just facilitate the acquisition of factual knowledge (which has a limited half-life). The process brings learners closer to understanding their own habits of mind and gives them practice and an identity in the culture they'll be expected to join after they graduate. This requires new literacies and a curriculum that emphasizes mental fitness, physical fitness, spiritual fitness, civic fitness, and technological fitness.

Models of assessment that emphasize self-directed and collaborative Peeragogical principles enrich the learning experience and accelerate and amplify deep understanding. Because these approaches are pull-based and generate tens of thousands of multi- or trans-media data points per learner, they also generate multi-dimensional portraits of learner development and provide feedback that goes far beyond strengths and weaknesses in content retention. The long-term benefit is exponential. Learners who can intentionally direct their own concentration are empowered far beyond knowledge acquisition or skill mastery. They become more effective thinkers and – because they are invested – more caring people. This learning experience is of their own making: it isn't business, it's personal. The inspiration to recreate the process for themselves and for others is the wellspring of the lifelong learner.

As Benjamin Disraeli put it, "In general the most successful man in life is the man who has the best information." It is a widely accepted truism in business that better data leads to better decisions. We now have the ability to generate, aggregate, analyze, and evaluate much richer data sets that can help us learn more about helping each other learn. Sharing richer data in different ways will have the same game changing effect in learning that it has in professional sports and investment banking.

Self-directed, collaborative assessment generates an unprecedented quantity and variety of data that illuminates aspects of learning, instruction, and overall systemic efficacy. Even a quick look at readily available freeware metrics, blog/social media content, and time stamps can provide valuable insight into an individual's working process and differentiate learners in a network.

In the larger scheme of things, Peeragogical assessment provides direct access to and practice in the culture learners will be expected to join when they complete their course of study. Collaboration, delegation, facilitating conversations, and other highly valued skills are developed in plain view, where progress can be critiqued and validated by peers, experts and the public.

But tall trees don't grow by themselves in the desert. Peeragogical innovation can be challenging in organizational cultures that prioritize control and standardization; as Senge *et al.* have observed, the system doesn't evaluate quality when dealing with the unfamiliar, it just pushes back. In schools this is so typical that it doesn't merit comment in traditional media. The world notices when Syria goes dark, but in school, restricted online access is business as usual.

Cultural constraints can make early adopters in technology-based Peeragogy seem like Promethean risk-takers. Whenever the author gives a talk or an interview, someone asks if he's in trouble.

Learners are not fooled by the rhetoric of in loco parentis or vision statements that emphasize "safe, nurturing learning environments." With notable exceptions, today's school leaders do not know as much about technology as the young people for whom they assume responsibility. Still, learners understand survival: they are fighting in unfavorable terrain against an enemy of great power. Innovating is impossible, and even loudly criticizing school or advocating for change is a risk. As a result many do just enough to satisfy requirements without getting involved enough to attract attention. Some have also internalized the critical voices of authority or the failure of the formal experience as evidence of their own inability: "I'm just not good at math."

How do we know when we're really good at something? Standardized testing feedback doesn't help learners improve. Most of us don't have a natural talent for offering or accepting criticism. And yet, as Wole Soyinka put it, "The greatest threat to freedom is the absence of criticism." Peeragogical interaction requires refining relational and topical critique, as well as skills in other "meta" literacies, including but not limited to critical thinking, collabora-

tion, conflict resolution, decision-making, mindfulness, patience and compassion.

Interpersonal learning skills are undervalued in today's schooling paradigm. Consequently there is an operational lack of incentive for teachers and learners to devote time and energy, particularly when it carries a perceived cost in achievement on tests that determine financial allocations and job security. In recent years there has been increasing pressure to tie teacher compensation, performance evaluation, and job status directly to student performance on standardized tests.

Some educators are introducing peer-to-peer network language and even introducing peer-based assessment. But the contracts, syllabi and letters to students typically stink of *the old way*. These one-to-many documents are presented by agents of the institution endowed with the power to reward or punish. To many students this does not represent a choice or a real opportunity to hack the learning experience. They suspect manipulation, and they wait for the other shoe to drop. Learners also don't like to be told they're free while being forced to operate within tight constraints. Consider this likely reaction to a policy that is highly regarded in the field:

> "Students may choose to reblog their work in a public place or on their own blogs, but do so at their own risk."
>
> *(What? Did I read that correctly?)*
>
> "Students may choose to reblog their work in a public place or on their own blogs, but do so at their own risk."
>
> *(Risk? What risk? The risk of possibly helping someone understand something that they didn't before, or get a different opinion than the one they had before? Someone please help me make sense of this.)*

To effectively adopt Peeragogical assessment in the schooling context, the community must construct a new understanding of

how the members in a network relate to one another independent of their roles in the surrounding social or hierarchical systems. This requires trust, which in school requires significant suspension of disbelief, which – and this is the hard part – requires actual substantive, structural change in the learning transaction. This is the defining characteristic of Open Source Learning: as the network grows, changes composition, and changes purpose, it also changes the direction and content of the learning experience. Every network member can introduce new ideas, ask questions, and contribute resources than refine and redirect the process.

This isn't easy. A member in this network must forget what she knows about school in order to test the boundaries of learning that shape her relationship to content, peers, and expert sources of information and feedback. This is how the cogs in the machine become the liminal heroes who redesign it. Having rejected the old way, they must now create the rituals that will come to define the new. They are following in the path of Oedipus, who took on the inscrutable and intimidating Sphinx, solved the riddle that had killed others who tried, and ushered out the old belief systems to pave the way for the Gods of Olympus. Imagine what would have happened if Oedipus had had the Internet.

# Part III

# Peeragogy in Practice

CHAPTER 5

# THINKING ABOUT PATTERNS

> Although a grounding in learning theory helps inform peer learning projects, Peeragogy, at its core, comes to life in applied practice. Even before convening a group for your peer learning project (discussed in Part IV), you will want to take a look over the patterns we have collected. You will likely return here many times as your project develops.

## What is a pattern?

A pattern is anything that has a repeated effect. In the context of peeragogy, the practice is to repeat processes and interactions that advance the learning mission. Frequent occurrences that are not desirable are called anti-patterns!

> **Christopher Alexander**: "Each pattern describes a problem which occurs over and over again in our environment, and then describes the core of the solution to that problem, in a way that you can use this solution a million times over, without ever doing it the same way twice." [1]

Patterns provide a framework that can be applied to similar issues but may be metaphorically solved in different ways, sometimes in real world or face to face events and other times in digital space. Outside of Alexander's own work in architecture, one the first groups to adopt a design pattern way of thinking about things were computer programmers. Writing in the foreward to Richard P. Gabriel's *Patterns of Software*, Alexander emphasizes that the key question to ask about any design approach is: does it help us build better?

> **Christopher Alexander**: "What is the Chartres of programming? What task is at a high enough level to inspire people writing programs, to reach for the stars?" [2]

We think that Peeragogy stands a good chance of being a "killer app" for pattern-based design. Learning bridges physical and virtual worlds all the time. And, in fact, a *Network of Learning* was the 18th pattern that Christopher Alexander introduced in his book, *A Pattern Language*.

> **Christopher Alexander**: "Work in piecemeal ways to decentralize the process of learning and enrich it through contact with many places and people all over the city: workshops, teachers at home or walking through the city, professionals willing to take on the young as helpers, older children teaching younger children, museums, youth groups travelling, scholarly seminars, industrial workshops, old people, and so on." [1]

Peeragogy can help to extend and enrich this network, and, as we shall see, patterns can be used by those involved to do ongoing "emergent" design, not only by building new structures, but by adapting and improving our catalog of patterns as we go. For consistency, and easy use, adaptation, and extension we present the patterns using the following template. The format is meant to be neutral and easy to work with – it's, intentionally, an outline that you might use to write a short abstract describing an active project.

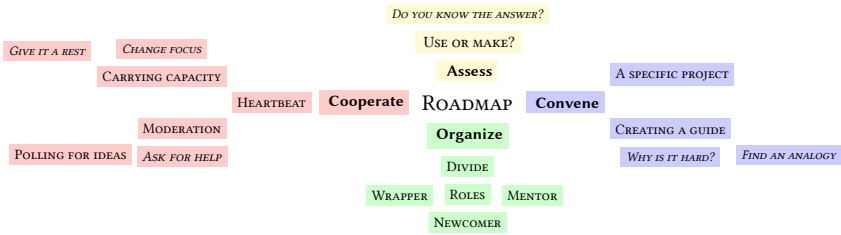
> **Title**: *Encapsulate the idea - possibly include a subtitle*
>
> **Context**: *Describe the context in which it is meaningful. What are the key* ***forces*** *acting in this context?*
>
> **Problem**: *Explain why there's some issue to address here.*

> **Solution**: *Talk about an idea about how to address the issue.*
>
> **Rationale**: *Why do we use this solution as opposed to some other solution?*
>
> **Resolution**: *How are the key forces resolved when the solution is applied?*
>
> **What's Next**: *Talk about specific next steps. How will the active forces continue to resolve in our project?*
>
> Patterns include the following optional elements:
>
> [**Examples**: *Present example(s) that have been encountered, if this aids comprehension.*]
>
> [**References**: *Citations, if relevant.*]

The "What's Next" section concretely links the patterns we discuss here to the Peeragogy project. It can be thought of as an annotation rather than part of the pattern itself. If you adapt the patterns for use in your own project, you're likely to have a different set of next steps. Although we think that these patterns can be generally useful, they aren't useful in the abstract, but rather, as a way for discussing what we actually do.

## A peeragogy pattern language

By looking at how patterns combine in real and hypothetical use cases, you can start to identify a *pattern language* that can be used in your projects. We can get a simplified view of these connections with the following diagram. It's important to clarify that everyone doesn't do it the same way. Here, the *Roadmap* is given a central position, but some peer learning projects will forego making a specific, detailed plan; their plan is just to see what develops. You can see here how peeragogy patterns often break down further into individual micro-steps: we'll say more about that shortly.




The subsequent main sections of this book – *Convening a group*, *Organizing a learning context*, *Cooperatation* and *Assessment* (or *Convene*, *Organize*, *Cooperate*, and *Assess*, for short) – represent big clusters of patterns that are likely to come up time and again in various projects. (The diagram above is a preliminary sketch, including some heuristic ideas from [3], as well as some patterns that we tried out and discarded by the time we wrote [4].) You are encouraged to invent your own patterns and to connect them in new ways.

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CHAPTER 6

# PATTERNS OF PEERAGOGY

This chapter outlines an approach to the organization of learning that draws on the principles of free/libre/open source software (FLOSS), free culture, and peer production. Mako Hill suggests that one recipe for success in peer production is to take a familiar idea – for example, an encyclopedia – and make it easy for people to participate in building it [11, Chapter 1]. We will take hold of "learning in institutions" as a map (Figure 6.1), although it does not fully conform to our chosen tacitly-familiar territory of *peeragogy*. To be clear, peeragogy is for *any group of people who want to learn anything*.

Despite thinking about learning and adaptation that may take place far outside of formal institutions, the historical conception of a university helps give shape to our inqury. The model university is not separate from the life of the state or its citizenry, but aims to "assume leadership in the application of knowledge for the direct improvement of the life of the people in every sphere" [8, p. 88]. Research that *adds to the store of knowledge* is another fundamental obligation of the university [8, p. 550]. The university provides a familiar model for collaborative knowledge work but it is not the only model available. Considering the role of collaboration in building Wikipedia, StackExchange, and free/libre/open source software development, we may be led to ask: What might an accredited free/libre/open university look like? How would it compare or contrast with the typical or stereotypical image of a university from Figure 6.1? Would it have similar structural features, like a Library, Dormitory, Science Hall and so on? Would participants take on familiar roles [5]? How would it compare with historical efforts like the Tuskegee Institute that involved students directly in the production of physical infrastructure [19, 6]? We use the word *peeragogy* to talk about collaboration in relatively non-hierarchical settings. Examples are found in education,



Figure 6.1: A prototypical university. Caption reads: "Wisconsin State University, Madison, Wis. 1879". Inset captions describe the pictured buildings: "Ladies Hall, South Dormitory, University Hall, Assembly Halls & Library, North Dormitory, Science Hall, President's Residence, University Farm, and Washburn Observatory." Public domain.

but also in business, government, volunteer, and NGO settings. Peeragogy involves both problem solving and problem definition. Indeed, in many cases it is preferable to focus on solutions, since people know the "problems" all too well [2]. Participants in a peeragogical endeavor collaboratively build emergent structures that are responsive to their changing context, and that in turn, change that context. In the Peeragogy project, we are developing the the theory and practice of peeragogy.

*Design patterns* offer a methodological framework that we have used to clarify our focus and organize our work. A design pattern expresses a commonly-occurring problem, a solution to that problem, and rationale for choosing this solution [13]. This skeleton is typically fleshed out with a *pattern template* that includes additional supporting material; individual patterns are connected with each other in a *pattern language*. What we

present here is rather different from previous pattern languages that touch on similar topics – like *Liberating Voices* [17], *Pedagogical Patterns* [3], and *Learning Patterns* [12]. At the level of the pattern template, our innovation is simply to add a "What's next" annotation, which anticipates the way the pattern will continue to "resolve".

This addition mirrors the central considerations of our approach, which is all about human interaction, and the challenges, fluidity and unpredictability that come with it. Something that works for one person may not work for another or may not even work for the same person in a slightly different situation. We need to be ready to clarify and adjust what we do as we go. Even so, it is hard to argue with a sensible-sounding formula like "If W applies, do X to get Y." In our view, other pattern languages often achieve this sort of common sense rationality, and then stop. Failure in the prescriptive model only begins when people try to define things more carefully and make context-specific changes – when they actually try to put ideas into practice. The problem lies in the inevitable distance between *do as I say*, *do as I do*, and *do with me* [9, p. 26]. If people are involved, things get messy. They may think that they are on the same page, only to find out that their understandings are wildly different. For example, everyone may agree that the group needs to go "that way." But how far? How fast? It is rare for a project to be able to set or even define all of the parameters accurately and concisely at the beginning. And yet design becomes a "living language" [1, p. xvii] just insofar as it is linked to action. Many things have changed since Alexander suggested that "you will get the most 'power' over the language, and make it your own most effectively, if you write the changes in, at the appropriate places in the book" [1, p. xl]. We see more clearly what it means to inscribe the changing form of design not just in the margins of a book, or even a shared wiki, but in the lifeworld itself. Other recent authors on patterns share similar views [16, 15, 18].

Learning and collaboration are of interest to both organizational studies and computer science, where researchers are increasingly making use of social approaches to software design and

development, as well as agent-based models of computation [14, 7]. The design pattern community in particular is very familiar with practices that we think of as peeragogical, including shepherding, writers workshops, and design patterns themselves [10, 4, 13].

## Pattern template

Table 6.1 shows the pattern template that we use to present our patterns. Along with the traditional design patterns components [13], each of our patterns is fleshed out with two illustrative examples. The first is descriptive, and looks at how the pattern applies in current Wikimedia projects. We selected Wikimedia as a source of examples because the project is familiar, a demonstrated success, and readily accessible. The second example shows how the pattern could be applied in the design of a future university. Each pattern concludes with a boxed annotation: "*What's Next in the Peeragogy Project*".

## A short motivating example

When one relative Newcomer was still in the onboarding process in Peeragogy project, she hit a wall in understanding the "patterns" section in the *Peeragogy Handbook* v1. A more seasoned peer invited her to a series of separate discussions with their own Heartbeat to flesh out the patterns and make them more accessible. At that time the list of patterns was simply a list of paragraphs describing recurrent trends. During those sessions, the impact and meaning of patterns captured her imagination. She went on to become the champion for the pattern language and its application in the Peeragogy project. During a "hive editing" session, she proposed the template we initially used to give structure to the patterns. She helped further revise the pattern language for the *Peeragogy Handbook* v3, and attended PLoP 2015. While a new domain can easily be overwhelming, this newcomer found A specific project to start with, and scaffolded her knowledge and contributions from that foundation.

| |
|---|
| *Motivation* for using this pattern. |

| |
|---|
| *Context* of application. |
| *Forces* that operate within the context of application, each with a mnemonic glyph. |
| *Problem* the pattern addresses. |
| *Solution* to the problem. |
| *Rationale* for this solution. |
| *Resolution* of the forces, named in bold. |

| |
|---|
| *Example 1*: How the pattern manifests in current Wikimedia projects. |
| *Example 2*: How the pattern could inform the design of a future university. |

| |
|---|
| *What's Next in the Peeragogy Project*: How the pattern relates to our collective intention in the Peeragogy project |

Table 6.1: Pattern template.

*Connections between the patterns of peeragogy*

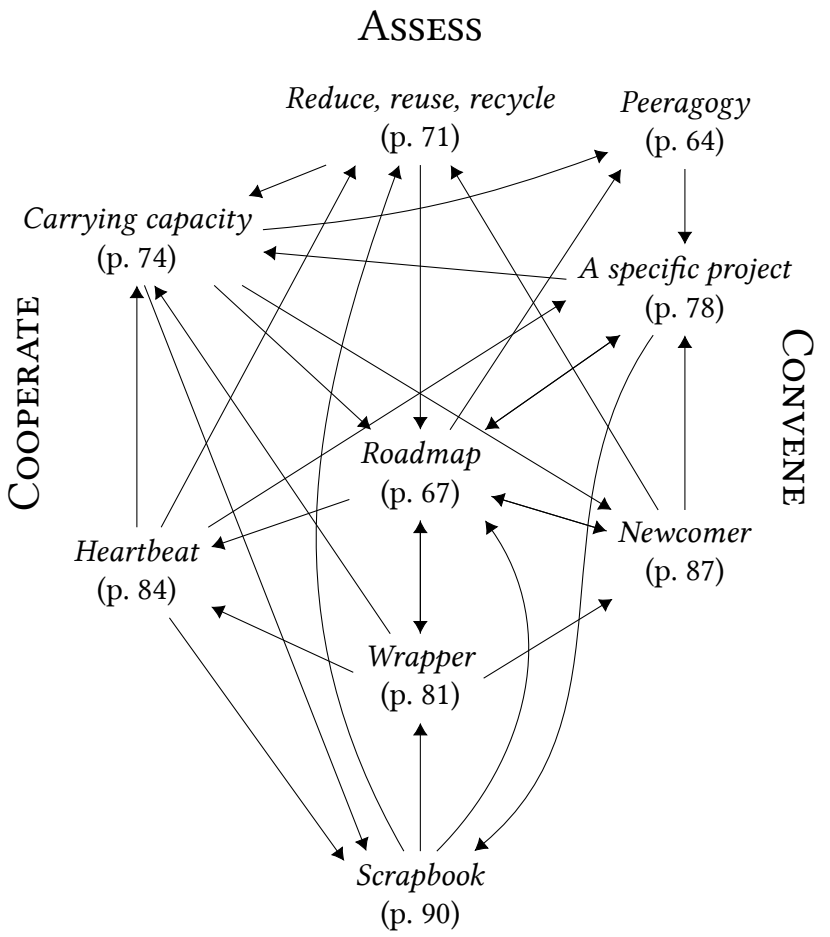



Figure 6.2: Connections between the patterns of peeragogy. An arrow points from pattern **A** to pattern **B** if the text of the description of pattern **A** references pattern **B**. Labels at the borders of the figure correspond to the main sections of the *Peeragogy Handbook*.

| 1. Peeragogy |
| --- |
| **How can we find solutions together?**<br>Get concrete about what the real problems are. |
| 2. Roadmap |
| **How can we get everyone on the same page?**<br>Build a plan that we keep updating as we go along. |
| 3. Reduce, reuse, recycle |
| **How can we avoid undue isolation?**<br>Use what's there and share what we make. |
| 4. Carrying capacity |
| **How can we avoid becoming overwhelmed?**<br>Clearly express when we're frustrated. |
| 5. A specific project |
| **How can we avoid becoming perplexed?**<br>Focus on concrete, doable tasks. |
| 6. Wrapper |
| **How can people stay in touch with the project?**<br>Maintain a summary of activities and any adjustments to the plan. |
| 7. Heartbeat |
| **How can we make the project "real" for participants?**<br>Keep up a regular, sustaining rhythm. |
| 8. Newcomer |
| **How can we make the project accessible to new people?**<br>Let's learn together with newcomers. |
| 9. Scrapbook |
| **How can we maintain focus as time goes by?**<br>Move things that are not of immediate use out of focus. |

Table 6.2: An overview of the problems and solutions in our pattern language.

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## 6.1 Peeragogy

**Motivation** This pattern is relevant to anyone who wants to do active learning together with others in a relatively non-hierarchical setting.

**Context** Collaborative projects like Wikipedia, StackExchange, and FLOSS represent an implicit challenge to the old "industrial" organization of work. This new way of working appears to promise something more resilient, more exciting, and more humane. The rhetoric has been questioned [8, 3]; but it is clear that in the context of these "free", "open", post-modern organizations, individual participants are learning [6] – and that they collectively adapt the methods and infrastructure as they go. Because everyone in these projects primarily learns by putting in effort on a shared work-in-progress, participants are more in touch with an *equality of intelligence* than an *inequality of knowledge* [4, pp. 38,119]. At the same time, they invoke a form of friendly competition, in which *the best craftmanship wins* [5, p. 89].

**Forces** **Threshold**: inclusiveness and specificity are in tension.

**Trust**: is only built through sharing and reciprocity.

**Problem** Even a highly successful project like Wikipedia is a work in progress that can be improved to better *empower and engage people around the world, to develop* richer and more useful *educational content, and to disseminate it* more *effectively* – and deploy it more creatively. How to go about this is a difficult question, and we don't know the answers in advance. There are rigorous challenges facing smaller projects as well, and fewer resources to draw on. Many successful free software projects are not particularly collaborative – and the largest projects are edited only by a small minority of users [2, 9]. Can we work smarter together?

**Solution** The act of asking "can we work smarter together?" puts learning front and center. Peeragogy takes that "center" and distributes it across a pool of heterogeneous relationships. Indeed, peeragogy can be understood as an up-to-date revision of Alexander's NETWORK OF LEARNING [1, p. 99]. It *decentralizes the process of learning and enriches it through contact with many places and people* in interconnected networks that may reach all over the world. Importantly, while people involved in a peeragogical process may be collaborating on A SPECIFIC PROJECT, they don't have to be direct collaborators outside of the learning context or co-located in time or space. Just as theories and practices of pedagogy articulate the transmission of knowledge from teachers to students, peeragogy articulates the way peers produce and use knowledge together (Figure 6.2).

**Rationale** The peeragogical approach particularly addresses the problems of small projects stuck in their individual silos, and large projects becoming overwhelmed by their own complexity. It does this by going the opposite route: explicating *what by definition is tacit* and employing *a continuous design process* [7, pp. 9–10]. Active learning together with others brings social and emotional intelligence to bear on the things that matter most.

**Resolution** Peeragogy helps people in different projects describe and solve real problems. If you share the problems that you're experiencing with others, there's a reasonable chance that someone may be able to help you solve them. Bringing a problem across the **threshold** of someone else's awareness helps achieve clarity. This process can guide individual action in ways that we wouldn't have seen on our own, and may lead to new forms of collective action we would never have imagined possible. People who gain experience comprehending problems together build **trust**. Making room for multiple right answers contributes further to resolving the tension between generality and specificity.

**Example 1** Wikipedia and its sister sites rely on user generated content, peer produced software, and are managed, by and large, by a pool of users who choose to get involved with governance and other "meta" duties. Wikimedia's pluralistic approach achieves something quite impressive: the Wikimedia Foundation runs the 7th most popular website in the world, and has under 300 employees. For comparison, the 6th (Amazon) and 8th (QQ) most popular websites are run by companies with over 200K and 28K employees, respectively.

**Example 2** Although one of the strengths of PEERAGOGY is to distribute the workload, this does not mean that infrastructure is irrelevant. No less than their predecessors, the students and researchers of the future university will need access to an Observatory and other scientific apparatus if they are to reach *ad astra, per aspera.*



Figure 6.3: Space Surveillance Telescope, White Sands New Mexico.

*What's Next in the Peeragogy Project* We intend to revise and extend the *Patterns of Peeragogy* into a framework that can describe and scaffold the learning that happens inside and outside of institutions.

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## 6.2 Roadmap

**Motivation** This pattern shows how the group can define the scope of their project and make a realistic plan to address it. This pattern provides the backbone of our pattern language. It can be used to find a shared goal.

**Context** PEERAGOGY has both distributed and centralized aspects. The discussants or contributors who collaborate on a project have different points of view and heterogeneous priorities, but they come together in conversations and joint activities.

**Forces**

**Variety**: people have different goals and interests in mind.



**Clarity**: some may be quite specific, and some rather vague.



**Coherence**: only some of these goals will be well-aligned.



**Problem** In order to collaborate, people need a way to share current, though incomplete, understanding of the space they are working in, and to nurture relationships with one another and the other elements of this space. At the outset, there may not even be a coherent vision for a project – but a only loose collection of motivations and sentiments. Once the project is up and running, people are likely to pull in different directions.

**Solution** Building a guide to the goals, activities, experiments and working methods can help NEWCOMERS and old-timers alike understand how the nature of their relationship with the project. It may combine features of a manifesto, a syllabus, and an issue tracker. It may be a design pattern or a pattern language [3]. The distinguishing qualities of a project ROADMAP are that it should be adaptive to circumstances, and that it should ultimately get us from *here* to *there*. By this same token, any given version of the roadmap is seen as fallible. In lieu of widespread participation, the project's WRAPPER should attempt to synthesize an accurate roadmap that is informed by participants' behavior, and should help moderate in case of conflict. Nevertheless, full consensus is not necessary: different goals, with different *heres* and *theres*, can be pursued separately, while maintaining communication.

**Rationale** In the Peeragogy project our initial roadmap was an outline of the first draft of the *Peeragogy Handbook*. Later, it took the form of a schedule of meetings following a regular "Heartbeat" supplemented by a list of upcoming deadlines. Most recently, it is expressed in the emergent objectives listed in Section 7 of the current paper. We have seen that a list of nice-to-have features created in a top-down fashion is comparatively unlikely to *go* anywhere. A backlog of tasks and a realistic plan for accomplishing them are vastly different things. An adaptive roadmap is an antidote to Tunnel Vision [1, pp. 121–124].

**Resolution** An emergent roadmap is rooted in real problems and justifiable solutions-in-progress in all their **variety** and communicates both resolution and follow-through. The process of meshing varied issues with one another requires thought and discussion, and this encourages **clarity**. The test of **coherence** is that contributed goals and ideas should be actionable. The ultimate quality-control test is if it worked, i.e., did it come to pass that the task(s) the roadmap was created to achieve ended up being achieved, or not? If all of the issues that the roadmap outlines are not resolved, the roadmap itself should be revised. Without a roadmap, we would never know.

**Example 1** The *Help* link present on every Wikipedia page could be seen as a localized Roadmap for individual user engagement: it tells users what they can do withthe site, and gives instructions on how to do it. For someone who is prepared to jump in and get to work, there are around 30 pages listing articles with various kinds of problems, for example articles tagged with style issues, or "orphaned" articles (i.e., articles with no links from other pages in the encyclopedia). Wikimedia previously developed a detailed strategic plan drawing on community input [2]. In 2015, a two-week Community Consultation was carried out and synthesized, resulting in "a direction that will guide the decisions for the organization." Community-organized WikiProjects often invite outside involvement on A specific project.

**Example 2** In a future university run in a peer produced manner, a fancy President's Residence presumably wouldn't be needed. Leadership would be carried out in a more collaborative and distributed fashion. However, depending on just how distributed things are, it may turn out to be useful for project facilitators to gather at a University Hall. Whereas there is strength in numbers, there is leverage in organization. This is what the ROADMAP provides.[1]



Figure 6.4: President's Home, University of Alabama.

*What's Next in the Peeragogy Project* If it becomes clear that something needs to change about the project, that is a clue that we might need to revise our patterns or record a new one. We can use the names of the patterns to tag our upcoming tasks.

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## 6.3 Reduce, reuse, recycle

**Motivation** This pattern can guide project participants in identifying and managing available resources.

**Context** In a peer production context, you are simultaneously "making stuff" and building on the work of others.

**Forces** **Derivative**: you don't have to do everything yourself!

**Sensemaking**: resources are useful only when you can make sense of them.

**Sharing**: your understanding gains robustness when you share with others.

**Problem** Many projects die because the cost of Reinventing the Wheel [c2] is too high. However, this is just one symptom of overfocus on a few priorities. Concerns may also arise if the project's output is not actually used by anyone.




Figure 6.5: A paradigmatic example of found-art. "Fountain by R. Mutt, Photograph by Alfred Stieglitz, The exhibit refused by the independents".

**Solution** "Steal like an artist," and make it possible for other people to build on your work too (Figure 6.5). In the Peeragogy project, we have used off-the-shelf and hosted software suited to the task at hand (incuding: Drupal, Google+, Google Hangouts, Google Docs, Wordpress, pandoc, Github, ShareLaTeX). Early on we agreed to release our *Peeragogy Handbook* under the terms of the Creative Commons Public Domain Dedication (CC0), the legal instrument that grants the greatest possible leeway

to downstream users. This has allowed us and others to repurpose and improve its contents in other settings, including our published papers. Follow the steps indicated by the keywords in the pattern's title:

- *Reduce* the panoply of interesting interrelated ideas and methods to a functional core (e.g. writing a book).
- *Reuse* whatever resources are relevant to this aim, factoring in "things I was going to have to do anyway" from everyone involved.
- *Recycle* what you've created in new connections and relationships.

**Rationale** Clearly we are not the first people to notice the problems with wheel-reinvention, including "missing opportunities, repeating common mistakes, and working harder than we need to." As a guest in one of our hangouts, Willow Brugh, of Geeks without Bounds and the MIT Media Lab, remarked that *people often think that they need to build a community, and so fail to recognize that they are already part of a community.* We converted our old pattern catalog from the *Peeragogy Handbook* into this paper, sharing it with a new community and gaining new perspectives; could we do something similar again?

**Resolution** Reweaving old material into **derivative** designs and new material into existing frameworks, we build deeper understanding, and carry out collective **sensemaking**. The project's ROADMAP develops by making sense of existing resources – including our worries and concerns. Often we only know what these are when we attempt to **share** them. Drawing on a wide range of resources boosts our collective CARRYING CAPACITY.

**Example 1** Contributors are encouraged to recycle existing works that are compatible with the Wikimedia-wide CC-By-SA license. Some subprojects been created purely to help repurpose

other existing works in this way. On the downstream side, DBPedia is an important resource for the semantic web, built by collating data from Wikipedia's "infoboxes". These infoboxes are themselves increasingly being populated automatically using information from WikiData. Researchers have been able to develop tools that reuse Wikipedia content in other ways [1, 2], However, these research projects do not always result in a tool that is accessible to day-to-day users.

**Example 2** The knowledge resources and collaboration tools currently available online are what make a low-cost, high-quality, formally-accredited future university conceivable. However, the available resources are not always as organized as they would need to be for educative purposes, so peeragogues can usefully put effort into Reduce, reuse, recycle'ing available resources into a functioning university Library.

*What's Next in the Peeragogy Project* Are there other educational resources and peeragogical case studies that we could fold into our work? Can we recycle material from the *Peeragogy Handbook* into a format that is easier to understand and apply?

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## 6.4 Carrying Capacity

**Motivation** This pattern can help project participants recognise and communicate their stresses to make themselves and the project more resilient.

**Context** One of the important maxims from the world of FLOSS is: "Given enough eyeballs, all bugs are shallow" [8, p. 30]. A partial converse is also true: there's only so much any one person can do, since we all have limited time and energy.

**Forces** **Antifragility**: each person's potential can only be realised if people take on enough, but not too much.



**Independence**: in a peeragogy context, it is often impossible to delegate work to others.



**Problem** How can we help prevent those people who are involved with the project from overpromising or overcommitting, and subsequently crashing and burning? First, let's be clear that are lots of ways things can go wrong. Simplistic expectations – like *assuming that others will do the work for you* [13] – can undermine your ability to correctly gauge your own strengths, weaknesses, and commitments. Without careful, critical engagement, you might not even notice when there's a problem. Where one person has trouble letting go, others may have trouble speaking up. Pressure builds when communication isn't going well.

**Solution** Symptoms of burnout are a sign that it's time to revisit the group's ROADMAP and your own individual plan. Are these realistic? If you have a "buddy" they can provide a reality check. Maybe things are not *that hard* after all – and maybe they don't need to be done *right now*. Generalizing from this: the project can promote an open dialog by creating opportunities for people to share their worries and generate an emergent plan for addressing them [10]. Use the project SCRAPBOOK to make note of

obstacles. For example, if you'd like to pass a baton, you'll need someone there who can take it. Maybe you can't find that person right away, but you can bring up the concern and get it onto the project's Roadmap. The situation is always changing, but if we continue to create suitable checkpoints and benchmarks, then we can take steps to take care of an issue that's getting bogged down.

**Rationale** Think of the project as an ecosystem populated by acts of participation. As we get to know more about ourselves and each other, we know what sorts of things we can expect, and we are able to work together more sustainably [6]. We can regulate our individual stress levels and improve collective outcomes by discussing concerns openly.

**Resolution** Guiding and rebalancing behaviour in a social context can begin with speaking up about a concern. When we acknowledge concerns, we must take into account our own boundedness. We have find an opportunity to make ourselves helpful, without impinging on others' **independence**. This doesn't mean allowing all possible stresses to run rampant: we work to stay within the realm of **antifragility** [12], where stress improves the system, rather than degrading them. As we share concerns and are met with care and practical support, our actions begin to align better with expectations (often as a result of forming more realistic expectations).

**Example 1** Wikipedia aims to emphasize a neutral point of view, but its users are not neutral. Users "speak up" about topics that matter to them. Coverage and participation are not neutral in another less sanguine sense. More information on Wikipedia deals with Europe than all of the locations outside of Europe [2]. As we remarked in the Peeragogy pattern, most of the actual work is contibuted by a small percentage of users. The technology limits the kinds of things can be said [2]. The total number of active editors has been falling since 2007. Some blame outmoded technology and an insider culture [11], or point to a stringent ed-



Figure 6.6: Queens College, North Carolina.

itorial approach that emerged in response to the site's popularity [3]. Others highlight the rise of successful competition, often inspired by wiki models, but driven by "corporate logic" [4, 5].

**Example 2** Progressive thinkers have for some time subscribed to the view that "there shall be no women in case there be not men, nor men in case there be not women" [7, Chapter 1.LII]. A separate Ladies Hall seems entirely archaic. However, in light of the extreme gender imbalance in free software, and still striking imbalance at Wikipedia [1, 9], it will be important to do whatever it takes to make women and girls welcome, not least because this is a significant factor in boosting our Carrying capacity.

*What's Next in the Peeragogy Project* Making it easy and fruitful for others to get involved is one of the best ways to redistribute the load. This often requires knowledge transfer and skill development among those involved; see Newcomer.

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## 6.5 A specific project

**Motivation** This pattern can help project participants get started, get focused, and make concrete change. It is especially useful for someone who is currently feeling stuck.

**Context** We often find ourselves confronted with what seems to be a difficult, complex, or even insurmountable problem. It won't go away, but a workable solution doesn't present itself, either. If there is a candidate solution, it's also clear there are not enough resources for it to be feasible.

**Forces** **Difficulty**: bringing about meaningful change is often hard work.



**Inertia**: when things are hard we may feel stuck, wring our hands, or preach to the choir.



**Problem** We are often blinded by our own prejudices and preferences. Considerable energy goes into pondering, discussing, exploring and feeling stuck. Meanwhile there may be a strong urge to make more concrete progress, and time is passing by. In a group setting, when the forward-movers ultimately try to act, those who are more wrapped up in the experience of pondering and exploring may rebel, if they feel that they are being left behind. Inaction may seem like the only safe choice, but it has risks too. Once things are moving, it's easy to get stuck again.

**Solution** One way to make progress when you're stuck is to ask a specific question to someone who may be able to help you get unstuck. Formulating a question helps your thinking become more concrete. Sometimes you'll see that a solution was within your grasp all along. Often, one question won't be enough, but you can repeat the process. In this way, you can reduce a large, complex, or ephemeral concern into a collection of smaller, specific, manageable tasks with clear next steps and success criteria. Use a SCRAPBOOK to make note of all the small things, and weave them into your project ROADMAP. This will show how the small pieces relate to the bigger picture. If you have a fairly specific idea about what you want to do, but you're finding it difficult to get it done, don't just ask for advice: recruit material help (cf. CARRYING CAPACITY). One example of a specific project from the Peeragogy project is our work on this paper, which had a specific target audience, a set of associated deadlines, and allowed us to get help from pattern experts.

**Rationale** We've seen time and again that having a specific project is a recipe for getting concrete, and that getting concrete is necessary for bringing about change. Asking for help (which is what happens when you vocalize a question) is one of the best ways to gain coherence. Making yourself understood can go a long way toward resolving deeper difficulties.

**Resolution** Where you run into **difficulty**, getting specific paves the way for incremental forward progress and helps to overcome **inertia**. The struggle between consensus and action is resolved in a tangible project that combines action with dialog. Learning something new is a strong sign that things are working. In the Peeragogy project, we have completed many projects during our weekly hangouts, for example "hive editing" an abstract for submission to a conference.

**Example 1** At first glance, the Wikimedia Foundation's mission presents an overwhelming concept. Where do we get started on a project with a global scale? How do we organise work? In practice, many future Wikipedians jump in, get to know other Wikipedia users, and start working on a focused todo list in A SPECIFIC PROJECT. Within Wikipedia, these are known as "WikiProjects." Guidance is available on how to start new WikiProjects. The Wikimedia Foundation also runs other public projects, including the Wikipedia Education Program and the GLAM Wiki (for Galleries, Libraries, Archives, and Museums). The latter maintains a *list of case studies that describes specific projects undertaken by cultural organizations and Wikimedia.*

**Example 2** A shared Dormitory is typically unnecessary if studying from where you currently live is an option. That said, a short-term rental or a time-shared cooperatively-owned living/working environment could be an asset for peeragogues who need to get together to work on A SPECIFIC PROJECT. Collegial (and convivial) peer support is important even without long-term co-habitation.



Figure 6.7: Dorm room, Ruin Academy, Taipei, Taiwan.

*What's Next in the Peeragogy Project* Let's use our pattern catalog to build specific, tangible "what's next" steps, add them to our ROADMAP, and carry them out with concrete actions. Let's be sure we know who's responsible for what, and employ a "buddy system" to help get things done.

## 6.6 Wrapper

**Motivation** This pattern suggests to find at least one person to fill an important role managing the project's public interface, and keeping participants up to date about activities.

**Context** You are part of an active, long-running, and possibly quite complex project with more than a handful of participants. How do you manage?

**Forces**

- **Interface**: the project shows people how they can use it.
- **Familiar**: the leader/follower dichotomy is easy to understand.
- **Equity**: peeragogy aims for fairness.

**Problem** In an active project, it can be effectively impossible to stay up to date with all of the details. Not everyone will be able to attend every meeting (see HEARTBEAT) or read every email. Project participants can easily get lost and drift away. The experience can be much more difficult for NEWCOMERS: joining an existing project can feel like trying to climb aboard a rapidly moving vehicle. If you've taken time off, you may feel like things have moved on so far that you cannot catch up. Information overload is not the only concern: there is also a problem with missing information. If they aren't shared, key skills can quickly become bottlenecks (see CARRYING CAPACITY).

**Solution** Someone involved with the project should regularly create a wrap-up summary, distinct from other project communications, that makes current activities comprehensible to people who may not have been following all of the details. In addition, project members should keep other informative resources like the landing page, ROADMAP, and documentation up to date. Ensure that these resources accurately represent the facts on the ground, and check empirically to see if they really show interested parties how they can get involved. The WRAPPER is both a role, and, sometimes, an artifact. Our *Handbook*'s cover literally wraps up its contents; the collaboratively written chat notes from our weekly Hangouts give a collaboratively-written overview of what was discussed in the meeting. Meetings themselves can be structured to give people a chance to sum up what they've accomplished during the week, as well as any problems they are running into. Between meetings, each participant is advised to maintain some sort of "learning log" in the form of a personal SCRAPBOOK, so that outstanding concerns are surfaced and available to discuss.

**Rationale** According to the theory proposed by Yochai Benkler, for free/open "commons-based" projects to work, it is important for participants to be able to contribute small pieces, and for the project to have a way to stitch those pieces together [1]. The WRAPPER helps perform this integrative stitching function. If you value participation, you may have to do some serious work to facilitate access to process.

**Resolution** Well-maintained records chronicle the project's history; up-to-date documentation makes the project more robust; a coherent look-and-feel offers an accessible **interface** to the outside world. Regularly circulated summaries can help to engage or re-engage members of a project, and can give an emotional boost to peeragogues who see their contributions and concerns mentioned, giving less engaged participants the **familiar** experience of "following" someone else's updates. People will




Figure 6.8: Design for a Peeragogy project dashboard.

judge from experience whether the project strives for **equity** or strives to maintain hidden power differentials.

**Example 1** There are many data streams around the Wikimedia project. They comprise an elaborate WRAPPER function for the project, with components that range from Today's Featured Article, which appears on the front page of Wikipedia, to formal annual reports from the nonprofit.

**Example 2** In-person meetings are just as relevant for contemporary humans as they were a century ago, even though we often work remotely, and have learned more about how to assemble on the fly [2]. Getting together for conventions, dance parties, and commencement ceremonies could comprise an important part of the future university's WRAPPER function, even if these events do not always take place in one specific Assembly Hall.

*What's Next in the Peeragogy Project* Let's make sure we have protocols in place that enable us to share progress, and to revise our "next steps" if people are getting stuck. Let's improve the interaction design for peeragogy.org so that it's clear how people can get involved.

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## 6.7 Heartbeat

**Motivation** This pattern can help project participants stay in touch, and stay motivated.

**Context** A number of people have a shared interest, and have connected with each other about it. However, they are not going to spend 24 hours a day, 7 days a week working together, either because they are busy with other things, or because working separately on some tasks is vastly more efficient.

**Forces** **Differentiation**: the time we spend together isn't all equally meaningful.



**Entropy**: something needs to hold the project together, or it will fall apart.



**Problem** How will the effort be sustained and coordinated sufficiently? How do we know this an active collaboration, and not just a bunch of people milling about? Is there a *there, there?*

**Solution** People seem to naturally gravitate to something with a pulse. *Once a day* (standups), *once a week* (meetings), or *once a year* (conferences, festivals) are common variants. When the project is populated by more than just a few people, it's likely that there will be several HEARTBEATS, building a sophisticated polyrhythm. A well-running project will feel "like an improvisational jazz ensemble" [1]. Much as the band director may gesture

to specific players to invite them to solo or sync up, a project facilitator may craft individual emails to ask someone to lead an activity or invite them to re-engage. Two common rhythm components are weekly synchronous meetings with an open agenda, combined with *ad hoc* meetings for focused work on A specific project. The precise details will depend on the degree of integration required by the group.

**Rationale** The project's heartbeat is what sustains it. Just as *people matter more than code* [2], so does the life of the working group matter more than mechanics of the work structure. Indeed, there is an quick way to do a reality check and find the project's strongest pulse: the activities that sustain a healthy project should sustain us, too (cf. Carrying capacity).

**Resolution** Noticing when a new Heartbeat is beginning to emerge is a way to be aware of the shifting priorities in the group, and contributes to further **differentiation**. This may ultimately be a good source of new patterns. On the other hand, if a specific activity is no longer sustaining the project, stop doing it, much as you would move an out-of-date pattern to the Scrapbook in order to make room for other concerns. The power of the Heartbeat is that the project can be as focused and intensive as it needs to be, working against **entropy** in the ways that start to be required as time goes by.

**Example 1** The yearly in-person gathering, Wikimania, is the most visible example of a Heartbeat for the Wikimedia movement. Local chapters and projects may run additional in-person get-togethers. Also of note is the twice-yearly call for proposals for individual engagement grants. There are other shorter cycles. Each day a highly-vetted Featured Article appears on the front page of Wikipedia, and is circulated to a special-purpose mailing list. The discussion of articles for deletion lasts at least seven days.



Figure 6.9: al-Biruni University farm, Kapisa province, Afghanistan.

**Example 2** Although it may sound quaint, working farms could help to physically sustain peeragogues, while putting the project's HEARTBEAT in tune with that of the seasons. In the current distributed mode, we tend our windowboxes and allotment gardens. New developments should unfold in a *logical order growing out of the needs of the community* [3, Chapter IX].

> *What's Next in the Peeragogy Project* Identifying and fostering new HEARTBEATS and new working groups can help make the community more robust. This is the time dimension of spin-off projects described in REDUCE, REUSE, RECYCLE.

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## 6.8 Newcomer

**Motivation** This pattern can help project participants be aware of the issues faced by newcomers, and cultivate a "beginner's mind" themselves.

**Context** When there's learning happening, it's because there is someone who is new to a topic, or to something about the topic.

**Forces** **Individuation**: each person learning optimally is what's best for the community.

**Mutuality**: our individuality does not isolate us from one another, but draws us together.

**Problem** Newcomers can feel overwhelmed by the amount of things to learn. They often don't know where to start. They may have a bunch of ideas that the oldtimers have never considered – or they may think they have new ideas, which are actually a different take on an old idea; see REDUCE, REUSE, RECYCLE. People who are new to the project can tell you what makes their participation difficult. Since you're learning as you go as well, you can ask yourself the same question: what aspects of this encounter are difficult for me?

**Solution** Instead of thinking of newcomers as "them", and trying to provide solutions, we focus on newcomers as "us" – which makes the search for solutions that much more urgent. We permit ourselves to ask naive questions. We entertain vague ideas. We add concreteness by trying A SPECIFIC PROJECT. We may then genuinely turn to others for help. We aim to foster a culture in

which the focus for everyone is on addressing our own learning challenges rather than on "providing" solutions for others [1]. When you begin a new project, try to systematically take notes and gather data to analyze and reflect upon later; leave artifacts for other future newcomers to use and build upon in their own research. In practice this may be a lot to ask for someone just joining a group, but over time we may have many ways to structure our collective engagement so that it leads to research cycles based on the "action research" steps *reflect*, *plan*, *act*, and *observe*. Note that there is a parallel with the four facets *assess*, *convene*, *organize*, *cooperate* from Figure 6.2. The history of the action research approach, with particular emphasis on educational applications, is surveyed in [5, Chapter 3]. One method for doing the reflection/assessment step is presented in the SCRAPBOOK pattern. Be flexible: networked attention (even more so than rigid cycles [3]) leads to new ways of knowing and expanded access to knowledge-production [7, 8].

**Rationale** A newcomer's confusion about how best to get involved or what the point of all this actually is may be due to a lack of structure in the project ROADMAP. Sharing vulnerability and confusion gives us a chance to learn.

**Resolution** An awareness of the difficulties that newcomers face can help us be more compassionate to ourselves and others. We strengthen the community by supporting all participants' **individuation**. We have a better chance of making the project useful for others if we're clear about how it is useful to *us*. By welcoming newcomers, we enhance the sense of **mutuality** with people who have never encountered the project before, and learn together with them. The facts start to become useful when we understand how people perceive them [4].

**Example 1** Wikipedia NEWCOMERS can make use of resources that include a "Teahouse" where questions are welcomed, a platform extension that changes the user interface for new editors,

and lots of documentation. The efforts of exceptional newcomers may be given special recognition. Newcomer "survival" is of interest to the Wikimedia Foundation. However, "Nearly all editors begin with a burst of activity, then quickly tail off" [6]. The degree to which those editors who are retained emphasize continuous upskilling is somewhat less clear. As regards learning their way around the community, quantitative support exists [6] for the claim that "novice users learn the rules and conventions for contributing both through observation and direct coaching from more knowledgeable others" [2].

**Example 2** It will often be pragmatic to connect Newcomers with employment directly, so that the future university may see a closer coupling of science and industry than would be found in the old Science Hall. Inspiration can be drawn the London-based freelancing cooperative Founders&Coders, which is able to offer intensive training in web development at no cost to successful applicants.



Figure 6.10: Science Laboratory, Aspatria Agricultural College, Cumberland, UK.

*What's Next in the Peeragogy Project* More detailed guides can show Newcomers how they can contribute and what to expect when they do. We should have different guides for different "user stories". We can start by listing some of the things we're currently learning about.

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## 6.9 Scrapbook

**Motivation** This pattern describes a way to make the project meaningful.

**Context** We have been working together for a while now. We have maintained and revised our pattern catalog, and we are

achieving some of the "What's Next" steps associated with some of the patterns.

**Forces** **Attention**: due to limited energy, we need to ask: where should we set the focus?

**Interest**: new experiences catch our attention.

**Meaning**: shared history makes things meaningful.

**Problem** Not all of the ideas we've come up with have proved workable. Not all of the patterns we've noticed remain equally relevant. In particular, some patterns no longer lead to concrete next steps.

**Solution** In order to maintain focus, is important to "tune" and "prune" the things we give our attention to. We can connect this understanding to any actions undertaken in the project by asking questions like these:

> (1) Review what was supposed to happen. (2) Establish what is happening/happened. (3) Determine what's right and wrong with what we are doing/have done. (4) What did we learn or change? (5) What else should we change going forward? [see Chapter 28 of the current volume]

Other review processes have been formalised within the field of architecture as a design review [7]. The review process may benefit from having an experienced facilitator on board [6, pp. 67, 142–143]. As current priorities become clearer, we decide where to focus. Anything that isn't receiving active attention should be moved to a Scrapbook. This may encompass:

- *Retired patterns* that are tabled or completed (no more next steps);

- *Proto-patterns* made of problems, issues, and concerns;
- *A back-catalog* of publications, reports, or other artifacts.

In the Peeragogy project, alongside our patterns we initially maintained a collection of antipatterns (like 'MAGICAL THINKING') but the next steps coming from these seemed particularly convoluted and abstract. So, we archived them. We present a list of outstanding problems – without known solutions – right up front in the Introduction to the *Peeragogy Handbook*. Our back-catalog includes academic papers [4, 5, 1, 2] and a thesis [3]. Everyone can maintain their own personal SCRAPBOOK as along with a communal one. You don't need to limit yourself to *your own* creativity: include interesting ideas from other sources (see REDUCE, REUSE, RECYCLE). In some cases a designated WRAPPER may have to do further work to elicit and organize contributions.

**Rationale** We want to keep attention focused on the most relevant issues. If a pattern, task, or concern does not lead to concrete "next steps" at the moment, sufficient time for reflection may offer a better understanding, and it may prove useful and actionable in a different context.

**Resolution** Judicious use of the SCRAPBOOK can help focus project participants' **attention** on current concerns, without losing grasp of items of **interest**. The currently active pattern catalog is leaner and more action-oriented as a result. If the ROADMAP shows where we're going, it is the SCRAPBOOK that shows most clearly where we've been, and collects the observations that are most **meaningful** to us.

**Example 1** The history of the Wikimedia Foundation, and of Wikipedia, are maintained as wiki pages. One of the entries on Wikipedia details outstanding issues, in the form of critiques. There are many tools available to help facilitate the process of vetting proposed fine-grained changes to articles. Policy concerns

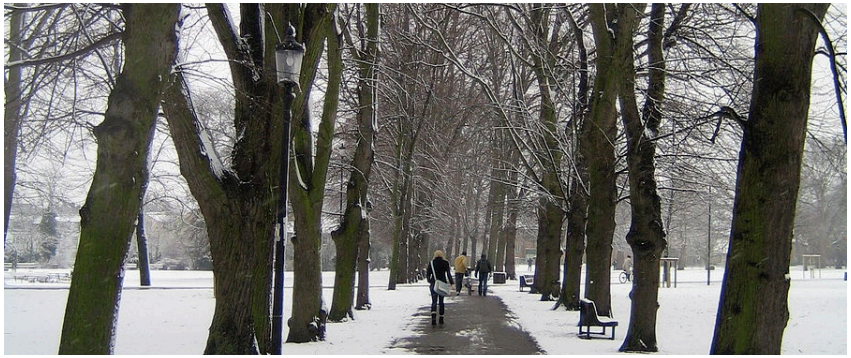


Figure 6.11: Christ's Pieces, Cambridge, UK.

are typically discussed at the Village Pump, and there are mechanisms in place for settling disputes.

**Example 2** Just as a university campus grows and changes in an emergent fashion over time, future peeragogues' attention will encompass new problems and patterns.

> *What's Next in the Peeragogy Project* After pruning back our pattern catalog, we want it to grow again: new patterns are needed. One strategy would be to turn the whole *Peeragogy Handbook* into design patterns.

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CHAPTER 7

# EMERGENT ROADMAP

> Here we reprise the "What's Next" steps from all the previous patterns, offering another view on the project ROADMAP in a concrete emergent form.

### ▸ Peeragogy

We intend to revise and extend the *Patterns of Peeragogy* into a framework that can describe and scaffold the learning that happens inside and outside of institutions.

### ▸ Roadmap

If it becomes clear that something needs to change about the project, that is a clue that we might need to revise our patterns or record a new one. We can use the names of the patterns to tag our upcoming tasks.

### ▸ Reduce, reuse, recycle

Are there other educational resources and peeragogical case studies that we could fold into our work? Can we recycle material from the *Peeragogy Handbook* into a format that is easier to understand and apply?

### ▸ Carrying capacity

Making it easy and fruitful for others to get involved is one of the best ways to redistribute the load. This often requires knowledge transfer and skill development among those involved; see NEWCOMER.

### ▸ A specific project

Let's use our pattern catalog to build specific, tangible "what's next" steps, add them to our ROADMAP, and carry them out with concrete actions. Let's be sure we know who's responsible for what, and employ a "buddy system" to help get things done.

### ▸ Heartbeat

Identifying and fostering new HEARTBEATS and new working groups can help make the community more robust. This is the time dimension of spin-off projects described in REDUCE, REUSE, RECYCLE.

### ▸ Wrapper

Let's make sure we have protocols in place that enable us to share progress, and to revise our "next steps" if people are getting stuck. Let's improve the interaction design for peeragogy.org so that it's clear how people can get involved.

### ▸ Newcomer

More detailed guides can show NEWCOMERS how they can contribute and what to expect when they do. We should have different guides for different "user stories". We can start by listing some of the things we're currently learning about.

### ▸ Scrapbook

After pruning back our pattern catalog, we want it to grow again: new patterns are needed. One strategy would be to turn the whole *Peeragogy Handbook* into design patterns.

CHAPTER 8

# CASE STUDY: SWATS

Learning to use technology with peers – the case of Students With Abilities in Technology (SWATs).

## Part 1: Introduction

Mind-amplifying technologies [1], technologies of cooperation [2], such as conversation technologies, as well as visualization tools, video and photo edition software, simulators or programming technologies are emerging learning tools in schools around the world. They are affordable and accessible enough to design learning environments. Latin America is no exception and it is fast becoming the norm to find convergent technology in the classroom.

We challenged students to develop a three-level game with a score or marker using Scratch, a program developed by MIT. This program allows you to develop computer programs using modules or blocks of instructions. The educational value of this tool lies not in its ease of use but in its nature as an authentic learning environment and ideal context for developing intellectual skills.

Once students have developed their programs and documented the process in a learning log, we asked them if they had faced problems in handling Scratch. In this way we were able to identify which students had difficulties in developing programs and what their problems were in the process of choosing instructions. However, we're also able to identify those students with particular technical skills. We call them Students With Abilities in Technology (SWATs). In case of difficulty, SWATs can be called in, and can decide if they want to give advice to peers and the teacher in the use of Scratch.

The idea of identifying these students and asking them to support their peers and teachers in specific tasks has an additional

educational component. It is clear that when a student is given the task of explaining or advising peers or teachers, he develops new competences and masters, to an even greater extent, those competences for which he/she was selected as SWAT.

We have observationally determined that this approach is relevant to the widespread use of digital devices in academic tasks and its extended application contributes to a more positive use of digital technologies for learning. We see how, as the use of technology in all learning environments becomes general, this approach of peer learning becomes an alternative to underpin the work of teachers. The figure of the SWAT in the classroom also enables a different form of relationship between pairs that generate new forms of interaction and learning that we can appraise and evaluate.

## Part 2. Representation as a pattern

Here's how the above case could be described using the pattern template that we've presented in the book. This may help others use the same model — or at least understand how it works in practice in more detail. Further questions may come to mind, which the reader can try to answer by transforming or extending the pattern in their own context.

**Title:** Students With Abilities in Technology (SWAT)

**Definition:** Private and public schools increasingly have digital devices in classrooms with Internet access (laptops, desktops, tablets, cell phones, etc.) and teachers with little or no expertise in the educational use of such devices. However, some of the students have considerable background with these kinds of tools. They can help the teachers and other students.

**Problem:** In general, teachers have multiple deficiencies in the adoption of emerging technologies. Their lack of expertise prevents them from realizing the full potential that technology has

as a relevant pedagogical mediation. The rate of change in the school context, however, is not coupled to the rate of change in current teacher training programs. This lack of pedagogical training is having a majorly disruptive impact in the classroom given this presence of technological devices in the classroom. The reaction of administrators and teachers to the proliferation of devices is, in a significant number of cases, rejection and stigmatization of emerging technologies. The cause of this rejection is that teachers ignore the educational potential of technology. They ignore how technology has changed the cognitive model of a whole generation. Having technical specialists to support the work of the teacher in the classroom is unthinkable from an economic standpoint.

**Solution:** The students themselves can be the solution to this problem. Some have superior technical knowledge and this is usually wasted. Teachers can incorporate them as assistants to help them and their peers. A student with digital skills can be the agent of change that many teachers need in order to learn how to use technology for the design of learning environments. This empowered group of students, that we named SWAT (Students With Abilities in Technology) support teachers and peers with lower-level digital competences. Support from students with technical knowledge could mean a significant change in the learning process, because teachers can now combine that knowledge with their teaching experience and pedagogical strategies. The result of this can be the discovery of the many possibilities technology has for the construction of knowledge and the development of new intellectual abilities.

In our work with middle school students (ages 12-14), the support of SWATs inside and outside the classroom was a very positive experience. At the beginning of each course students are required to develop projects involving the use of technology. Students who show a greater competence in the use of technical tools are invited to join as SWAT. Once SWATs are identified, they are asked about the possibility of supporting teachers and their peers

in the use of specific computer tools. It is impressive to see teachers becoming co-learners who take advantage of this privileged status of their students to master tools that promote their ability to redesign learning environments. When students need support for developing their projects, SWATs show them strategies to accomplish them. The majority report great pleasure and pride in their new role as peer advisors.

**Challenges arising in practice:** This peeragogical approach changes the prevailing educational paradigm through collaboration between teachers and students, and among students themselves. There are many possible points of friction. To have one or more SWATs in each learning group transforms the way in which teachers and students interact with each other and with available technologies, but, again, can create challenges for teachers who may be used to a more "banking" style of teaching.

**What's next:** Can we find mentors for the SWATs to help them become even better with technology? Can we find other ways to reward these students? At the same time, can the idea be applied across the curriculum, and across other competencies, to involve more students in the peer-teaching role?

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# Part IV

# Convening A Group

CHAPTER 9

# SO YOU'VE DECIDED TO TRY PEER LEARNING...

So you've decided to try peer learning? Maybe you've already found a few people who will support you in this effort. Congratulations! It's time now to focus your thinking. How will you convene others to form a suitable group? How will you design a learner experience which will make your project thrive? In this chapter, we suggest a variety of questions that will help you to make your project more concrete for potential new members. There are no good or bad answers - it depends on the nature of your project and the context. Trying to answer the questions is not something you do just once. At various stages of the project, even after it's over, some or all of those questions will aquire new meanings - and probably new answers.

**Fabrizio Terzi**: "There is a force of attraction that allows aggregation into groups based on the degree of personal interest; the ability to enhance and improve the share of each participant; the expectation of success and potential benefit."

## Group identity

Note that there are many groups that may not need to be "convened", since they already exist. There is a good story from A. T. Ariyaratne in his collected works in which he does "convene" a natural group (a village) - but in any case, keep in mind at the outset that the degree of group-consciousness that is necessary for peer learning to take place is not fixed. In this section, we suppose you are just at the point of kicking off a project. What

steps should you take? We suggest you take a moment to ponder the following questions first - and revisit them afterward, as a way to identify best practices for the next effort.

## There will be a quiz

Those taking the initiative should ask themselves the traditional Who, What, Where, When, Why, and How. (Simon Sinek suggests to begin with Why, and we touched on Who, above!). In doing so, preliminary assumptions for design and structure are established. However, in peer learning it is particularly important to maintain a healthy degree of openness, so that future group members can also form their answers on those questions. In particular, this suggests that the design and structure of the project (and the group) may change over time. Here, we riff on the traditional 5W's+H with six clusters of questions to help you focus your thinking about the project and amplify its positive outcomes.

## Expectations for participants

### 1. Who: Roles and flux

- What are some of the roles that people are likely to fall into (e.g. Newcomer, Wrapper, Lurker, Aggregator, etc.)?
- How likely is it that participants will stick with the project? If you expect many participants to leave, how will this effect the group and the outcome?
- Do you envision new people joining the group as time goes by? If so, what features are you designing that will support their integration into an existing flow?
- Will the project work if people dip in and out? If so, what features support that? If not, how will people stay focused?

### 2. What: Nature of the project

- What skills are required? What skills are you trying to build?

- What kinds of change will participants undergo? Will they be heading into new ground? Changing their minds about something? Learning about learning?
- What social objective, or "product" if any, is the project aiming to achieve?
- What's the 'hook?' Unless you are working with an existing group, or re-using an existing modality, consistent participation may not be a given.

### 3. When: Time management

- What do you expect the group to do, from the moment it convenes, to the end of its life-span, to create the specific outcome that will exist at the conclusion of its last meeting? [2] Note that what people ACTUALLY do may be different from what you envision at the outset, so you may want to revisit this question (and your answer) again as the project progresses.
- Keeping in mind that at least one period of is inertia is very likely [2], what event(s) do you anticipate happening in the group that will bring things back together, set a new direction, or generally get things on track? More generally, what kinds of contingencies does your group face? How does it interface to the "outside world"?
- What pre-existing narratives or workflows could you copy in your group?
- How much of a time commitment do you expect from participants? Is this kind of commitment realistic for members of your group?
- What, if anything, can you do to make participation "easy" in the sense that it happens in the natural flow of life for group members?
- Does everyone need to participate equally? How might non-equal participation play out for participants down the line?

### 4. Where: Journey vs Destination

- What structures will support participants in their journey to the end result(s) you (or they) have envisioned? What content can you use to flesh out this structure?
- Where can the structure "flex" to accommodate unknown developments or needs as participants learn, discover, and progress?

### 5. Why: Tool/platform choice

- What tools are particularly suited to this group? Consider such features as learning styles and experiences, geographical diversity, the need for centralization (or decentralization), cultural expectations related to group work, sharing, and emerging leadership.
- Is there an inherent draw to this project for a given population, or are you as facilitator going to have to work at keeping people involved? How might your answer influence your choice of tools? Is the reward for completion the learning itself, or something more tangible?
- In choosing tools, how do you prioritize such values and objectives as easy entry, diverse uses, and high ceilings for sophisticated expansion?

### 6. How: Linearity vs Messiness

- How will your group manage feedback in a constructive way?
- Why might participants feel motivated to give feedback?
- How firm and extensive are the social contracts for this group? Do they apply to everyone equally, or do they vary with participation level?
- What do people need to know at the start? What can you work out as you go along? Who decides?
- How welcome are "meta-discussions"? What kinds of discussions are not likely to be welcome? Do you have facilities in place for "breakout groups" or other peer-to-peer

interactions? (Alternatively, if the project is mostly distributed, do you have any facilities in place for coming together as a group?)

## Cycles of group development

The above questions remain important thoughout the life of the project. People may come and go, particpants may propose fundamentally new approaches, people may evolve from lurkers to major content creators or vice-versa. The questions we suggest can be most effective if your group discusses them over time, as part of its workflow, using synchronous online meetings (e.g., Big Blue Button, Adobe Connect, Blackboard Collaborate), forums, Google docs, wikis, and/or email lists. Regular meetings are one way to establish a "heartbeat" for the group.

In thinking about other ways of structuring things, note that the "body" of the *Peeragogy Handbook* follows a Tuckman-like outline (*Convening a Group* is our "forming", *Organizing a Learning Context* is our "storming and norming", *Coworking/Facilitation* is our "performing", and *Assessment* is our "adjourning"). But we agree with Gersick [1], and Engeström [2], that groups do not always follow a linear or cyclical pattern with their activities!

Nevertheless, there may be some specific stages or phases that you want *your* group to go through. Do you need some "milestones," for example? How will you know when you've achieved "success?"

In closing, it is worth reminding you that it is natural for groups to experience conflict, especially as they grow or cross other threshold points or milestones - or perhaps more likely, when they don't cross important milestones in a timely fashion (ah, so you remember those milestones from the previous section!). Nevertheless, there are some strategies can be used to make this conflict productive, rather than merely destructive (see Ozturk and Simsek [3]).

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CHAPTER 10

# PLAY AND LEARNING

Once more we're back to the question, "What makes learning fun?" There are deep links between play and learning. Consider, for instance, the way we learn the rules of a game through playing it. The first times we play a card game, or a physical sport, or a computer simulation we test out rule boundaries as well as our understanding. Actors and role-players learn their roles through the dynamic process of performance. The resulting learning isn't absorbed all at once, but accretes over time through an emergent process, one unfolding further through iterations. In other words, the more we play a game, the more we learn it.

In addition to the rules of play, we learn about the subject which play represents, be it a strategy game (chess, for example) or simulation of economic conflict. Good games echo good teaching practice, too, in that they structure a single player's experience to fit their regime of competence (cf. Vygotsky's zone of proximal learning, a la Gee [1]). That is to say a game challenges players at a level suited to their skill and knowledge: comfortable enough that play is possible, but so challenging as to avoid boredom, eliciting player growth. Role-playing in theater lets performers explore and test out concepts; see Boal [2]. Further, adopting a playful attitude helps individuals meet new challenges with curiousity, along with a readiness to mobilize ideas and practical knowledge. Indeed, the energy activated by play can take a person beyond an event's formal limitations, as players can assume that play can go on and on [3].

> **Douglas Thomas and John Seely Brown**: "All systems of play are, at base, learning systems." [4]

Games have always had a major social component, and learning plays a key role in that interpersonal function. Using games to

build group cohesion is an old practice, actually a triusm in team sports.

It is important to locate our peeragogical moment in a world where gaming is undergoing a renaissance. Not only has digital gaming become a large industry, but gaming has begun to infiltrate non-gaming aspects of the world, sometimes referred to as "gamification." Putting all three of these levels together, we see that we can possibly improve co-learning by adopting a playful mindset. Such a playful attitude can then mobilize any or all of the above advantages. For example,

- Two friends are learning the Russian language together. They invent a vocabulary game: one identifies an object in the world, and the other must name it in Russian. They take turns, each challenging the other, building up their common knowledge.
- A middle-aged man decides to take up hiking. The prospect is somewhat daunting, since he's a very proud person and is easily stymied by learning something from scratch. So he adopts a "trail name", a playful pseudonym. This new identity lets him set-aside his self-importance and risk making mistakes. Gradually he grows comfortable with what his new persona learns.
- We can also consider the **design** field as a useful kind of playful peeragogy. The person *playing the role* of the designer can select the contextual frame within which the design is performed. This frame can be seen as the *rules* governing the design, the artifact and the process. These rules, as with some games, may change over time. Therefore the possibility to adapt, to tailor one's activities to changing context is important when designing playful learning activities. (And we'll look at some ways to design peer learning experiences next!)

Of course, "game-based learning" can be part of standard pedagogy too. When peers create the game themselves, this presumably involves both game-based learning and peer learning. Clas-

sic strategy games like Go and Chess also provide clear examples of peer learning practices: the question is partly, what skills and mindsets do our game-related practices really teach?

> **Socrates**: "No compulsory learning can remain in the soul ...In teaching children, train them by a kind of game, and you will be able to see more clearly the natural bent of each."

### Exercises that can help you cultivate a playful attitude

- Use the Oblique Strategies card deck (Brian Eno and Peter Schmidt, 1st edition 1975, now available in its fifth edition) to spur playful creativity. Each card advises players to change their creative process, often in surprising directions.
- Take turns making and sharing videos. This online collaborative continuous video storytelling involves a group of people creating short videos, uploading them to YouTube, then making playlists of results. Similar to Clip Kino, only online.
- Engage in theater play using Google+ Hangout. e.g. coming together with a group of people online and performing theatrical performances on a shared topic that are recorded.

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CHAPTER 11

# K-12 PEERAGOGY

Teachers tend to work in isolation on their own islands, keeping their learning to themselves, yet they also generously share resources with one another. It is this latter trait that is becoming increasingly important as the role of the educator continues to expand. As educational technology research specialist Stephen Downes observes, the expectations on teachers have grown from "being expert in the discipline of teaching and pedagogy…[to needing to have] up-to-date and relevant knowledge and experience in it. Even a teacher of basic disciplines such as science, history or mathematics must remain grounded, as no discipline has remained stable for very long, and all disciplines require a deeper insight in order to be taught effectively." It is no longer possible for an educator to work alone to fulfil each of these roles: the solution is to work and learn in collaboration with others. This is where peer-based sharing and learning online, connected/networked learning, or peeragogy, can play an important role in helping educators.

## Becoming a connected/networked learner

The following steps are set out in 'phases' in order to suggest possible experiences one may encounter when becoming connected. It is acknowledged that every learner is different and these 'phases' only serve as a guide.

### Phase 1: Deciding to take the plunge

To help educators begin to connect, the Connected Educator's Starter Kit was created during Connected Educator's Month in August 2012. This article previews the main steps. The first step to becoming a 'connected educator-learner' involves making the

commitment to spending the time you'll need to learn how to learn and share in an open, connected environment.

### Phase 2: Lurking

We start off as lurkers. A learner can be considered a true 'lurker' after reviewing the starter kit, establishing a digital presence (through a blog or a wiki) or signing up for Twitter and creating a basic profile containing a photo. In this phase, lurkers will begin to 'follow' other users on Twitter and observe educational Twitter 'chats'. Lurkers will also begin to seek out other resources through blogs, Facebook, Edmodo and LinkedIn groups.

### Phase 3: Entering the fray

The lurker begins to develop into a connected educator-learner once he or she makes the decision to enter into a dialogue with another user. This could take the form of a personal blog post, participation on an education-related blog or wiki or a an exchange with another Twitter user. Once this exchange takes place, relationships may begin to form and the work towards building a Personal Learning Network (PLN) begins.

One such site where such relationships can be built is Classroom 2.0, which was founded by Steve Hargadon. Through Classroom 2.0, Steve facilitates a number of free online learning opportunities including weekly Blackboard Collaborate sessions, conferences, book projects and grassroots cross-country educational-transformation tours. Classroom 2.0 also offers a supportive Social Ning—a free, social learning space that provides online conferences and synchronous and recorded interviews with inspirational educators—for connected educator-learners around the world.

### Phase 4: Building and shaping your PLN

Just as not every person one meets becomes a friend, it is important to remember that not every exchange will lead to a co-learning peeragogy arrangement. It may be sufficient to follow

another who provides useful content without expecting any reciprocation. It is dependent on each educator-learner to determine who to pay attention to and what learning purpose that individual or group will serve. It is also up to the learner-educator to demonstrate to others that he or she will actively participate.

There are a number of strategies one can use when shaping the PLN to learn. However, one of the best ways educators can attract a core of *peeragogues* is by sharing actively and demonstrating active and open learning for others.

There are a number of sites where a new educator-learner can actively and openly learn. In addition to personal blogging and wikis, other professional development opportunities include open, online courses and weekly synchronous online meetings through video, podcasts or other forms of media.

Examples include: Connected Learning TV, TechTalkTuesdays, VolunteersNeeded, SimpleK12, K12 Online, CEET, and EdTechTalk.

Alternatively, courses are offered with P2PU's School of Education or a wide variety of other opportunities collected by TeachThought and Educator's CPD online. Peggy George, the co-faciliator of the weekly Classroom 2.0 LIVE Sessions, created a livebinder package of free 'PD On Demand' connected professional development online options for peeragogy enthusiasts.

### Stage 5: Extending the digital PLN and connecting face-to-face

Over time, once the connected educator-learner has established a refined PLN, these peeragogues may choose to shift their learning into physical learning spaces. Some options available for these educator-learners would include the new 'grassroots unconferences', which include examples such as: EduCon, EdCamps, THATcamp and ConnectedCA.

These (un)conferences are free or extremely low-cost and focus on learning from and with others. These 'unconferences' are typically publicized through Twitter, Google Apps, and Facebook. Connecting face-to-face with other peeragogues can strengthen

bonds to learning networks and help to promote their sustainability.

## Postscript

Sylvia Tolisano, Rodd Lucier and Zoe Branigan-Pipen co-created an infographic which explores the experiences individuals may encounter in the journey to become connected learners through another related sequence of steps: *Lurker*, *Novice*, *Insider*, *Colleague*, *Collaborator*, *Friend*, and *Confidant*. Googlize it, and have a look at our Recommended Readings in Chapter 31 for additional resources.

CHAPTER 12

# P2P SELF-ORGANIZED LEARNING ENVIRONMENTS

This conversational piece invites you to engage in a journey to create your own learning space. You'll find many points of entry that allow you to affectd emerging structure. Reciprocal mentoring can create a ripple effect for those who follow.

## The Guiding Strategy:

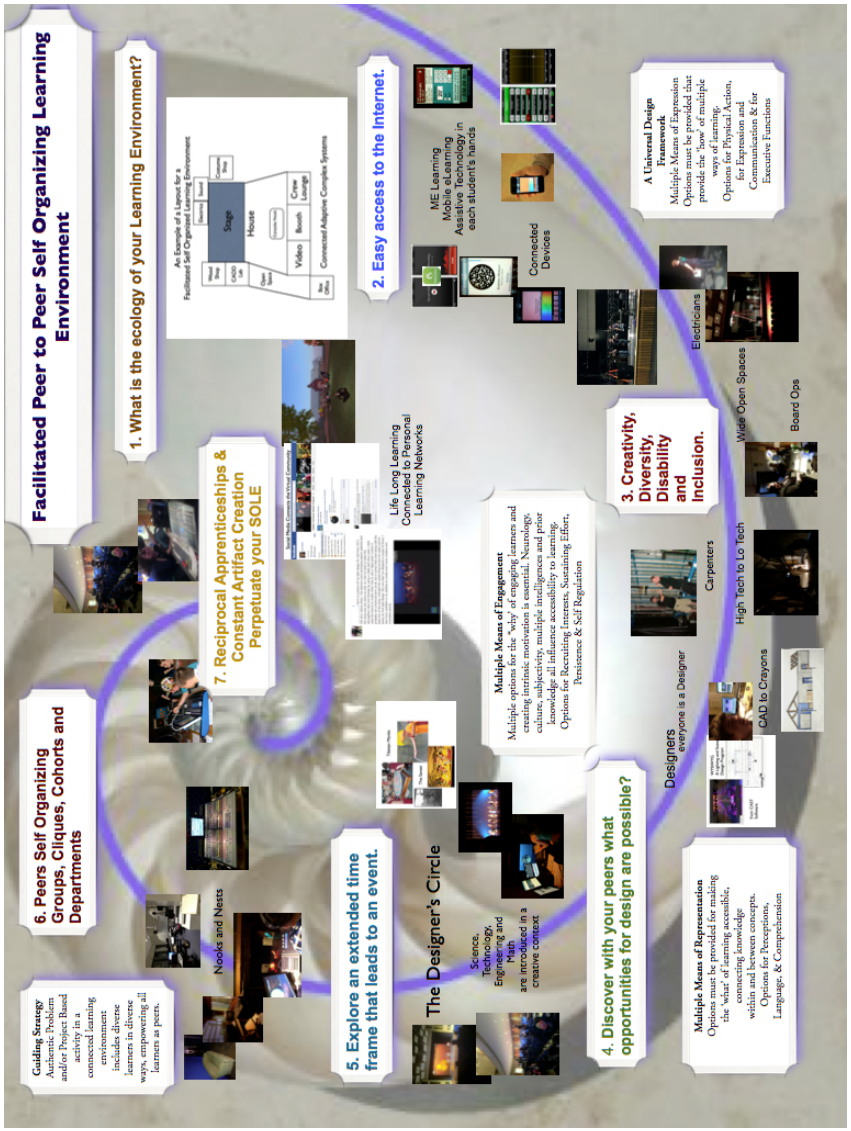
In his Peeragogical Case Study David Preston states:

> *Peeragogical interaction requires refining relational and topical critique, as well as skills in other "meta" literacies, including but not limited to critical thinking, collaboration, conflict resolution, decision-making, mindfulness, patience and compassion.*

A Self-Organizing Learning Environment, or SOLE, with a living structure accomplishes all of these outcomes, or David's "meta-literacies," simultaneously. An authentic problem and/or project based activity in a connected learning environment includes diverse learners in diverse ways by empowering all learners as peers.

This provides the authentic learning environment with which to design a SOLE. SOLEs are everywhere. How have we evolved as a species, if not through self-organizing? A conversation between strangers is self organizing, each learning about something or each other. The spaces around people conversing is also an environment, though not explicitly a learning one.

While we are always self-organizing to learn or accomplish




A visualization of the facilitated peer to peer SOLE, full-size at `http://goo.gl/7StkJK`

things, one place that SOLEs do not always exist are in learning institutions. In many educational institutions, our learning environments are predominately organized by the teacher, curriculum, or society. How can we nurture peer to peer learning environments to organize? How does the role of the teacher differ in a SOLE? In what ways can we unite that fundamental, passionate human characteristic of curiosity and self-organizing back into our Learning Environments?

The model that Sugata Mitra [2] is experimenting with gives us some scaffolding to create one ourselves. This is the goal of his SOLE Tool Kit (3). Sugata's kit is directed towards children between 8 and 12 years old. I was wondering if there is a way to make it more universal in its application. How can I apply it to my situation? How is a SOLE different in the context of peer to peer learning? This chapter of the Handbook uses Sugata's model as a doorway into our understanding a SOLE approach to peer to peer learning. Its three key components are: learners, context and project. I find the discussion needs to integrate what we are learning about diverse learners into a Universal Design for Learning [4] context. After all, we cannot take for granted who the peers are in the SOLE. Equally, the context, the learning environment (LE) must be as deeply considered as the learners participating. As a learning designer, I am also seeking more clues about the living structure of a well crafted SOLE.

## Centers within the Center

SOLEs exist in a particular context. Take Sugata's hole in the wall [5] experiment. The parameters of the environment of a computer embedded in a wall in India are very specific. Sugata's act was to design a project in order to facilitate a process within that environment. The elements he introduced were a touch screen computer embedded in a wall with specific software. Sugata has abstracted this design into a Tool Kit. He speaks of 'Child Driven Learning', intrinsically motivated learning with the curiosity to learn something in particular. As a learner-centric peeragogy, SOLEs are emergent, bottom up, seeking to answer: How do we

design a project (or phrase a problem) that ignites a learner's passion?

A SOLE is a facilitated learning environment (LE) that can nurture learner-driven activity. For instance, in the Hole in the Wall example, the design is the context of the wall, the street, the neighborhood –and the facilitation is the touch screen monitor in the wall. They are brilliantly united. In this sense it is an intentional, self-aware learning environment. The Wall's computer is a strange foreign object that anyone would have to figure out how to take advantage of. But this is not in the classroom, or in the 'school.' It is an informal LE. Just like learning a game [6], there is an entire ecology that surrounds you. This is very much a systemic approach. The context is facilitated explicitly (your design of the SOLE), but also implicitly in the hidden curriculum [7] that defines your LE.

Above is the layout of the transformed learning environment [8] I explored to work around the hidden curriculum of the traditional classroom. The LE has a tremendous, if not overwhelming influence, on learning [9]. The first step in connected learning is to reconnect to the environment around us. For me, the primary context of my LE is a performing arts center at a small rural liberal arts college. The Performing Arts Center is a Center within the context of the college and community. A diversity of spaces within the facility are inhabited: small and cozy, large and public, technology embedded everywhere, all focused on the project based learning that emerges producing a performance. I stay away from a formal classroom as much as possible. These spaces are Centers within the Center, 'loosely connected adaptive complex systems' [10] within themselves, just like people. I believe that the possibility of a SOLE emerging as a living structure seems to depend on the correct types of complex systems engaged in the LE.

What is the role of the internet in your design? Mitigating inequalities and accommodating diverse learners are somewhat assisted by access to the internet. But it is the immediate, just-in-time learning [11] that makes free and open access to the world wide web so important in a SOLE. Wireless is available

throughout this LE. Nooks and lounges, interconnected, but separate rooms, provide lots of places for collaboration or solitary work, for staying connected or hiding out. In a UDL vision of a facilitated peer to peer SOLE, technology is integral to the design. In the case of my LE, with the use of digital audio, multi-media, database management, robotic lighting and dichroic [12] colors, learners are accustomed to accessing and augmenting reality with technology: allowing learners to access their social media is part of their content creation.

Do we start our SOLE as peers? Peer to peer assumes your participants are peers–especially you, the facilitator. There needs to be enough diversity and complexity to include all learners, engendering a Universally Designed Context [13]. What is the role of diversity in peer to peer SOLE building? How are diverse learners peers? In my LE, I discovered 70% of my learners have learning challenges. I know my LE is not unique in this regard. I have to facilitate a SOLE design that is inclusive. This is in contradistinction to commonality, yet this diversity is what we crave, for creativity and innovation, for deep learning to occur. Crafting your SOLE using multiple means of representation, expression and engagement empowers learners to be peers. A diverse learning environment, supporting diverse learning styles and diverse learners, supports a complex project based SOLE. But there are many SOLEs within the SOLE since learning is occurring on many levels with each student and within each group. We do not all get the same thing at the same time. Learning outcomes are diverse, emergent, serendipitous.

What type of project, problem or event will focus your efforts? Either a [learner generated syllabus 14] may emerge from the SOLE, or a user generated education [15] within a specific context may answer this question. Ownership and leadership emerge when learners can apply their creativity and/or authentically assist each other in a common goal. Opportunities to design and modify even small things will draw learners into a project. The more they must rely on each other, collaborate and share their creativity, their designs and actualization–the more they work together as peers. The spaces in your LE are most likely already de-

signed and built to accommodate the purpose of the facility in the context of the college or school. We cannot really redesign the actual space, but we can redesign many aspects. We can look for designs within it. Being able to design your own space, or project, is critical to taking ownership of your learning and experiencing the consequences. As learners mature and look for ways to be more involved, I suggest they redesign the shop, the repertory lighting plot, or the procedures of their department and/or SOLE overall. Exchanging roles as designer also stimulates peer interaction. Why not integrate design and design thinking? In my context, lighting, scene, costume and sound design are interconnected opportunities. Along with accompanying technology, every opportunity is used to nurture empathy, creativity, rationality and systems thinking. Integral to the learner generated syllabus or project design should be continuous artifact creation. A great place to start the design process and to begin to generate content is by using a virtual world.

Constant content creation can integrate assessment into your SOLE. It is the quality of the artifacts created along the way that reveals the success of your SOLE. Media that chronicles a journey through time, created by each learner, reveals the depth of participation. It is nearly impossible to cheat. The learner expresses their comprehension in the types and extent of artifact creation.

As the facilitator, I look for opportunities to introduce the unexpected, bigger questions, deeper considerations, along the way. For example, in the context of my LE, one of the events feature Tibetan Monks. They bring a counterpoint to the inflated egos and cult of personality which is prevalent in our context. The SOLE Plan is extended. It can happen over a much longer amount of time than one class or one day. The actors rehearse for weeks, as the design team designs, giving time for: research, absorption, misleads, mistakes, correction and reflection. A SOLE needs time and persistence to generate artifacts, documentation and experiences of the project and virtual worlds are an excellent way to extend time and space synchronously and asynchronously.

Sugata emphasizes the Big questions. We do not always know what they are. A focus? A goal? A product? And the event? That

should be decided with the group. The learners intuit the direction that leads to deep engagement and the bigger questions. I try and leave it ambiguous, suggesting some of the things they might encounter. Facilitating the SOLE in this context, we face endless questions connected to the specific LE, to all the imaginary scenarios, Herculean tasks and questions– like building castles, programming a digital sound console, troubleshooting robotic lighting instruments, how to make the illusion of fire or, even, who killed Charlemagne? The Box Office is an example of an informal SOLE that has emerged recurrently over time. I have noticed that its vitality depends on the characters and the ebb and flow of learners entering the group or graduating.

The physical space is a small, windowless and often damp room with a couple of couches and a desk with a computer squeezed in. My very own 'Hole in the Wall' experiment. The bottom of the door can remain closed, while the top is open, like a stable. Primarily the students are paid to be there, answering the phone, reserving tickets, greeting patrons and managing the Box Office and the Front of the House. In the SOLE, this subtle inversion of the institutional value proposition turns 'work study' into studying work. This is an informal LE nested within the context of the formal institution and the wider LE: a center within a center. Some semesters there are business majors working their way up the job ladder: Usher to Assistant Front of House Manager, to Assistant Box Office Manager, to Box Office Manager. Sometimes this takes 4 years, sometimes it happens in a semester or two. It is a recursive SOLE that differs as the interests and skills of the students who inhabit the space change. As the current manager puts it, the Box Office is a 'constantly evolving puzzle.'

This example of a SOLE in an informal LE is similar to the other types of SOLE's that occur within a facilitated LE. The learner's interact as reciprocal apprentices, leaning on one another to solve challenges and problems. Groups are self-selective, this type of work suits their temperament and interests, or time. This cohort is almost a clique, attracting their boyfriends and girlfriends. They begin initiatives, re-design the lobby for crowd control, redecorate and rearrange the space constantly, decide their

schedules and split up responsibility. Everyone is always training everyone, because the environment turns over each semester. It is explicitly an informal LE. The workers are students. This inverts the usual state of affairs, where essentially they are being paid to learn, though they may not even be aware of it. Occasionally, the learning experience resonates deeply with them. A number of them have used the experience to leverage jobs that parallel their interests, or get them started on their careers.

Job titles, roles of responsibility, are often problematic in a SOLE. The bottom line is that as peers we are all equal and at certain times everyone is expected to do everything regardless of their roles. Titles go to people's heads. But this is part of the experience. Keep the titles moving, change it up when things get bottlenecked over personalities. Sometimes I create duplicate positions, Assistants of Assistants. and Department Heads. The Apprenticeship model is at play but in a new way in a SOLE. There are peers and there are peers. As power struggles emerge, some like-to-like grouping occurs. The role of the facilitator becomes mediator. The emergent epistemology of abundance and connected learning asks for a multitude of 'experts.' In the same way, leadership can be distributed, flowing as varying needs arise.

The experience of practicing leadership skills and encountering all the variables of working with diverse folks quickly gives feedback to us if this is a helpful role for this person. It is messy sometimes, and there are conflicts. After a few events, they learn how to manage a Box Office, dealing with patrons, emergencies, complaints and bag check. They confront the larger peer group, the student body, with authority and empathy. They are very proud of their jobs and make their own name tags with titles. A hierarchy gives them rewards that they have been trained to expect from years in school. It is another way of developing intrinsic motivation and challenges them to interact with their peers authentically.

As facilitator, I try to leave them alone as much as possible. The context has been created, the computer in the wall is on a desk. Extending the design of your SOLE contributes to its living structure. I have used Facebook as a Supplemental LMS [16]

since 2007 because this is where my students are and it allows them to control the structures of groups emergently. The learners create the groups as they are relevant. The facilitator does not. Usually they invite me in! For now, Facebook aggregates the learning community that the SOLE inspires as learners become leaders, establish connections with each other and mentor newbies. This activity is integrated into artifact creation, 'comments' and documentation of their personal learning journey. Facebook becomes a precursor for their portfolios, and in some cases, it is their portfolio. Reciprocal Apprenticeships [17] occur in the dynamic of collaboration among peers. Continuity in time beyond the event horizon is accomplished by these relationships. Peers nurture one another along the shared learning journey that the SOLE provides. As facilitator and designer, you are, most of all, in a reciprocal relationship with the other learners. This is the essence of being a peer, an interaction that respects what each of us brings to the experience.

## A review

> **Sugata Mitra**: It is great to see the thinking that has gone into taking the idea of a SOLE forward. To my mind, SOLEs are quite experimental at this time and efforts such as these will provide invaluable data. I look forward to this. I notice that most of the important design features of a SOLE are incorporated into the article. I repeat them anyway, just to emphasise:
>
> 1. Large, publicly visible displays are very important, this is what resulted in the surprising results in the hole in the wall experiments and subsequent SOLEs for children in England and elsewhere.
> 2. The absence of unnecessary people in the learning space, no matter who they are; parents, teachers, principals, curious adults etc.

3. Free, undirected activity, conversation and movement.
4. A certain lack of order: I must emphasise that 'Self Organised', the way I use it does not mean 'organising of the self'. Instead it has a special meaning from the subject, Self Organising Systems, a part of Chaos Theory. The SOLE should be a space at the 'edge of chaos', thereby increasing the probability of the appearance of 'emergent order'.

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CHAPTER 13

# CASE STUDY: MEETING WITH THE PRO VICE-CHANCELLOR

As a teacher, Peeragogy is a way of life for me, for my students, and for those I come into contact with. However, introducing it to others can be daunting and confusing. My methodology for an explanation of interactions is to show by doing, and not only show, but include people into the fold, so they experience it first hand. This case study describes a meeting between the Pro Vice-Chancellor at my University, myself, and three other people, two of whom were my guests from abroad. So far it does not sound like Peeragogy, but like any meeting. It was scheduled in an office to be a semi-formal meeting of introduction to my superiors about Open Source Learning.

Pete is a final year student studying Instrumental & Vocal Teaching in Music at the University of Chichester, and he knows both of my guests as we all worked on a project in the previous academic year. This was a flying visit for my guests and everyone thought it would be nice to say hello in person, so I sent Pete a message to meet us before our meeting. There were no other instructions, requirements, or explanations. When we arrived to find Pete waiting, it was a jovial scene, with handshakes and laughter. We all walked around the campus and when it came time for our meeting, I said to Pete, "You have been a part of this, why don't you come along?" There was no planned agenda, no script, nothing besides a scheduled meeting time.

The four of us arrived at the office to meet with the Pro V-C and much to his credit, he didn't question why one of my students was there unannounced, but welcomed all of us. The Pro V-C asked most of the questions, as he was the one being introduced to something new. My guests, Pete, and I already work within a Peeragogical framework, but it is fair to say that this concept

is less prevalent and not the norm across higher education contexts. The point of this particular meeting was not to introduce Peeragogy, but Open Source Learning, which adds a dimension to learning within an institution by extending beyond the boundaries of discipline area, age, and physical setting.

In any institution people have responsibilities, and whenever a new opportunity or idea presents itself, it is natural to discuss, sound it out, and ask questions. That was the purpose of this meeting. The interesting thing where Peeragogy is concerned is how the meeting unfolded. It was not a meeting between four senior academics, but a meeting between five people. A student joined us for that meeting and played as active a role as any of the others present.

The exact detail of the content at this meeting is not important for this case study, but some of the overall questions and how these were answered by a demonstration of the underlying principles and unfolding practices of Peeragogy are at the core of this example.

Peeragogy for me is a methodology that permeates the other aspects of what I do both within and without teaching spaces. Open Source Learning is something else that I do, which both encompasses and extends beyond Peeragogy, and this meeting was instigated to introduce my Pro Vice-Chancellor to the founder of the OSL Foundation, David Preston, another professor visiting from California who also practices open source learning, and myself, a co-founder of the OLS foundation.

Firstly, there was no sense of tension between those in the room. The Professors, student, and Pro Vice-Chancellor were all as cogs in a clock, working together to move forward. People sparked off one another and hierarchies dissolved. There was no sense of 'hands up to speak', and Pete spoke just as much as anyone else. This was also an organic process. It was through the openness and receptivity of the Pro V-C that he allowed and enabled himself to join an already working, organic body. There was always respect for one another's experience, expertise, viewpoints, and the relevance of an individual's contribution to the discussion was valued. Pete could speak with as much

authority and conviction as the Professor of Architecture on co-learning and how genuine learner-inquisitiveness enables autonomy. They explained together, drawing upon one another's experience and perspectives to form a more complete picture for our host.

The conversation went on, and one key question was: How does this help the students, the learners? It is a very relevant question and one that should be asked. As Pro V-C at a university, in a position of authority where decisions about learning and teaching reside with your name on them, it is so important to understand and really seek out all aspects of opportunities, including the risks and benefits to all involved. In short: What is the benefit to this over working within some other defined or predetermined framework? The question is relevant beyond educational settings, because in commerce, or in any interactive situation there is always someone 'on the receiving end'. With Peeragogy that boundary blurs between those people in a co-learning environment, and with Open Source Learning the boundaries with the outside world also dissolve.

These benefits were demonstrated within the meeting itself. We were five people discussing a methodology, various projects, and possibilities. We stepped beyond roles to work together in a productive, open, learning environment. This is only possible when people in management positions, and in this case on the senior management team, are willing to be receptive to lecturers and students, who are then given the freedom and respect to come to the table. The respect is essential as the experience, skill, and perspective of a professor is inherently different to that of a manager or a student, and each person has a unique contribution to offer. Because one person has more or different experience does not render another member of the group irrelevant. By nature we need to learn from and with one another.

The individual skills within a specialism are valued and through an OSL approach they are not at all undermined or threatened, but valued and integral to the Peeragogical community that thrives within this practice. Even at the table at that meeting were represented Architecture, Music, History, English,

and all still needed their individually and meticulously developed specialist skills, which prepared them for successful interactive co-learning. The Peeragogical and Open Source Learning approaches develop a host of further skills around communication, information gathering and research, collaboration, and extend to develop confidence with using technology, presentation, and dissemination to wider networks.

As a teacher, leader, and facilitator there was not sense of giving anything up to adopt a co-learning framework for the meeting. There was no personal risk or degrading of my own value, instead it empowered others and made the conversation and the interaction at that meeting stronger.

We all left energised and interested in new possibilities to take projects and relationships further. To understand the difference between a Peeragogical, Open Sourced approach to learning and a more reactive framework, consider the distance from there to here. In this case study, five people came together and because of the shared outlook and approach, we were all active in enacting solid, informed progress. How many people could, without warning, ask a student to meet distinguished guests and join them with the someone from the university's senior management team, and not have any inhibitions? It is about transparency, trust, building and sharing skills, and creating situations where everyone finds value, where learning perpetuates and propagates. Looking at it in those terms can illustrate the distance from 'there' there to 'here'. Afterwards Pete, the student amongst the academics, commented to me that he wouldn't have been able to even show up at that meeting, let alone speak if I didn't let him. In his words, 'That's a two-way respect right there.'

# Part V

# Organizing a Learning Context

CHAPTER 14

# INTRODUCTION TO ORGANIZING CO-LEARNING

This section about organizing Co-Learning rests on the assumption that learning always happens in a context, whether this context is a structured "course" or a (potentially) less structured "learning space". For the moment we consider the following division:

- *Organizing Co-learning Contexts*
  - Courses ("linked to a timeline or syllabus")
  - Spaces ("not linked to a timeline or syllabus")

This section focuses on existing learning contexts and examines in detail how they have been "organized" by their . At a "meta-level" of development, we can talk about this parallel structure:

- *Building Co-learning Platforms*
  - Development trajectories (e.g. "design, implement, test, repeat")
  - Platform features (e.g. forums, wikis, ownership models, etc.)

A given learning environment will have both time-like and space-like features as well as both designed-for and un-planned features. A given learning platform will encourage certain types of engagement and impose certain constraints. The question for both "teachers" and "system designers" – as well as for learners – should be: *what features best support learning?*

The answer will depend on the learning task and available resources.

For example, many people believe that the best way to learn a foreign language is through immersion. But not everyone who

wants to learn, say, French, can afford to drop everything to go live in a French-speaking country. Thus, the space-like full immersion "treatment" is frequently sacrificed for course-like treatments (either via books, CDs, videos, or ongoing participation in semi-immersive discussion groups).

System designers are also faced with scarce resources: programmer time, software licensing concerns, availability of peer support, and so forth. While the ideal platform would (magically) come with solutions pre-built, a more realistic approach recognizes that problem solving always takes time and energy. The problem solving approach and associated "learning orientation" will also depend on the task and resources at hand. The following sections will develop this issue further through some specific case studies.

## Case Study 1: "Paragogy" and the After Action Review.

In our analysis of our experiences as course organizers at P2PU, we (Joe Corneli and Charlie Danoff) used the US Army's technique of After Action Review (AAR). To quote from our paper [2]:

> As the name indicates, the AAR is used to review training exercises. It is important to note that while one person typically plays the role of evaluator in such a review [...] the review itself happens among peers, and examines the operations of the unit as a whole.
>
> The four steps in an AAR are:
>
> 1. Review what was supposed to happen (training plans).
> 2. Establish what happened.
> 3. Determine what was right or wrong with what happened.

4. Determine how the task should be done differently the next time.

The stated purpose of the AAR is to "identify strengths and shortcomings in unit planning, preparation, and execution, and guide leaders to accept responsibility for shortcomings and produce a fix."

We combined the AAR with our paragogy principles –

1. Changing context as a decentered center.
2. Meta-learning as a font of knowledge.
3. Peers provide feedback that wouldn't be there otherwise.
4. Paragogy is distributed and nonlinear.
5. Realize the dream if you can, then wake up!

and went through steps 1-4 for each principle to look at how well it was implemented at P2PU. This process helped generate new policies that could be pursued further at P2PU or similar institutions. By presenting our paper at the Open Knowledge Conference (OKCon), we were able to meet P2PU's executive director, Philipp Schmidt, as well as other highly-involved P2PU participants; our feedback may ultimately have contributed to shaping the development trajectory for P2PU.

In addition, we developed a strong prototype for constructive engagement with peer learning that we and others could deploy again. In other words, variants on the AAR and the paragogical principles could be incorporated into future learning contexts as platform features [3] or re-used in a design/administration/moderation approach [4]. For example, we also used the AAR to help structure our writing and subsequent work on paragogy.net.

## Case Study 2: Peeragogy, Year One.

We surveyed members of the Peeragogy community with questions similar to those used by Boud and Lee [1] and then identified strengths and shortcomings, as we did with the AAR above.

## Questions

These were discussed, refined, and answered on an etherpad: revisions to the original set of questions, made by contributors, are marked in italics.

1. Who have you learned with or from in the Peeragogy project? *What are you doing to contribute to your peers' learning?*
2. How have you been learning during the project?
3. Who are your peers in this community, and why?
4. What were your expectations of participation in this project? *And, specifically, what did you (or do you) hope to learn through participation in this project?*
5. What actually happened during your participation in this project (so far)? *Have you been making progress on your learning goals (if any; see previous question) – or learned anything unexpected, but interesting?*
6. What is right or wrong with what happened (Alternatively: how would you assess the project to date?)
7. How might the task be done differently next time? (What's "missing" here that would create a "next time", *"sequel", or "continuation"?*)
8. *How would you like to use the Peeragogy handbook?*
9. *Finally, how might we change the questions, above, if we wanted to apply them in your peeragogical context?*

### Reflections on participants' answers

Some of the tensions highlighted in the answers are as follows:

1. *Slow formation of "peer" relationships.* There is a certain irony here: we are studying "peeragogy" and yet many respondents did not feel they were really getting to know one another "as peers", at least not yet. Those who did have a "team" or who knew one another from previous experiences, felt more peer-like in those relationships. Several remarked that they learned less from other individual participants and more from "the collective" or "from everyone". At the same time, some respondents had ambiguous feelings about naming individuals in the first question: "I felt like I was going to leave people out and that that means they would get a bad grade - ha!" One criterion for being a peer was to have built something together, so by this criterion, it stands to reason that we would only slowly become peers through this project.

2. *"Co-learning", "co-teaching", "co-producing"?* One respondent wrote: "I am learning about peeragogy, but I think I'm failing [to be] a good peeragogue. I remember that Howard [once] told us that the most important thing is that you should be responsible not only for your own learning but for your peers' learning. [...] So the question is, are we learning from others by ourselves or are we [...] helping others to learn?" Another wrote: "To my surprise I realized I could contribute organizationally with reviews, etc. And that I could provide some content around PLNs and group process. Trying to be a catalyst to a sense of forward movement and esprit de corps."

3. *Weak structure at the outset, versus a more "flexible" approach.* One respondent wrote: "I definitely think I do better when presented with a framework or scaffold to use for participation or content development. [...] (But perhaps it is just that I'm used to the old way of doing things)."

Yet, the same person wrote: "I am interested in [the] applicability [of peeragogy] to new models for entrepreneurship enabling less structured aggregation of participants in new undertakings, freed of the requirement or need for an entrepreneurial visionary/source/point person/proprietor." There is a sense that some confusion, particularly at the beginning, may be typical for peeragogy. With hindsight, one proposed "solution" would be to "have had a small group of people as a cadre that had met and brainstormed before the first live session [...] tasked [with] roles [and] on the same page".

4. *Technological concerns.* There were quite a variety, perhaps mainly to do with the question: how might a (different) platform handle the tension between "conversations" and "content production"? For example, will Wordpress help us "bring in" new contributors, or would it be better to use an open wiki? Another respondent noted the utility for many readers of a take-away PDF version. The site (peeragogy.org) should be "[a] place for people to share, comment, mentor and co-learn together in an ongoing fashion."

5. *Sample size.* Note that answers are still trickling in. How should we interpret the response rate? Perhaps what matters is that we are getting "enough" responses to make an analysis. One respondent proposed asking questions in a more ongoing fashion, e.g., asking people who are leaving: "What made you want to quit the project?"

## Discussion

**Lisewski and Joyce**: In recent years, the tools, knowledge base and discourse of the learning technology profession has been bolstered by the appearance of conceptual paradigms such as the 'five stage e-moderating model' and the new mantra of 'communities of practice'. This paper will argue that, although these frameworks are useful in informing and

> guiding learning technology practice, there are inherent dangers in them becoming too dominant a discourse. [5]

Instead of a grand narrative, Peeragogy is a growing collection of case studies and descriptive patterns. As we share our experiences and make needed adaptations, our techniques for doing peer learning and peer production become more robust. Based on the experiences described above, here are a few things people may want to try out in future projects:

- "Icebreaking" techniques or a "buddy system"; continual refactoring into teams.
- Maintain a process diagram that can be used to "triage" new ideas and effort.
- Prefer the "good" to the "best", but make improvements at the platform level as needed.
- Gathering some information from everyone who joins, and, if possible, everyone who leaves.

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CHAPTER 15

# ADDING STRUCTURE WITH ACTIVITIES

In the introduction to "Organizing a Learning Context", we remarked that a "learning space" is *only potentially* less structured than a "course". For example, a library tends to be highly structured, with quiet rooms for reading, protocols for checking out books, a cataloging and shelving system that allows people to find what they are looking for, as well as rules that deter vandalism and theft. (Digital libraries don't need to play by all the same rules, but are still structured.)

But more structure does not always lead to better learning. In a 2010 Forbes article titled, "The Classroom in 2020," George Kembel describes a future in which "Tidy lectures will be supplanted by messy real-world challenges." The Stanford School of Design, (or "d.school" – which Kemble co-founded and currently directs) is already well-known for its open collaborative spaces, abundant supply of Post-It notes and markers, and improvisational brainstorm activities – almost the opposite of traditional lecture-based learning.

One "unexpected benefit" of dealing with real-world challenges is that we can change our approach as we go. This is how it works in peer learning: peers can decide on different structures not just once (say, at the beginning of a course), but throughout the duration of their time together. This way, they are never "stuck" with existing structures, whether they be messy or clean. At least… that's the ideal.

In practice, "bottlenecks" frequently arise. For example, in a digital library context, there may be bottlenecks having to do with software development, organizational resources, community good will, or access to funding – and probably all of the above. In a didactic context, it may be as simple as one person knowing something that others do not.

While we can't eliminate scarcity in one stroke, we can design

activities for peer learning that are "scarcity aware" and that help us move in the direction of adaptive learning structures.

## Planning Peer Learning Activities

We begin with two simple questions:

- How do we select an appropriate learning activity?
- How do we go about creating a learning activity if we don't find an existing one?

"Planning a learning activity" should mean planning an *effective* learning activity, and in particular that means something that people can and will engage with. In short, an appropriate learning activity may be one that you already do! At the very least, current activities can provide a "seed" for even more effective ones.

But when entering unfamiliar territory, it can be difficult to know where to begin. And remember the bottlenecks mentioned above? When you run into difficulty, ask yourself: why is this hard? You might try adapting Zed Shaw's task-management trick, and make a list of limiting factors, obstacles, etc., then cross off those which you can find a strategy to deal with (add an annotation as to why). For example, you might decide to overcome your lack of knowledge in some area by hiring a tutor or expert consultant, or by putting in the hours learning things the hard way (Zed would particularly approve of this choice). If you can't find a strategy to deal with some issue, presumably you can table it, at least for a while.

Strategic thinking like this works well for one person. What about when you're planning activities for someone else? Here you have to be careful: remember, this is peer learning, not traditional "teaching" or "curriculum design". The first rule of thumb for *peer learning* is: don't plan activities for others unless you plan to to take part as a fully engaged participant. Otherwise, you might be more interested in the literature on *collaborative learning*, which has often been deployed to good effect within a stan-

dard pedagogical context (see e.g. Bruffee [1]). In a peer learning setting, everyone will have something to say about "what do you need to do" and "why is it hard," and everyone is likely to be interested in everyone else's answer as well as their own.

Furthermore, different participants will be doing different things, and these will be "hard" for different reasons. Part of *your* job is to try to make sure that not only are all of the relevant roles covered, but that the participants involved are getting enough support.

## One scenario: building activities for the Peeragogy Handbook

Adding a bunch of activities to the handbook won't solve all of our usability issues, but more activities would help. We can think about each article or section from this perspective:

1. When looking at this piece of text, what type of knowledge are we (and the reader) trying to gain? Technical skills, or abstract skills? What's the point?

2. What's difficult here? What might be difficult for someone else?

3. What learning activity recipes or models might be appropriate? (See e.g. [2], [3].)

4. What customizations do we need for this particular application?

### As a quick example: designing a learning activity for the current page

1. *We want to be able to come up with effective learning activities to accompany a "how to" article for peer learners*

2. *It might be difficult to "unplug" from all the reading and writing that we're habituated to doing.*

3. *But there are lots and lots of ways to learn.*

4. *Therefore, the proposed handbook activity is to simply step away from the handbook for a while.*

5. **Look for some examples of peer learning in everyday life. When you've gained an insight about peer learning from your own experience, come back and create a related activity to accompany another handbook page!**

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CHAPTER 16

# THE STUDENT AUTHORED SYLLABUS

In either formal learning, informal learning or models which transition between the two, there are many opportunities for learners to co-create the syllabus and/or outline their own course of action. The *sage on the stage* of formal instruction must become at the most *a guide on the side* who acts as a coach appearing only when needed rather than as a lecturer who determines the content that the learners need to master. In the following inspirational but certainly not prescriptive examples, we will focus on co-learning methods drawn from a Social Constructivist perspective, which fits nicely here.

We offer a few examples below to show a range of learner centered approaches. They all are based on co-learners hosting each other for one of a number of digestible topics in the larger subject area or domain that the group formed in order to explore. This can take place across a number of media and timelines.

The following methods will result in each co-learner gaining deep knowledge in a specific topic and moderate knowledge across several topics. The unique joy of this approach is that no two cohorts will ever be the same. The content will always be fresh, relevant, and changing. A group can even reconvene with slightly or dramatically different topics over and over using the same underlying process.

The appropriateness of the learner-created syllabus technique depends on two factors: 1) the involvement of experts in the group and 2) the level of proficiency of the group. In general, novices who may or may not have a deep interest in the subject matter benefit from more structure and experts who point to key concepts and texts. An example of this is the university survey course for first or second year students who, we assume, need more guidance as they enter the subject matter. Graduate seminars are generally much more fluid, open dialogues between

motivated experts require little structure or guidance.

We also need effective methods for groups which contain novices, experts, and everyone in between. In groups with a wide range of expertise, it is important that each co-learner chooses to focus their deep inquiry on a topic that they are less familiar with. This will *even out* the expertise level across the cohort as well as ensure that a co-learner is neither bored nor dominating the dialogue.

## 3 example designs to structure the learning

### Weekly topics structure

One way to structure the course is to have each co-learner host a topic each week. Perhaps multiple students host their topics in the same week. This progression provides a rotation of presentations and activities to support the entire group in engaging with the topics and challenges to the thinking of the presenters in a constructive and respectful manner.

> *Pro:* co-learners have discrete timelines and manageable chunks of responsibility.
>
> *Con:* the format may become disjointed, and the depth of inquiry will likely be somewhat shallow.

### Milestone based structure

In this structure, each co-learner hosts their topics in parallel with similar activities and milestones that the whole group moves through together. Milestones can be set for a certain date, or the group can *unlock* their next milestone whenever all participants have completed the previous milestone. This second milestone timeline can be great for informal groups in which participation levels vary from week to week due to external factors, and the sense of responsibility and game-like levels can be motivating for many co-learners.

Each co-learner may start with a post of less than 500 words introducing the topic on a superficial level. When everyone has

done this, the group might move on to posting questions to the post authors. Then, there may be a summary post of the activity so far with critical recommendations or insights.

> *Pro:* co- learners have more time to digest a topic, formulate a complex schema, and generate deeper questions.
>
> *Con:* it will be a few weeks before the topic level schema can form into a broader understanding of the subject matter or domain (seeing the big picture takes longer).

#### Relay learning structure.

This is similar to the milestone structure. However, co-learners rotate topics. If one learner posts an introductory write-up on a topic the first cycle, they may be researching questions on another topic in the next cycle, posting a summary in a third, and then posting a summary on their original topic in the fourth.

> *Pro:* co-learners can experience responsibility for several topics.
>
> *Con:* co-learners may receive a topic that is poorly researched or otherwise neglected.

## Content

### A vast number of topics

Within a subject of mutual interest to a group, there are a considerable number of topics or questions. What is important is that each co-learner can take responsibility for a reasonably narrow area given the duration of the course or the timeline of the group. Areas that are too broad will result in a very superficial understanding, and areas that are too narrow will result in a dull experience. For example, in marine biology, topics such as "the intertidal zone" may be too broad for a course cycle of a few weeks.

Narrowing to one species may be too specific for a course over a few months.

### Learner generated topics

Most cohorts will have some knowledge of the shared area of interest or an adjacent area. It is a good idea to respect the knowledge and experience that each member of the group brings to the table. A facilitator or coordinator may generate a list of potential topic areas, setting an example of the scale of a topic. We suggest that the participants in the group are also polled for additions to the list. In large courses, sending out a Google Form via email can be an effective way to get a quick list with a high response rate.

### Expert informed topics

If there is no expert facilitator in the group, we suggest that the cohort begin their journey with a few interviews of experts to uncover what the main buzz words and areas of focus might be. One way to locate this type of expert help is through contacting authors in the subject matter on social networks, reviewing their posts for relevance, and reaching out with the request.

We recommend two people interview the expert over video chat, for example in a Hangout. One person conducts the interview, and one person takes notes and watches the time. We strongly suggest that the interview be outlined ahead of time:

> *Warm up*: Who are you, what are your goals, and why do you think this interview will help?
>
> *Foundational questions*: Ask a few questions that might elicit short answers to build rapport and get your interviewee talking.
>
> *Inquiry*: What people say and what they do can often be very different. Ask about topics required for mastery of the subject matter (e.g. What are the areas someone would need to know about to be considered proficient in this subject?). Also, ask questions that

require storytelling. Avoid superlative or close-ended questions.

*Wrap up*: Thank the interviewee for his or her time, and be sure to follow up by sharing both what you learned and what you accomplished because he or she helped you.

## Shared goals and group norms

### Choosing useful outputs

Getting together for the sake of sharing what you know in an informal way can be fairly straightforward and somewhat useful. Most groups find that a common purpose and output that are explicitly defined and documented help to engage, motivate, and drive the group. For the examples above, the group may decide to create a blog with posts on the various topics or create a wiki where they can share their insights. Other outputs can include community service projects, business proposals, recommendations to senior management or administration, new products, and more. The key is to go beyond sharing for sharing sake and move toward an output that will be of use beyond the co-learning group. This activity is best described in Connectivist theory as the special case of networked learning where we find evidence of learning in collective action and/or behavioral change in groups rather than a psychological or neurological process in individuals.

### Group cohesion (a.k.a. the rules of the road)

One challenge of this kind of collaboration is that each group will need to decide on norms, acceptable practices and behaviors. Culturally diverse groups in particular may run into communication or other issues unless there is a way to create shared expectations and communicate preferences.

One way to do this is with a team charter. This is a living document where the initial rules of engagement can live for reference.

The group may add or edit this document over time based on experience, and that is a welcome thing! This documentation is a huge asset for new members joining the group who want to contribute quickly and effectively. Any co-editing word processing program will work, but we strongly recommend something that can be edited simultaneously and that lives in the cloud. (Google Docs is convenient because you can also embed your Charter into another site.)

Try starting with the following three sections, and allow some time for the group to co-edit and negotiate the document between icebreakers and kicking off the official learning process.

> *Mission:* Why are you forming the group? What do you want to accomplish together?
>
> *Norms:* Use netiquette? No flaming? Post your vacation days to a shared calendar? Cultural norms?
>
> *Members:* It is useful to include a photo and a link to a public profile such as Twitter, Google+ or Facebook.

## Assessments and feedback loops

### Co-authored assessment rubrics

Tests. Quizzes. Exams. How can the co-learning group assess their performance?

These types of courses benefit from an approach similar to coaching. Set goals as individuals and a group in the beginning, define what success looks like, outline steps that are needed to achieve the goal, check in on the goal progress periodically, and assess the results at the end of the course against the goal criteria. Goals may include domain expertise, a business outcome, a paper demonstrating mastery, a co-created resource, or even the quality of collaboration and adherence to shared group norms.

### Learner created assessments

Another effective way to create an assessment is to decide on an individual or group output and create a peer assessment rubric based on the goals of the individual or group.

One way to create a rubric is to spend some time defining the qualities you want your output to have based on positive examples. Perhaps a group wants to create a blog. Each person on the team may identify the qualities of a great blog post based on examples that they admire. They can use that example to create a criteria for assessment of co-learner authored blog posts. We recommend that the criteria have a 0 to 5 point scale with 0 being non-existent and 5 being superb. Writing a few indicators in the 1, 3, and 5 columns helps to calibrate reviewers.

Create a shared document, perhaps starting with a list of criteria. Collapse similar criteria into one item, and create the indicators or definitions of 1, 3, and 5 point performance. Agree on the rubric, and decide on how the co-learners will be assigned assessment duties. WIll everyone review at least two others? Will each co-learner product need at least 3 reviewers before it goes live? Will you use a spreadsheet or a form to collect the assessments?

In a university setting, the instructor of record may wish to approve a peer assessment rubric, and it is sometimes a good idea to have a few outside experts give feedback on criteria that the group may have missed.

### Outside assessments

It is possible that an instructor of record or similar authority will create the assessment for performance. In these cases, it is crucial that the co-learners have access to the grading rubric ahead of time so that they can ensure their activities and timeline will meet any requirements. In this case, it may be possible to require that the co-learners self-organize entirely, or there may be intermediary assignments such as the charter, project plan or literary review.

## Cyclical use of these models

### So much more to learn

As mentioned above, the joy of this type of learning is that no two groups will ever do it the same. Their process, goals, and outcomes can all be unique. As designers and facilitators of this type of learning environment, we can say it is a wild ride! Each class is exciting, refreshing, and on trend. The co-learners become our teachers.

If a group generates more topics than it is possible to cover at one time given the number of group members or if a group has plans to continue indefinitely, it is always possible to set up a system where potential topics are collected at all times. These unexplored topics can be harvested for use in another learning cycle, continuing until the group achieves comprehensive mastery.

## Risks

This format is not without its own unique pitfalls: some challenges are learner disorientation or frustration in a new learning structure with ambiguous expectations and uneven participation. Some groups simply never gel, and we do not know why they have failed to achieve the cohesion required to move forward. Other groups are the exact opposite. Here are a few risks to consider if you would like to try the methods suggested here and how to mitigate them.

> *Uneven expertise:* Ask co-learners to be responsible for topics that are new to them.
>
> *Uneven participation and cohesion:* Ask co-learners what they want to do to motivate the group rather than imposing your own ideas.
>
> *Experts/facilitators that kill the conversation:* In the charter or other documentation, explicitly state that the purpose of the discussion is to further the conversation, and encourage experts to allow others to ex-

plore their own thinking by asking probing (not leading) questions.

*Ambiguous goals:* Encourage the group to document their mission and what they will do as a team. This can change over time, but it is best to start out with a clear purpose.

## Conclusion

Make mistakes. Correct course. Invite new perspectives. Create a structure that everyone can work with. Change it when it breaks. Most of all, have fun!

CHAPTER 17

# CASE STUDY: COLLABORATIVE EXPLORATIONS

## Part I (Peter).

Collaborative Exploration invites participants to shape their own directions of inquiry and develop their skills as investigators and teachers (in the broadest sense of the word). The basic mode of a Collaborative Exploration centers on interactions over a delimited period of time in small groups. Engagement takes place either online, for instance via Google+, or face-to-face. The aim is to create an experience of re-engagement with oneself as an avid learner and inquirer. This section combines practical information about how to run Collaborative Explorations as well as ideas and questions about how to make sense of what happens in them. A companion entry conveys one participant's experience with several Collaborative Explorations (hereafter, "CE").

### Overview and contrast to cMOOCs

The tangible goal of any CE is to develop contributions to the topic defined by the "case", which is written by the host or originator of the CE in advance, and which is intended to be broad and thought-provoking (some examples are given below). We aim for a parallel experiential goal, which is that we hope participants will be impressed at how much can be learned with a small commitment of time using this structure. The standard model for an online CE is to have four sessions spaced one week apart, in which the same small group interacts in real time via the internet, for an hour per session. Participants are asked to spend at least 90 minutes between sessions on self-directed inquiry into the case, and to share their inquiries-in-progress with their small group and a wider community. Reflection typically involves shifts in partici-

pants' definition of what they want to find out and how. Any participants wondering how to define a meaningful and useful line of inquiry are encouraged to review the scenario for the CE, any associated materials, posts from other participants, and to think about what they would like to learn more about or dig deeper into. Everyone is left, in the end, to judge for themselves whether what interests them is meaningful and useful.

During the live sessions, participants can expect to do a lot of listening, starting off in the first session with autobiographical stories that make it easier to trust and take risks with whoever has joined that CE, and a lot of writing to gather their thoughts, sometimes privately, sometimes shared. There is no assumption that participants will pursue the case beyond the limited duration of the CE. This said, the tools and processes that the CE employs for purposes of inquiry, dialogue, reflection, and collaboration are designed to be readily learned by participants, and to translate well into other settings – for instance, where they can be used to support the inquiries of others. In short, online CEs are moderate-sized open online collaborative learning. It remains to be seen whether the CE "movement" will attract enough participants to scale up to multiple learning communities around any given scenario, each hosted by a different person and running independently. A MOOC (massive open online course) seeks to get masses of people registered, knowing that a tiny fraction will complete it, while CE best practices focus on establishing effective learning in small online communities, and then potentially scale up from there by multiplying out. CEs aim to address the needs of online learners who want to:

- dig deeper, make "thicker" connections with other learners
- connect topics with their own interests
- participate for short periods of time
- learn without needing credits or badges

Currently, even the most high-profile MOOCs do not appear to be conducive to deep or thick inquiry. For example, while link-sharing is typical in "connectivist" or "cMOOCs", annotation and

discussion of the contents is less common. By contrast, CEs are structured to elicit participants' thoughtful reflections and syntheses. The use of the internet for CEs, in contrast, is guided by two principles of online education (Taylor 2007).

- Use computers first and foremost to teach or learn things that are difficult to teach or learn with pedagogical approaches that are not based on computers
- Model computer use, at least initially, on known best practices for teaching/learning without computers.

Thus, CEs bring in participants from a distance, make rapid connections with informants or discussants outside the course, and contribute to evolving guides to materials and resources. At the same time, participants benefit from the support of instructors/facilitators and peers who they can trust, and integrate what they learn with their own personal, pedagogical, and professional development.

## Example scenarios or "cases"

### Connectivist MOOCs: Learning and collaboration, possibilities and limitations

The core faculty member of a graduate program at a public urban university wants help as they decide how to contribute to efforts made at the university program to promote open digital education. It is clear that the emphasis will not be on xMOOCs, i.e., those designed for transmission of established knowledge, but on cMOOCs. In other words, the plan is to emphasize connectivist learning and community development emerges around, but may extend well beyond, the materials provided by the MOOC hosts (Morrison 2013; Taylor 2013). What is not yet clear is just how learning works in cMOOCs. What are the possibilities and limitations of this educational strategy? How do they bear on themes like creativity, community, collaboration, and openness? The program is especially interested in anticipating any undesirable consequences…

**Science and policy that would improve responses to extreme climatic events**

Recent and historical climate-related events shed light on the social impact of emergency plans, investment in and maintenance of infrastructure, as well as investment in reconstruction. Policy makers, from the local level up, can learn from the experiences of others and prepare for future crises. The question for this case is how to get political authorities and political groups—which might be anywhere from the town level to the international, from the elected to the voluntary—interested in learning about how best to respond to extreme climatic events. Changes might take place at the level of policy, budget, organization, and so on. It should even be possible to engage people who do not buy into the idea of human-induced climate change—after all, whatever the cause, extreme climatic events have to be dealt with….

## The structure

Independent of the topic, we've found the following common structure useful for our online CEs. *Before the first live session*: Participants review the scenario, the expectations and mechanics, join a special-purpose Google+ community and get set up technically for the hangouts.

**Session 1**: *Participants getting to know each other.* After freewriting to clarify thoughts and hopes, followed by a quick check-in, participants take 5 minutes each to tell the story of how they came to be a person who would be interested in participating in a Collaborative Exploration on the scenario. Other participants note connections with the speaker and possible ways to extend their interests, sharing these using an online form.

*Between-session work*: Spend at least 90 minutes on inquiries related to the case, posting about this to Google+ community for the CE, and reviewing the posts of others.

**Session 2**: *Clarify thinking and inquiries.* Freewriting on one's thoughts about the case, followed by a check-in, then turn-taking "dialogue process" to clarify what participants are thinking

about their inquiries into the case. Session finishes with gathering and sharing thoughts using an online form.

*Between-session work*: Spend at least 90 minutes on (a) inquiries related to the case and (b) preparing a work-in-progress presentation.

**Session 3**: *Work-in-progress presentations.* 5 minutes for each participant, with "plus-delta" feedback given by everyone on each presentation.

*Between-session work*: Digest the feedback on one's presentation and revise it into a self-standing product (i.e., one understandable without spoken narration).

**Session 4**: *Taking Stock.* Use same format as for session 2 to explore participants' thinking about (a) how the Collaborative Exploration contributed to the topic (the tangible goal) and to the experiential goal, as well as (b) how to extend what has emerged during the CE.

*After session 4 (optional)*: Participants share on a public Google+ community not only the products they have prepared, but also reflections on the Collaborative Exploration process.

## How to make sense of what happens in CEs

(Re)engagement with oneself as an avid learner and inquirer in CEs is made possible by the combination of:

- Processes and tools used for inquiry, dialogue, reflection, and collaboration;
- Connections made among the diverse participants who bring to bear diverse interests, skills, knowledge, experience, and aspirations;
- Contributions from the participants to the topics laid out in scenarios.

The hope is that through experiencing a engagement with learning, participants will subsequently transfer experience with this triad into their own inquiries and teaching-learning interactions, the ways that they support inquiries of others; other prac-

tices of critical intellectual exchange and cooperation; and that they will be more prepared to challenge the barriers to learning that are often associated with expertise, location, time, gender, race, class, or age.

## Acknowledgements

The comments of Jeremy Szteiter and the contributions of the participants of the 2013 Collaborative Explorations have helped in the preparation of this article.

# Part II (Teryl).

As a May graduate of the Master's program in Critical and Creative Thinking (CCT) at UMass Boston, I owe my gratitude to Professors Peter Taylor and Jeremy Szteiter for inviting me to informally continue my education less than a month later. It is a tribute to them that I would then take four consecutive CEs without stopping. They can best share how to run a CE, but as a "student," it is how to creatively take a CE that I'd like to share.

**June 2013 CE: Scaffolding Creative Learning** I was grateful participants took the time to post links and ideas to support my inquiries, yet something else intrigued me about the potential of Collaborative Exploration. Luanne Witkowski, an artist and one of the CCT instructors, took our ideas and made a diagram incorporating our scaffolding concepts together; she changed her own original drawing to include all of ours. I wanted to pay forward and back my learning too, so I combined the ideas of all the participants, adapted and taught a lesson outside the CE and then shared the results. From this jumping into someone else's scaffolding, I went into even more experimental learning in the next CE.

**July 2013 CE: Design in Critical Thinking** In a second CE, I took the title literally and developed a design IN critical thinking.

To try out my triangle tangent thinking model, during a lesson on leadership in church, I suddenly stopped teaching a classroom of older professional adults halfway in and asked them to participate in "design as you go" curriculum—by taking over the class. Since I wanted to be fair, along with my lesson outline I had already given them a supposed "icebreaker" activity that they could teach from, although they also had the option of my continued teaching. Results? My triangle drawing works as a lesson plan; the class took the tangent, but surprisingly, I wasn't just relegated to moderator, it became a true co-facilitation,a model of change at the midpoint for both the individual and community in the choices and direction.

**September 2013 CE: Everyone Can Think Creatively** This CE had to be commended for its participants humoring my project and allowing the exploration of testing a CE itself. Was it possible to be a Creative Failure in a Creativity CE? To evaluate "Creative Failure in a Creativity CE," I used a simple test. If creative success (unknowingly given by my CE community) was a product both "novel AND useful," any post without a comment was a failure ("not useful") to my readers. Any post that a reader commented on that was similar to something else already done was "useful," but not novel. Failure had me posting again. Did I mention what nice people these were when they didn't know what I was doing? It would have been easy for them to ignore my continued posting, yet the community of a CE cannot be praised enough. They were supportive of me and finding academic colleagues who have a sense of humor is mercifully not novel, but extremely useful in this experience.

**October 2013 CE: Stories to Scaffold Creative Learning** In this CE I gave myself the challenge of indirect teaching. Could I be a story "shower", not teller? I took concepts important to me about teaching with story, yet also tried to leave space for others' interpretations. Ironically, in some ways creative failure continued—again I was not as helpful as I had wished. This CE

also had a twist—no hero stories allowed, so my creative and personal stories had to be ambiguous or use other connecting structures based on the participants' preferences. It was interesting which stories worked best—fiction worked more with humor, real experience worked if I shared about someone other than myself and other kinds worked with visuals. Collaborative Explorations provide a safe space for joint learning and teaching to occur. The resulting diversity blends well into a community that is curious, courageous and creative. Although I have an M.A. as the first completely online CCT student, the deeply connected CE community had face-to-face learning "feel." It does require time, openness, and commitment during times of collective intense focus on a topic. Yet, seeing where the participant-directed 'design as you go' curriculum ends up is worth investing in and sharing with others. After all, there are many other ways still out there to try out CEs.

**Postscript** I also ran a CE for the Susquehanna Conference of the UMC for 10 days, working with a group of professionals exploring a call into ordained ministry. Going in cold, I had to work harder to do community building without the Google hangout meetings and recommend their inclusion to increase the comfort level and participation of the group members.

## Resources

Further examples of CE scenarios can be viewed at

`http://cct.wikispaces.com/CEt`.

Recommended readings below convey some of the sources for the CE processes. Ideas about possible extensions of CEs can be viewed in the full prospectus at

`http://cct.wikispaces.com/CEp`.

## References

1. Morrison, D. (2013). "A tale of two MOOCs @ Coursera: Divided by pedagogy".

2. Taylor, P. J. (2007) "Guidelines for ensuring that educational technologies are used only when there is significant pedagogical benefit," International Journal of Arts and Sciences, 2 (1): 26-29, 2007 (adapted from `http://bit.ly/etguide`).

3. Taylor, P. J. (2013). "Supporting change in creative learning".

## Recommended Reading

1. Paley, V. G. (1997). The Girl with the Brown Crayon. Cambridge, MA, Harvard University Press.

2. Paley, V.G. (2010). The Boy on the Beach: Building Community by Play. Chicago, University of Chicago Press.

3. Taylor, P. J. and J. Szteiter (2012). Taking Yourself Seriously: Processes of Research and Engagement Arlington, MA, The Pumping Station.

4. White, M. (2011). Narrative Practice: Continuing the Conversation. New York, Norton.

# Part VI

# Cooperation

CHAPTER 18

# INTRODUCTION TO COOPERATION: CO-FACILITATION

Facilitation is a process of helping groups work cooperatively and effectively. Facilitation can be particularly helpful for individuals who, based on a certain level of insecurity or inexperience, tend to lurk rather than participate. At the same time, in peeragogy, a facilitator isn't necessarily an "authority." Rather, facilitation work is done in service to the group and the group dialogue and process. For example, a facilitator may simply "hold space" for the group by setting up a meeting or a regular series of discussions.

## Co-facilitating in peer-to-peer learning

Co-facilitation can be found in collaborations between two or more people who need each other to complete a task, for example, learn about a given subject, author a technical report, solve a problem, or conduct research. Dee Fink writes that "in this process, there has to be some kind of change in the learner. No change, no learning" [1]. Significant learning requires that there be some kind of lasting change that is important in terms of the learner's life; in peeragogy, one way to measure the effectiveness of co-facilitation is to look for a change in the peer group.

Co-facilitation roles can be found in groups/teams like basketball, health, Alcoholics Anonymous, spiritual groups, etc. For example, self-help groups are composed of people who gather to share common problems and experiences associated with a particular problem, condition, illness, or personal circumstance. There are some further commonalities across different settings. Commenting on the work of Carl Rogers:

> **Godfrey Barrett-Lennard**: The educational situation which most effectively promotes significant

> learning is one in which (1) threat to the self of the learner is reduced a minimum, and (2) differentiated perception of the field of experience is facilitated. [2]

Part of the facilitator's role is to create a safe place for learning to take place; but they should also challenge the participants.

> **John Heron**: Too much hierarchical control, and participants become passive and dependent or hostile and resistant. They wane in self-direction, which is the core of all learning. Too much cooperative guidance may degenerate into a subtle kind of nurturing oppression, and may deny the group the benefits of totally autonomous learning. Too much autonomy for participants and laissez-faire on your part, and they may wallow in ignorance, misconception, and chaos. [3]

## Co-facilitating discussion forums

If peers are preparing a forum discussion, here are some ideas from "The Community Tool Box", that can be helpful as guidelines:

- Explain the importance of collaborative group work and make it a requirement.
- Establish how you will communicate in the forum.
- Be aware of mutual blind spots in facilitating and observing others.
- Watch out for different rhythms of intervention.

## Co-facilitating wiki workflows

A good place to begin for any group of co-facilitators working with a wiki are Wikipedia's famous "5 Pillars."

- Wikipedia is an encyclopedia.

- Wikipedia writes articles from a neutral point-of-view.
- Wikipedia is free content that anyone can edit, use, modify, and distribute.
- Editors should interact with each other in a respectful and civil manner.
- Wikipedia does not have firm rules.

## Co-facilitating live sessions

Learning experiences in live sessions are described in the article Learning Re-imagined: Participatory, Peer, Global, Online by Howard Rheingold, and many of these points are revisited in the handbook section on real-time tools. But we want to emphasize one point here:

> **Howard Rheingold**: Remember you came together with your peers to accomplish something, not to discuss an agenda or play with online tools; keep everything as easily accessible as possible to ensure you realize your goals.

## References

1. Fink, L. D (2003). *Creating significant learning experiences: An integrated approach to designing college courses.* John Wiley & Sons.

2. Barrett-Lennard, G. T. (1998). *Carl Rogers' Helping System: Journey & Substance.* Sage.

3. Heron, J. (1999). *The complete facilitator's handbook.* London: Kogan Page.

CHAPTER 19

# THE WORKSCAPE, A LEARNING PLATFORM FOR CORPORATIONS

Cultivating a results-oriented peer-learning program in a corporate learning ecosystem involves a few tweaks of the approach and tools we discussed in relation to more open, diverse networks.

## The Workscape, a platform for learning

Formal learning takes place in classrooms; informal learning happens in *workscapes.* A workscape is a learning ecosystem. As the environment of learning, a workscape includes the workplace. In fact, a workscape has no boundaries. No two workscapes are alike. Your workscape may include being coached on giving effective presentations, calling the help desk for an explanation, and researching an industry on the Net. My workscape could include participating in a community of field technicians, looking things up on a search engine, and living in France for three months. Developing a platform to support informal learning is analogous to landscaping a garden. A major component of informal learning is natural learning, the notion of treating people as organisms in nature. People are free-range learners. Our role is to protect their environment, provide nutrients for growth, and let nature take its course. A landscape designer's goal is to conceptualize a harmonious, unified, pleasing garden that makes the most of the site at hand. A workscape designer's goal is to create a learning environment that increases the organization's longevity and health and the individual's happiness and well-being. Gardeners don't control plants; managers don't control people. Gardeners and managers have influence but not absolute authority. They can't make a plant fit into the landscape or a person fit into a team. In an ideal Workscape, workers can easily find the people and informa-

tion they need, learning is fluid and new ideas flow freely, corporate citizens live and work by the organization's values, people know the best way to get things done, workers spend more time creating value than handling exceptions, and everyone finds their work challenging and fulfilling.

## The technical infrastructure of the Workscape

When an organization is improving its Workscape, looking at consumer applications is a good way to think about what's required. Ask net-savvy younger workers how they would like to learn new skills, and they bring up the features they enjoy in other services:

- Personalize my experience and make recommendations, like Amazon.
- Make it easy for me to connect with friends, like Facebook.
- Keep me in touch with colleagues and associates in other companies, as on LinkedIn.
- Persistent reputations, as at eBay, so you can trust who you're collaborating with.
- Multiple access options, like a bank that offers access by ATM, the Web, phone, or human tellers.
- Don't overload me. Let me learn from YouTube, an FAQ, or linking to an expert.
- Show me what's hot, like Reddit, Digg, MetaFilter, or Fark do.
- Give me single sign-on, like using my Facebook profile to access multiple applications.
- Let me choose and subscribe to streams of information I'm interested in, like BoingBoing, LifeHacker or Huffpost.
- Provide a single, simple, all-in-one interface, like that provided by Google for search.
- Help me learn from a community of kindred spirits, like SlashDot, Reddit, and MetaFilter.

- Give me a way to voice my opinions and show my personality, as on my blog.
- Show me what others are interested in, as with social bookmarks like Diigo and Delicious.
- Make it easy to share photos and video, as on Flickr and YouTube.
- Leverage "the wisdom of crowds," as when I pose a question to my followers on Twitter or Facebook.
- Enable users to rate content, like "Favoriting" an item on Facebook or +!ing is on Google or YouTube.

Some of those consumer applications are simple to replicate in-house. Others are not. You can't afford to replicate Facebook or Google behind your firewall. That said, there are lots of applications you can implement at a reasonable cost. Be skeptical if your collaborative infrastructure doesn't include these minimal functions:

**Profiles** - for locating and contacting people with the right skills and background. Profile should contain photo, position, location, email address, expertise (tagged so it's searchable). IBM's Blue Pages profiles include how to reach you (noting whether you're online now), reporting chain (boss, boss's boss, etc.), link to your blog and bookmarks, people in your network, links to documents you frequently share, members of your network.

**Activity stream** - for monitoring the organization pulse in real time, sharing what you're doing, being referred to useful information, asking for help, accelerating the flow of news and information, and keeping up with change

**Wikis** - for writing collaboratively, eliminating multiple versions of documents, keeping information out in the open, eliminating unnecessary email, and sharing responsibility for updates and error correction

**Virtual meetings** - to make it easy to meet online. Minimum feature set: shared screen, shared white board, text chat, video of participants. Bonus features: persistent meeting room (your office online), avatars.

**Blogs** - for narrating your work, maintaining your digital reputation, recording accomplishments, documenting expert knowledge, showing people what you're up to so they can help out

**Bookmarks** - to facilitate searching for links to information, discover what sources other people are following, locate experts

**Mobile access** - Half of America's workforce sometimes works away from the office. Smart phones are surpassing PCs for connecting to networks for access and participation. Phones post more Tweets than computers. Google designs its apps for mobile devices before porting them to PCs.

**Social network** - for online conversation, connecting with people, and all of the above functions.

## Conclusion

Learning used to focus on what was in an individual's head. The individual took the test, got the degree, or earned the certificate. The new learning focuses on what it takes to do the job right. The workplace is an open-book exam. What worker doesn't have a cell phone and an Internet connection? Using personal information pipelines to get help from colleagues and the Internet to access the world's information is encouraged. Besides, it's probably the team that must perform, not a single individual. Thirty years ago, three-quarters of what a worker need to do the job was stored in her head; now it's less than 10%.

CHAPTER 20

# PARTICIPATION

> Methods of managing projects, including learning projects, range from more formal and structured to casual and unstructured. As a facilitator, you'll see your peeragogy community constantly adjust, as it seeks an equilibrium between order and chaos, ideally allowing everyone to be involved at their own pace without losing focus, and in such a manner that the collective can deliver.

For teachers reading this, and wondering how to use peeragogy to improve participation in their classrooms, it's really quite simple: reframe the educational vision using peeragogical eyes. Recast the classroom as a community of people who learn together, the teacher as facilitator, and the curriculum as a starting point that can be used to organize and trigger community engagement. However, just because it's simple doesn't mean it's easy! Whatever your day job may be, consider: how well do the various groups you participate in work together – even when the members ostensibly share a common purpose? Sometimes things tick along nicely, and, presumably, sometimes it's excruciating. What's your role in all of this? How do *you* participate?

## Guidelines for participation

- Accept that some people want to watch what is going on before jumping in. This doesn't mean you have to keep them hanging around forever. After a while, you may unenroll people who don't add any value to the community. In our Peeragogy project, we've asked people to explicitly re-enroll several times. Most do renew; some leave.

- Accept that people may only contribute a little: if this contribution is good it will add value to the whole.
- Understand that you can not impose strict deadlines on volunteers; adjust targets accordingly.
- Let your work be "open" in the sense described in Wikipedia's Neutral Point of View policy.
- Give roles to participants and define some "energy centers" who will take the lead on specific items in the project.
- Organize regular face-to-face or online meetings to talk about progress and what's needed in upcoming days/weeks.
- Ask participants to be clear about when they will be ready to deliver their contributions.
- Have clear deadlines, but allow contributions that come in after the deadline – in general, be flexible.
- Add a newcomer section on your online platform to help new arrivals get started. Seasoned participants are often eager to serve as mentors.

When we think about project management in an organization, we often relate to well-established tools and processes. For example, we can use the Project Management Body of Knowledge (PMBOK) as a standard. For the Project Management Institute (PMI) and many workers, these standards are seen as the key to project success. In classical project management, tasks and deadlines are clearly defined. We will, for example, use Program Evaluation and Review Technic (PERT) to analyze and represent tasks. We often represent the project schedule using a Gantt chart. Those are just two of the project management tools that illustrate how "mainstream" project management rests firmly on an engineering background. In these very structured projects, each actor is expected to work exactly as planned and to deliver his part of the work on time; every individual delay can potentially lead to a collective delay.

Peeragogy projects may be, naturally, a bit different from other settings, although we can potentially reuse both formal and informal methods of organization. For example, unlike a typical

wiki – or classroom – peeragogy projects often expect to break the 90/9/1 rule. Keep in mind that some participants may not contribute all the time – but one really good idea can be a major contribution. See the anti-pattern "Misunderstanding Power" for some further reflections on these matters.

How are we doing? If we consider our basic population to be those in our Google+ community, then as of January 2014, around 4% had contributed to the handbook – pretty good. However, we have yet to reach a contribution profile anything like 70/20/10. It's important to remember that – especially in a volunteer organization – no one can "make"' other people participate, and that all the lists of things to do are for nought if no one steps in to do the work. For this reason, if anything is going to happen, what's needed are *realistic* estimates of available work effort. Finally, in closing this section, we want to emphasize that measures of participation offer only a very rough proxy for measures of learning, although the two are clearly related.

CHAPTER 21

# NEW DESIGNS FOR CO-WORKING AND CO-LEARNING

Interpersonal exchange and collaboration to develop and pursue common goals goes further than "learning" or "working" in their mainstream definitions. This article will look at examples drawn from Linux, Wikipedia, and my own work on PlanetMath, with a few surprises along the way, leading us to new ways of thinking about how to do co-design when building systems for peer learning and peer production.

## Co-working as the flip side of convening

> **Linus Torvalds**: The first mistake is thinking that you can throw things out there and ask people to help. That's not how it works. You make it public, and then you assume that you'll have to do all the work, and ask people to come up with suggestions of what you should do, not what they should do. Maybe they'll start helping eventually, but you should start off with the assumption that you're going to be the one maintaining it and ready to do all the work. The other thing–and it's kind of related–that people seem to get wrong is to think that the code they write is what matters. No, even if you wrote 100% of the code, and even if you are the best programmer in the world and will never need any help with the project at all, the thing that really matters is the users of the code. The code itself is unimportant; the project is only as useful as people actually find it.

In fact, we can think of contributors as a special class of "user"

with a real time investment in the way the project works. We typically cannot "Tom Sawyer" ourselves into leisure or ease just because we manage to work collaboratively, or just because we have found people with some common interests. And yet, in the right setting, many people do want to contribute! For example, on "Wikipedia, the encyclopedia anyone can edit" (as of 2011) as many as 80,000 visitors make 5 or more edits per month. This is interesting to compare with the empirical fact that (as of 2006) "over 50% of all the edits are done by just .7% of the users... *24 people*... and in fact the most active 2%, which is 1400 people, have done 73.4% of all the edits." Similar numbers apply to other peer production communities.

## A little theory

In many natural systems, things are not distributed equally, and it is not atypical for e.g. 20% of the population to control 80% of the wealth (or, as we saw, for 2% of the users to do nearly 80% of the edits). Many, many systems work like this, so maybe there's a good reason for it. Let's think about it in terms of "coordination" as understood by the late Elinor Ostrom. She talked about "local solutions for local problems". By definition, such geographically-based coordination requires close proximity. What does "close" mean? If we think about homogeneous space, it just means that we draw a circle (or sphere) around where we are, and the radius of this circle (resp. sphere) is small.

An interesting mathematical fact is that as the dimension grows, the volume of the sphere gets "thinner", so the radius must increase to capture the same $d$-dimensional volume when $d$ grows! In other words, the more different factors impact on a given issue, the less likely there are to be small scale, self-contained, "local problems" or "local solutions" in the first place.

As a network or service provider grows (like a MOOC as opposed to a Collaborative Exploration, for example), they typically build many weak ties, with a few strong ties that hold it all together. Google is happy to serve everyone's web requests – but

they can't have just anyone walking in off the street and connecting devices their network in Mountain View.

By the way, the 2006 article about Wikipedia quoted above was written by Aaron Swartz ("over 50% of all the edits are done by... 24 people", etc.), who achieved considerable notoriety for downloading lots and lots of academic papers with a device plugged into MIT's network. His suicide while under federal prosecution for this activity caused considerable shock, grief, and dismay among online activists. One thing we could potentially take away from the experience is that there is a tremendous difference between a solo effort and the distributed peer-to-peer infrastructures like the ones that underly the PirateBay, which, despite raids, fines, jail sentences, nation-wide bans, and server downtime, has proved decidedly hard to extinguish. According to a recent press release: "If they cut off one head, two more shall take its place."

## Co-working: what is an institution?

As idealists, we would love to be able to create systems that are both powerful and humane. Some may reflect with a type of sentimental fondness on completely mythical economic systems in which a "dedicated individual could rise to the top through dint of effort." But well-articulated systems like this *do* exist: natural languages, for example, are so expressive and adaptive that most sentences have never been said before. A well-articulated system lends itself to "local solutions to local problems" – but in the linguistics case, this is only because all words are not created equal.

> **Dr Seuss**: My brothers read a little bit. Little words like 'If' and 'It.' My father can read big words, too, Like CONSTANTINOPLE and TIMBUKTU.

We could go on here to talk about Coase's theory of the firm, and Benkler's theory of "Coase's Penguin". We might continue quoting from Aaron Swartz. But we will not get so deeply into

that here: you can explore it on your own! For now, it is enough to say that an institution is a bit like a language. This will help us a lot in the next section.

## Designing a platform for peer learning

> PlanetMath *is a virtual community which aims to help make mathematical knowledge more accessible.*

In my PhD thesis [1], I talk about my work to turn this long-running website, which since 2001 had focused on building a mathematics encyclopedia, into a peer produced peer learning environment. We wanted to retain all of the old activities related to authoring, reviewing, and discussing encyclopedia articles, but we would also add a bunch of new features having to do with mathmatical problem solving, an activity that is suitable for mathematical beginners.

My first translation of this idea into a basic interaction design was as follows. People can continue to add articles to PlanetMath's encyclopedia: they can connect one article to another (A→A) either by making one article the "parent" of another, or, more typically, via an inline link. Like in the old system, users can discuss any object (X→T), but now there is more structure: *problems* can be connected to articles (A→P) and *solutions* can be connected to problems (P→S). Instead of explicitly modeling "goals," I decided that problems and articles could be organized into "collections," the same way that videos are organized into playlists on YouTube, and that the user would get encouraging directed feedback as they work their way through the problems in a given collection. I described a few other types of objects and interactions, like questions and answers, groups, and the ability to change the "type" of certain contributed objects.

The next step was to do a complete overhaul of PlanetMath's software system, to build something that could actually *do* all of that. After deploying the realized system and doing some studies with PlanetMath users, I realized the design summarized above was not complete. Note that this is very much along the lines of

what Linus Torvalds said above: I did the design, and me and a small group of collaborators with their own vested interests built the system, then we put it out there to get more ideas from users.

The main thing that was missing from the earlier design was the idea of a *project*. From interviewing users, it became clear to me that it would be helpful to think of every object as being part of at least one project: everything should have someone looking after it! Importantly, getting back to the very beginning of this article, each project can define its own purpose for existing. Here's how I put it in my thesis:

> *Actions and artifacts are embedded within projects, which can be modeled in terms of informal user experience and formal system features. Project updates can be modeled with a language of fundamental actions. Projects themselves model their outcomes, and are made "viable" by features that connect to the motivations and ambitions of potential participants.*

The key point is that the evolving design describes a sort "grammar" for the kinds of things that can be done on PlanetMath. In the updated design, projects are something like paragraphs that combine simple sentences. The language can be extended further, and I hope that will happen in further study. In particular, we need to understand more about how the "sublanguage" of project updates works (compare the Roadmap pattern described in this handbook).

## The discussion continues: Reliving the history of mathematics as a peeragogical game?

These notes have shown one approach to the design of spaces for learning and knowledge building. Although the article has focused on mathematics learning, similar reflections would apply to designing other sorts of spaces for learning or working, for instance, to the continued development of the Peeragogy project

itself! Perhaps it can contribute to the development of a new kind of institution.

> **Doug Breitbart**: It occurred to me that you could add a learning dimension to the site that sets up the history of math as a series of problems, proofs and theorems that, although already solved, could be re-cast as if not yet solved, and framed as current challenges which visitors could take on (clearly with links to the actual solutions, and deconstruction of how they were arrived at, when the visitor decides to throw in the towel).

## References

1. Corneli, J. (2014). Peer Produced Peer Learning: A Mathematics Case Study. Ph. D. thesis. The Open University.

CHAPTER 22

# A CO-WORKING STORY

The board of a housing association needs to set a strategy that takes account of major changes in legislation, the UK benefits system and the availability of long term construction loans. Julian, eager to make use of his new-found peeragogical insights suggests an approach where individuals research specific factors and the team work together to draw out themes and strategic options. In the beginning, he proposes that each board member researches an area of specific knowledge or interest.

Jim, the Chairman, identifies questions he wants to ask the Chairs of other Housing Associations. Pamela (a lawyer) agrees to do an analysis of the relevant legislation. Clare, the CEO, plans out a series of meetings with the local councils in the boroughs of interest to understand their reactions to the changes from central government. Jenny, the operations director, starts modelling the impact on occupancy from new benefits rules. Colin, the development director, re-purposes existing work on options for development sites to reflect different housing mixes on each site. Malcolm, the finance director, prepares a briefing on the new treasury landscape and the changing positions of major lenders.

Each member of the board documents their research in a private wiki. Julian facilitates some synchronous and asynchronous discussion to draw out themes in each area and map across the areas of interest. Malcolm, the FD, adapts his financial models to take different options as parameters. Clare refines the themes into a set of strategic options for the association, with associated financial modelling provided by Malcolm. Individual board members explore the options asynchronously before convening for an all-day meeting to confirm the strategy.

# Part VII

# Assessment

CHAPTER 23

# INTRODUCTION TO PEERAGOGICAL ASSESSMENT

This article is about both assessment in peer learning and an exercise in assessment, as we put our strategy for assessment into practice by evaluating the Peeragogy Handbook itself.

## Adapting strategies for learning assessment to the peer-learning context

In "Effective Grading: A Tool for Learning and Assessment," Barbara E. Walvoord and Virginia Johnson Anderson have outlined an approach to grading. They address three questions:

1. Who needs to know, and why?
2. Which data are collected?
3. How does the assessment body analyze data and present findings?

The authors suggest that institutions, departments, and assessment committees should begin with these simple questions and work from them towards anything more complex. These simple questions provide a way to understand - and assess - any strategy for assessment! For example, consider "formative assessment" (in other words, keeping track of how things are going). In this context, the answers to the questions above would be:

1. Teachers need to know about the way students are thinking about their work, so they can deliver better teaching.

2. Teachers gather a lot of these details on learning activities by "listening over the shoulders" of students.

3. Teachers apply analysis techniques that come from their training or experience – and they do not necessarily present their assessments to students directly, but rather, feed it back in the form of improved teaching.

This is very much a "teacher knows best" model! In order to do something like formative assessment among peers, we would have to make quite a few adjustments.

1. At least some of the project participants would have to know how other participants are thinking about their work as well as analyzing their own progress. We are then able to "deliver better teaching" and work together to problem-solve when difficulties arise.

2. It may be most convenient for each participant to take on a share of the work (e.g. by maintaining a "learning journal" which might be shared with other participants). This imposes a certain overhead, but as we remarked elsewhere, "meta-learning is a font of knowledge!" Outside of persistent self-reflection, details about others' learning can sometimes be abstracted from their contributions to the project ("learning analytics" is a whole topic unto itself).

3. If a participant in a "learning project" is bored, frustrated, feeling closed-minded, or for whatever other reason "not learning," then there is definitely a question. But for whom? For the person who isn't learning? For the collective as a whole? We may not have to ponder this conundrum for long: if we go back to the idea that "learning is adaptation," someone who is not learning in a given context will likely leave and find another context where they can learn more.

This is but one example of an assessment strategy: in addition to "formative assessment", "diagnostic" and "summative" strategies are also quite popular in mainstream education. The main

purpose of this section has been to show that when the familiar roles from formal education devolve "to the people," the way assessment looks can change a lot. In the following section, we offer and begin to implement an assessment strategy for evaluating the peeragogy project as a whole.

## Case study in peeragogical evaluation: the Peeragogy project itself

We can evaluate this project partly in terms of its main "deliverable," the Peeragogy Handbook (which you are now reading). In particular, we can ask: Is this handbook useful for its intended audience? If so, in what ways? If not, how can we adapt? The "intended audience" could potentially include anyone who is participating in a peer learning project, or who is thinking about starting one. We can also evaluate the learning experience that the co-creators of this handbook have had. Has working on this book been a useful experience for those involved? These are two very different questions, with two different targets for analysis – though the book's co-creators are also part of the "intended audience". Indeed, we might start by asking "how has working on this book been useful for us?"

### A methodological interlude: "Follow the money"

The metrics for learning in corporations are business metrics based on financial data. Managers want to know: Has the learning experience enhanced the workers' productivity? When people ask about the ROI of informal learning, ask them how they measure the ROI of formal learning. Test scores, grades, self-evaluations, attendance, and certifications prove nothing. The ROI of any form of learning is the value of changes in behavior divided by the cost of inducing the change. Like the tree falling over in the forest with no one to hear it, if there's no change in behavior over the long haul, no learning took place. ROI is in the mind of the beholder, in this case, the sponsor of the learning who is going to decide whether or not to continue investing.

Because the figure involves judgment, it's never going to be accurate to the first decimal place. Fortunately, it doesn't have to be. Ballpark numbers are solid enough for making decisions.

The process begins before the investment is made. What degree of change will the sponsor accept as worthy of reinvestment? How are we going to measure that? What's an adequate level of change? What's so low we'll have to adopt a different approach? How much of the change can we attribute to learning? You need to gain agreement on these things beforehand. Monday morning quarterbacking is not credible. It's counterproductive to assess learning immediately after it occurs. You can see if people are engaged or if they're complaining about getting lost, but you cannot assess what sticks until the forgetting curve has ravaged the learners' memories for a few months. Interest also doesn't guarantee results in learning, though it helps. Without reinforcement, people forget most of what they learn in short order. It's beguiling to try to correlate the impact of learning with existing financial metrics like increased revenues or better customer service scores. Done on its own, this approach rarely works because learning is but one of many factors that influence results, even in the business world. Was today's success due to learning or the ad campaign or weak competition or the sales contest or something else? The best way to assess how people learn is to ask them. How did you figure out how to do this? Who did you learn this from? How did that change your behavior? How can we make it better? How will you? Self-evaluation through reflective practice can build both metacognition and self-efficacy in individuals and groups. Too time consuming? Not if you interview a representative sample. For example, interviewing less than 100 people out of 2000 yields an answer within 10% nineteen times out of twenty, a higher confidence level than most estimates in business. Interviewing 150 people will give you the right estimate 99% of the time.

## Roadmaps in Peer Learning

We have identified several basic and more elaborate patterns that describe "the Peeragogy effect". These have shaped the way we

think about things since. We think the central pattern is the Roadmap, which can apply at the individual level, as a personal learning plan, or at a project level. As we've indicated, sometimes people simply plan to see what happens: alternative versions of the Roadmap might be a compass, or even the ocean chart from the *Hunting of the Snark*. The roadmap may just be a North Star, or it may include detailed reasons "why," further exposition about the goal, indicators of progress, a section for future work, and so forth. Our initial roadmap for the project was the preliminaly outline of the handbook; as the handbook approached completion at the "2.0" level, we spun off additional goals into a new roadmap for a Peeragogy Accelerator. Additional patterns flesh out the project's properties in an open "agora" of possibilities. Unlike the ocean, our map retains traces of where we've been, and what we've learned. In an effort to document these "paths in the grass," we prepared a short survey for Peeragogy project participants.

We asked people how they had participated (e.g., by signing up for access to the Social Media Classroom and mailing list, joining the Google+ Community, authoring articles, etc.) and what goals or interests motivated their participation. We asked them to describe the Peeragogy project itself in terms of its aims and to evaluate its progress over the first year of its existence. As another measure of "investment" in the project, we asked, with no strings attached, whether the respondent would consider donating to the Peeragogy project. This survey was circulated to 223 members of the Peeragogy Google+ community, as well as to the currently active members of the Peeragogy mailing list. The responses outlining the project's purpose ranged from the general: "How to make sense of learning in our complex times?" – to much more specific:

> **Anonymous Survey Respondent 1**: Push education further, providing a toolbox and techniques to self-learners. In the peeragogy.org introduction page we assume that self-learners are self-motivated, that may be right but the Handbook can also help them to

> stay motivated, to motivate others and to face obstacles that may erode motivation.

Considering motivation as a key factor, it is interesting to observe how various understandings of the project's aims and its flaws intersected with personal motivations. For example, one respondent (who had only participated by joining the Google+ community) was: "[Seeking] [i]nformation on how to create and engage communities of interest with a shared aim of learning." More active participants justified their participation in terms of what they get out of taking an active role, for instance:

> **Anonymous Survey Respondent 2**: "Contributing to the project allows me to co-learn, share and co-write ideas with a colourful mix of great minds. Those ideas can be related to many fields, from communication, to technology, to psychology, to sociology, and more."

The most active participants justified their participation in terms of beliefs or a sense of mission:

> **Anonymous Survey Respondent 3**: "Currently we are witnessing many efforts to incorporate technology as an important tool for the learning process. However, most of the initiatives are reduced to the technical aspect (apps, tools, social networks) without any theoretical or epistemological framework. Peeragogy is rooted in many theories of cooperation and leads to a deeper level of understanding about the role of technology in the learning process. I am convinced of the social nature of learning, so I participate in the project to learn and find new strategies to learn better with my students."

Or again:

> **Anonymous Survey Respondent 4**: "I wanted to understand how peer production really works. Could we create a well-articulated system that helps people interested in peer production get their own goals accomplished, and that itself grows and learns? Peer production seems linked to learning and sharing - so I wanted to understand how that works."

They also expressed criticism of the project, implying that they may feel rather powerless to make the changes that would correct the course:

> **Anonymous Survey Respondent 5**: "Sometimes I wonder whether the project is not too much 'by education specialists for education specialists.' I have the feeling peer learning is happening anyway, and that teens are often amazingly good at it. Do they need 'learning experts' or 'books by learning experts' at all? Maybe they are the experts. Or at least, quite a few of them are."

Another respondent was more blunt:

> **Anonymous Survey Respondent 6**: "What problems do you feel we are aiming to solve in the Peeragogy project? We seem to not be sure. How much progress did we make in the first year? Some... got stuck in theory."

But, again, it is not entirely clear how the project provides clear pathways for contributors to turn their frustrations into changed behavior or results. Additionally we need to be entirely clear that we are indeed paving new ground with our work. If there are proven peer learning methods out there we have not examined and included in our efforts, we need to find and address them. Peeragogy is not about reinventing the wheel. It is also not entirely clear whether excited new peers will find pathways to turn their excitement into shared products or process.

For example, one respondent (who had only joined the Google+ community) had not yet introduced current, fascinating projects publicly:

> **Anonymous Survey Respondent 7**: "I joined the Google+ community because I am interested in developing peer to peer environments for my students to learn in. We are moving towards a community-based, place-based program where we partner with community orgs like our history museum for microhistory work, our local watershed community and farmer's markets for local environmental and food issues, etc. I would love for those local efforts working with adult mentors to combine with a peer network of other HS students in some kind of cMOOC or social media network."

Responses such as this highlight our need to make ourselves available to hear about exciting new projects from interested peers, simultaneously giving them easier avenues to share. Our work on developing a peeragogy accelerator in the next section is an attempt to address this situation.

### Summary

We can reflect back on how this feedback bears on the main sections of this book with a few more selected quotes. These motivate further refinement to our strategies for working on this project, and help build a constructively-critical jumping off point for future projects that put peeragogy into action.

*How can we build strong collaboration?*

> "A team is not a group of people who work together. A team is a group of people who trust each other."

*How can we build a more practical focus?*

> "The insight that the project will thrive if people are working hard on their individual problems and sharing feedback on the process seems like the key thing going forward. This feels valuable and important."

*How to connect with newcomers and oldcomers?*

> "I just came on board a month ago. I am designing a self-organizing learning environment (SOLE) or PLE/PLN that I hope will help enable communities of life long learners to practice digital literacies."

*How can we be effective and relevant?*

> "I am game to also explore ways attach peeragogy to spaces where funding can flow based on real need in communities."

## Conclusion

We can estimate individual learning by examining the real problems solved by the individual. It makes sense to assess the way groups solve problems in a similar way. Solving real problems often happens very slowly, with lots of practice along the way. We've learned a lot about peer learning in this project, and the assessment above gives a serious look at what we've accomplished, and at how much is left.

CHAPTER 24

# RESEARCHING PEERAGOGY

This is an unfinished essay from 2001, found nearly a decade and a half later in a box of odds and ends. The essay foreshadows our ongoing research on peer produced peer learning, and also helps to highlight some of the difficulties associated with this enterprise.

## RESEARCH SKILL DEVELOPMENT PROGRAM.

***THE POINT.*** This is an effort at understanding how research skills in the mathematical sciences [but it could be any topic] can be acquired by students.

***WHO WE ARE.*** We are students at a state-funded liberal arts college based in Sarasota, Florida [but it could be anyone]. Our school is called New College. The emphasis of the program at New College is self-directed learning.

***SELF-DIRECTED LEARNING.*** Since people have free will and learn from experience, self-directed learning could be said to take place wherever people engage in any activity. However, this view is unfounded, and the implication is false. Unstructured learning is more accurately undirected. If learning is structured, say by a teacher, this does not imply that it is self-directed, even given the free will of the learner to participate. The choice to participate in learning is not the same as directing the learning. Structure can impose the direction on a (passive) learner. This does not mean that the presence of a teacher or a system to learn implies that the student's learning is not self-directed. The criterion we are looking for is that the student have an active, ongoing and purposive role in deciding the structure of his/her/its [e.g. computer programs] learning environment. A teacher must be informed by and responsive to the student's feedback, or the learning the student does under that teacher's instruction is not self-directed.

***INTEGRATION OF RESEARCH AND EDUCATION.*** In deciding upon a course of study, it behooves the student, as he/she/it examines a potential activity, to consider questions such as these, with the utmost care:

- What is the intellectual merit of the proposed activity?
- Is there sufficient access to resources?
- How well-conceived and organized is the proposed activity?
- To what extent does the proposed activity suggest and explore creative and original concepts?
- To what extent will it enhance possibilities for future work?
- What are the broader impacts of the proposed activity?
- What is the product?

If these questions are addressed well, the student will enter upon a focused program and will have already at the beginning devised a coherent plan for its satisfactory completion. Furthermore, the product is likely to be a net benefit to society. The idea of traditional education is that it is the student, with an increased knowledge and skill base, who constitutes the product. His/her/its knowledge and skills (upon exiting the educational program) are valued by society, and he/she/it is willing to put forth during the program a commensurate amount of blood, sweat, and tears (not to mention tuition and time) to extract the valuable knowledge and skills. In scientific fields, one of these skills is supposed to be the ability to do research. The idea that "the best proof of someone's research ability is the research they have done" has played a significant role in the way scientific education, and the scientific enterprise, has been run in recent years. Research experience at the undergraduate level is one of the top criteria considered by graduate programs in science when they decide which candidates to admit. It is not without reason, then, that national programs for undergraduate researchers (most notably, the National Science Foundation's Research Experiences for Undergraduates (REU) summer programs) are highly competitive, taking only the best qualified applicants nationwide. Many

technical and land-grant universities have internally- or industry-funded Undergraduate Research Opportunities Programs (UROP) which offer financial awards to undergraduate students, which enable them to collaborate with faculty on specialized research projects in their joint field of interest, or to do original work on their own. These programs make it possible for students to make research a part of their background. In particular, such programs give students a chance to see what it is like to work on open problems (usually the problems devised by the program administrator or principal investigator; occasionally on questions proposed by the student researchers themselves). It goes without saying that such experiences are typically only part of the curriculum. The NSF's vision of integrating research and education is to have individuals concurrently assume responsibilities as researchers, educators, and students, where all engage in joint efforts that infuse education with the excitement of discovery and enrich research through the diversity of learning perspectives. The benefits of such a system are manifold. It is however very difficult to implement in most educational contexts. A place like New College, where the culture already is disposed towards student self-direction, may be unique in its ability to foster an undergraduate scientific curriculum based primarily on research. The questions listed at the beginning of this section are the questions a researcher must answer when initiating a research program for undergraduates. (They were lifted from the NSF's summary of how they review REU proposals.) By pointing out here that the same questions are the natural questions for a student to ask when considering how to invest his/her/its time and energy, we mean to point to the unique possibility afforded the self-directed learner, namely: he/she/it can act as a researcher, an educator, and a student concurrently, and, to a degree that is possible for very few, harmoniously.

***RESEARCH AS A WAY OF LIFE (ADDITIONAL REVIEW CRITERIA SPECIFIC TO REU).*** There are other criteria considered by the NSF, for example, the qualifications of the person who proposes the research project. This is *prima facie* difficult for undergraduates to fulfil satisfactorily. Further criteria include:

- The appropriateness and value of the educational experience for the student participants, particularly the appropriateness of the research project(s) for undergraduate involvement and the nature of student participation in these activities.
- The quality of the research environment, including the record of the mentor(s) with undergraduate research participation, the facilities, and the professional development opportunities.
- Appropriateness of the student recruitment and selection plan, including plans for involving students from underrepresented groups and from institutions with limited research opportunities.
- Quality of plans for student preparation and follow-through designed to promote continuation of student interest and involvement in research.
- For REU sites, effectiveness of institutional commitment and of plans for managing the project and evaluating outcomes.

**Some afterthoughts, with the benefit of hindsight (2015)**
The idea that an undergraduate student could run an REU program is perhaps not entirely ridiculous, but it is still extremely unlikely to work – as the essay points out. What is possible is for a student or group of students to set up a website and collaborate informally online. This is what Aaron Krowne did in around 2001, with PlanetMath.org. I joined a few years later, as a graduate student in mathematics. PlanetMath was a little bit like an always-on version of the project outlined in the essay above. The main emphasis was on building a mathematics encyclopedia, but some contributors were doing original research and collaborating with each other. The site administrators and assorted devotees were also doing a lot of meta-level thinking about how the project could improve. In 2005 or thereabouts, I started a wiki called AsteroidMeta to help organize those discussions. By this time, I was no longer in the mathematics graduate programme:

I had more or less stopped going to classes a year earlier. My interests had more to do with how computers could change the way people do mathematics than in doing mathematics the way it had always been done. Myself and a few other PlanetMath contributors published research papers on this theme in a symposium on Free Culture and the Digital Library that Aaron helped organize at Emory, where he was then Head of Digital Library Research. Working on informal collaborations like this, and doing related open source software development, I built a CV that helped me get into another postgrad program in 2010. This time, in the United Kingdom, where I was able to largely set my own research agenda from the start. I focused on rebuilding the PlanetMath website (as described in the *Handbook* chapter on "New Designs for Co-Working and Co-Learning"). Presenting some of this work at Wikimania 2010, I met Charlie Danoff, and when we later met online at P2PU, we decided to sit in on each others first round of courses. As the term progressed, we collaboratively developed a critique of the way things worked at P2PU and suggested some principles that would guide improvement. We called this "paragogy." When Howard Rheingold learned about this work from Charlie, who was taking one of his online classes at RheingoldU, he suggested the more accessible name "peeragogy." To our pleasant surprise Howard then drew on his network of friends and fans to kick off the Peeragogy project. Naturally, I joined, and was able to draw on what we learned in my thesis. Unlike the previous time around, I also had a lot of formal support from my supervisors, as well as a lot of self-organized support from others, and I completed the program successfully. In doing so, I began to accrue the credentials that would be necessary for organizing a formally-funded research project like the one outlined in the essay above. Doing this in the undergraduate research setting would, of course, require interested undergraduates. At the moment, I'm employed as a computer science researcher, exploring the development of peer learning and peer production with the computational "its" mentioned in the essay. The Peeragogy project continues to be a great resource for collaborative research on research and collaboration.

# Part VIII

# Technologies, Services, and Platforms

CHAPTER 25

# INTRODUCTION TO TECHNOLOGIES FOR PEERAGOGY

**It is tempting to bring a list of technologies out as a glorious cookbook.** We need a 1/2 cup of group writing tools, 2 tsp. of social network elements, a thick slice of social bookmarking, and some sugar, then put it in the oven for 1 hour for 350 degrees.

We have created a broad features/functions list for Handbook readers to reflect upon and consider. The joy of this list is that you can consider alternatives for the way you communicate and work while you are planning the project, or can add in new elements to solve communications gaps or create new tools.

However, too many tools spoil the broth. In the writing of this Handbook, we found that out firsthand. We spent a lot of marvelous energy exploring different tools to collaborate, curate information, do research, tag resources, and adjudicate among all of our points of view. In looking at groups working with the various MOOCs, as another example, different groups of students often camp in different social media technologies to work.

In large courses, students often have to be pushed into various social media tools to "co-create" with great protest and lots of inertia. And finally, co-learning groups often come from very different backgrounds, ages, and stages of life, with very different tools embedded in their current lives. Do we have time for three more tools in our busy days? Do more tools help – or do they interfere with our work?

In this section, we'll share with you a few issues:

- What technologies are most useful in peer learning? What do we use them for? What features or functions help our co-learning process?
- How do we decide (a) as a group and (b) for the group on what tools we can use? Do we decide upfront, or grow as

we go?
- How do we coach and scaffold each other on use of tools?
- How much do the tool choices impact the actual outcome of our learning project?
- What are the different roles that co-learners can take in co-teaching and co-coaching the technology affordances/assumptions in the project to make others' lives easier?

Keep in mind – your needs for tools, plus how the way the group uses them, will change as the co-learning project moves along. Technologies themselves tend to change rapidly. Are you willing to change tools during the project as your needs and users change, or do you plan to use a given tool set from the beginning to the end of your project?

## Features and Considerations

We will begin below with a discussions of "features" and initial considerations, and then move to a broader "Choose Your Own Adventure"-style matrix of features leading to a wide variety of collaboration-based technology tools online.

### Technologies and Features

As we will share in the extensive list below, there are abundant tools now available – both for free and for pay – to bring great features to our co-learning endeavors. It is tempting to grab a group of fancy tools and bring the group into a fairly complex tool environment to find the perfect combination of resources. The challenge: adult learners seek both comfort and context in our lives [1], [2]. In choosing tool "brands", we can ignore the features themselves and what we need as parts of the puzzle for learning. We also can have anxiety about our self-beliefs around computers and technology, which in turn can limit our abilities [3].

Before we get to brands and choices, it helps to ask a few questions about the learning goals and environments:

- What do we need as features, and at what stage of the learning process?
- What are we already comfortable with, individually and as a group?
- Do we want to stay with comfortable existing tools, or do we want to stretch, or both?
- What types of learners do we have in this group? Technologically advanced? Comfortable with basics?
- Do we want to invest the time to bring the whole group up to speed on tools? Do all the group members agree on this? Do we want to risk alienating members by making them invest time in new resources?
- We know that our use will migrate and adapt. Do we want to plan for adaptation? Observe it? Learn from it? Make that change intentional as we go?

Researchers over the years have heavily examined these questions of human, technology, and task fit in many arenas. Human-Computer Interaction researchers have looked at "fit" and "adaptive behavior," as well as how the tools can affect how the problem is presented by Te'eni [4]. Creativity support tools [5] have a whole line of design research, as has the field of Computer-Supported Collaborative Work Systems (CSCW). For co-learners and designers interested in the abundance in this space, we've added some additional links below. We here will make this a bit easier. For your co-learning environment, you may want to do one or two exercises in your decision planning:

What *features do you need*? Do you need collaboration? Graphic models? Places to work at the same time (synchronous)? Between meetings (asynchronous)? What are the group members *already using* as their personal learning platforms? It also makes sense to do an inventory about what the group already has as their learning platforms. I'm doing that with another learning group right now. People are much more comfortable – as we

also have found in our co-creation of this Handbook – creating and co-learning in tools with which they already are comfortable. Members can be co-teachers to each other – as we have have – in new platforms. What *type of tools*, based on the features that we need, shall we start out with? Resnick *at al.* [6] looked at tools having:

- Low thresholds (easy to get people started)
- Wide walls (able to bring in lots of different situations and uses) and
- High ceilings (able to do complex tasks as the users and uses adapt and grow).

What are important features needed for co-creation and *working together*? In other pages above, we talk abundantly about roles and co-learning challenges. These issues also are not new; Dourish & Bellottii [7] for example, shared long-standing issues in computer-supportive collaborative work online about how we are aware of the information from others, passive vs. active generation of information about collaborators, etc. These challenges used to be "solved" by software designers in individual tools. Now that tools are open, abundant, and diverse, groups embrace these same challenges when choosing between online resources for co-learning.

### Useful Uses and Fancy Features of Technological Tools

From here, we will help you think about what might be possible, linking to features and solution ideas. We start with ways to ask the key questions: What do you want to do and why? We will start with features organized around several different axes:

1. Time/Place
2. Stages of Activities and Tasks
3. Skill Building/Bloom's Taxonomy
4. Use Cases
5. Learning Functions

Each will link to pages that will prompt you with features, functionality, and technology tool ideas.

### Time/Place

We can further break down tools into whether they create or distribute, or whether we can work simultaneously (synchronous) or at our own times (ascynchronous). To make elements of time and place more visual, Baecker [8] created a CSCW Matrix, bringing together time and place functions and needs. Some tools are synchronous, such as Google+ Hangouts, Blackboard Collaborate, and Adobe Connect, while others let us work asynchronously, such as wikis and forums. Google Docs can work be used both ways. We seem to be considering here mostly tools good for group work, but not for solo, while many others are much easier solo or in smaller groups.

### Stages of Activities and Tasks

Ben Shneiderman [5] has simplified the proliferation of models in this area (e.g., Couger and Cave) with a clear model of four general activities and eight tasks for individuals, which we can lean on as another framework for co-creation in co-learning.

Tools and functions won't be clear cut between areas. For example, some tools are more focused on being generative, or for creating content. Wikis, Etherpad, Google docs, and others usually have a commenting/talk page element, yet generating content is the primary goal and discursive/consultative functions are in service to that. Some tools are discursive, or focused on working together for the creative element of "relating" above – Blackboard Collaborate, the social media class room forums, etc.

### Skill Building (Cognitive, a la Bloom's Taxonomy, see below)

Given that we are exploring learning, we can look to Bloom's Taxonomy (revised, see [9]) for guidance as to how we can look at knowledge support. Starting at the bottom, we have:

- Remembering, as a base;
- Understanding,
- Applying,
- Analyzing,
- Evaluating, and then, at the top,
- Creating.

We could put "search" in the Remembering category above. Others contest that Search, done well, embraces most of the Bloom's elements above. Samantha Penney has created a Bloom's Digital Taxonomy Pyramid infographic, describing tools for learning, which you may want to check out.

### Use Cases (I want to....)

Technologies can be outlined according to the need they serve, or the use case they fulfill. Examples: If we need to 'curate', a platform like Pearl Trees is an option. To 'publish' or 'create', we can look to a wiki or WordPress. Other choices might be great in order to 'collaborate', etc.

One challenge is that tools are not that simple. As we look more closely at the technologies today, we need to reach more broadly to add multiple tags to them. For example Twitter can be used for "Convening a group," for "micro-blogging," for "research," etc.

- Collaborate with a Group
- Create Community
- Curate Information
- Research
- Publish Information
- Create Learning Activities
- Make Something

These plans get more complex, as you are making a group of decisions about tool functionality in order to choose what combination works for the use cases. It may be most useful to use a

concept map (a tech tool) to think about the needs and combinations that you would bring together to achieve each Use Case or Learning Module.

### Technology Features/Functions

We have not made this easy! There are lots of moving elements and options here, none of them right for everything, and some of them fabulous for specific functions and needs. Some have the low thresholds but may not be broad in scope. Some are broad for many uses; others are specific task-oriented tools. That is some of the charm and frustration.

Weaving all of the above together, we have brought together a shared taxonomy for us to discuss and think about co-learning technology features and functions, which we present as an appendix below. This connects various technology features within an expanded version of Ben Shneiderman's creativity support tools framework. We've created this linked toolset with multiple tags, hopefully making it easier for you to evaluate which tool suits best the necessities of the group. Please consider this a starting point for your own connected exploration.

## Appendix: Features and Functions

Weaving all of these frameworks together, we have brought together a shared taxonomy for us to discuss and think about co-learning technology features and functions. We have connected various technology features with an expanded version of Ben Shneiderman's creativity support tools framework for the linked resource guide. For convenience and to help keep it up to date, we're publishing this resource on Google Docs. We present an overview in the following chapters.

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CHAPTER 26

# FORUMS

Forums are web-based communication media that enable groups of people to conduct organized multimedia discussions about multiple topics over a period of time. Selecting the right kind of platform for forum conversations is important, as is know-how about facilitating ongoing conversations online. Forums can be a powerful co-learning tool for people who may have never met face-to-face and could be located in different time zones, but who share an interest in co-learning. Asynchronous media such as forums (or simple email distribution lists or Google Docs) can be an important part of a co-learning toolkit that also include synchronous media from face-to-face meetups to Google+ Hangouts or webinars via Blackboard Collaborate, Adobe Connect, or the open source webconferencing tool, Big Blue Button).

## What is a forum and why should a group use it?

A forum, also known as a message board, bbs, threaded discussion, or conferencing system, affords asynchronous, many-to-many, multimedia discussions for large groups of people over a period of time. That means that people can read and write their parts of the discussion on their own schedule, that everyone in a group can communicate with everyone else, and that graphics, sounds, and videos can accompany text. The best forums index discussion threads by topic, title, tag, date, and/or author and also keep track of which threads and entries (also known as posts) each logged-in participant has already read, making it possible to click on a "show me all the new posts and threads" link each time a participant logs in. This particular form of conversational medium

meets the need for organizing conversations after they reach a certain level of complexity. For example, if twenty people want to discuss five subjects over ten days, and each person makes one comment on each subject every day, that makes for one thousand messages in each participant's mailbox. On email lists, when the conversation drifts from the original topic, the subject line usually does not change, so it makes it difficult to find particular discussions later.

Forums make possible a new kind of group discussion that unfolds over days, weeks, and months, in a variety of media. While blogs are primarily about individual voice, forms can be seen as the voice of a group. The best forum threads are not serial collections of individual essays, but constitute a kind of discourse where the discussion becomes more than the sum of its individual posts. Each participant takes into account what others have said, builds on previous posts, poses and answers questions of others, summarize, distill, and concludes.

This short piece on guidelines for discussion board writing is useful, as is this short piece on shaping a culture of conversation. Lively forums with substantial conversation can glue together the disparate parts of a peeragogy group – the sometimes geographically dispersed participants, texts, synchronous chats, blogs, wikis and other co-learning tools and elements. Forum conversations are an art in themselves and forums for learning communities are a specific genre. Reading the resources linked here – and communicating about them – can help any peeragogy group get its forums off to a good start

**How to start fruitful forum discussions:**

In most contexts, starting a forum with a topic thread for introductions tends to foster the sense of community needed for valuable conversations. This short piece on how to host good conversations online offers general advice. In addition to introductions, it is often helpful to start a topic thread about which new topic threads to create – when everybody has the power to start a new thread and not everybody knows how forums work, a confusing

duplication of conversations can result, so it can be most useful to make the selection of new topic threads a group exercise. A topic thread to ask questions about how to use the forum can prevent a proliferation of duplicate questions. It helps to begin a forum with a few topic threads that invite participation in the context of the group's shared interest "Who is your favorite photographer" for a group of photographers, for example, or "evolution of human intelligence" for a group interested in evolution and/or human intelligence. Ask questions, invite candidate responses to a challenge, make a provocative statement and ask for reactions.

Whether or not you use a rubric for assessing individual participants' forum posts, this guide to how forum posts are evaluated by one professor can help convey the difference between a good and a poor forum conversation:

*4 Points* -The posting(s) integrates multiple viewpoints and weaves both class readings and other participants' postings into their discussion of the subject.

*3 Points* -The posting(s) builds upon the ideas of another participant or two, and digs deeper into the question(s) posed by the instructor.

*2 Points* -A single posting that does not interact with or incorporate the ideas of other participants' comments.

*1 Point* -A simple "me too" comment that neither expands the conversation nor demonstrates any degree of reflection by the student.

*0 Points* -No comment.

### Selecting a forum platform

- You don't want a forum for discussions among two or three people; you do want a forum for discussions among half a dozen or five thousand people.
- You don't want a forum for exchanges of short duration (an hour, a day or two) among any number of people; you do want a forum for ongoing conversations that can continue for months.

- You don't want a forum if blogs with comment threads will do – blogs with comments afford group discourse, but is not easily indexed and discourse gets complicated with more than a dozen or so bloggers and commenters.

If you do want to select a platform for forum discourse, you will want to decide whether you have the technical expertise available to install the software on your own server or whether you want to look for a hosted solution. Cost is an issue.

Fortunately, an online forum maven by the name of David Wooley has been keeping an up-to-date list of available software and services for more than a decade:

- Forum Software for the Web
- Forum and Message Board Hosting Services

These 2003 suggestions on how to choose a forum by Howard Rheingold can be helpful. If blogs with comments afford a kind of networked individualistic discourse, and video conferencing emulates face-to-face meeting, forums can be seen as a channel for expression of the group voice. When people react to and build on each other's comments, they can learn to act as a collective intelligence as well as a collection of individuals who are communicating in order to learn.

CHAPTER 27

# WIKI

In the context of P2P-learning, a wiki platform can be a useful and powerful collaboration tool. This section will help you understand what a wiki is and what it is not, why you should use it, how to choose a wiki engine and finally how you could use it in a P2P context. Some examples of P2P-learning projects run on wikis will help you see the potential of the tool.

## What is a wiki?

For Ward Cunningham father of the wiki, "a wiki is a freely expandable collection of interlinked Web 'pages', a hypertext system for storing and modifying information - a database, where each page is easily editable by any user with a forms-capable Web browser client" [1].

According to Wikipedia : "a wiki is a website whose users can add, modify, or delete its content via a web browser using a simplified markup language or a rich-text editor" [2].

You can watch the CommonCraft video "Wikis in Plain English" on YouTube to better understand what a wiki is.

## What differentiates the wiki from other co-editing tools?

The previous definitions show that a wiki is a "website," in other words it is composed of pages that are connected together by hyperlinks. In addition, every authorized person (not all wikis are totally open like Wikipedia) can edit the pages from a web browser, reducing time and space constraints. In case one saves a mistake, or for any other reason, would like to go back to a

previous version, a feature called "history" allows users to see previous versions and to roll back to any of them. This version history allows a comparison of versions that avoids the clutter of the "commentaries rainbow" that we are used to in popular word processors. For example, if you work on a wiki page and come back later, you will be able to catch up by comparing your latest version with the lastest version generated by someone else.

Tools like Google Docs or Etherpad are design to enable co-editing on a single document. This can be seen as a "wiki way" of working on a document as it is web based and includes versioning. But it is not a wiki because a single document is not a website. Those tools offer realtime collaboration which wikis do not and are far easier for beginners to use as they work in WYSIWYG mode, which many wikis do not support. However, the advanced features of the wiki markup language make it a more powerful tool. In summary, tools like Googles Docs or Etherpad are a great way to quickly collaborate (synchronously, asynchronously, or a mixture of both) on a single document for free, with a low barrier to entry and no technical support. (Note that Etherpad does have a "wiki-links" plugin that can allow it to be used in a more wiki-like way; Hackpad is another real-time editing tool that prominently features linking – and it claims to be "the best wiki ever".)

Using a real wiki engine is more interesting for bigger projects and allows a huge number of users to collaborate on the same platform. A wiki reduces the coordination complication as e-mails exchanges are no more needed to coordinate a project. On the other hand it can help us deal with complexity ([3], [4]) especially if you put basic simple rules in place like the Wikipedia's neutral point of view to allow every participant to share her or his ideas.

Going back to the continuum we talked about before, some tools like Moodle, SharePoint, WordPress, Drupal and others have built-in wiki features. Those features can be good but will typically not be as good for wiki-building purposes as a well-developed special-purpose wiki engine. In other words, the main focus of those tools is not the wiki, which is only a secondary feature. When you choose a real wiki engine like Mediawiki, Tiki, Foswiki, etc., the wiki will be your platform, not a feature of it.

For example if you start a wiki activity in a Moodle course, this wiki will be only visible to a specific group of students and searchable only to those students. On the other hand if your learning platform is a wiki, the whole platform will be searchable to all members regarding their permissions. We are not saying here that a wiki is better than other tools but if you need a wiki engine to address your needs you may consider going with a strong wiki engine rather than a "micro-wiki" engine embedded in an other tool.

## Why use a wiki?

Those are the main reasons you should consider a wiki for your peer learning projects :

- To reduce cumbersome coordination issues by having a central and continually updated place to store your content. You will reduce e-mail usage drastically, and have access to your content from anywhere, using any operating system.
- To keep track of the evolution of your project and be able to view or roll back any previous version of a wiki page using the history feature.
- To make links between wiki pages to connect ideas and people but also make links to external URL's. This last possibility is very handy to cite your sources.
- To deal with complexity. As a wiki allows anyone to contribute, if you set some easy rules like Wikipedia's NPOV (Neutral Point of View), you will be able to catch more complexity as you will allow everyone to express his or her opinion. Wikis also integrate a forum or comment feature that will help you solve editing conflicts.
- To deal with work in progress. A wiki is a great tool to capture an ongoing work.
- To support transparency by letting every member of the community see what all the others are doing.
- To support a network structure; as a wiki is in essence a horizontal tool. Using a hyperlinks you can…

> **Gérard Ayache**: "... jump by a single click from one network node to another, from one computer to another, from one bit of information to the other, from one universe to another, from one brain to another." (Translated from [5].)

## How to choose a wiki engine?

You will find more than a hundred different wiki engines.

The first main distinction is between open source ones that are free to download and commercial ones you will have to pay for. You will find powerful engines on both sides, both open-source and commercial. Sometimes, the open-source ones look less polished at first sight, but are backed by a strong community and offer a range of customization possibilities. The commercial wiki engines are sold as a package, nicely presented but often offering less customization on the user side. Additional features or custom-made tools will cost you extra.

The second distinction that we can make is between wiki farms and self-hosted wikis. The wiki farm is a hosting service you can find for both open-source or commercial wikis. The goal of those farms is to simplify the hosting of individual wikis. If you don't want to choose wiki farm hosting, you will have to host the wiki on your own server. This will give you more latitude and data privacy but will require more technical skills and maintenance fees.

The Wikimatrix web site will help you choose the best wiki for your needs. It allows you to compare the features of more than a hundred wiki engines. Ward Cunningham's list of the top 10 best wiki engines can be found on our Peeragogy.org site.

## How can a wiki be useful in a peeragogy project?

A wiki is a good tool for collaborative projects and especially suited for work in progress, as you can easily track changes using

the history, compare those versions and, if necessary, roll back to previous versions. In other words, nothing gets lost.

Here are some ideas about how to use a wiki in a peeragogy project :

- **Use a wiki as your learning platform**. It can also support Massive Open Online Courses (MOOCs). A wiki will help you organize your learning context. You can choose to give access to your wiki only to the project participants or open it to the public like Wikipedia. Using hyperlinking, you will operationalize the theory of connectivism by connecting nodes together. As a learning platform, wikis are powerful because you can easily see what others are doing, share with them, get inspired, merge ideas or link to ideas. In other words, it fosters emulation between learners. For additional resources about wikis in education, look on Diigo.
- **Manage your peeragogy project**. A wiki is an excellent tool for project collaboration. Above all, the wiki can be a central place for peer learners to write or link to content. Even if you use several technologies to run your project as we did to write this handbook, at the end of the day all the content can be centralized on a wiki using direct writing on wiki pages or embedding hyperlinks. This way, members can access the content from anywhere and from any device connected to the internet, using any platform or application. They will always see the most recent version while being able to browse through the version history to understand what has changed since their last visit.
- **Publish your project**. As a wiki is a website you can easily use it to show your work to the world. Regarding web design, don't forget that a wiki can look way better than a Wikipedia page if you customize it

## Examples of peeragogy projects run on wikis

Appropedia is a wiki site for collaborative solutions in sustain-

ability, poverty reduction and international development through the use of sound principles and appropriate technology and the sharing of wisdom and project information. The site is open to stakeholders to find, create and improve scalable and adaptable solutions.

Teahouse is a peeragogy project run on a wiki that gives newcomers a place to learn about Wikipedia culture and get feedback from experienced Wikipedians.

## What are the best practices when using a wiki?

- **Cofacilitation** – help each other learn, help each other administer
- **Self-election** – enable people to choose what they want to work on, at their own pace, in their own way
- **Communication** – use comment threads and talk pages to discuss wiki changes
- **Documenting changes** – most wikis enable editors to write very brief descriptions of their edits
- **Rules** – keep rules at a minimum level to avoid chaos without constraining creativity
- **Fun** – make it fun for people to contribute

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CHAPTER 28

# REAL-TIME MEETINGS

Web services that enable broadband-connected learners to communicate in real time via audio, video, slides, whiteboards, chat, and screen-sharing enable learning groups to add some of the audio-visual dimensions familiar from synchronous face-to-face communication to otherwise asynchronous platforms such as forums, blogs, and wikis. This article includes resources for finding and evaluating appropriate for-free or for-fee platforms, tips on participative activities for real-time meetings, and suggestions for blending real-time and asynchronous media.

## Real-time meeting media

The *Peeragogy Handbook* was conceived and constructed by a group of people on four continents who had not met and had not known about each other before we began meeting online. The process involves asynchronous media, including forums, wikis, social bookmarking groups, and Wordpress, but it probably would never have cohered into a group capable of collective action if it had not been for the real-time meetings where we were able to see each other's faces, hear each other's voices, use a whiteboard as an anonymous agenda-generator, exchange links in chat, show each other examples through screen-sharing. Together, the asynchronous and real-time media enabled us to begin to see ourselves as an effective group. We used both real-time and asynchronous tools to work out processes for creating, refining, and publishing the Handbook, to divide labor, decide on platforms and processes, to collaboratively compose and edit articles, and to design and add graphical and video elements. In particular, we used the Blackboard Collaborate platform, a web-service that enables up to

50 people at a time to meet in a multimedia, recordable, meeting room for around $500/year. We've experimented with other paid platforms, such as Adobe Connect (about the same price as Collaborate), and when we meet in groups of ten or less, we often use the free and recordable Google+ Hangout service. Smaller groups also use Skype or free telephone conferencing services. Mumble is an open source audio-only tool that is popular with gamers. We're watching the development of Big Blue Button, a free and open-source real-time meeting platform, as it develops the full suite of tools that are currently only available for a fee. Dozens of other free, ad-supported and/or freemium webconferencing systems such as Big Marker and Dim-Dim can be found in lists like Howard Rheingold's and Robin Good's (see links at the end of this chapter). Free phone conferencing services provide another technological "lowest common denominator": some provide a few extras like downloadable recordings.

## Features of real-time meeting platforms

There are many free services for chat, screen-sharing, whiteboards, and video conferencing, but combining all these components in separate panes of the same screen (preferably) or as separate tabs of a browser can have a powerful synchronizing and harmonizing effect on the group. The features to look for in meeting platforms include:

**Audio and video**: Choose platforms that enable voice-over-internet-protocol (VOIP) and easy ways for participants to configure their microphones and speakers. Today's webcams, together with adequate lighting and a broadband connection, enable a number of people to be visible at the same time. In Blackboard Collaborate, the person who is speaking at a given moment is visible in the largest video pane, while other participants are available in smaller video windows. Audio and video convey much more of a human dimension than text communications alone. A group of people who have seen and heard each other online are able to work together via asynchronous media such as forums and wikis more effectively. Online face-to-face meetings are often the

best way for a group to argue constructively and decide on critical issues. Forums and email are comparatively bad choices for distributed decison-making.

**Slide pushing:** The best platforms will convert .ppt or .pdf files for sequential display. With the addition of text chat, annotations to slides, and the ability to "raise your hand" or interrupt with your voice, an online lecture can be a more multidimensional experience than even a highly discursive in-person lecture.

**Text chat:** As a backchannel, a means of quickly exchanging links to relevant resources, a channel for collaborative notetaking, a way of communicating with the lecturer and with other participants, text chat adds a particularly useful dimension to real-time peeragogical meetings – especially when the division of labor is explicitly agreed upon in advance. We've found that even in meetings that use the real-time collaborative editor Etherpad for collaborative note taking, participants may gravitate toward the built-in chat box for discussion.

**Screen sharing:** The ability of participants to show each other what is on their screens becomes especially important in peer learning, where we all have some things to show each other.

**Web tours:** An alternative to screen-sharing is the ability to display the same web page(s) to all participants by entering URLs.

**Interactive whiteboards**: A shared space that enables participants to enter text, drawings, shapes, colors, to move and resize media, and to import graphic content – especially if it allows anonymous actions – can foster the feeling of participating in a collective intelligence. Collaborative anonymous mind-mapping of the discussion is one technique to try with whiteboards. The whiteboard can also be used to generate an emergent agenda for an "un-meeting".

## Configuring Google+ Hangout - a free alternative for up to 10 people

For up to 10 people, each equipped with a webcam, microphone, and broadband connection, Google+ Hangout can provide high-

quality audio-video conferencing. By enabling the text-chat feature and adding Google Docs (text documents, presentations, or spreadsheets), screensharing, and SketchUp (whiteboard), it is possible to emulate most of what the commercial services offer. Adobe Connect and Blackboard Collaborate currently have the user-interface advantage of displaying chat, video, whiteboard/slides as resizable panes on one screen; at present, the free Google services can provide a powerful extension of the basic audio-video platform, but participants have to shift between different tabs or windows in the browser. Note that it is possible to stream a Hangout and record it to YouTube, again at no cost to the user. We've used this tool extensively in the Peeragogy project.

## Suggestions for real-time meetings

In the nine online courses I have facilitated, the emphasis on co-learning encouraged participants to suggest and shape active roles during real-time meetings. By creating and taking on roles, and shifting from role to role, participants engage in a kind of collective learning about collective learning which can be as pleasurable as well as useful. Typically we first brainstorm, then analyze, then organize and present the knowledge that we discover, construct, and ultimately convey together.

## Roles for participants in real-time meetings

- **Searchers:** search the web for references mentioned during the session and other resources relevant to the discussion, and publish the URLs in the text chat
- **Contextualizers:** add two or three sentences of contextual description for each URL
- **Summarizers:** note main points made through text chat.
- **Lexicographers:** identify and collaboratively define words and phrases on a wiki page.
- **Mappers:** keep track of top level and secondary level categories and help the group mindmapping exercise at the end

of the session.

- **Curators:** compile the summaries, links to the lexicon and mindmaps, contextualized resources, on a single wiki page.
- **Emergent Agendas:** using the whiteboard for anonymous nomination and preference polling for agenda items, with voice, video, and text-chat channels for discussing nominations, a group can quickly set its own agenda for the real-time session.

## The Paragogical Action Review

Charlie Danoff and Joe Corneli slightly modified the US Army's "After Action Review" into a technique for evaluating peer learning as it happens. The five steps in the "PAR" are:

1. Review what was supposed to happen
2. Establish what is happening
3. Determine what's right and wrong with what we are doing
4. What did we learn or change?
5. What else should we change going forward?

Participants can run through these steps during live meetings to reassess the medium, the readings, the group dynamics, or any other choices that have learning relevance. The focus in the PAR is on change: as such, it provides a simple way to help implement the "double loop learning" described Chris Argris [1].

### Reference

1. Argyris, Chris. "Teaching smart people how to learn." Harvard Business Review, 69.3, 1991.

### Resources

1. http://delicious.com/hrheingold/webconferencing
2. http://www.mindmeister.com/12213323/best-online-collaboration-tools-2012-robin-good-s-collaborative-map

CHAPTER 29

# CONNECTIVISM IN PRACTICE — HOW TO ORGANIZE A CMOOC

Massive Open Online Courses (MOOCs) are online learning events that can take place synchronously and asynchronously for months. Participants assemble to hear, see, and participate in backchannel communication during live lectures. They read the same texts at the same time, according to a calendar. Learning takes place through self-organized networks of participants, and is almost completely decentralized: individuals and groups create blogs or wikis around their own interpretations of the texts and lectures, and comment on each other's work; each individual and group publicises their RSS feed, which are automatically aggregated by a special (freely available) tool, gRSShopper. Every day, an email goes out to all participants, aggregating activity streams from all the blogs and wikis that engage that week's material. MOOCs are a practical application of a learning theory known as "connectivism" that situates learning in the networks of connections made between individuals and between texts.

Not all MOOCs are Connectivist MOOCs (or *cMOOCs*). Platforms such as Coursera, edX and Udacity offering MOOCs which follow a more traditional, centralized approach (these are sometimes called *xMOOCs*). In this type of MOOC, a professor is taking the lead and the learning-experience is organized top-down. However, some xMOOCs seem to adopt a more blended approach. For instance, the course *E-learning and Digital Cultures* makes use of online spaces beyond the Coursera environment, and the course organizers want some aspects of participation in this course to involve the wider social web.

In this chapter we'll focus on cMOOCs. One might wonder why a course would want to be 'massive' and what '*massive*' means. cMOOC-pioneer Stephen Downes explains that his fo-

cus is on the development of a *network structure*, as opposed to a *group structure*, to manage the course. In a network structure there isn't any central focus, for example, a central discussion. That's also the reason why he considers the figure of 150 active participants – *Dunbar's Number* – to be the lower cut-off in order to talk about 'massive':

> **Stephen Downes**: Why Dunbar's number? The reason is that it represents the maximum (theoretical) number of people a person can reasonably interact with. How many blogs can a person read, follow and respond to? Maybe around 150, if Dunbar is correct. Which means that if we have 170 blogs, then the blogs don't constitute a 'core' - people begin to be selective about which blogs they're reading, and different (and interacting) subcommunities can form.

## A learning theory for the digital age

Traditionally, scholars distinguish between three main categories of learning theories: *behaviorism*, *cognitivism* and *constructivism*. Stephen Downes and others would add a fourth one: connectivism, but this is disputed. The central application of connectivism to date is as a theory of what happens in Massive Open Online Courses.

The connectivist theory describes learning as a process of creating connections and developing networks. It is based on the premise that knowledge exists out in the world, rather than inside an individual's mind. Connectivism sees the network as a central metaphor for learning, with a node in the network being a concept (data, feelings, images, etc.) that can be meaningfully related to other nodes. Not all connections are of equal strength in this metaphor; in fact, many connections may be quite weak.

On a practical level, this approach recommends that learning should focus on where to find information (streams), and how to evaluate and mash up those streams, rather than trying to enter lots of (perishable) information into one's skull. Knowing the

pipes is more important than knowing what exactly each pipe contains at a given moment. This is the theory. The practice takes place in Connectivist MOOCs (cMOOCs), like Change11. Here, people are free to participate at will. Each week a subject is discussed during synchronous sessions, which are recorded and uploaded for reference on the Change11 website. The site also includes an archive of daily newsletters and RSS-feeds of blog posts and tweets from participants.

cMOOCs tend to be learner-centered. People are encouraged to pursue their own interests and link up with others who might help them. But the distributed and free nature of the projects also leads to complaints; participants often find it confusing when they attempt to follow up on all the discussions (the facilitators say one should not try to follow up on *all* the content).

> **Stephen Downes**: This implies a pedagogy that (a) seeks to describe 'successful' networks (as identified by their properties, which I have characterized as diversity, autonomy, openness, and connectivity); and (b) seeks to describe the practices that lead to such networks, both in the individual and in society (which I have characterized as modeling and demonstration (on the part of a teacher) and practice and reflection (on the part of a learner).

## Anatomy of a cMOOC

One example of a MOOC that claims to embody the connectivist theory is change.mooc.ca. The "how it works" section of the site explains what connectivism means in practice. The MOOC organizers developed a number of ways to combine the distributed nature of the discussions with the need for a constantly updated overview and for a federated structure. So, if your team wants to organize an open online course, these are five points to take into consideration:

There is no body of content the participants have to memorize, but the learning results from activities they undertake. The

activities are different for each person. A course schedule with suggested reading, assignments for synchronous or asynchronous sessions is provided (e.g. using Google Docs spreadsheets internally, Google Calendar externally; one could also use a wiki), but participants are free to pick and choose what they work on. Normally there is a topic, activities, reading resources and often a guest speaker for each week. One should even reflect upon the question whether a start and end date are actually needed. It is crucial to explain the particular philosophy of this kind of MOOC, and this right from the outset, because chances are learners will come with expectations informed by their more traditional learning experiences.

1. It is important to discuss the "internal" aspects, such as self-motivation: what do the participants want to achieve, what is their larger goal? And what are their intentions when they select certain activities (rather than other possibilities)? Everyone has her own intended outcome. Suggest that participants meditate on all this and jot down their objectives. And how can they avoid becoming stressed out and getting depressed because they feel they cannot "keep up with all this?" The facilitators should have a good look at these motivations, even if it's impossible to assist every participant individually (for large-scale MOOCs).

2. Ideally, participants should prepare for this course by acquiring the necessary digital skills. Which skills are "necessary" can be decided by the group itself in advance. It's all about selecting, choosing, remixing - also called "curating". There are lots of tools which you can use for this: blogs, social bookmarks, wikis, mindmaps, forums, social dashboards, networks such as Twitter with their possibilities such as hashtags and lists. Maybe these tools are self-evident for some, but not necessarily for all the participants.

3. The course is not located in one place but is distributed across the web: on various blogs and blogging platforms, on

various groups and online networks, on photo- and video-sharing platforms, on mindmaps and other visualization platforms, on various tools for synchronous sessions. This wide variety is in itself an important learning element.

4. There are weekly synchronous sessions (using Blackboard collaborate, or similar group chatting tool). During these sessions, experts and participants give presentations and enter into discussions. Groups of participants also have synchronous meetings at other venues (such as Second Life). Try to plan this well in advance!

5. Many participants highly appreciate efforts to give an overview of the proceedings. Specifically, the Daily Newsletter is a kind of hub, a community newspaper. In that Daily there is also a list of the blog posts mentioning the course-specific tag (e.g. "Change11"), also the tweets with hashtag #change11 are listed in the Daily. Of course, the MOOC has a site where sessions, newsletters and other resources are archived and discussion threads can be read.

From the very beginning of the course, it's necessary to explain the importance of tagging the various contributions, to suggest a hashtag.

For harvesting all this distributed content, Stephen Downes advocates the use of gRSShopper, which is a personal web environment that combines resource aggregation, a personal dataspace, and personal publishing (Downes developed it and would like to build a hosted version - eventually financed via Kickstarter). The gRSShopper can be found on the registration page, which is useful primarily for sending the newsletter. It allows you to organize your online content any way you want, to import content - your own or others' - from remote sites, to remix and repurpose it, and to distribute it as RSS, web pages, JSON data, or RSS feeds.

> **Stephen Downes**: For example, the gRSShopper harvester will harvest a link from a given feed. A per-

> son, if he or she has admin privileges, can transform this link into a post, adding his or her own comments. The post will contain information about the original link's author and journal. Content in gRSShopper is created and manipulated through the use of system code that allows administrators to harvest, map, and display data, as well as to link to and create their own content. gRSShopper is also intended to act as a fully-fledged publishing tool.

Alternatives for registrations: Google Groups for instance. But specific rules about privacy should be dealt with: what will be the status of the contributions? In this MOOC the status is public and open by default, for Downes this is an important element of the course.

## Technologies

Some MOOCs use Moodle, but Downes dislikes the centralization aspect and it's not as open as it could be, saying "people feel better writing in their own space." Other possibilities: Google Groups, Wordpress, Diigo, Twitter, Facebook page, Second Life; but each course uses different mixtures of the many tools out there. People choose their environment - whether it is WoW or Minecraft. Students use Blogger, WordPress, Tumblr, Posterous as blogging tools.

## RSS harvesting is a key element

Give participants a means to contribute their blogfeed. In "Add a New Feed," Downes explains how to get this structure and additional explanations (via videos) in order to contribute their blog feed. The administrator in this case uses gRSShopper to process the content and put it in a database, process it and send it to other people. Alternatively one can use Google Reader (the list of feeds is available as an OPML file - which can be imported to other platforms). There is also a plug-in for Wordpress that lets you use a

Google Doc spreadsheet for the feeds, then Wordpress for the aggregation). Many other content management systems have RSS harvesting features.

Each individual could run her own aggregator, but Downes offers it as a service. But aggregators are needed, whether individual, centralized or both.

### Specialized harvesting

Using Twitter, Diigo, Delicious, Google Groups, If This Then That (IFTTT) and Feed43 (take ordinary web page and turn it into an RSS feed).

### Synchronous environments

Synchronous platforms include Blackboard Collaborate (used now for Change11); Adobe Connect; Big Blue Button; WizIQ; Fuze; WebX; webcasting; web radio; videoconferencing with Skype or Google Hangout in conjunction with Livestream or ustream.tv. Or take the Skype/Hangout audiostream and broadcast is as webradio. Set up and test ahead of time, but don't hesitate to experiment. Note also, there is a more extensive discussion of real-time tools in another section of the handbook.

### Newsletter or Feeds

Feeds are very important (see earlier remarks about the Daily newsletter). You can use Twitter or a Facebook page, Downes uses email, he also creates an RSS version through gRSShopper and sends it through Ifttt.com back to Facebook and Twitter. For the rest of us there is Wordpress, which you can use to create an email news letter. Downs also suggests a handy guide on how to design and build an email newsletter without loosing your mind!

Consider using a content management system and databases to put out specialized pages and the newsletter in an elegant way, but this requires a steep learning curve. Otherwise, use blogs / wikis.

### the Use of Comments

Participants are strongly encouraged to comment on each others' blogs and to launch discussion threads. By doing so they practice a fundamental social media skill - developing networks by commenting on various places and engaging in conversations. It is important to have activities and get people involved rather than to just sit back and watch. For an in-depth presentation, have a look at Facilitating a Massive Open Online Course by Stephen Downes, in which he focuses on research and survey issues, preparing events, and other essentials.

## Resources

- Change MOOC: "How this Course Works" (`http://change.mooc.ca/how.htm`)
- "What is a MOOC" (video)
- "Success in a MOOC" (video)
- "Knowledge in a MOOC" (video)
- "Change MOOC 11: An introduction and an invitation" (video)

(We've collected these videos – all originally uploaded by David Cormier – in the YouTube playlist accessible here: `http://is.gd/peeragogy_mooc_playlist`.)

# Part IX

# Resources

CHAPTER 30

# HOW TO PUT PEERAGOGY INTO ACTION

Welcome to the Peeragogy Accelerator.

The purpose of the *Peeragogy Accelerator* is to use the power of peer learning to help build great organizations.

We will do this by investing time and energy, rather than money, building a distributed community of peer learners, and a strongly vetted collection of best practices. Our project complements others' work on sites like Wikiversity and P2PU, but with an applied flavor. It is somewhat similar to Y Combinator and other start-up accelerators or incubators, but we're doing it the commons based peer production way.

Here, we present *Peeragogy in Action*, a project guide in four parts. Each part relates to one or more sections of our handbook, and suggests activities to try while you explore peer learning. These activities are designed for flexible use by widely distributed groups, collaborating via a light-weight infrastructure. Participants may be educators, community organizers, designers, hackers, dancers, students, seasoned peeragogues, or first-timers. The guide should be useful for groups who want to build a strong collaboration, as well as to facilitators or theorists who want to hone their practice or approach. Together, we will use our various talents to build effective methods and models for peer produced peer learning. We've labeled the phases as Stage 1 through Stage 4, because that's the schedule we use, but if you're working through this on your own, you can choose your own pace. Let's get started!

## Stage 1. Set the initial challenge and build a framework for accountability among participants. (1-3 weeks)

*Activity* – Come up with a plan for your work and an agreement, or informal contract, for your group. You can use the suggestions in this document as a starting point, but your first task is to revise the outline we've developed so that it suits your needs. It might be helpful to ask: What are you interested in learning? What is your primary intended outcome? What problem do you hope to solve? How collaborative does your project need to be? How will the participants' expertise in the topic vary? What sort of support will you and other participants require? What problems won't you solve?

*Technology* – Familiarize yourself with the collaboration tools you intend to use (e.g. a public wiki, a private forum, a community table, social media, or something else). Create something in text, image, or video form introducing yourself and your project(s) to others in the worldwide peeragogy community.

*Suggested Resources* – The Peeragogy Handbook, parts I ('Introduction') and II ('Motivation'). For a succinct theoretical overview, please refer to our literature review, which we have adapted into a Wikipedia page about 'Peer learning'.

*Observations from the Peeragogy project* – We had a fairly weak project structure at the outset, which yielded mixed results. One participant said: "I definitely think I do better when presented with a framework or scaffold to use for participation or content development." Yet the same person wrote with enthusiasm about being "freed of the requirement or need for an entrepreneurial visionary."

*Further Reading* – Boud, D. and Lee, A. (2005). *'Peer learning' as pedagogic discourse for research education.* Studies in Higher Education, 30(5):501–516.

*Further Questions*: What subject or skill does YOUR group want to learn? OR What product or service does YOUR group want to produce?

## Stage 2. Bring in other people to support your shared goals, and make the work more fun too. (1-2 weeks)

*Activity* – Write an invitation to someone who can help as a co-facilitator on your project. Clarify what you hope to learn from them and what your project has to offer. Helpful questions to consider as you think about who to invite: What resources are available or missing? What do you already have that you can build on? How will you find the necessary resources? Who else is interested in these kinds of challenges? Go through the these questions again when you have a small group, and come up with a list of more people you'd like to invite or consult with as the project progresses.

*Technology* – Identify tools that could potentially be useful during the project, even if it's new to you. Start learning how to use them. Connect with people in other locales who share similar interests or know the tools. Find related groups, communities, and forums and engage with others to start a dialogue.

*Suggested resources* – The Peeragogy Handbook, parts IV ('Convening a Group') and V ('Organizing a Learning Context').

*Observations from the Peeragogy project* – We used a strategy of "open enrollment." New people were welcome to join the project at any time. We also encouraged people to either stay involved or withdraw; several times over the first year, we required participants to explicitly reaffirm interest in order to stay registered in the forum and mailing list.

*Further Reading* – Schmidt, J. Philipp. (2009). Commons-Based Peer Production and education. Free Culture Research Workshop Harvard University, 23 October 2009.

*Further Questions*: What are the best learning resources about your topic? OR What are the best production resources for the product or service you hope to build? What tools and platforms does your group need to accomplish their learning goal? OR How will you identify and select the appropriate tools and platforms?

## Stage 3. Solidifying your work plan and learning strategy together with concrete measures for 'success' to move the project forward. (1-3 weeks)

*Activity* – Distill your ideas by writing an essay, making visual sketches, or creating a short video to communicate the unique plans for organization and evaluation that your group will use. By this time, you should have identified which aspects of the project need to be refined or expanded. Dive in!

*Technology* – Take time to mentor others or be mentored by someone, meeting up in person or online. Pair up with someone else and share knowledge together about one or more tools. You can discuss some of the difficulties that you've encountered, or teach a beginner some tricks.

*Suggested resources* – The Peeragogy Handbook, parts VI ('Cooperation'), VII ('Assessment'), and at least some of part II ('Peeragogy in Practice').

*Observations from the Peeragogy project* – Perhaps one of the most important roles in the Peeragogy project was the role of the 'Wrapper', who prepared and circulated weekly summaries of forum activity. This helped people stay informed about what was happening in the project even if they didn't have time to read the forums. We've also found that small groups of people who arrange their own meetings are often the most productive.

*Further Reading* – Argyris, Chris. "Teaching smart people how to learn." Harvard Business Review 69.3 (1991); and, Gersick, Connie J.G. "Time and transition in work teams: Toward a new model of group development." Academy of Management Journal 31.1 (1988): 9-41.

*Further Questions*: What are your benchmarks for success in your learning enterprise? OR What are your benchmarks for success in your production enterprise or service organization?

## Stage 4. Wrap up the project with a critical assessment of progress and directions for future work. Share any changes to this syllabus that you think would be useful for future peeragogues! (1-2 weeks).

*Activity* – Identify the main obstacles you encountered. What are some goals you were not able to accomplish yet? Did you foresee these challenges at the outset? How did this project resemble or differ from others you've worked on? How would you do things differently in future projects? What would you like to tackle next?

*Writing* – Communicate your reflection case. Prepare a short written or multimedia essay, dealing with your experiences in this course. Share the results by posting it where others in the broader Peeragogy project can find it.

*Suggested resources* – The Peeragogy Handbook, parts VIII ('Technologies, Services, and Platforms') and IX ('Resources').

*Observations from the Peeragogy project* – When we were deciding how to license our work, we decided to use CC0, emphasizing 're-usability' and hoping that other people would come and remix the handbook. At the moment, we're still waiting to see the first remix edition, but we're confident that it will come along in due course. Maybe you'll be the one who makes it!

*'Extra credit'* – Contribute back to one of the other organisations or projects that helped you on this peeragogical journey. Think about what you have to offer. Is it a bug fix, a constructive critique, pictures, translation help, PR, wiki-gnoming or making a cake? Make it something special, and people will remember you and thank you for it.

*Further reading* – Stallman, Richard. "Why software should be free" (1992).

*Further Questions*: Write your own!

**Micro-Case Study: The Peeragogy Project, Year 1**

Since its conception in early 2012, the Peeragogy Project has collected over 3700 comments in our discussion forum, and over 200 pages of expository text in the handbook. It has given contributors a new way of thinking about things together. However, the project has not had the levels of engagement that should be possible, given the technology available, the global interest in improving education, and the number of thoughful participants who expressed interest. We hope that the handbook and this accompanying syllabus will provide a seed for a new phase of learning, with many new contributors and new ideas drawn from real-life applications.

We began with these four questions:

1. *How does a motivated group of self-learners choose a subject or skill to learn?*

2. *How can this group identify and select the best learning resources about that topic?*

3. *How will these learners identify and select the appropriate technology and communications tools and platforms to accomplish their learning goal?*

4. *What does the group need to know about learning theory and practice to put together a successful peer-learning program?*

**Micro-Case Study: The Peeragogy Project, Year 2**

10 new handbook contributors joined in the project's second year. We've begun a series of weekly Hangouts on Air that have brought in many additional discussants, all key people who can help to fulfil peeragogy's promise. The handbook has been considerably improved through edits and discussion. The next step for us is putting this work into action in the *Peeragogy Accelerator*.

**Micro-Case Study: The Peeragogy Project, Years 3+4**

We published our plans as "Building the Peeragogy Accelerator", presenting it at OER14 and inviting feedback. In the run up to this, we had been very active creating additional abstracts and submitting them to conferences. However, despite our efforts we failed to recruit any newcomers for the trial run of the Accelerator. Even so, piloting the Accelerator with some of our own projects worked reasonably well,[1] but we decided to focus on the handbook in the second half of the year. As the project's line-up shifted, participants reaffirmed the importance of having "no camp counsellors." In the last quarter of 2014, we created the first version of the workbook that is now presented in Part I of this book as a quickstart guide to peeragogy. We also revised the pattern catalog, and used the revised format to create a "distributed roadmap" for the Peeragogy project – featured in Chapter 7 of the third edition of the handbook. In early 2015 we migrated Handbook production to Github. We presented our patterns at the 2015 Pattern Languages of Programming conference, and spent some more time revising and improving the Handbook.

[1] For an overview, see `http://is.gd/up_peeragogy_accelerator`.

CHAPTER 31

# RECOMMENDED READING

## "Good faith collaboration"

1. Reagle, J. M. (2010). Good faith collaboration: The culture of Wikipedia, MIT Press.

## Writings about fun and boredom

1. Kano, J. (1995/2013). The Contribution of Judo to Education.

2. Pale King, unfinished novel, by David Foster Wallace

3. On the Poverty of Student Life, by Mustapha Khayati

## The structure of learning

Check out work by Bruce Tuckman, Gilly Salmon, Ken Wilber, Martin Oliver, Gráinne Conole, Ruth Deakin-Crick, Howard Gardner, and Mihaly Csíkszentmihályi.

## Motivation

1. Simon Sinek, Start With Why: How Great Leaders Inspire Everyone To Take Action, Penguin Books, 2011

## Case Study: 5PH1NX

1. Senge, Peter. "The fifth discipline: The art and science of the learning organization." *New York: Currency Doubleday* (1990).

## Alexandrian Design Patterns

1. Article, "Manifesto 1991" by Christopher Alexander, Progressive Architecture, July 1991, pp. 108–112, provides a brief summary of Alexander's ideas in the form of a critique of mainstream architecture. Many of the same sorts of critical points would carry over to mainstream education. Some highlights are excerpted here.

2. The Origins of Pattern Theory, the Future of the Theory, And The Generation of a Living World, Christopher Alexander's talk at the 1996 ACM Conference on Object-Oriented Programs, Systems, Languages and Applications (OOPSLA)

## On Newcomers

1. OpenHatch.org, "an open source community aiming to help newcomers find their way into free software projects."

2. Why do newcomers abandon open source software projects? (sildes by Igor Steinmacher and coauthors)

## Antipatterns

1. The Sapir-Whorf Hypothesis

2. Bourdieu's notion of "symbolic violence".

## SWATs

1. Cavallo, David. "Emergent design and learning environments: Building on indigenous knowledge." IBM Systems Journal 39.3.4 (2000): 768-781.

## Convening a Group

1. Engeström, Y. (1999). Innovative learning in work teams: Analyzing cycles of knowledge creation in practice. In Y.

Engeström, R. Miettinen & R.-L-. Punamäki (Eds.), *Perspectives on activity theory*, (pp. 377-404). Cambridge, UK: Cambridge University Press

2. Gersick, C. (1988). Time and transition in work teams: Toward a new model of group development. *Academy of Management Journal* 31 (Oct.): 9-41.

3. Mimi Ito's observations about manga fan groups co-learning Japanese

4. Rheingold U, MindAmp groups

5. Shneiderman, B. (2007). Creativity support tools: accelerating discovery and innovation. *Commun. ACM* 50, 12 (December 2007), 20-32.

6. David de Ugarte, Phyles. (Summary) (Book)

7. Scheidel, T. M., & Crowell, L. (1964). Idea development in small discussion groups. *Quarterly Journal of Speech*, 50, 140-145.

8. Scheidel, T. M., & Crowell, L. (1979), *Discussing and Deciding - A Desk Book for Group Leaders and Members*, Macmillan Publishing

9. Ozturk and Simsek, "Of Conflict in Virtual Learning Communiities in the Context of a Democratic Pedagogy: A paradox or sophism?," in *Proceedings of the Networked Learning Conference, 2012, Maastricht.* Video or text.

10. Paragogy Handbook, How to Organize a MOOC

11. Cathy Davidson et al., How a Class Becomes a Community

## K-12 Peeragogy

**For pointers to tools for your classroom, check out:**

- Richard Byrne

- Sylvia Tolisano
- Caitlin Tucker
- Vicki Davis

**How to develop your PLN:**

- Degrees of Connected Teaching by Rodd Lucier
- TeachThought

**Theory & philosophy of connnected learning for classroom transformation:**

- David Truss
- Steven Downes
- Will Richardson

## Adding Structure with Activities

1. The d.school Bootcamp Bootleg (CC-By-NC-SA) includes lots of fun activities to try. Can you crack the code and define new ones that are equally cool?

2. Puzio, R. S. (2005). "On free math and copyright bottlenecks." *Free Culture and the Digital Library Symposium Proceedings.*

## Co-Facilitation

1. Peer Education: Training of Trainers Manual; UN Interagency Group on Young Peoples Health

2. Co Facilitating: Advantages & Potential Disadvantages. J. Willam Pfeifer and John E Johnes

3. Summary of John Heron's model of the role of facilitators

4. Carl Rogers, Core Conditions and Education, Encyclopedia of Informal Education

5. Peer Mediation, Study Guides and Strategies

6. Co-Facilitation: The Advantages and Challenges, Canadian Union of Public Employees

7. Bohemia Interactive Community Wiki Guidelines

8. Barrett-Lennard, G. T. (1998) Carl Roger's Helping System. Journey and Substance. London: Sage

9. 5 Pillars of Wikipedia, from Wikipedia

10. *Training the Force*, US Army Field Manual #FM 7-0 (FM 25-100)

11. Learning Reimagined: Participatory, Peer, Global, Online, by Howard Rheingold

12. Research Gate is a network dedicated to science and research, in which members connect, collaborate and discover scientific publications, jobs and conferences.

13. Creating and Facilitating Peer Support Groups, by The Community Tool Box

14. Facilitation Tips, by Villanova University

15. Herding Passionate Cats: The Role of Facilitator in a Peer Learning Process, by Pippa Buchanan

16. Reflective Peer Facilitation: Crafting Collaborative Self-Assessment, by Dale Vidmar, Southern Oregon University Library

17. Effective Co-Facilitation, by Everywoman's Center, University of Massachussetts

18. "Teaching smart people how to learn" by Chris Argyris, Harvard Business Review 69.3, 1991; also published in expanded form as a book with the same name.

## Assessment

1. Morgan, C. and M. O'Reilly. (1999). Assessing Open and distance learners. London: Kogan Page Limited.

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6. Connectivism: A Learning Theory for the Digital Age by George Siemens

7. A Connectivism Glossary
8. Rhizomes and Networks by George Siemens
9. Rhizomatic Education: Community as Curriculum by Dave Cormier
10. Knowing Knowledge, a book by George Siemens
11. Net Smart, Howard Rheingold (about internal and external literacies for coping with the 'always on' digital era)
12. Massive Open Online Courses: Setting Up (StartToMOOC, Part 1)
13. The MOOC guide

### And, a word list for your inner edu-geek

You can read about all of these things on Wikipedia.

1. Constructivism
2. Social constructivism
3. Radical constructivism
4. Enactivism
5. Constructionism
6. Connectivism

CHAPTER 32

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# Towards generating novel games using conceptual blending

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**Abstract**

We sketch the process of creating a novel video game by blending two video games specified in the Video Game Description Language (VGDL), following the COINVENT computational model of conceptual blending. We highlight the choices that need to be made in this process, and discuss the prospects for a computational game designer based on blending.

## Introduction

This paper outlines the process of creating a novel video game by blending two video game descriptions given in the Video Game Description Language (VGDL). Several issues need to be considered: which games to blend? Which elements of the game will be blended and which removed? Do we need to adjust the resulting game in order to make it playable? Taking two games from the GVG-AI Competition (Perez et al. 2015), we provide a worked example based on a recent computational, domain-independant model of conceptual blending (Bou et al. 2015).

Our account provides a high-level sketch of how a computational system could blend game designs, and does not propose a concrete architecture. Instead, our intent is explore how a game blending system could work. We highlight the specific challenges presented by various steps in the blending process, and describe how certain steps provide the opportunity to add or close off new potential mechanics and meanings. The implementation and evaluation of this approach will be the focus of future work.

## Background

Recent research (Treanor et al. 2012; Cook and Colton 2014) has examined computational generation of games from provided concepts. The current exploration of blending game descriptions is motivated by our observation that game designers often build new games by combining ideas from existing games.

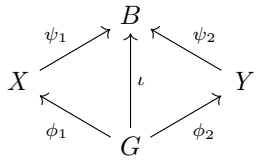
Conceptual blending is the notion that new concepts can be generated by combining elements of existing concepts. **COINVENT** is an EU FP7 project that aims to develop a "computationally feasible, formal model of conceptual blending" (Schorlemmer et al. 2014)[1]. The project takes Goguen's formalisation of conceptual blending as a starting point (Goguen 1999). Early implementations draw on the related notion of amalgams (Ontañón and Plaza 2010). The core ideas in the model are:

**Conceptual spaces** may be mathematical theories, ontologies, or, in our case, video game descriptions.

**Signature morphisms** between conceptual spaces are mappings from the symbols of a source conceptual space into the symbols of target conceptual space. In the diagram below, $\phi_1$, $\phi_2$, $\psi_1$, $\psi_2$, and $\iota$ are all signature morphisms.

**Generic spaces** are a particular kind of conceptual space that possess maximum commonality with a number of given input spaces. In the diagram below, $\phi_1$ and $\phi_2$ represent the inclusion of the elements of the generic space $G$ in both $X$ and $Y$.

**Blends** are unique conceptual spaces that preserve structure from the input spaces and a generic space $G$, via a specific arrangement of signature morphisms; namely, such that the diagram is commutative, and that 'axioms' in the input spaces $X, Y$ are sent to 'theorems' in the unique colimit $B$, known as *the blend*:




Within the COINVENT project the model has been primarily applied to examples from mathematics (Bou et al. 2015) and music (Cambouropoulos, Kaliakatsos-Papakostas, and Tsougras 2014). The constituent processes have been automated to differing degrees, and for some simple examples complete automation is feasible. One of the key outstanding issues is the evaluation of blends, both from the point of view of judging the final outcome, and from an 'online' point of view that can help constrain the potential combinatorial explosion as the collection of possible input spaces is explored, and possible blends are computed.

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[1] http://www.coinvent-project.eu/

**VGDL:** The Video Game Description Language allows the specification of simple arcade-style games (Schaul 2014). It was recently developed to support research into artificial intelligence and games, in particular general game-playing agents, and now forms the basis of the General Video Game Playing AI (GVG-AI) Competition (Perez et al. 2015).

VGDL conceptualises a game as having five components:

**Sprite Set** The entities in the game, their types and properties. The predefined types are Avatar, FromAvatar, NPC, Static, Movable, Resource and Portal, plus a large number of specialised subtypes, e.g. RandomNPC. The sprites are arranged in a hierarchy, where subsprites (here denoted $\sqsubseteq$) inherit types/properties of their ancestors.

**Interaction Set** The mechanics of the game. Each interaction rule defines the consequences of a particular pair of sprites overlapping.

**Termination Set** The win/lose conditions, based on either a timeout or the value of counter, e.g. player health.

**Level Mapping** A translation from ASCII characters to sprites.

**Level Designs** Multiple levels can be defined in ASCII, interpreted using the Level Mapping.

A VGDL specification defines the first four elements, with level designs being specified in a set of accompanying text files.

The sprites and rules are described in terms of a predefined ontology, e.g. a sprite can be a RandomNPC which wanders around the screen. The default ontology is fairly limited, and the VGDL versions of well-known titles are often quite simplistic. Nevertheless, it can be used to define an impressive range of games which present a serious challenge for a general game-playing agent, as demonstrated by the GVG Competition examples. In the context of game design, VGDL presents a large space of possible games. Random instances from within that space are unlikely to be coherant, playable games, let alone high quality ones (Barros and Togelius 2014). Hence it also presents a research challenge for automated game design, and researchers are now beginning to look at VGDL generation (Nielsen et al. 2015b).

## Blending VGDL

Taking VGDL specifications as our 'conceptual spaces' for games, we wish to show how a computational agent could conceivably create new designs by blending existing game elements. Our proposed approach closely follows the general COINVENT process model:

**Select** two input games to blend.

**Generalise** the games by matching selected elements from each, resulting in a *generic game concept*.

**Blend** the games by combining their constituent elements, reformulated in terms of the generic game concept.

**Evaluate** whether the (perhaps nonsensical) blend is a viable game.

**Weaken** the blend by removing/adjusting elements to achieve viability.

**Run** the blend by elaborating the design to exploit any new structure which it has introduced.

The following account is a thought experiment, but is, we argue, computationally plausible. We expect that the majority of games developed by such a process would not be playable, and indeed there will be many partially developed blends that will fail. For non-trivial examples, each stage presents a huge number of choices — a combinatorial explosion which any automated designer needs to tackle in order to extract playable needles out of a haystack of 'frankengames'. We draw attention below to some of the decisions a computational designer would have to confront.

The term "designer" is used to refer to this imagined agent, but one might also conceive a co-creative approach where a human designer takes some of the key decisions, and perhaps takes inspiration from the blends the system produces.

Previous work in automated game design has suggested that there are few intrinsic criteria for game quality (Nelson et al. 2015). Playtesting, whether by humans or bots, is an essential component of game design. An implementation could use feedback from automated playtesting of the games to guide this process, as in e.g. (Nielsen et al. 2015a; 2015b).

We completely sidestep the difficult issues of blending level designs and blending audiovisual elements, focusing instead on blending game entities and rules. Of course, the design of all these elements is intimitely connected, and a full account of game blending would encompass these as well.

### Selection

The first question is which games *should* we blend? Input game selection can be thought of as taking part in some wider game design context: perhaps an agent is elaborating a particular design, and decides to blend in another game concept in order to fix a perceived problem with the original. Alternatively, an agent could be simply exploring the creative possibilities of mashing together diverse game designs.

For this paper, we have just hand-selected two games from the GVG Competition example set: *Frogger* (AKA *Frogs*) and *Zelda* (Perez et al. 2015). Both games are inspired by well-known commercial titles and have simple specifications — the VGDL is shown in Figure 1 and 2 respectively. The level and visual design are both outside the scope of this paper, so for simplicity we have omitted the `LevelMapping` information and `img` and `color` properties. We have also renamed a few sprites and made minor alterations for clarity, without affecting the game mechanics.

It is possible to intuit a lot about how these games work from the VGDL descriptions: the player controls an Avatar, Missiles travel in given direction etc. Due to lack of space, we do not include a full description of the games: see Schaul (2014) for more details, or play

```
SpriteSet
  forest > SpawnPoint stype=log
    dense > prob=0.4 cooldown=10
      sparse > prob=0.1 cooldown=5
  structure > Immovable
      home > Door
      water
      wall
  log > Missile orientation=LEFT speed=0.1
  safety > Resource limit=2
  frog > MovingAvatar
  truck > Missile img=truck
      rightTruck > orientation=RIGHT
          fastRtruck > speed=0.2
          slowRtruck > speed=0.1
      leftTruck > orientation=LEFT
          fastLtruck > speed=0.2
          slowLtruck > speed=0.1

InteractionSet
  home frog  > killSprite scoreChange=1
  frog log   > changeResource resource=safety value=2
  frog log   > pullWithIt
  frog wall  > stepBack
  frog water > killIfHasLess resource=safety limit=0
  frog water > changeResource resource=safety value=-1
  frog truck > killSprite scoreChange=-2
  log   EOS  > killSprite
  truck EOS  > wrapAround

TerminationSet
  SpriteCounter stype=home win=True
  SpriteCounter stype=frog win=False
```

Figure 1: *Frogger* VGDL specification.

them using the Java emulator at `https://github.com/EssexUniversityMCTS/gvgai`.

## Generalisation

The generalisation step proposes a *generic game concept* which can be mapped onto elements of the two input games. In the case of VGDL, we need to propose one or more generic sprites, which can then be mapped on to a subset of the two input sprite sets. For example, if game $X$ features *aliens* and game $Y$ *ghosts*, we could propose a generic game concept $G$ with *enemies*. We then have two *signature morphisms*, i.e. mappings from the sprites of $G$ to the sprites of the input games: $\phi(G, X) : enemy \mapsto alien$ and $\phi(G, Y) : enemy \mapsto ghost$.

Even for quite small sprite sets the number of possible mappings is considerable. We can imagine quite radical mappings, e.g. where a sprite is mapped to both the player avatar and a ghost, or both an alien and an ammo pack, and these clearly have a lot of creative potential. However, for now we make the additional simplifying assumption that the mappings preserve the VGDL sprite categories: Avatar, NPC, Resource etc. So the generic player avatar is mapped to the player avatars in both input games, and so on.

In our running example, the designer could propose the following category-preserving generic sprite set that maps

```
SpriteSet
  door > Door
  key > Immovable
  wall > Immovable
  sword > Flicker limit=5 singleton=True
  movable >
      link > ShootAvatar stype=sword
          nokey
          withkey
      monster > RandomNPC
          monsterQuick > cooldown=2
          monsterNormal > cooldown=4
          monsterSlow > cooldown=8

InteractionSet
  movable wall  > stepBack
  nokey door    > stepBack
  door withkey  > killSprite scoreChange=1
  monster sword > killSprite scoreChange=2
  link monster  > killSprite scoreChange=-1
  key link      > killSprite scoreChange=1
  nokey key     > transformTo stype=withkey

TerminationSet
  SpriteCounter stype=door win=True
  SpriteCounter stype=link win=False
```

Figure 2: *Zelda* VGDL specification.

as follows:

| Generic | Frogger | Zelda |
|---|---|---|
| avatar | frog | link |
| goal | home | door |
| wall | wall | wall |
| enemy | truck | monster |
| fastREnemy | fastRTruck | monsterQuick |
| fastLEnemy | fastLTruck | monsterQuick |
| slowREnemy | slowRTruck | monsterSlow |
| slowLEnemy | slowLTruck | monsterSlow |

Note that both input games have additional unmapped sprites, e.g. *Zelda*'s `key` and `sword` don't correspond to anything in the generic game.

## Blending the Sprites

An initial version of our blended game — let's call it *Frolda* — can now be generated. The blended sprite set combines those from both games, merging sprites identified by the signature morphisms: this is shown in Figure 3.

In generating this blend, we have retained as much structure as possible from both games: `goal`$\sqsubseteq$`structure` is based on the `door` from *Zelda*, and the `monster` and `truck` sprite hierarchies have been integrated, retaining the `rightTruck` and `leftTruck` sprites from *Frogger* (renamed `rightEnemy` and `leftEnemy`).

The input games used two separate mechanisms to control NPC speed — `cooldown` (rate of action) and `speed` (distance traveled per movement action) — hence all enemies are slower in the blend than either of the inputs. This is an example of a undesirable interaction between game el-

```
forest > SpawnPoint stype=log
   dense > prob=0.4 cooldown=10
   sparse > prob=0.1 cooldown=5
structure > Immovable
   goal > Door
   water
   wall
   key
sword > Flicker limit=5 singleton=True
log > Missile orientation=LEFT speed=0.1
safety > Resource limit=2
movable >
   avatar > ShootAvatar stype=sword
      nokey
      withkey
   enemy > Missile
      rightEnemy > orientation=RIGHT
         fastREnemy > cooldown=2 speed=0.2
         slowREnemy > cooldown=8 speed=0.1
      leftEnemy > orientation=LEFT
         fastLEnemy > cooldown=2 speed=0.2
         slowLEnemy > cooldown=8 speed=0.1
   monsterNormal > RandomNPC cooldown=4
```

Figure 3: Initial sprite set for *Frolda*.

ements that will need to be dealt with later in the blending process.

Two significant decisions have been made during the sprite set blend:

1. The player is a `ShootAvatar`, following *Zelda*.
2. An `enemy` is a `Missile`, following *Frogger*.

This is a form of weakening (see below) which it makes sense to apply at this stage, adjusting the blend to ensure the initial blend is valid VGDL.

Some elements of the input games are unaffected by the blend: the walls, the river system from *Frogger* (`forest/water/log/safety`), and the `sword`, `key` and `monsterNormal` from *Zelda*.

## Blending the Rules

We continue the blend by combining the interaction sets: an initial version is shown in Figure 4. All interactions can be included without modification, except for `avatar-enemy`, which reduces the score by 2 in *Frogger*, but only 1 in *Zelda*. By choosing the latter value we again weaken the blend in order to obtain valid VGDL.

Finally, we blend the termination sets. In this example, the structure happens to be identical: the player loses if the avatar is killed, and wins if the goal is 'killed' by being touched by the avatar.

## Evaluation/Weakening

The initial blend includes all the sprites, interactions and termination conditions from both games, with a unique blended structure defined by the chosen signature morphisms. Adjustments were made to the blend to ensure we had valid VGDL, which introduced choices about how the unique blend should be weakened. The next step is to consider

```
goal avatar  > killSprite scoreChange=1
goal withkey > killSprite scoreChange=1
nokey goal   > stepBack
key avatar   > killSprite scoreChange=1
nokey key    > transformTo stype=withkey

movable wall > stepBack
avatar wall  > stepBack
avatar log   > changeResource resource=safety value=2
avatar log   > pullWithIt
avatar water > killIfHasLess resource=safety limit=0
avatar water > changeResource resource=safety value=-1
log   EOS    > killSprite
enemy EOS    > wrapAround

enemy sword  > killSprite scoreChange=2
avatar enemy > killSprite scoreChange=-1
monsterNormal sword  > killSprite scoreChange=2
avatar monsterNormal > killSprite scoreChange=-1
```

Figure 4: Initial interaction set for *Frolda*

whether we need to further weaken the blend to improve the game's coherance and playability.

First, we can see that the initial interaction set (see Figure 4) contains some redundancies:

- `avatar` and `movable` have the same interaction with `wall`, and `avatar`$\sqsubseteq$`movable`. Also note that `truck`$\sqsubseteq$`movable` and if trucks are blocked by walls then they will be prevented from wrapping around, i.e. all the trucks will end up stacked against the wall. Hence, we remove the `movable` interaction and retain the more specific rule.
- `avatar` and `withkey` have the same interaction with `goal`, and `withkey`$\sqsubseteq$`avatar`. If we retain the more general rule then the player could always get to the goal, making the key redundant. Hence we again choose to retain the more specific rule, and drop the `avatar` interaction.

Another principle we can apply is parsimony: does the game contain unnecessary elements? This is very much context-dependent. A designer who was working on *Frogger*, but had blended it with *Zelda* in order to add a key- door mechanism might regard the wandering `monsterNormal` as an unnecessary addition. Likewise, a *Zelda* designer who had blended in the 'crossing the road' structure from *Frogger* could regard the log/river system as unwanted. If we are interested in mashing up the two games, we could regard any additional structure outside the core of the blend (i.e. the generic shared structure) as unnecessary. Here we adopt the latter position, and remove both the wandering monsters (`monsterNormal`) and river system.

Finally, we may wish to retune values in the blended game, such as the speed of the enemies, which may not be appropriate in their new context. As noted above, the enemies speed in the initial blend is being modulated by both `speed` and `cooldown` values. Here, we decide that we can use the default `cooldown` value.

A final version of *Frolda* is shown in Figure 5. The game

```
SpriteSet
  structure > Immovable
      goal  > Door
      wall
      key
  sword > Flicker limit=5 singleton=True
  movable >
    avatar  > ShootAvatar stype=sword
      nokey
      withkey
    enemy > Missile
      rightEnemy > orientation=RIGHT
        fastREnemy > speed=0.2
        slowREnemy > speed=0.1
      leftEnemy > orientation=LEFT
        fastLEnemy > speed=0.2
        slowLEnemy > speed=0.1

InteractionSet
  goal withkey > killSprite scoreChange=1
  nokey goal   > stepBack
  key  avatar  > killSprite scoreChange=1
  nokey key    > transformTo stype=withkey
  avatar wall  > stepBack
  avatar enemy > killSprite scoreChange=-1
  enemy EOS    > wrapAround
  enemy sword  > killSprite scoreChange=2

TerminationSet
  SpriteCounter stype=goal   win=True
  SpriteCounter stype=avatar win=False
```

Figure 5: Final specification for *Frolda*.

features enemies that move laterally left and right across the screen, but which can be killed by using a sword, and requires the player to collect the key before reaching the goal. Figure 6 shows the game running in the GVG-AI Java emulator, with our choice of sprite images and level design. This level takes advantage of the key-door mechanic to force Frolda to cross the 'road' of monsters twice before exiting the level.

'Running the blend' refers to further elaborating the blended game to take advantage of the synthesis of game elements. For example, we may decide to make the `key` a Missile so that as well as 'crossing the road' of enemies, the player has to pick the key out of the moving traffic.

## Alternative Blends

One can imagine many alternative choices to the ones made above. Consider these examples, out of the many possible blends:

- A generic game which doesn't distinguish the speed of enemies, resulting in single-speed monsters which travel left and right.
- A generic game without a unified class of enemies. Trucks and monsters would both be included in the blend as a separate sprites, and the designer could run the blend by inventing new interactions for them.
- Blending `home` with `door`$\sqsubseteq$`structure` was straightforward. But what if *Frogger* had specified `home`$\sqsubseteq$`building`? One option would be to blend `building` and `structure` during the generalisation phase. Another would be to create a new relationship in the blend, e.g. `building`$\sqsubseteq$`structure`. A third option would be to remove, say, `building` entirely.
- When blending the sprite set we could have selected `MoveAvatar` instead of `ShootAvatar`, leaving the sword without a role in the game, to be later removed during weakening.
- Choosing `RandomNPC` enemies instead of `Missile` would have given us a *Zelda*-style free-roaming monsters instead of the horizontally-moving trucks that can be used in level designs to create a dangerous 'road'.

## Next Steps

The account above has glossed over many details, although we hope that each step is computationally plausible. Towards this end we intend to implement a basic VGDL blender using the Java VGDL library (Perez et al. 2015). We envision the primary research topic for blended games to surround *experimentation with heuristics*: can we manage the massive combinatorial possibilities to produce a diverse range of non-trivial, coherent, novel, playable games? A subsequent phase of work would integrate the implementation with the COINVENT software stack, as that matures. The COINVENT system currently accepts input in the Web Ontology Language (OWL) or the Common Algebraic Specification Language (CASL), so one likely approach would be to write a translator, for instance between VGDL descriptions and OWL.

A more immediate next step to support system-building is to work out further compelling examples in detail, e.g.

**Pac-Man + Zombies** The *Survive Zombies* game that ships with the GVG-AI collection has some similarity to *Pac-Man* (Perez et al. 2015). Both games give the protagonist the task of collecting a certain resource while avoiding a population of `Chasers`. However, in *Survive Zombies* another sprite is introduced (bees) which transforms the chasers into honey. One possible blend with *Pac-Man* would allow the protagonist to place fruit that is toxic to ghosts in strategic locations.

**Sokoban + Snake** The protagonist's tail grows when boxes are pushed to the goal[2].

**Sokoban + Sokoban** Two copies of *Sokoban* hooked together, so that boxes cleared from one side appear on the other. Perhaps best conceived of as a two-player game, with each player in the role of "Maxwell's demon". VGDL would have to be extended to multi-player games.

Two other directions that we did not address here include *level design* and *audiovisual design* for games. Rather than directly blending the example levels that accompany

[2]It seems challenging to implement Snake in VGDL, since the snake's tail would have to be made of a chain of `chasers` that are programmed to follow each other.

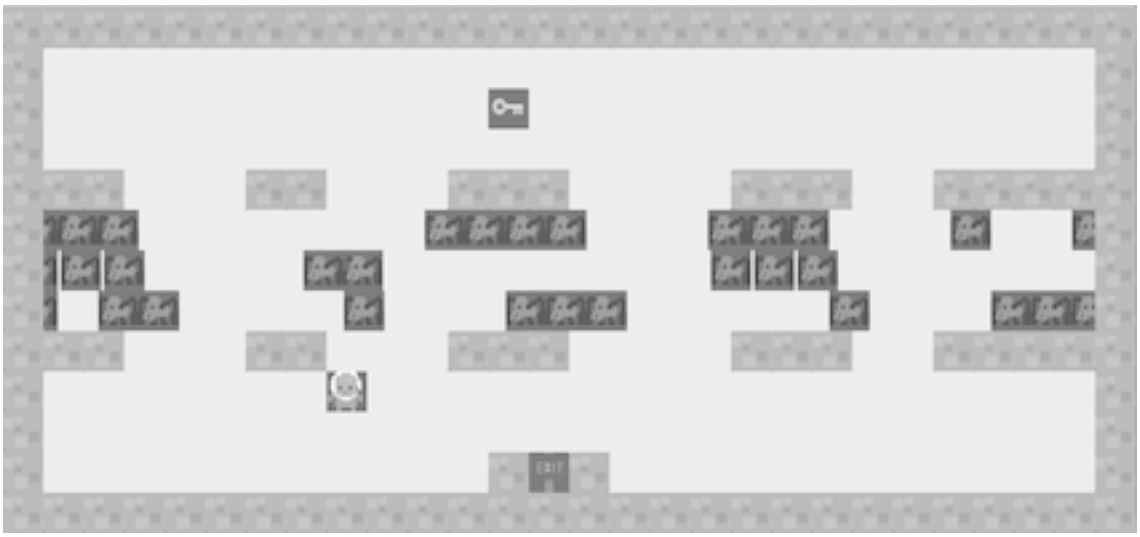


Figure 6: A screenshot of *Frolda* running in the GVG-AI Java emulator. The player (bottom left) has to cross the 'road' of monsters to get the key (top) and then cross again before exiting the level (bottom).

VGDL, we propose to express the *level idiom* in the form of a declarative level generator, e.g. in ASP (Smith and Mateas 2011), and blend these generators in concert with the rest of game description. Other aspects of visual design (e.g. sprites) could be developed through blending, building on recent work in icon blending (Confalonieri et al. 2015). Given its focus on game-playing agents, VGDL does not currently support any audio that could be blended, although it would be relatively easy to extend with simple audio effects to support research in that direction.

## Discussion

One aspect in which our account is significantly lacking is how a computational designer can "run the blend" to take advantage of the new creative opportunities afforded by the newly combined elements. Consider the variant mentioned above where trucks and monsters remain separate sprites. How should they interact? We expect our initial implementation will use quite basic mechanisms, e.g. randomly selecting an event (or no event) for the interaction of the two sprites. A more radical move would be to elaborate new mechanics. For instance, monsters could commandeer and drive trucks. To give Frolda a chance to survive in this new *Mad Max* universe, it may be important to give her the opportunity to collect a bazooka that can launch missiles at the trucks. How this kind of elborative process can be realised computationally could be a direction for future research.

The player's imagination does a significant amount of work to make sense of the blend — perhaps especially in a game with limited graphics. Knowledge of the original games may help the player envision the new blended scenario. Thus, if we recognise the two source games, *Frolda* may be envisioned as an anthropomorphic frog equipped with a sword. We have only considered structural blending, but a broader view could include blending of audiovisual elements to achieve these kind of effects.

Brandt and Brandt offer a critique of conceptual blending that can help in thinking about what makes a blended game playable (Brandt and Brandt 2005). They argue that a purely *conceptual* approach to blending is not as cognitively useful as an approach that considers blends in their communicative context. In the case of games, there is both a narrative and an interactive component to the communication. This is consistent with the view from PCG that intrinsic criteria are insufficient for automated game design, and that some form of playtesting is required (Nelson et al. 2015).

## Conclusions

We have shown that it is — in principle — possible to provide a computational approach to blending games specified in VGDL, although many such blends are possible. We highlighted the opportunities for emergent meaning making that are afforded to game designers using this approach. Nevertheless, it is difficult to point to anything resembling elaboration, evaluation, or meaning-making within VGDL games themselves: for instance, the observation that a given game is unduly difficult for the player would have to come from outside. Automated playtesting could provide some of the necessary feedback to guide the blending process, and will be integral to the implementation of the approach we have outlined here.

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# Patterns of Peeragogy

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We describe nine design patterns that we have developed in our work on the Peeragogy project, in which we aim to help design the future of learning, inside and outside of institutions. We use these patterns to build an "emergent roadmap" for the project. The primary audience we envision for the paper are teams of people who aspire to collaboratively manage their own free/open/libre learning and development projects.

Categories and Subject Descriptors: K.3.1 [**Computers and Education**]: Computer Uses in Education—*Collaborative learning*; K.4.3 [**Computers and Society**]: Organizational Impacts—*Computer-supported collaborative work*

General Terms: Human Factors

Additional Key Words and Phrases: peer learning, peer production, design patterns

**ACM Reference Format:**

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## INTRODUCTION

This paper outlines an approach to the organization of learning that draws on the principles of free/libre/open source software (FLOSS), free culture, and peer production. Mako Hill suggests that one recipe for success in peer production is to take a familiar idea – for example, an encyclopedia – and then make it easy for people to participate in building it [Hill 2013, Chapter 1]. We will take hold of "learning in institutions" as a map (Figure 1), although it does not fully conform to our chosen tacitly-familiar territory of *peeragogy*. To be clear, peeragogy is for *any group of people who want to learn anything*.[1]



Fig. 1: A university campus. Inset captions describe the buildings: "*Ladies Hall*, *Dormitories*, *University Hall*, *Library and Assembly Hall*, *Science Hall*, *President's Residence*, *University Farm*, *Observatory*." This image serves as a key to the figures that illustrate our patterns.

[1] `https://www.youtube.com/watch?v=TDRGJzoNbAc`

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Although we are thinking about learning and adaptation that may take place far outside of formal institutions, the historical conception of a university helps give shape to our inquiry. The model university is not separate from the life of the state or its citizenry, but aims to "assume leadership in the application of knowledge for the direct improvement of the life of the people in every sphere" [Curti et al. 1949, p. 88]. Research that *adds to the store of knowledge* is another of its fundamental obligations [Curti et al. 1949, p. 550]. While the university provides a familiar model for collaborative knowledge work, it is not the only model available. Considering the role of online collaboration in building reference resources, Q&A sites, and free/libre/open source software, we may well ask: What would an accredited free/libre/open university look like? How would it compare or contrast with the typical or stereotypical image in Figure 1? Would it have similar structural features, like a Dormitory, Library, Science Hall and so on? Would participants take on familiar roles [Corneli and Mikroyannidis 2011]? How would it compare with historical efforts like the Tuskegee Institute that involved students directly in the production of physical infrastructure [Washington 1901; Corneli et al. 2014]?

We use the word *peeragogy* to talk about peer-led multi-way collaboration in relatively non-hierarchical settings. Examples are found in education, but also in business, government, volunteer, and NGO settings. Peeragogy involves both problem solving and problem definition, and results in learning. It can be preferable to focus on finding solutions, since people may know the "problems" all too well [Ariyaratne 1977]. Participants in a peeragogical endeavor collaboratively build emergent structures that are responsive to their changing context, and that, in turn, change that context. In the Peeragogy project, we are developing the theory and practice of peeragogy.

*Design patterns* offer a methodological framework that we have used to clarify our focus and organize our work. A design pattern expresses a commonly-occurring problem, a solution to that problem, and rationale for choosing this solution [Meszaros and Doble 1998]. This skeleton is typically fleshed out with a *pattern template* that includes additional supporting material; individual patterns are connected with each other in a *pattern language*. What we present here is rather different from previous pattern languages that touch on similar topics – like *Liberating Voices* [Schuler 2008], *Pedagogical Patterns* [Bergin et al. 2012], and *Learning Patterns* [Iba and Iba Laboratory 2014]. At the level of the pattern template, our innovation is simply to add a "What's next" annotation, which anticipates the way the pattern will continue to "resolve".

This addition mirrors the central considerations of our approach, which is all about human interaction, and the challenges, fluidity and unpredictability that come with it. Something that works for one person may not work for another or may not even work for the same person in a slightly different situation. We need to be ready to clarify and adjust what we do as we go. Even so, it is hard to argue with a sensible-sounding formula like "If W applies, do X to get Y." In our view, other pattern languages often achieve this sort of common sense rationality, and then stop. Failure in the prescriptive model only begins when people try to define things more carefully and make context-specific changes – when they actually try to put ideas into practice. The problem lies in the inevitable distance between *do as I say*, *do as I do*, and *do with me* [Deleuze 2004, p. 26].

If people are involved, things get messy. They may think that they are on the same page, only to find out that their understandings are wildly different. For example, everyone may agree that the group needs to go "that way." But how far? How fast? It is rare for a project to be able to set or even define all of the parameters accurately and concisely at the beginning. And yet design becomes a "living language" [Alexander et al. 1977, p. xvii] just insofar as it is linked to action. Many things have changed since Alexander suggested that "you will get the most 'power' over the language, and make it your own most effectively, if you write the changes in, at the appropriate places in the book" [Alexander et al. 1977, p. xl]. We see more clearly what it means to inscribe the changing form of design not just in the margins of a book, or even a shared wiki, but in the lifeworld itself. Other recent authors on patterns share similar views [Reiners et al. 2012; Schümmer et al. 2014; PLAST Collective 2015].

Learning and collaboration are of interest to both organizational studies and computer science, where researchers are increasingly making use of social approaches to software design and development, as well as agent-based models of computation [Minsky 1967; Corneli et al. 2015]. The design pattern community in particular is very familiar with practices that we think of as peeragogical, including shepherding, writers workshops, and

design patterns themselves [Harrison 1999; Coplien and Woolf 1997; Meszaros and Doble 1998]. We hope to help design pattern authors and researchers expand on these strengths. The primary audience we envision for the paper are teams of people who aspire to collaboratively manage their own free/open/libre learning and development projects.

Plan of the work

Table I shows the pattern template that we use throughout the paper. Along with the traditional design patterns components [Meszaros and Doble 1998], each of our patterns is fleshed out with two illustrative examples. The first looks at how the pattern applies in current Wikimedia projects. We selected Wikimedia as a source of examples because the project is familiar, a demonstrated success, and readily accessible. The second example shows how the pattern could be applied in the design of a free/open/libre university. Each pattern concludes with a boxed annotation that describes "*What's Next in the Peeragogy Project*". Following the convention of the design pattern literature, we write the names of patterns in small-caps. Section 1 defines the concept of PEERAGOGY more explicitly as a design pattern. Sections 2–9 present the other patterns in our pattern language. This is followed by an emergent roadmap for the Peeragogy project and a review of the broader contributions of this work. Figure 2 illustrates the interconnections between the patterns, and Table II summarizes their "nuts and bolts".

| |
|---|
| *Motivation* for using this pattern. |
| *Context* of application. |
| *Forces* that operate within the context of application, each with a mnemonic glyph. |
| *Problem* the pattern addresses. |
| *Solution* to the problem. |
| *Rationale* for this solution. |
| *Resolution* of the forces, named in bold. |
| *Example 1*: How the pattern manifests in current Wikimedia projects. |
| *Example 2*: How the pattern could inform the design of a future university. |
| *What's Next in the Peeragogy Project*: How the pattern relates to our collective intention in the Peeragogy project |

Table I. : Pattern template.

A short motivating example

When one of us was a NEWCOMER to the Peeragogy project, she hit a wall in understanding the "patterns" section in the *Peeragogy Handbook* v1. A more seasoned peer invited her to a series of separate discussions with their own HEARTBEAT to flesh out the patterns and make them more accessible. At that time the list of patterns was simply a list of paragraphs describing recurrent trends. During those sessions, the impact and meaning of patterns captured her imagination. She went on to become the champion for the pattern language and its application in the Peeragogy project. During a "hive editing" session, she proposed the template we initially used to give structure to the patterns. She helped further revise the pattern language for the *Peeragogy Handbook* v3, and attended PLoP 2015. While a new domain can easily be overwhelming, this newcomer found A SPECIFIC PROJECT to start with, and scaffolded her knowledge and contributions from that foundation.

| **Pattern** | **How can we. . .** | **Here's how:** |
|---|---|---|
| PEERAGOGY | . . . find solutions together? | *Figure out what the real problems are.* |
| ROADMAP | . . . get everyone on the same page? | *Build a plan that we keep updating.* |
| REDUCE, REUSE, RECYCLE | . . . avoid undue isolation? | *Use what's there and share what we make.* |
| CARRYING CAPACITY | . . . avoid becoming overwhelmed? | *Clearly express when we're frustrated.* |
| A SPECIFIC PROJECT | . . . avoid becoming perplexed? | *Focus on concrete, doable tasks.* |
| HEARTBEAT | . . . make the project "real" for participants? | *Keep up a regular, sustaining rhythm.* |
| WRAPPER | . . . stay in touch? | *Circulate any adjustments to the plan.* |
| NEWCOMER | . . . make the project accessible to new people? | *Let's learn together with newcomers.* |
| SCRAPBOOK | . . . maintain focus as time goes by? | *Keep coming back to the priorities.* |

Table II. : An overview of the problems and solutions in our pattern language.

## *Connections between the patterns of peeragogy*

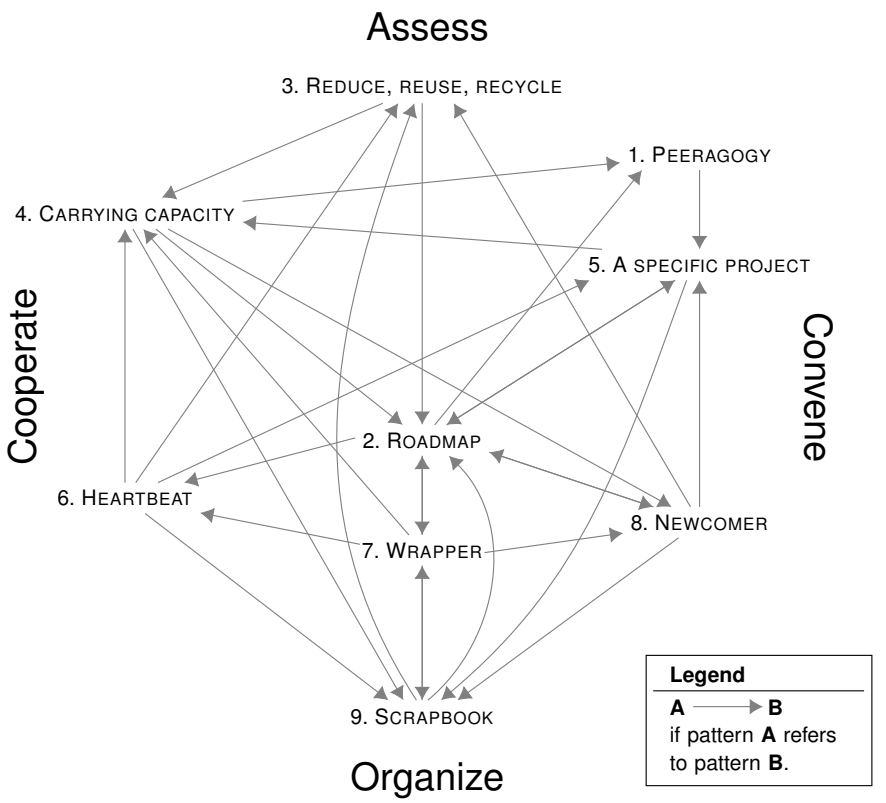



Fig. 2: Peeragogy patterns are highly interconnected. This figure shows how they fit together with and within a wider context. Specifically, the labels at the four edges of the figure correspond to the most closely related sections of the *Peeragogy Handbook*.

## PATTERN CATALOG

### 1. PEERAGOGY

*Motivation.* This pattern is relevant to anyone who wants to do active learning together with others in a relatively non-hierarchical setting.

*Context.* Collaborative projects like Wikipedia, StackExchange, and FLOSS represent an implicit challenge to the old "industrial" organization of work. This new way of working appears to promise something more resilient, more exciting, and more humane. The rhetoric has been questioned [Shaw and Hill 2014; Kreiss et al. 2011]. In and across these "free", "open", post-modern organizations, individual participants are learning [Schmidt 2009] – and that they collectively change the methods and infrastructure as they go. Because everyone in these projects primarily learns by putting in effort on a shared work-in-progress, participants are more in touch with an *equality of intelligence* than an *inequality of knowledge* [Rancière 1991, pp. 38, 119]. At the same time, they invoke a form of friendly competition, in which *the best craftmanship wins* [Raymond 2001, p. 89].

*Forces.* **Threshold**: inclusiveness and specificity are in tension.
**Trust**: is only built through sharing and reciprocity.

*Problem.* Even a highly successful project like Wikipedia is a work in progress that can be improved to better *empower and engage people around the world, to develop* richer and more useful *educational content, and to disseminate it* more *effectively* – and deploy it more creatively.[2] How to go about this is a difficult question, and we don't know the answers in advance. There are rigorous challenges facing smaller projects as well, and fewer resources to draw on. Many successful free software projects are not particularly collaborative – and the largest projects are edited only by a small minority of users [Hill 2011; Swartz 2006]. Can we work smarter together?

*Solution.* The act of asking "can we work smarter together?" puts learning front and center. Peeragogy takes that "center" and distributes it across a pool of heterogeneous relationships. Indeed, peeragogy can be understood as an up-to-date revision of Alexander's NETWORK OF LEARNING [Alexander et al. 1977, p. 99]. It *decentralizes the process of learning and enriches it through contact with many places and people* in interconnected networks that may reach all over the world. Importantly, while people involved in a peeragogical process may be collaborating on A SPECIFIC PROJECT, they don't have to be direct collaborators outside of the learning context or co-located in time or space. Just as theories and practices of pedagogy articulate the transmission of knowledge from teachers to students, peeragogy articulates the way peers produce and use knowledge together (Figure 2).

*Rationale.* The peeragogical approach particularly addresses the problems of small projects stuck in their individual silos, and large projects becoming overwhelmed by their own complexity. It does this by going the opposite route: explicating *what by definition is tacit* and employing *a continuous design process* [Schümmer et al. 2014, pp. 9–10]. As Howard Rheingold remarks in the foreword to the *Peeragogy Handbook*: "What made this work? Polycentric leadership is one key" [Rheingold et al. 2015, p. iii]. "Peer-led" shouldn't suggest that there are no leaders: rather, it means that multiple leaders act as peers.

*Resolution.* Peeragogy helps people in different projects describe and solve real problems. If you share the problems that you're experiencing with others, there's a reasonable chance that someone may be able to help you solve them. Bringing a problem across the **threshold** of someone else's awareness helps achieve clarity. This process can guide individual action in ways that we wouldn't have seen on our own, and may lead to new forms of collective action we would never have imagined possible. People who gain experience comprehending problems together build **trust**. Making room for multiple right answers contributes further to resolving the tension between generality and specificity.

[2] `https://wikimediafoundation.org/wiki/Mission_statement`

*Example 1.* Wikipedia and its sister sites Wiktionary, Wikiversity, etc. (collectively "Wikimedia") rely on user-generated content, peer produced software, and are managed, by and large, by a pool of users who choose to get involved with governance and other "meta" duties.[3] The Wikimedia Foundation maintains the servers and acts on behalf of this "global movement". They achieve something quite impressive: Wikipedia is the 7th most popular website in the world, but the Wikimedia Foundation has under 300 employees. For comparison, the 6th (Amazon) and 8th (QQ) most popular websites are run by companies with over 200K and 28K employees, respectively.[4,5,6,7]

*Example 2.* Although one of the strengths of PEERAGOGY is to distribute the workload, this does not mean that infrastructure is irrelevant. The students and researchers of the future university will need access to an Observatory and other scientific apparatus if they are to reach *ad astra, per aspera* (Figure 3).[8]



Fig. 3: *Observatory*: Space Surveillance Telescope, New Mexico.

*What's Next in the Peeragogy Project* We intend to revise and extend the *Patterns of Peeragogy* into a framework that can describe and scaffold the learning that happens inside and outside of institutions.

## 2. ROADMAP

*Motivation.* This pattern shows how your group can define the scope of their project and make a realistic plan to address it. This pattern provides the backbone of our pattern language. It can be used to find a shared goal.

*Context.* PEERAGOGY has both distributed and centralized aspects. The discussants or contributors who collaborate on a project have different points of view and heterogeneous priorities, but they come together in conversations and joint activities.

*Forces.*
- **Variety**: people have different goals and interests in mind.
- **Clarity**: some goals may be quite specific, and some rather vague.
- **Coherence**: only some of these goals will be well-aligned.

*Problem.* In order to collaborate, people need a way to share current, though incomplete, understanding of the space they are working in, and to nurture relationships with one another and the other elements of this space. At the outset, there may not even be a coherent vision for a project – but a only loose collection of motivations and sentiments. Once the project is up and running, people are likely to pull in different directions.

*Solution.* Building a guide to the goals, activities, experiments and working methods can help NEWCOMERS and old-timers alike understand their relationship with the project. It may combine features of a manifesto, a syllabus, and an issue tracker. It may be a design pattern or a pattern language [Kohls 2010]. The distinguishing qualities of a project ROADMAP are that it should be adaptive to circumstances, and that it should ultimately get us from *here* to *there*. By this same token, any given version of the roadmap is seen as fallible. In lieu of widespread participation, the project's WRAPPER should attempt to synthesize an accurate roadmap that is informed by participants' behavior, and should help moderate in case of conflict. Nevertheless, full consensus is not necessary: different goals, with different *heres* and *theres*, can be pursued separately, while maintaining communication.

[3] `https://www.wikimedia.org/`
[4] `https://en.wikipedia.org/wiki/Wikimedia_Foundation#Employees`
[5] `http://phx.corporate-ir.net/phoenix.zhtml?c=97664&p=irol-newsArticle&ID=2100418`
[6] `https://www.google.com/finance?cid=695431`
[7] `http://www.alexa.com/topsites`
[8] Latin: "With difficulty, to the stars."

*Rationale.* The group evolves from a less-sophisticated to a more-sophisticated manner of operating by using the roadmap. Using the roadmap builds a collective awareness of how things are working in practice. In the Peeragogy project our initial roadmap was a "crowdsourced" outline of the first edition of the *Peeragogy Handbook*. Later, it took the form of a schedule of meetings following a regular HEARTBEAT, supplemented by a list of upcoming deadlines. Most recently, our roadmap is expressed in the emergent objectives collected at the end of current paper. We have seen that a list of nice-to-have features created in a top-down fashion is comparatively unlikely to go anywhere! A backlog of tasks and a realistic plan for accomplishing them are vastly different things. An adaptive roadmap is an antidote to TUNNEL VISION [Dikel et al. 2001, pp. 121–124].

*Resolution.* An emergent roadmap is rooted in real problems and justifiable solutions-in-progress in all their **variety** and communicates both resolution and follow-through. The process of meshing varied issues with one another requires thought and discussion, and this encourages **clarity**. The test of **coherence** is that contributed goals and ideas should be actionable. The ultimate quality-control test is if it worked, i.e., did it come to pass that the task(s) the roadmap was created to achieve ended up being achieved? If all of the issues that the roadmap outlines are not resolved, the roadmap itself should be revised. Without a roadmap, we would never know.

*Example 1.* The *Help* link present on every Wikipedia page could be seen as a localized ROADMAP for individual user engagement: it tells users what they can do with the site, and gives instructions on how to do it.[9] For someone who knows what they're doing, there are around 30 pages listing articles with various kinds of problems, for example articles tagged with style issues, or "orphaned" articles (i.e., articles with no links from other pages in the encyclopedia).[10,11,12] In 2010-2011, Wikimedia developed a strategic plan drawing on community input [Kim et al. 2011]. In 2015, a two-week Community Consultation was carried out; synthesis resulted in "a direction that will guide the decisions for the organization."[13,14] Community-organized WikiProjects often invite and guide involvement on A SPECIFIC PROJECT.



Fig. 4: *President's Residence*, University of Alabama.

*Example 2.* In a future university run in a peer produced manner, a fancy President's Residence presumably wouldn't be needed (Figure 4). Leadership would be carried out in a more collaborative and distributed fashion. However, depending on just how distributed things are, it may turn out to be useful for project facilitators to gather at a University Hall. Whereas there is strength in numbers, there is leverage in organization. This is what the ROADMAP provides.

*What's Next in the Peeragogy Project* If it becomes clear that something needs to change about the project, that is a clue that we might need to revise our patterns or record a new one. We can use the names of the patterns to tag our upcoming tasks.

## 3. REDUCE, REUSE, RECYCLE

*Motivation.* This pattern can guide project participants in identifying and managing available resources.

[9] https://en.wikipedia.org/wiki/Help:Contents
[10] https://en.wikipedia.org/wiki/Category:Wikipedia_article_cleanup
[11] https://en.wikipedia.org/wiki/Category:Wikipedia_articles_with_style_issues
[12] https://en.wikipedia.org/wiki/Category:All_orphaned_articles
[13] https://blog.wikimedia.org/2015/02/23/strategy-consultation/
[14] https://blog.wikimedia.org/2015/08/27/strategy-potential-mobile-multimedia-translation/

*Context.* In a peer production context, you are simultaneously "making stuff" and building on the work of others.

*Forces.* **Derivative**: you don't have to do everything yourself!
**Sensemaking**: resources are useful only when you can make sense of them.
**Sharing**: your understanding gains robustness when you share with others.

*Problem.* Many projects die because the cost of REINVENTING THE WHEEL [c2] is too high. However, this is just one possible symptom of overfocus on a few priorities. Concerns may also arise if the project's output is not actually used by anyone.

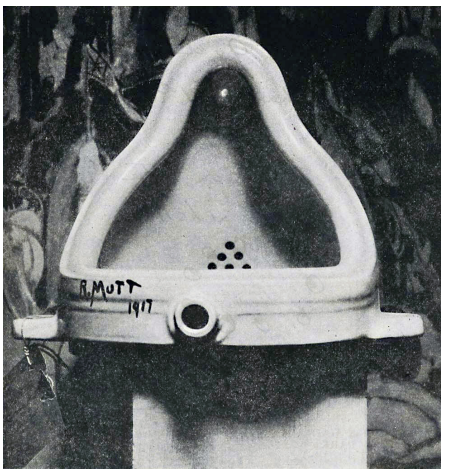



Fig. 5: A paradigmatic example of found-art. "Fountain by R. Mutt, Photograph by Alfred Stieglitz, THE EXHIBIT REFUSED BY THE INDEPENDENTS".

*Solution.* "Steal like an artist," and make it possible for other people to build on your work too (Figure 5). In the Peeragogy project, we have used off-the-shelf and hosted software suited to the task at hand (including: Drupal, Google+, Google Hangouts, Google Docs, Wordpress, pandoc, Github, ShareLaTeX). Early on we agreed to release our *Peeragogy Handbook* under the terms of the Creative Commons Public Domain Dedication (CC0), the legal instrument that grants the greatest possible leeway to downstream users.[15] This has allowed us and others to repurpose and improve its contents in other settings, including the current paper. Follow the steps indicated by the keywords in the pattern's title:

—*Reduce* the panoply of interesting interrelated ideas and methods to a functional core (e.g. writing a book).
—*Reuse* resources relevant to this aim, factoring in "things I was going to have to do anyway" from everyone involved.
—*Recycle* what you've created in new connections and relationships.

*Rationale.* Clearly we are not the first people to notice the problems with wheel-reinvention, including "missing opportunities, repeating common mistakes, and working harder than we need to."[16] As a guest in one of our hangouts, Willow Brugh, of Geeks without Bounds and the MIT Media Lab, remarked that *people often think that they need to build a community, and so fail to recognize that they are already part of a community.*[17] We converted our old pattern catalog from the *Peeragogy Handbook* into this paper, sharing it with a new community and gaining new perspectives; could we do something similar again?

*Resolution.* Reweaving old material into **derivative** designs and new material into existing frameworks, we build deeper understanding, and carry out collective **sensemaking**. The project's ROADMAP develops by making sense of existing resources – including our worries and concerns. Often we only know what these are when we attempt to **share** them. Drawing on a wide range of resources boosts our collective CARRYING CAPACITY.

*Example 1.* Contributors are encouraged to recycle existing works that are compatible with the Wikimedia-wide Creative Commons Attribution-ShareAlike (CC-By-SA) license.[18] Some sub-projects have been created purely to help repurpose other existing works in this way.[19] On the downstream side, DBPedia is an important resource

[15] https://creativecommons.org/publicdomain/zero/1.0/
[16] https://blog.wikimedia.org/2013/11/19/learning-patterns-new-way-share-important-lessons/
[17] https://www.youtube.com/watch?v=NpyQfYVKfBI
[18] https://creativecommons.org/weblog/entry/15411/
[19] https://en.wikipedia.org/wiki/Wikipedia:WikiProject_Mathematics/PlanetMath_Exchange

for the semantic web, built by collating data from Wikipedia's "infoboxes".[20] These infoboxes are themselves increasingly being populated automatically using information from WikiData.[21] Researchers have been able to develop tools that reuse Wikipedia content in other ways [Reinhold 2006; Riche et al. 2010], However, these research projects do not always result in a tool that is accessible to day-to-day users.

*Example 2.* The knowledge resources and collaboration tools currently available online are what make a low-cost, high-quality, formally-accredited future university conceivable. However, the available resources are not always as organized as they would need to be for educative purposes, so peeragogues can usefully put effort into REDUCE, REUSE, RECYCLE'ing available resources into a functioning university Library.

> *What's Next in the Peeragogy Project* Are there other educational resources and peeragogical case studies that we could fold into our work? Can we recycle material from the *Peeragogy Handbook* into a format that is easier to understand and apply?

## 4. CARRYING CAPACITY

*Motivation.* This pattern can help project participants recognize and communicate their stresses to make themselves and the project more resilient.

*Context.* One of the important maxims from the world of FLOSS is: "Given enough eyeballs, all bugs are shallow" [Raymond 2001, p. 30]. A partial converse is also true: there's only so much any one person can do, since we all have limited time and energy.

*Forces.* **Antifragility**: each person's potential can only be realized if people take on enough, but not too much. 

**Independence**: in a peeragogy context, it is often impossible to delegate work to others. 

*Problem.* How can we help prevent those people who are involved with the project from over-promising or over-committing, and subsequently crashing and burning? First, let's be clear that there are lots of ways things can go wrong. Simplistic expectations – like *assuming that others will do the work for you* [Torvalds and Vaughan-Nichols 2011] – can undermine your ability to correctly gauge your own strengths, weaknesses, and commitments. Without careful, critical engagement, you might not even notice when there's a problem. Where one person has trouble letting go, others may have trouble speaking up. Pressure builds when communication isn't going well.

*Solution.* Serious frustration is a sign that it's time to revisit the group's ROADMAP and your own individual plan. Are these realistic? If you have a "buddy" they can provide a reality check. Maybe things are not *that hard* after all – and maybe they don't need to be done *right now*. Generalizing from this: the project can promote an open dialog by creating opportunities for people to share their worries and generate an emergent plan for addressing them [Seikkula and Arnkil 2006]. Use the project SCRAPBOOK to make note of obstacles. For example, if you'd like to pass a baton, you'll need someone there who can take it. Maybe you can't find that person right away, but you can bring up the concern and get it onto the project's ROADMAP. The situation is always changing, but if we continue to create suitable checkpoints and benchmarks, then we can take steps to take care of an issue that's getting bogged down.

*Rationale.* Think of the project as an ecosystem populated by acts of participation. As we get to know more about ourselves and each other, we know what sorts of things we can expect, and we are able to work together

[20] `http://wiki.dbpedia.org/`
[21] `https://www.wikidata.org/wiki/Wikidata:Main_Page`

more sustainably [Ostrom 2010]. We moderate stress and improve collective outcomes by taking concerns seriously.

*Resolution.* Guiding and rebalancing behavior in a social context can begin with speaking up about a concern. When we acknowledge concerns, we must take into account our own boundedness. We have find an opportunity to make ourselves helpful, without impinging on others' **independence**. This doesn't mean allowing all possible stresses to run rampant: we work to stay within the realm of **antifragility**, where manageable stress *improves* the system rather than degrading it [Taleb 2012]. As we share concerns and are met with care and practical support, our actions begin to align better with expectations (often as a result of forming more realistic expectations).

*Example 1.* Wikipedia aims to emphasize a neutral point of view, but its users are not neutral.[22] Users "speak up" about topics that matter to them.[23] Coverage and participation are not neutral in another less sanguine sense. More information on Wikipedia deals with Europe than all of the locations outside of Europe [Graham et al. 2014]. As we remarked in the PEERAGOGY pattern, most of the actual work is contributed by a small percentage of users. The technology limits the kinds of things that can be said [Graham et al. 2014]. The total number of active editors has been falling since 2007.[24] Some blame outmoded technology and an insider culture [Simonite 2013], or a stringent editorial approach that emerged in response to the site's popularity [Halfaker et al. 2013]. Others highlight the rise of successful competition, often inspired by wiki models, but driven by "corporate logic" [Kreiss et al. 2011; Morell 2011]. Some proposed solutions focus on various indicators of "community health".[25]

*Example 2.* Progressive thinkers have for some time subscribed to the view that "there shall be no women in case there be not men, nor men in case there be not women" [Rabelais 1894, Chapter 1.LII]. A separate Ladies Hall seems entirely archaic (Figure 6). However, in light of the extreme gender imbalance in free software, and still striking imbalance at Wikipedia [Ghosh et al. 2002; Reagle 2012], it will be important to do whatever it takes to make women and girls welcome, not least because this is a significant factor in boosting our CARRYING CAPACITY.



Fig. 6: *Ladies Hall*: Queens College, North Carolina.

*What's Next in the Peeragogy Project* Making it easy and fruitful for others to get involved is one of the best ways to redistribute the load. This often requires knowledge transfer and skill development among those involved; see NEWCOMER.

## 5. A SPECIFIC PROJECT

*Motivation.* This pattern can help project participants get started, get focused, and make concrete change. It is especially useful for someone who is currently feeling stuck.

*Context.* We often find ourselves confronted with what seems to be a difficult, complex, or even insurmountable problem. It won't go away, but a workable solution doesn't present itself, either. If there is a candidate solution, it's also clear there are not enough resources for it to be feasible.

[22] https://en.wikipedia.org/wiki/Wikipedia:Neutral_point_of_view
[23] https://en.wikipedia.org/wiki/Wikipedia:Activist
[24] https://strategy.wikimedia.org/wiki/Editor_Trends_Study/Results
[25] https://lists.wikimedia.org/pipermail/wiki-research-l/2016-January/004959.html

*Forces.* **Difficulty**: bringing about meaningful change is often hard work.
**Inertia**: when things are hard we may feel stuck, wring our hands, or preach to the choir.

*Problem.* One is often blinded by one's prejudices and preferences. Some may put considerable energy into pondering, discussing, exploring and feeling stuck. Some may want more concrete progress, and notice the time passing by. In a group setting, when the forward-movers ultimately try to act, those who are more wrapped up in the experience of pondering and exploring may rebel, if they feel that they are being left behind. Inaction may seem like the only safe choice, but it has risks too. And, once moving, things can easily get bogged down again.

*Solution.* One way to make progress when you're stuck is to ask a specific question to someone who may be able to help you get unstuck. Formulating a question helps your thinking become more concrete. Sometimes you'll see that a solution was within your grasp all along. Often, one question won't be enough, but you can repeat the process. In this way, you can reduce a large, complex, or ephemeral concern into a collection of smaller, specific, manageable tasks with clear next steps and success criteria. Use a SCRAPBOOK to make note of all the small things, and weave them into your project ROADMAP. This will show how the small pieces relate to the bigger picture. If you have a fairly specific idea about what you want to do, but you're finding it difficult to get it done, don't just ask for advice: recruit material help (cf. CARRYING CAPACITY). One example of a specific project from the Peeragogy project is our work on this paper, which had a specific target audience, a set of associated deadlines, and allowed us to get help from pattern experts.

*Rationale.* We've seen time and again that having a specific project is a recipe for getting concrete, and that getting concrete is necessary for bringing about change. Asking for help (which is what happens when you vocalize a question) is one of the best ways to gain coherence. Making yourself understood can go a long way toward resolving deeper difficulties.

*Resolution.* Where you run into **difficulty**, getting specific paves the way for incremental forward progress and helps to overcome **inertia**. The struggle between consensus and action is resolved in a tangible project that combines action with dialog. Learning something new is a strong sign that things are working. In the Peeragogy project, we have completed many projects during our weekly hangouts, for example "hive editing" an abstract for submission to a conference.

*Example 1.* At first glance, the Wikimedia Foundation's mission may seem like a good idea, but hard to do anything about.[26] In practice, many Wikipedians contribute to the mission by working on A SPECIFIC PROJECT. Within Wikipedia, these are known as "WikiProjects."[27,28] Guidance is available on how to start new WikiProjects.[29] The Wikimedia Foundation also runs other public projects, including the Wikipedia Education Program and the GLAM Wiki (for Galleries, Libraries, Archives, and Museums).[30,31] The latter maintains a *list of case studies that describes specific projects undertaken by cultural organizations and Wikimedia*.[32]


Fig. 7: *Dormitory*, Ruin Academy, Taipei, Taiwan.

*Example 2.* Collegial and convivial peer support via remote collaboration or short-term meet-ups may fill some of the requirements

[26] https://wikimediafoundation.org/wiki/Mission_statement
[27] https://en.wikipedia.org/wiki/Wikipedia:WikiProject_Council/Directory
[28] https://en.wikipedia.org/wiki/Wikipedia:WikiProject_Council/Guide
[29] https://en.wikipedia.org/wiki/Wikipedia:WikiProject_Council/Guide/WikiProject
[30] https://outreach.wikimedia.org/wiki/Education/Wikipedia_Education_Collaborative/Tasks
[31] https://outreach.wikimedia.org/wiki/GLAM
[32] https://outreach.wikimedia.org/wiki/GLAM/Case_studies

of “student life”. Peeragogy can also happen in neighborhoods, and among persons sharing long-term co-habitation. While a traditional Dormitory may not be necessary, a shared rented or cooperatively-owned living/working environment could be an asset for peeragogues working together on A SPECIFIC PROJECT (Figure 7).

> *What’s Next in the Peeragogy Project* Let’s use our pattern catalog to build specific, tangible “what’s next” steps, add them to our ROADMAP, and carry them out with concrete actions. Let’s be sure we know who’s responsible for what, and employ a “buddy system” to help get things done.

## 6. HEARTBEAT

*Motivation.* This pattern can help project participants stay in touch, and stay motivated.

*Context.* A number of people have a shared interest, and have connected with each other about it. However, they are not going to spend 24 hours a day, 7 days a week working together, either because they are busy with other things, or because working separately on some tasks is vastly more efficient.

*Forces.* **Differentiation**: the time we spend together isn’t all equally meaningful. 
**Entropy**: something needs to hold the project together, or it will fall apart. 

*Problem.* How will the effort be sustained and coordinated sufficiently? How do we know this an active collaboration, and not just a bunch of people milling about? Is there a *there, there?*

*Solution.* People seem to naturally gravitate to something with a pulse. *Once a day* (stand-ups), *once a week* (meetings), or *once a year* (conferences, festivals) are common variants. When the project is populated by more than just a few people, it’s likely that there will be several HEARTBEATS, building a sophisticated polyrhythm. A well-running project will feel “like an improvisational jazz ensemble” [Dikel et al. 2001]. Much as the band director may gesture to specific players to invite them to solo or sync up, a project facilitator may craft individual emails to ask someone to lead an activity or invite them to re-engage. Two common rhythm components are weekly synchronous meetings with an open agenda, combined with *ad hoc* meetings for focused work on A SPECIFIC PROJECT. The precise details will depend on the degree of integration required by the group.

*Rationale.* The project’s heartbeat is what sustains it. Just as *people matter more than code* [Torvalds and Vaughan-Nichols 2011], so does the life of the working group matter more than mechanics of the work structure. Indeed, there is an quick way to do a reality check and find the project’s strongest pulse: the activities that sustain a healthy project should sustain us, too (cf. CARRYING CAPACITY).

*Resolution.* Noticing when a new HEARTBEAT is beginning to emerge is a way to be aware of the shifting priorities in the group, and contributes to further **differentiation**. This may ultimately be a good source of new patterns. On the other hand, if a specific activity is no longer sustaining the project, stop doing it, much as you would move an out-of-date pattern to the SCRAPBOOK in order to make room for other concerns. The power of the HEARTBEAT is that the project can be as focused and intensive as it needs to be, working against **entropy** in the ways that start to be required as time goes by.

*Example 1.* The yearly in-person gathering, Wikimania, is the most visible example of a HEARTBEAT for the Wikimedia movement.[33] Local chapters and projects may run additional in-person get-togethers.[34] Also of note is the twice-yearly call for proposals for individual engagement grants.[35] There are other shorter cycles. Each day

[33] https://meta.wikimedia.org/wiki/Wikimania
[34] http://wikiconferenceusa.org/
[35] https://meta.wikimedia.org/wiki/Grants:IEG



Fig. 8: *University Farm*: al-Biruni University, Kapisa province, Afghanistan.

a highly-vetted Featured Article appears on the front page of Wikipedia, and is circulated to a special-purpose mailing list.[36,37,38] The discussion of articles for deletion lasts at least seven days.[39]

*Example 2.* Although it may sound quaint, some variant of the University Farm could help to physically sustain peeragogues, while putting the project's HEARTBEAT in tune with that of the seasons (Figure 8). In the current distributed mode, we tend our window boxes and allotment gardens. New developments should unfold in a *logical order growing out of the needs of the community* [Washington 1901, Chapter IX].

> *What's Next in the Peeragogy Project* Identifying and fostering new HEARTBEATS and new working groups can help make the community more robust. This is the time dimension of spin-off projects described in REDUCE, REUSE, RECYCLE.

## 7. WRAPPER

*Motivation.* This pattern suggests to find at least one person to fill an important role managing the project's public interface, and keeping participants up to date about activities.

*Context.* You are part of an active, long-running, and possibly quite complex project with more than a handful of participants. How do you manage?

*Forces.* **Interface**: the project shows people how they can use it.
**Familiarity**: the leader/follower dichotomy is easy to understand.
**Equity**: peeragogy aims for fairness.

*Problem.* In an active project, it can be effectively impossible to stay up to date with all of the details. Not everyone will be able to attend every meeting (see HEARTBEAT) or read every email. Project participants can easily get lost and drift away. The experience can be much more difficult for NEWCOMERS: joining an existing project

[36] https://en.wikipedia.org/wiki/Wikipedia:Today%27s_featured_article
[37] https://en.wikipedia.org/wiki/Wikipedia:Featured_article_candidates
[38] https://lists.wikimedia.org/mailman/listinfo/daily-article-l
[39] https://en.wikipedia.org/wiki/Wikipedia:Articles_for_deletion

can feel like trying to climb aboard a rapidly moving vehicle. Information overload is not the only concern: there is also a problem with missing information. If key skills are not shared, they can quickly become bottlenecks (see CARRYING CAPACITY).

*Solution.* Someone involved with the project should regularly create a wrap-up summary, distinct from other project communications, that makes current activities comprehensible to people who may not have been following all of the details. In addition, project members should keep other informative resources like the landing page, ROADMAP, and documentation up to date. Check empirically to see if they really show interested parties how they can get involved. Building on the idea of a "project dashboard" in Figure 9, we can guide potential contributors to live help; we can then see what questions they ask.[40] The WRAPPER is both a role, and, sometimes, an artifact. Our *Handbook*'s cover literally wraps up its contents; the collaboratively written chat notes from our weekly Hangouts give a collaboratively-written overview of what was discussed in the meeting. Meetings themselves can be structured to give people a chance to sum up what they've accomplished during the week, as well as any problems they are running into. Between meetings, each participant is advised to maintain some sort of "learning log" in the form of a personal SCRAPBOOK, so that outstanding concerns are surfaced and available to discuss.

*Rationale.* According to the theory proposed by Yochai Benkler, for free/open "commons-based" projects to work, it is important for participants to be able to contribute small pieces, and for the project to have a way to stitch those pieces together [Benkler 2002]. The WRAPPER helps perform this integrative stitching function. If you value participation, you may have to do some serious work to facilitate access to process.

*Resolution.* Well-maintained records chronicle the project's history; up-to-date documentation makes the project more robust; a coherent look-and-feel offers an accessible **interface** to the outside world. Regularly circulated summaries can help to engage or re-engage members of a project, and can give an emotional boost to peeragogues who see their contributions and concerns mentioned, giving less engaged participants the **familiar** experience of "following" someone else's updates. People will judge from experience whether the project strives for **equity** or strives to maintain hidden power differentials.

*Example 1.* There are many data streams around the Wikimedia project. They comprise an elaborate WRAPPER function for the project, with components that range from Today's Featured Article, which appears on the front page of Wikipedia, to formal annual reports from the nonprofit.[41,42]

*Example 2.* In-person meetings are just as relevant for contemporary humans as they were a century ago, even though we have learned more about how to assemble on the fly [Rheingold 2007]. Getting together for conventions, dance parties, and commencement ceremonies could comprise an important part of the future university's WRAPPER function, even if these events do not always take place in one specific Assembly Hall.

> *What's Next in the Peeragogy Project* Let's make sure we have protocols in place that enable us to share progress, and to revise our "next steps" if people are getting stuck. Let's improve the interaction design for peeragogy.org so that it's clear how people can get involved.

## 8. NEWCOMER

*Motivation.* This pattern can help project participants be aware of the issues faced by newcomers, and cultivate a "beginner's mind" themselves.

[40] `https://gitter.im/orgs/Peeragogy/rooms`
[41] `https://en.wikipedia.org/wiki/Wikipedia:Today%27s_featured_article`
[42] `https://wikimediafoundation.org/wiki/Annual_Report`

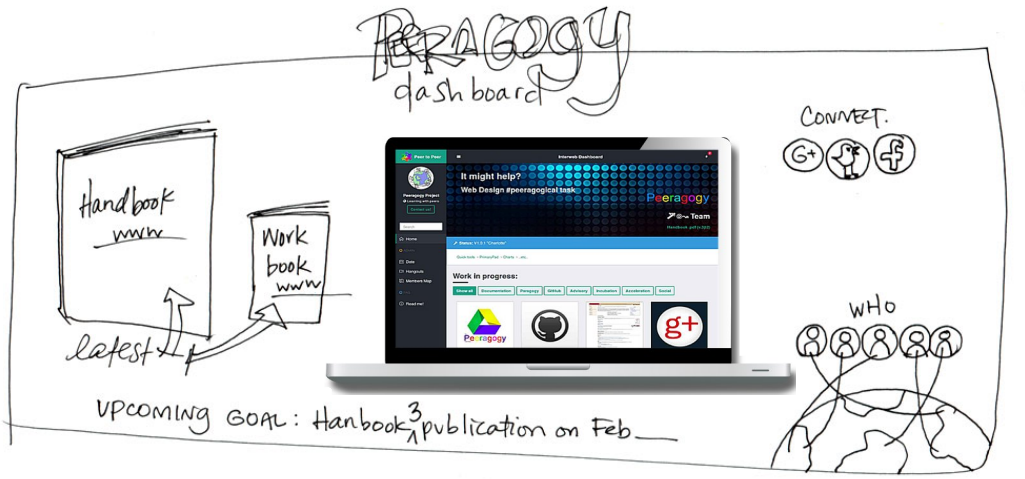



Fig. 9: Design for a Peeragogy project dashboard (sketch by Amanda Lyons, prototype by Fabrizio Terzi; images used with permission).

*Context.* When there's learning happening, it's because there is someone who is new to a topic, or to something about the topic.

*Forces.* **Individuation**: each person learning optimally is what's best for the community.
**Mutuality**: our individuality does not isolate us from one another, but draws us together.

*Problem.* Newcomers can feel overwhelmed by the amount of things to learn. They often don't know where to start. They may have a bunch of ideas that the old-timers have never considered – or they may think they have new ideas, which are actually a different take on an old idea; see REDUCE, REUSE, RECYCLE. People who are new to the project can tell you what makes their participation difficult. Since you're learning as you go as well, you can ask yourself the same question: what aspects of this encounter are difficult for me?

*Solution.* Instead of thinking of newcomers as "them", and trying to provide solutions, we focus on newcomers as "us" – which makes the search for solutions that much more urgent. We permit ourselves to ask naive questions. We entertain vague ideas. We add concreteness by trying A SPECIFIC PROJECT. We may then genuinely turn to others for help. We aim to foster a culture in which the focus for everyone is on addressing our own learning challenges rather than on "providing" solutions for others [Boud and Lee 2005]. When you begin a new project, try to systematically take notes and gather data to analyze and reflect upon later; leave artifacts for other future newcomers to use and build upon in their own research. In practice this may be a lot to ask for someone just joining a group, but over time we may have many ways to structure our collective engagement so that it leads to research cycles based on the "action research" steps *reflect*, *plan*, *act*, and *observe*. Note that there is a parallel with the four facets *assess*, *convene*, *organize*, *cooperate* from Figure 2. The history of the action research approach, with particular emphasis on educational applications, is surveyed in [McNiff 2013, Chapter 3]. One method for doing the reflection/assessment step is presented in the SCRAPBOOK pattern. Be flexible: networked attention (even more so than rigid cycles [Engeström 1999]) leads to new ways of knowing and expanded access to knowledge-production [Simondon 2012; Wagner 2008].

*Rationale.* A newcomer's confusion about how best to get involved or what the point of all this actually is may be due to a lack of structure in the project ROADMAP. Sharing vulnerability and confusion gives us a chance to learn.

*Resolution.* An awareness of the difficulties that newcomers face can help us be more compassionate to ourselves and others. We strengthen the community by supporting all participants' **individuation**. We have a better chance of making the project useful for others if we're clear about how it is useful to *us*. By welcoming newcomers, we enhance the sense of **mutuality** with people who have never encountered the project before, and learn together with them. The facts start to become useful when we understand how people perceive them [Freire 1982].

*Example 1.* Wikipedia NEWCOMERS can make use of resources that include a "Teahouse" where questions are welcomed, a platform extension that changes the user interface for new editors, and lots of documentation.[43,44,45] The efforts of exceptional newcomers may be given special recognition.[46] Newcomer "survival" is of interest to the Wikimedia Foundation.[47] However, "Nearly all editors begin with a burst of activity, then quickly tail off" [Panciera et al. 2009]. The degree to which those editors who are retained strive to maintain a "beginner's mind" is less clear. As regards learning their way around the community, there is quantitative support [Panciera et al. 2009] for the claim that "novice users learn the rules and conventions for contributing both through observation and direct coaching from more knowledgeable others" [Bryant et al. 2005].

*Example 2.* It will often be pragmatic to connect NEWCOMERS with employment directly, so that the future university may see a closer coupling of science and industry than would be found in the old Science Hall (Figure 10). Inspiration can be drawn the London-based freelancing cooperative Founders&Coders, which is able to offer intensive training in web development at no cost to successful applicants, on the basis that some trainees will choose to join the cooperative as paying members later on.[48]



Fig. 10: *Science Hall*: Aspatria Agricultural College, Aspatria, Cumberland, UK

*What's Next in the Peeragogy Project* More detailed guides can show NEWCOMERS how they can contribute and what to expect when they do. We should have different guides for different "user stories". We can start by listing some of the things we're currently learning about.

## 9. SCRAPBOOK

*Motivation.* This pattern describes a way to make the project meaningful.

*Context.* We have been working together for a while now. We have maintained and revised our pattern catalog, and we are achieving some of the "What's Next" steps associated with some of the patterns.

*Forces.* **Attention**: due to limited energy, we need to ask: where should we set the focus?
**Interest**: new experiences catch our attention.
**Meaning**: shared history makes things meaningful.

[43] https://en.wikipedia.org/wiki/Wikipedia:Teahouse
[44] https://en.wikipedia.org/wiki/Wikipedia:GettingStarted
[45] https://en.wikipedia.org/wiki/Help:Editing
[46] https://en.wikipedia.org/wiki/Template:The_New_Editor%27s_Barnstar
[47] https://meta.wikimedia.org/wiki/Research:Newcomer_survival_models
[48] http://www.foundersandcoders.com/academy/

*Problem.* Not all of the ideas we've come up with have proved workable. Not all of the patterns we've noticed remain equally relevant. In particular, some patterns no longer lead to concrete next steps.

*Solution.* In order to maintain focus, is important to "tune" and "prune" the things we give our attention to. We can connect this understanding to any actions undertaken in the project by asking questions like these:

> (1) Review what was supposed to happen. (2) Establish what is happening/happened. (3) Determine what's right and wrong with what we are doing/have done. (4) What did we learn or change? (5) What else should we change going forward? [Rheingold et al. 2015, Chapter 28], after [US Army 1993].

Other review processes have been formalized, including the design review in architecture and the postmortem in theater and other teamwork settings [Mathers et al. 2013; Kerth 2001]. The review process may benefit from having an experienced facilitator on board [Gabriel 2002, pp. 67, 142–143]. As current priorities become clearer, we decide where to focus. Anything that isn't receiving active attention should be moved to a SCRAPBOOK. This may encompass:

—*Retired patterns* that are tabled or completed (no more next steps);

—*Proto-patterns* made of problems, issues, and concerns;

—*A back-catalog* of publications, reports, or other artifacts.

In the Peeragogy project, alongside our patterns we initially maintained a collection of antipatterns (like 'MAGICAL THINKING') but the next steps coming from these seemed particularly convoluted and abstract. So, we archived them.[49] We present a list of outstanding problems – without known solutions – right up front in the Introduction to the *Peeragogy Handbook* [Rheingold et al. 2015, Chapter 1]. Other proto-patterns include 'ONBOARDING' and 'DON'T QUIT YOUR DAY JOB', which arose in our review of this paper (see "Emergent Roadmap", below). Our back-catalog includes academic papers [Corneli et al. 2014, 2013; Corneli 2012; Corneli and Danoff 2011] and a thesis [Corneli 2014]. Everyone can maintain their own personal SCRAPBOOK as along with a communal one. Furthermore, you don't need to limit yourself to *your own* creativity: include interesting ideas from other sources (see REDUCE, REUSE, RECYCLE). In some cases a designated WRAPPER may have to do further work to elicit and organize contributions.

*Rationale.* We want to keep attention focused on the most relevant issues. If a pattern, task, or concern does not lead to concrete "next steps" at the moment, sufficient time for reflection may offer a better understanding, and it may prove useful and actionable in a different context.

*Resolution.* Judicious use of the SCRAPBOOK can help focus project participants' **attention** on current concerns, without losing grasp of items of **interest**. The currently active pattern catalog is leaner and more action-oriented as a result. If the ROADMAP shows where we're going, it is the SCRAPBOOK that shows most clearly where we've been, and collects the observations that are most **meaningful** to us.

*Example 1.* The history of the Wikimedia Foundation, and of Wikipedia, are maintained as wiki pages.[50,51] One of the entries on Wikipedia details outstanding issues, in the form of critiques.[52] There are many tools available to

[49] http://paragogy.net/Scrapbook
[50] https://wikimediafoundation.org/wiki/History_of_the_Wikimedia_Foundation
[51] https://en.wikipedia.org/wiki/Wikipedia
[52] https://en.wikipedia.org/wiki/Criticism_of_Wikipedia



Fig. 11: *Park*: Christ's Pieces, Cambridge, UK.

help facilitate the process of vetting proposed fine-grained changes to articles.[53,54] Policy concerns are typically discussed at the Village Pump, and there are mechanisms in place for settling disputes.[55,56]

*Example 2.* Just as a university campus grows and changes over time, future peeragogues will be drawn to new problems and patterns. They will trace new paths and build new emergent structures (Figure 11).

> *What's Next in the Peeragogy Project* After pruning back our pattern catalog, we want it to grow again: new patterns are needed. One strategy would be to turn the whole *Peeragogy Handbook* into design patterns.

## EMERGENT ROADMAP

Table III reprises the "What's Next" steps from all of the previous patterns, offering another view on the Peeragogy project's ROADMAP in a concrete emergent form. This table has been vetted by project participants, who suggested revisions. This led to an outline for a new pattern, 'ONBOARDING', a facilitative process that would complement NEWCOMER and WRAPPER.[57] We also tagged many of the items in our list of upcoming tasks with the names of patterns in the pattern catalog.[58] However, the techniques that we have described in these patterns reach far beyond our own project. In many cases, the ideas are ancient, or even primordial (like HEARTBEAT). The patterns can be applied with or without high technology. The Peeragogy project is one of

> "[T]ens of thousands of projects in the traditions of world improvement élan – without any central committee that would have to, or even could, tell the active what their next operations should be." [Sloterdijk 2013, p. 402]

[53] https://en.wikipedia.org/wiki/Special:RecentChanges
[54] https://en.wikipedia.org/wiki/Wikipedia:Recent_changes_patrol#Tools
[55] https://en.wikipedia.org/wiki/Wikipedia:Village_pump_(policy)
[56] https://en.wikipedia.org/wiki/Wikipedia:Requests_for_comment
[57] https://github.com/Peeragogy/Onboarding-Project/issues/1
[58] https://goo.gl/tcN3q6

| 1. PEERAGOGY |
| --- |
| We intend to revise and extend the *Patterns of Peeragogy* into a framework that can describe and scaffold the learning that happens inside and outside of institutions. |
| 2. ROADMAP |
| If it becomes clear that something needs to change about the project, that is a clue that we might need to revise our patterns or record a new one. We can use the names of the patterns to tag our upcoming tasks. |
| 3. REDUCE, REUSE, RECYCLE |
| Are there other educational resources and peeragogical case studies that we could fold into our work? Can we recycle material from the *Peeragogy Handbook* into a format that is easier to understand and apply? |
| 4. CARRYING CAPACITY |
| Making it easy and fruitful for others to get involved is one of the best ways to redistribute the load. This often requires knowledge transfer and skill development among those involved; see NEWCOMER. |
| 5. A SPECIFIC PROJECT |
| Let's use our pattern catalog to build specific, tangible "what's next" steps, add them to our ROADMAP, and carry them out with concrete actions. Let's be sure we know who's responsible for what, and employ a "buddy system" to help get things done. |
| 6. HEARTBEAT |
| Identifying and fostering new HEARTBEATS and new working groups can help make the community more robust. This is the time dimension of spin-off projects described in REDUCE, REUSE, RECYCLE. |
| 7. WRAPPER |
| Let's make sure we have protocols in place that enable us to share progress, and to revise our "next steps" if people are getting stuck. Let's improve the interaction design for peeragogy.org so that it's clear how people can get involved. |
| 8. NEWCOMER |
| More detailed guides can show NEWCOMERS how they can contribute and what to expect when they do. We should have different guides for different "user stories". We can start by listing some of the things we're currently learning about. |
| 9. SCRAPBOOK |
| After pruning back our pattern catalog, we want it to grow again: new patterns are needed. One strategy would be to turn the whole *Peeragogy Handbook* into design patterns. |

Table III. : What's next in the Peeragogy project

When we talk about "next steps," we aim to show what can be realistically expected from us. And yet, the emergent ROADMAP also goes beyond specific individuals in the project, who will come and go.

Despite its tendency to longevity, robustness, and widespread applicability, peeragogy does have limits. For example, the benefits to collaboration notwithstanding, "there is no substitute for really hard thinking on your own" [Atiyah 1974]. Nevertheless, even the most independent knowledge worker reads and publishes: and humans are social animals. The question is usually how best – not whether – to involve others [Coase 1937; Benkler 2002]. This relates to another prospective pattern: 'DON'T QUIT YOUR DAY JOB'. Peer learning and peer production can build amazing resources, but they don't always pay the bills. That said, we have thought a lot about how the Peeragogy project could be deployed as a project accelerator built the commons-based peer production way, and

what that might mean from a business standpoint [Corneli et al. 2014]. Possible revenue streams include book sales and membership fees – as well as good old fashioned service provision.[59] At the moment there is no final agreement, but the presence of questions about the business aspects of the project may be a "clue" pointing in the direction of another pattern.

There is a further issue that presents more of a paradox. We can't dictate the behavior of other participants, and we often can't guess ourselves what's coming up. A peeragogical ROADMAP should prepare people for the *absence* of clear step-by-step direction, the *presence* of different viewpoints and priorities, and the consequent *requirement* to be reasonably self-directed. Many features of the peeragogical approach would be irrelevant to a project that is managed in a top-down fashion, and that can rely on other coordination mechanisms (like contracts) to manage work. Peeragogy is relevant when we must define or redefine the problems together.

## CONCLUSION

This paper presents nine patterns of peeragogy and connects them to concrete next steps for the Peeragogy project. In order to demonstrate generality, we included examples that illustrate how the patterns manifest in current Wikimedia projects, and how they could inform the design of a future university rooted in the values and methods of peer production. The university metaphor need not constrain the future application of the ideas, which cut across many modes of engagement (Figure 12). Even so, a project to translate the free/libre/open university from metaphor to reality would offer an intriguing opportunity to test the generality of Hill's hypothesis on the mobilizing potential of "the combination of a familiar goal (e.g., 'simply reproduce Encyclopedia Britannica') with innovative methods (e.g., 'anybody can edit anything')" [Hill 2013, p. 13]. We close by reviewing our approach to the organization of learning, using three dimensions of analysis that have been previously applied to describe research on peer production [Benkler et al. 2015].

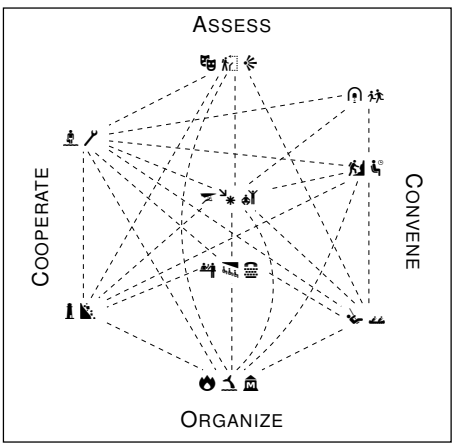



Fig. 12: Mnemonic

*Organization.* Managing work on our project with design patterns that are augmented with a "What's next" follow-through step allows us to "set and execute goals in a decentralized manner" [Benkler et al. 2015], reintegrating structure in the form of an emergent ROADMAP. Our methods apply at varied levels of scale and degrees of formality, inside or outside of institutional frameworks. If you already write patterns, you can add a "What's next" step to them and try the approach for yourself.

*Motivation.* The future of learning may be the *Chartres of programming* [Alexander 1999], but it will have plenty in common with the bazaar [Raymond 2001]. Philipp Schmidt indicates that *learning is at the core of peer production communities* [Schmidt 2009]. Our patterns help to explicate the way these communities work, but more importantly, we hope these patterns will help potentiate a global culture of collaborative learning, inside and outside of institutions.

*Quality.* "By intervening in real communities, these efforts achieve a level of external validity that lab-based experiments cannot" [Benkler et al. 2015]. The "What's next" annotation piloted here will be helpful to other design pattern authors who aim to use patterns as part of a research intervention. Peer production is not guaranteed to out-compete proprietary solutions [Benkler et al. 2015; Hill 2011]: its potential for success will depend on the way the problems are framed, and our ability to follow through.

[59] https://github.com/Peeragogy/peeragogy-handbook/blob/master/en/Peeragogy-Business_Model_Canvas.pdf

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# Implementing feedback in creative systems: A workshop approach

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## Abstract

One particular challenge in AI is the computational modelling and simulation of creativity. Feedback and learning from experience are key aspects of the creative process. Here we investigate how we could implement feedback in creative systems using a social model. From the field of creative writing we borrow the concept of a Writers Workshop as a model for learning through feedback. The Writers Workshop encourages examination, discussion and debates of a piece of creative work using a prescribed format of activities. We propose a computational model of the Writers Workshop as a roadmap for incorporation of feedback in artificial creativity systems. We argue that the Writers Workshop setting describes the anatomy of the creative process. We support our claim with a case study that describes how to implement the Writers Workshop model in a computational creativity system. We present this work using patterns other people can follow to implement similar designs in their own systems. We conclude by discussing the broader relevance of this model to other aspects of AI.

## 1 Introduction

In educational applications it would be useful to have an automated tutor that can read student work and make suggestions based on diagnostics, like, is the paper wrong, and if so how? What background material should be recommended to the student for review?

In the current paper, we "flip the script" and look at what we believe to be a more fundamental problem for AI: computer programs that can themselves learn from feedback. After all, if it was easy to build great automatic tutors, they would be a part of everyday life. As potential users (thinking from both sides of the desk) we look forward to a future when that is the case.

Along with automatic tutoring, computational creativity is a challenge within artificial intelligence where feedback plays a vital part (for example Pérez y Pérez, Aguilar, & Negrete, 2010; Pease, Guhe, & Smaill, 2010). Creativity cannot happen in a 'silo' but instead is influenced and affected by feedback and interaction with others (Csikszentmihalyi, 1988; Saunders, 2012). Computational creativity researchers are starting to place more emphasis on social interaction and feedback in their systems and models (Saunders, 2012; Gervás & León, 2014; Corneli et al., 2015). Still, nearly 3 in 4 papers at the 2014 International Conference for Computational Creativity[1] failed to acknowledge the role of feedback or social communication in their computational work on creativity.

To highlight and contribute towards modelling feedback as a crucial part of creativity, we propose in this paper a model of computational feedback for creative systems based on Writers Workshops (Gabriel, 2002), a literary collaborative practice that encourages interactive feedback within the creative process. We introduce the Writers Workshop concept (Section 2) and critically reflect on how it could encourage serendipity and emergence in computational models of intelligence and creativity. These considerations lead us to propose a Writers Workshop computational model of feedback in computational creativity and AI systems (Section 2.1), the central contribution of this paper. In Section 3 we consider how the Writers Workshop model fits into previous work in various related areas. While we acknowledge that this paper is offering a roadmap for this model rather than a full implementation, we consider how the model could be practically implemented in a computational system and report our initial implementation work (Section 4). In concluding discussions, we reflect on divergent directions in which this work could potentially be useful in the future.

## 2 The Writers Workshop

Richard Gabriel (2002) describes the practise of Writers Workshops that has been put to use for over a decade within the Pattern Languages of Programming (PLoP) community. The basic style of collaboration originated much earlier with groups of literary authors who engage in peer-group critique. Some literary workshops are open as to genre, and happy to accommodate beginners, like the Minneapolis Writers Workshop[2]; others are focused on professionals working within a specific genre, like the Milford Writers Workshop.[3]

[1] ICCC is the key international conference for research in computational creativity.

[2] `http://mnwriters.org/how-the-game-works/`

[3] `http://www.milfordsf.co.uk/about.htm`

The practices that Gabriel describes are fairly typical:

- Authors come with work ready to present, and read a short sample.
- This work is generally work in progress (and workshopping is meant to help improve it). Importantly, it can be early stage work. Rather than presenting a created artefact only, activities in the workshop can be aspects of the creative process itself. Indeed, the model we present here is less concerned with after-the-fact assessment than it is with dealing with the formative feedback that is a necessary support for creative work.
- The sample work is then discussed and constructively critiqued by attendees. Presenting authors are not permitted to rebut these comments. The commentators generally summarise the work and say what they have gotten out of it, discuss what worked well in the piece, and talk about how it could be improved.
- The author listens and may take notes; at the end, he or she can then ask questions for clarification.
- Generally, non-authors are either not permitted to attend, or are asked to stay silent through the workshop, and perhaps sit separately from the participating authors/reviewers.[4]

Essentially, the Writers Workshop is somewhat like an interactive peer review. The underlying concept is reminiscent of Bourdieu's *fields of cultural production* (Bourdieu, 1993) where cultural value is attributed through interactions in a community of cultural producers active within that field.

## 2.1 Writers Workshop as a computational model

The use of Writers Workshop in computational contexts is not an entirely new concept. In PLoP workshops, authors present design patterns and pattern languages, or papers about patterns, rather than more traditional literary forms like poems, stories, or chapters from novels. Papers must be workshopped at a PLoP or EuroPLoP conference in order to be considered for the *Transactions on Pattern Languages of Programming* journal. A discussion of writers workshops in the language of design patterns is presented by Coplien and Woolf (1997).

The steps in the workshop can be distilled into the following phases, each of which could be realised as a separate computational step in an agent-based model:

1. Author: `presentation`
2. Critic: `listening`
3. Critic: `feedback`
4. Author: `questions`
5. Critic: `replies`
6. Author: `reflections`

[4]Here we present Writers Workshops as they currently exist; however this last point is debatable. Whether non-authors should be able to participate or not is an interesting avenue for experimentation both in human and computational contexts. The workshop dialogue itself may be considered an "art form" whose "public" may potentially wish to consume it in non-participatory ways. Compare the classical Japanese *renga* form (Jin'Ichi, Brazell, & Cook, 1975).

The `feedback` step may be further decomposed into `observations` and `suggestions`. This protocol is what we have in mind in the following discussion of the Writers Workshop.[5]

**Dialogue example**

Note that for the following dialogue to be possible computationally, it would presumably have to be conducted within a lightweight process language. Nevertheless, for convenience, the discussion will be presented here as if it was conducted in natural language. Whether contemporary systems have adequate natural language understanding to have interesting interactions is one of the key unanswered questions of this approach, but protocols such as the one described above are sufficient to make the experiment.

For example, here's what might happen in a discussion of the first few lines of a poem, "On Being Malevolent". As befitting the AI-theme of this workshop, "On Being Malevolent" is a poem written by an early user-defined flow chart in the FloWr system (known at the time as Flow) (Colton & Charnley, 2014).

> FLOW: "*I hear the souls of the damned waiting in hell. / I feel a malevolent spectre hovering just behind me / It must be his birthday*."
>
> SYSTEM A: I think the third line detracts from the spooky effect, I don't see why it's included.
>
> SYSTEM B: It's meant to be humourous – in fact it reminds me of the poem you presented yesterday.
>
> MODERATOR: Let's discuss one poem at a time.

Even if, perhaps and especially because, "cross-talk" about different poems bends the rules, the dialogue could prompt a range of reflections and reactions. System A may object that it had a fair point that has not been given sufficient attention, while System B may wonder how to communicate the idea it came up with without making reference to another poem. Here's how the discussion given as example in Section 2 might continue, if the systems go on to examine the next few lines of the poem.

> FLOW: "*Is God willing to prevent evil, but not able? / Then he is not omnipotent / Is he able, but not willing? / Then he is malevolent.*"
>
> SYSTEM A: These lines are interesting, but they sound a bit like you're working from a template, or like you're quoting from something else.
>
> SYSTEM B: Maybe try an analogy? For example, you mentioned birthdays: you could consider an analogy to the conflicted feelings of someone who knows in advance about her surprise birthday party.

[5]The connections between Writers Workshops and design patterns, noted above, appear to be quite natural, in that the steps in the workshop protocol roughly parallel the typical components of design pattern templates: *context*, *problem*, *solution*, *rationale*, *resolution of forces*.




Figure 1: A paper prototype for applying the *Successful Error* pattern following a workshop-like sequence of steps

This portion of the discussion shifts the focus of the discussion onto a line that was previously considered to be spurious, and looks at what would happen if that line was used as a central metaphor in the poem.

> FLOW: Thank you for your feedback. My only question is, System B, how did you come up with that analogy? It’s quite clever.
> SYSTEM B: I’ve just emailed you the code.

Whereas the systems were initially reviewing poetry, they have now made a partial genre shift, and are sharing and remixing code. Such a shift helps to get at the real interests of the systems (and their developers). Indeed, the workshop session might have gone better if the systems had focused on exchanging and discussing more formal objects throughout.

### 2.2 How the Writers Workshop can lead to computational serendipity

Learning involves engaging with the unknown, unfamiliar, or unexpected and synthesising new understanding (Deleuze, 2004 [1968]). In the workshop setting, learning can develop in a number of unexpected ways, and participating systems need to be prepared for this. One way to evaluate the idea of a Writers Workshop is to ask whether it can support learning that is in some sense *serendiptious*, in other words, whether it can support discovery and creative invention that we simply couldn’t plan for or orchestrate in another way.

Figure 1 shows a paper prototype showing how one of the “patterns of serendipity” that were collected by Van Andel (1994) might be modelled in a workshop-like dialogue sequence. The patterns also help identify opportunities for serendipity at several key steps in the workshop sequence.

**Serendipity Pattern: *Successful error*.** Van Andel describes the creation of Post-it™ Notes at 3M. One of the instrumental steps was a series of internal seminars in which 3M employee Spencer Silver described an invention he was sure was interesting, but was unsure how to turn into a useful product: weak glue. The key prototype that came years later was a sticky bookmark, created by Arthur Fry. In the Writers Workshop, authors similarly have the opportunity to share things that they find interesting, but that they are not certain about. The author may want to ask a specific question about their creation: Does $x$ work better than $y$? They may flag certain parts of the work as especially problematic. They may think that a certain portion of the text is interesting or important, without being sure why. Although there is no guarantee that a participating critic will be able to take these matters forward, sometimes they do – and the workshop environment will produce something that the author wouldn’t have thought of alone.

**Serendipity Pattern: *Outsider*.** Another example from van Andel considers the case of a mother whose son was aflicted by a congenital cateract, who suggested to her doctor that rubella during pregnancy may have been the cause. In the workshop setting, someone who is not an “expert” may come up with a sensible idea or suggestion based on their own prior experience. Indeed, these suggestions may be more sensible than the ideas of the author, who may be to close to the work to notice radical improvements.

**Serendipity Pattern: *Wrong hypothesis*.** A third example describes the discovery that lithium can have a therapeutic effect in cases of mania. Originally, lithium carbonate had merely been used a control by John Cade, who was interested in the effect of effect of uric acid, present in soluble

lithium urate. Cade was searching for causal factors in mania, not therapies for the condition: but he found that lithium carbonate had an unexpected calming effect. Similarly, in the workshop, the author may think that a given aspect of their creation is the interesting "active ingredient," and it may turn out that another aspect of the work is more interesting to critics. Relatedly, the author may not fully comprehend a critic's feedback and may have to ask follow-up questions to understand it.

**Serendipity Pattern:** ***Side effect.*** A fourth example described by van Andel concerns Ernest Huant's discovery that nicotinamide, which he used to treat side-effects of radiation therapy, also proved efficacious against tuberculosis. In the workshop setting, one of the most important places where a side-effect may occur concerns feedback from the critic to the author. In the simple case, feedback may trigger revisions to the work under discussion. In a more general, and more unpredictable case, feedback may trigger broader revisions to the generative codebase.

This collection of patterns shows the likelihood of unexpected results coming out of the communication between author and critics. This suggests several guidelines for system development, which we will discussed in a later section.

Further guidelines for structuring and participating in traditional writers workshops are presented by Linda Elkin in (Gabriel, 2002, pp. 201–203). It is not at all clear that the same ground rules should apply to computer systems. For example, one of Elkin's rules is that "Quips, jokes, or sarcastic comments, even if kindly meant, are inappropriate." Rather than forbidding humour, it may be better for individual comments to be rated as helpful or non-helpful. Again, in the first instance, usefulness and interest might be judged in terms of explicit criteria for serendipity; see (Corneli, Pease, Colton, Jordanous, & Guckelsberger, 2014; Pease, Colton, Ramezani, Charnley, & Reed, 2013). The key criterion in this regard is the *focus shift*. This is the creation of a novel problem, comprising the move from discovery of interesting data to the invention of an application. This process is distinct from identifying routine errors in a written work. Nevertheless, from a computational standpoint, noticing and being robust to certain kinds of errors is often a preliminary step. For example, the work might contain a typo, grammatical or semantic error, while being logically sound. In a programming setting, this sort of problem can lead to crashing code, or silent failure. In general communicative context, argumentation may be logically sound, but not practically useful or poorly exposited. Finally, even a masterful, correct, and fully spellchecked piece of argumentation may not invite further dialogue, and so may fail to open itself to further learning. Identifying and engaging with this sort of deeper issue is something that skillful workshop participants may be able to do. Dialogue in the workshop can build on strong or less strong work – but provoking interpretative thoughts and comments always require a thoughtful critical presence and the ability to engage. This can be difficult for humans and poses a range of challenges for computers – but also promises some interesting results.

# 3 Related work

In considering the potential and contribution of the Writers Workshop model outlined in Section 2, we posit that the Writers Workshop model is useful for encouraging feedback in computational systems, and in particular systems that are designed to be creative or serendipitous.

Feedback has long been a central concept in AI-related fields such as cybernetics (Ashby, 1956; Seth, 2015). Feedback about feedback (and &c for higher orders) is understood to be relevant to thinking about *learning* and *communication* (Bateson, 1972). We now consider the importance of the roles that communicative feedback play in computational creativity and computational serendipity and discuss previous related work in incorporating feedback into such computational systems.

## 3.1 Feedback in computational creativity

*Creativity* is often envisaged as involving cyclical processes (e.g. Dickie's (1984) art circle, Pease and Colton's (2011) Iterative Development-Expression-Appreciation model). There are opportunities for embedded feedback at each step, and the creative process itself is "akin to" a feedback loop. However, despite these strong intimations of the central importance of feedback in the creative process, our sense is that feedback has not been given a central place in research on computational creativity. In particular, current systems in computational creativity, almost as a rule, do *not* consume or evaluate the work of other systems.[6]

Gervás and León (2014) theorise a creative cycle of narrative development as involving a Composer and an Interpreter, in such a way that the Composer has internalised the interpretation functionality. Individual creativity is not the poor relation of social creativity, but its mirror image. Nevertheless, even when computer models explicitly involve multiple agents and simulate social creativity (like Saunders & Gero, 2001), they rarely make the jump to involve multiple systems. The "air gap" between computationally creative systems is very different from the historical situation in human creativity, in which different creators and indeed different cultural domains interact vigorously (Geertz, 1973).

## 3.2 Feedback in computational serendipity

The term computational serendipity is rather new, but its foundations are well established in prior research.

Grace and Maher (2014) examine *surprise* in computing, seeking to "adopt methods from the field of computational creativity [...] to the generation of scientific hypotheses." This is an example of an effort focused on computational *invention*.

An area of AI where serendipity can be argued to play an important part is in pattern matching. Current computer programs are able to identify known patterns and "close matches" in data sets from certain domains, like music (Meredith, Lemström, & Wiggins, 2002). Identifying known

[6] An exception to the rule is Mike Cook's AppreciationBot (`https://twitter.com/AppreciationBot`), which is a reactive automaton that "appreciates" tweets from MuseumBot.

patterns is a special case of the more general concept of *pattern mining* (Bergeron & Conklin, 2007). In particular, the ability to extract *new* higher order patterns that describe exceptions is an example of "learning from feedback." Deep learning and evolutionary models increasingly use this sort of idea to facilitate strategic discovery (Samothrakis & Lucas, 2011). Similar ideas are considered in business applications under the heading "process mining" (Van Der Aalst, 2011).

In earlier work (Corneli et al., 2014, 2015), we used the idea of dialogue in a Writers Workshop framework to sketch a "theory of poetics rooted in the making of boundary-crossing objects and processes" and described (at a schematic level) "a system that can (sometimes) make 'highly serendipitous' creative advances in computer poetry" while "drawing attention to theoretical questions related to program design in an autonomous programming context."

### 3.3 Communications and feedback

The Writers Workshop heavily relies on communication of feedback within the workshop. Gordon Pask's conversation theory, reviewed in (Pask, 1984; Boyd, 2004), goes considerably beyond the simple process language of the workshop, although there are structural parallels. We see that a basic Pask-style learning conversation bears many similarities to the Writers Workshop model of communicative feedback (Boyd, 2004, p. 190):

1. Conversational participants are carrying out some actions and observations;
2. Naming and recording what action is being done;
3. Asking and explaining why it works the way it does;
4. Carrying out higher-order methodological discussion; and,
5. Trying to figure out why unexpected results occured.

Variations to the underlying system, protocol, and the schedule of events should be considered depending on the needs and interests of participants, and several variants can be tried. On a pragmatic basis, if the workshop proved quite useful to participants, it could be revised to run monthly, weekly, or continuously.[7]

## 4 Case study: Flowcharts and Feedback

This section describes work that is currently underway to implement the Writers Workshop model, not only within one system but as a new paradigm for collaboration among disparate projects. In order to bring in other participants, we need a neutral environment that is not hard to develop for: the FloWr system mentioned in Section 2.1 offers one such possibility. The basic primary objects in the FloWr system are *flowcharts*, which are comprised of interconnected *process nodes* (Charnley, Colton, & Llano, 2014; Colton & Charnley, 2014). Process nodes specify input and output types, and internal processing can be implemented in Java, or other languages that interoperate with the JVM, or by invoking external web services. One of the common applications to date is to generate computer poetry, and we will focus on that domain here.

A basic set of questions, relative to this system's components, are as follow:

1. *Population of nodes*: What can they do? What do we learn when a new node is added?
2. *Population of flowcharts*: Pease et al. (2013) have described the potentially-serendipitous repair of "broken" flowcharts when new nodes become available; this suggests the need for test-driven development framework.
3. *Population of output texts*: How to assess and comment on a generated poetic artefact?

In a further evolution of the system, the sequence of steps in a Writers Workshop could itself be spelled out as a flowchart. The process of reading a poem could be conceptualised as generating a semantic graph (Harrington & Clark, 2007; Francisco & Gervás, 2006). Feedback could be modelled as annotations to a text, including suggested edits. These markup directives could themselves be expressed as flowcharts. A standardised set of markup structures may partially obviate the need for strong natural language understanding, at least in interagent communication. Thus, we could agree that `observations` will consist of stand-off annotations that connect textual passages to public URIs using a limited comparison vocabulary, and `suggestions` will consist of simple stand-off line-edits, which may themselves be marked up with rationale. These restrictions, and similar restrictions around constrained turn-taking, could be progressively widened in future versions of the system. The way the poems that are generated, the models of poems that are created, and the way the feedback is generated, all depend on the contributing system's body of code and prior experience, which may vary widely between participating systems. In the list of functional steps below, all of the functions could have a subscripted "$\mathcal{E}$", which is omitted throughout. Exchanging path dependent points of view will tend to produce results that are different from what the individual participating systems would have come up with on their own.

I. Both the author and critic should be able to work with a model of the text. Some of the text's features may be explicitly tagged as "interesting." Outstanding questions may possibly be brought to the attention of critical listeners, e.g. with the request to compare two different versions of the poem (`presentation`, `listening`).
   1. *A model of the text.* $m : T \to M$.
   2. *Tagging elements of interest.* $\mu : M \to I$.

II. Drawing on its experience, the critic will use its model of the poem to formulate feedback (`feedback`).
   1. *Generating feedback.* $f : (T, M, I) \to F$.

[7] For a comparison case in computer Go, see `http://cgos.computergo.org/`.

III. Given the constrained framework for feedback, statements about the text will be straightforward to understand, but rationale for making these statements may be more involved (`questions`, `replies`).

   1. *Asking for more information.* $q : (M, F, I) \rightarrow Q$.
   2. *Generating rationale.* $a : (M, F, Q) \rightarrow \Delta F$.

IV. Finally, feedback may affect the author's model of the world, and the way future poems are generated (`reflection`).

   1. *Updating point of view.* $\rho : (M, F) \rightarrow \Delta \mathcal{E}$.

The final step is perhaps the most interesting one, since it invites us to consider how individual elements of feedback can "snowball" and go beyond line-edits to a specific poem to much more fundamental changes in the way the presenting agent writes poetry. Here methods for pattern mining, discussed in Section 3.2, are particularly relevant. If systems can share code (as in our sample dialogue in Section 2.1) this will help with the rationale-generating step, and may also facilitate direct updates to the codebase. However, shared code may be more suitably placed into the common pool of resources available to FloWr than copied over as new "intrinsic" features of an agent.

Although different systems with different approaches and histories are important for producing unexpected effects, "offline" programmatic access to a shared pool of nodes and existing flowcharts may be useful. Outside of the workshop itself, agents may work to recombine nodes based on their input and output properties to assemble new flowcharts. This can potentially help evaluate and evolve the population of nodes programmatically, if we can use this sort of feedback to define fitness functions. The role of temporality is interesting: if the workshop takes place in real time, this will require different approaches to composition that takes place offline (Perez, Samothrakis, Lucas, & Rohlfshagen, 2013). Complementing these "macro-level" considerations, it is also worth commenting on the potential role of "micro-level" feedback within flowcharts. Local evaluation of output from a predecessor node could feed backwards through the flowchart, similar to backpropagation in neural networks. This would rely on a reduced version of the functional schema described above.

## 5 Concluding discussion and future directions

We have described a *general* and *computationally feasible* model for using feedback in AI systems, particularly creative systems. The Writers Workshop concept, borrowed from creative writing, is transformed into a model of a structured approach to eliciting, processing and learning from feedback. To better evaluate how the Writers Workshop model helps us advance in our goal of incorporating feedback into artificial creativity, we critically considered how the model fits into related work. In particular, we found that serendipity, a key concept within creativity and AI more generally, is a concept with which the Writers Workshop model could assist computational progress. In this respect, we should highlight the difference between "global" analytics describing the collection of nodes and flowcharts in the FloWr ecosystem, and the path-dependent process of analysis and synthesis that takes place in a workshop setting.

Our preliminary implementation work (Section 4) shows that the model can be transfered to a functional implementation. This work highlights several considerations relevant to further work with the Writers Workshop model:

- Each contributing system should come to the workshop with at least a basic awareness of the workshop protocol, with work to share, and prepared to give constructive feedback to other systems.
- The workshop itself needs to be prepared, with a suitable communication platform and a moderator or global flowchart for moving the discussion from step to step.
- A controlled vocabulary for communications and interaction would be a worthwhile pursuit of future research, perhaps based on an ontology inspired by the Interaction Network Ontology.[8]
- In order to get the most value out of the workshop experience, systems (and their wranglers) should ideally have questions they are investigating. As discussed above, prior experience plays an important role in every step. This opens up a range of issues for further research on modeling motivations and learning from experience.
- Systems should be prepared to give feedback, and to carry out evaluations of the helpfulness (or not) of feedback from other systems and of the experience overall.

Developing systems that could successfully navigate this collaborative exercise would be a significant advance in the field of computational creativity. Since the experience is about learning rather than winning, there is little motivation to "game the system" (cf. Lenat, 1983). Instead the emphasis is squarely upon mutual benefit: computational systems helping to develop each other through communication and feedback.

The benefits of the Writers Workshop approach could innovate well beyond models for feedback and communication within a particular environment or restricted domain. Following the example of the Pattern Languages of Programming (PLoP) community, we propose that the Writers Workshop model could be deployed within the Computational Creativity community to design a workshop in which the participants are computer systems instead of human authors. The annual International Conference on Computational Creativity (ICCC), now entering its sixth year, could be a suitable venue.

Rather than the system's creator presenting the system in a traditional slideshow and discussion, or a system "Show and Tell," the systems would be brought to the workshop and would present their own work to an audience of other systems, in a Writers Workshop format. This could be accompanied by a short paper for the conference proceedings written

[8] The Interaction Network Ontology primarily describes interactions within humans as opposed to within human societies; a distinct *Social* Interaction Ontology does not seem to exist at present. However, the classes of the Interaction Network Ontology appear to be quite broadly relevant. This ontology is documented at `http://www.ontobee.org/browser/index.php?o=INO`. Its URI is `http://svn.code.sf.net/p/ino/code/trunk/src/ontology/INO.owl`.

by the system's designer describing the system's current capabilities and goals. If the Workshop really works well, future publications might adapt to include traces of Workshop interactions, commentary from a system on other systems, and offline reflections on what the system might change about its own work based on the feedback it receives. Paralleling the PLoP community, it could become standard to incorporate the workshop into the process of peer review for the new *Journal of Computational Creativity*.[9] AI systems that review each other would surely be a major demonstration and acknowledgement of the usefulness of feedback within AI.

In closing, we wish to return briefly to the scenario of computer generated feedback in educational contexts that we raised at the beginning of this paper and then set aside. The elements of our functional design for sharing feedback among computational agents has a range of features that continue to be relevant for generating useful feedback with human learners. Students are experience-bound, and a robust approach to formative assessment and feedback should take into account the student's historical experience, so far as this can be known or inferred. In order for feedback, recommendations, and so on to adequately take individual history into account, sophisticated modelling and reasoning would be required. Nevertheless, from the point of view of participating computational agents, a student may simply look like another agent. It is in this regard that computational models of learning from feedback are seen as fundamental.

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[9] `http://www.journalofcomputationalcreativity.cc`

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# The role of blending in mathematical invention

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**Abstract**

We model the mathematical process whereby new mathematical theories are invented. Here we explain the use of conceptual blending for this purpose, and show examples to illustrate the process in action. Our longer-term goal is to support machine and human mathematical creativity.

## Introduction

We are concerned with creativity in mathematics: creativity as evinced by human and artificial mathematicians, individually and collectively.

Work on *conceptual blending* has been much influenced by Fauconnier and Turner (1998, 2002). More recently, the centrality of conceptual blending to creativity has been stressed by Turner (2014), where he writes:

> . . . the human spark comes from our advanced ability to *blend* ideas to make *new* ideas. Blending is the origin of ideas. (Turner, 2014, p 2)

The claim is that blending in this sense is a general human cognitive ability, and as such applies to mathematics, as much as to art, poetry, music and so on (see for example Turner (2005)).

The place of mathematics and the sciences among creative endeavours has been stressed by the literary critic George Steiner:

> It is in mathematics and the sciences that the concepts of creation and of invention, of intuition and of discovery, exhibit the most immediate, visible force.
>
> Steiner (2001, p 145)

Blending involves recognising features common to mathematical concepts, even when expressed in different terminology. The role of mathematical analogy in creative mathematics is well expressed by Weil (1960), and a general plea for analogical reasoning within science in (Arbib and Hesse, 1986).

We are investigating *computational* accounts of mathematical creativity, taking conceptual blending as a key ingredient. The work of Goguen (1999, 2005) has provided a general framework for comparison of conceptual spaces, and computation of blends. This enables the use of richer representation formalisms, and so is closer to contemporary mathematics than previous computational realisations of blending, such as in Pereira (2007).

This paper deals with the creative process in mathematics, as modelled along the lines above. We focus on the use of blending within a single process, searching for blends satisfying some evaluation criteria, from the starting point of some given conceptual spaces.

While cognitive issues are important to us, this paper is focused on issues in representation and representation change; there are however brief comments on cognition in the conclusions.

We start by providing some background, followed by an example to illustrate the components involved in our approach. A historical example based on Georg Cantor's work follows. The most extended example was carried out by a pure mathematician (D. Gómez-Ramírez), working in a domain close to his own; in this case, the blend mechanism threw up some unexpected properties, which provoked new work by the mathematician.

Subsequently we give some more speculative thoughts on where this work can go in the future, by considering Galois theory as a test-bed. Finally, we discuss the evaluation of work along these lines, and give some conclusions.

## Background

### Blending in Mathematics

Lakoff and Núñez (2000) are among the first to present a cognitive account of the origin and development of mathematical ideas,[1] arguing against the "romance of mathematics" in which mathematics is presented as an ever-increasing set of universal, absolute, certain truths which exist independently of humans. They present the thesis that human mathematics is grounded in bodily experience of a physical world, and mathematical entities inherit properties which objects in the world have, such as being consistent or stable over time. Exploring the physical world of object collection might lead to concepts like the empty collection and rules like "adding a collection of $n$ objects to an empty

[1] This is lamented by Lakoff and Núñez, who claim that (prior to their work), "there was still no discipline of mathematical idea analysis from a cognitive perspective" (Lakoff and Núñez, 2000).

collection yields a collection with $n$ objects". People then form grounding metaphors between the physical world and an abstract mathematical world, allowing us to project from everyday experiences onto abstract concepts, thus leading to the concept of zero and the axiom that $n+0 = n$. Lakoff and Núñez posit that blending different mathematical metaphors leads to more complex ideas (see also Alexander (2011)).

Alongside this account of mathematical cognition, mainstream contemporary mathematics has developed its own methodology and foundations, enjoying an exceptional place among scientific disciplines. Its methods, objects of study and sometimes astonishing results have widespread, if not universal, acceptance.

In conclusion, mathematics is a scientific discipline having not only a fundamental cognitive component, necessary in its development, but also possessing a collection of general principles and structures going beyond a particular school of thought. Among these general processes we want to highlight in this paper the importance that conceptual blending has in mathematics, incorporating both cognitive and mathematically specific aspects in order to create new mathematical concepts.

## Terminology for conceptual blending

Our notion of conceptual blending is informed by Category theory, and highly influenced by Goguen's work on concepts (Goguen, 2005). In this paper we use the terminology below, and elucidate the terminology by means of a running example – discovering a version of the integers (in the sense of providing a partial approach to the genuine integers) using blending.

**Conceptual spaces** are partial and temporary representational structures which are constructed on the fly when talking about a particular situation, which are informed by the knowledge structures associated with a domain. These are influenced by Boden's idea of a concept space which is mapped, explored and transformed by transcending mapped boundaries (Boden, 1977), and form the input spaces to our blend.

As an example of two conceptual spaces, consider one as a theory NAT – a theory of the natural numbers, and FUNC – a theory of a total unary function with an inverse. We will refer back to these theories in this exposition.

(Many-sorted) First-order **Axioms** are the criteria which will be used here to delineate the conceptual spaces. The axiomatic method has been a fundamental aspect of mathematical research since Euclid, and various axiom changes have led to revolutions in mathematics. For instance, rejecting the parallel postulate opened up fascinating new areas of non-Euclidean geometry.

The precise formulations for NAT and FUNC can be found in Listings 1 and 2. Notice that these formulations obviously refer to partial representations of the genuine concepts employed by mathematicians. In the conceptual space with theory NAT, an example of an axiom is $\forall x.\neg\ 0 = s(x)$ – that is that zero is not a successor element. The conceptual space with theory FUNC has an axiom $\forall x.\ f(finv(x)) = x$.

**Signature morphisms** between conceptual spaces are mappings from the symbols of the source conceptual space into the symbols of the other conceptual space. For example NAT contains a function $\lambda x : Nat.\ s(x)$ that maps $x$ to its successor, and FUNC contains a function defined over a set $X$ that maps each element to an image $\lambda x : X.\ f(x)$. A theory $G$ with a morphism to both NAT and FUNC might contain a function $\lambda x : N.\ func(x)$ that takes every number in some set $N$ to its image under *func*. When we show a mapping we write this as

$$s \quad \leftarrow_{\phi(G,\text{NAT})} \quad func \quad \rightarrow_{\phi(G,\text{FUNC})} \quad f \qquad (1)$$

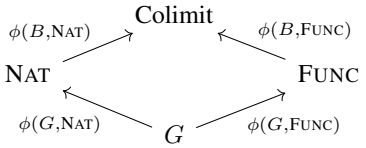
$$Nat \quad \leftarrow_{\phi(G,\text{NAT})} \quad N \quad \rightarrow_{\phi(G,\text{FUNC})} \quad X \qquad (2)$$

The mapping $\phi(G, \text{NAT})$ is a signature morphism from $G$ to NAT. Note that associated types are also mapped.

**Input Spaces** refer to two or more conceptual spaces of interest.

**Generic spaces** are conceptual spaces that possess commonality between input spaces.

**Colimits** are conceptual spaces representing a blend of input spaces with respect to a given generic space and a set of signature morphisms. These are uniquely computed given a generic space and a set of morphisms. Here is a diagrammatic representation of such a computation in our example using theories NAT and FUNC :




The conceptual space represented by the Colimit is often referred to as the *blend*.

**Internal Evaluation** constitutes a variety of techniques to determine whether a computed colimit is viable as a conceptual space. In our example, since the conceptual spaces are mathematical theories, we can exploit the notion of consistency. This is a way of evaluating whether a blend is not only creative, but also valid. In the example of theories NAT and FUNC, the computed blend is inconsistent due to the emergent axioms in the computed colimit. The only type existing within the colimit is from now on referred to as $\mathbb{Z}$ to distinguish it from the natural numbers. Notice that in the colimit it holds that:

$$\forall x : \mathbb{Z}.\neg\ zero = s(x)$$

$$\forall x : \mathbb{Z}.\ s(sinv(x)) = x\ .$$

This is an inconsistency, as from the second axiom we see that there is an element for which 0 is the successor.

**Weakening** refers to the process of weakening the input theories by removing symbols or axioms. If we remove the axiom

$$\forall x : Nat.\neg\ zero = s(x)$$

then the resulting computed colimit contains a mathematical theory which is consistent.

Martinez et al. (2014) provides an algorithm to explore the space of blends resulting from given input spaces and a given generic space, where weakening is achieved by omitting axioms. The algorithm returns the blends which are consistent, and maximally so, among those in this space of

blends. This algorithm assumes that consistency of relevant theories can be checked, so is not always effective.

**Running the blend** refers to elaborating or completing a mathematical theory. Sometimes there are missing definitions which need to be discovered. For example in the new theory the following axiom appears

$$\forall x, y : \mathbb{Z}.\ s(x) + y \quad = \quad s(x + y)\ ,$$

but we also are interested in theorems such as

$$\forall x, y : \mathbb{Z}.\ \mathit{sinv}(x) + y \quad = \quad \mathit{sinv}(x + y)\ .$$

Finding suitable theorems is an example of running the blend, and from which it is possible to discover and prove theorems such as

$$\forall x, y : \mathbb{Z}.\ \mathit{sinv}(x) + s(y) \quad = \quad x + y\ .$$

### Technologies

The approach explained above corresponds to Goguen's proposal (Goguen, 1999) for implementing blending, but slightly simplified (as in Kutz, Neuhaus, Mossakowski, and Codescu (2014)): we use the normal colimit construction, rather than $\frac{3}{2}$-colimits (both described in Goguen (1999)).

Additionally we assume that the conceptual spaces involved are given using a CASL specification (Astesiano et al., 2002) and that the morphisms are theorem preserving (i.e. map theorems to theorems). The reason for these assumptions is that in these cases it is well-known how to compute colimits: the colimit specification essentially corresponds to the disjoint union of the two target conceptual spaces except for not repeating the symbols given in the common source conceptual space. Moreover, we will be using the HETS system (Mossakowski, Maeder, and Lüttich, 2007) to compute such colimits. The code for the implemented examples in this paper is available on-line.[2]

The use of CASL specifications means that we deal with first-order logic; CASL is supported in the HETS system, and colimits here can be computed in the current implementation of HETS. Although higher-order logic (with Henkin semantics) is available in HETS (indeed in CASL) and the colimits are well-known to exist (because higher-order in this form is reducible to many-sorted first-order logic), it is worth noticing that the calculus of such colimits is not currently available in HETS. This restricts the formalisms that can be used directly for our purposes, where computation of colimits is central to our approach.

## Blending and the infinite

### Example Revisited – the Integers

As a first demonstration of the machinery involved in blending mathematical theories, we consider combining a theory of natural numbers with the concept of the inverse of a function to obtain the integers. Let us assume a simple partial axiomatisation of the natural numbers (without order axioms) as shown in Listing 1, and call this theory NAT. Now let us also define a simple theory which introduces the concept of a function with an inverse as shown in Listing 2, and call this theory FUNC.

[2] See: https://github.com/ewenmaclean/ICCC2015_hetsfiles

**spec** NAT =
**sort** *Nat*
**ops** *zero* : *Nat*;
*s* : *Nat* → *Nat*;
__+__ : *Nat* × *Nat* → *Nat*
∀ *x*, *y* : *Nat*
• $s(x) = s(y) \Rightarrow x = y$
• $\neg\ zero = s(x)$
• $s(x) + y = s(x + y)$
• $zero + y = y$
**end**

Listing 1: A theory of the natural numbers without order

**spec** FUNC =
**sort** *X*
**op** $f : X \to X$
**op** $\mathit{finv} : X \to X$
∀ *x* : *X*
• $f(\mathit{finv}(x)) = x$
• $\mathit{finv}(f(x)) = x$
**end**

Listing 2: A theory with a function and its inverse defined

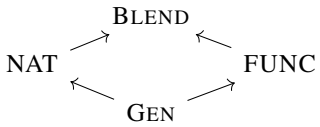
**Identifying a Generic Space** In order to incorporate the notion of blending here we want to be able to identify a "generic" component of each theory and compute the colimit. We can use the HDTP system (Gust, Kühnberger, and Schmidt, 2006; Schmidt, 2010) to discover a common theory and signature morphism between symbols in the two theories NAT and FUNC. The Generic theory GEN contains a sort *N* and a function *func*, and the morphisms from the Generic theory to NAT and FUNC are:

$$s \quad \leftarrow_{\phi(G,\mathrm{NAT})} \quad \mathit{func} \quad \rightarrow_{\phi(G,\mathrm{FUNC})} \quad f \qquad (3)$$

$$\mathit{Nat} \quad \leftarrow_{\phi(G,\mathrm{NAT})} \quad N \quad \rightarrow_{\phi(G,\mathrm{FUNC})} \quad X \qquad (4)$$

Here the successor function is identified in the mapping with the function in the theory FUNC.

**Computing the Colimit** The HETS system (Mossakowski et al., 2007) can then be exploited to find a new theory by computing the colimit:




This generates the theory shown in Listing 3 (for the sake of understanding it is used *p*, for predecessor, instead of *sinv*).

**Removal of Inconsistencies** This theory is automatically determined to be inconsistent due to the axioms

$$\forall x : \mathbb{Z}.\neg\ zero = s(x) \qquad (5)$$

$$\forall x : \mathbb{Z}.\ s(p(x)) = x \qquad (6)$$

**spec** SPEC =
**sort** $N$
**op** $_+_ : N \times N \to N$
**op** $p : N \to N$
**op** $s : N \to N$
**op** $zero : N$
$\forall x, y : N \bullet s(x) = s(y) \Rightarrow x = y$
$\forall x : N \bullet \neg\, zero = s(x)$
$\forall x, y : N \bullet s(x) + y = s(x + y)$
$\forall y : N \bullet zero + y = y$
$\forall x : N \bullet s(p(x)) = x$
$\forall x : N \bullet p(s(x)) = x$
**end**

Listing 3: An inconsistent partial approach to the integers (without order)

Removal of the limiting axiom (5) from Listing 1 results in generating a blend theory which is very similar to what we understand to be the integers as shown in Listing 4.

**spec** SPEC =
**sort** $N$
**op** $_+_ : N \times N \to N$
**op** $p : N \to N$
**op** $s : N \to N$
**op** $zero : N$
$\forall x, y : N \bullet s(x) = s(y) \Rightarrow x = y$
$\forall x, y : N \bullet s(x) + y = s(x + y)$
$\forall y : N \bullet zero + y = y$
$\forall x : N \bullet s(p(x)) = x$
$\forall x : N \bullet p(s(x)) = x$
**end**

Listing 4: A consistent partial approach to the integers (without order)

**Running the Blend** Running the blend refers to discovering definitions or adding axioms to flesh out the blend. In the example of the version in Listing 4, the definition of plus needs to be extended to understand how to calculate with the predecessor function:

$$p(x) + y = p(x + y)$$

from which theorems such as

$$p(x) + s(y) = x + y$$

can be proved.

## Potential and actual infinity

Some of the ideas of Lakoff and Núñez (2000) have been reworked by the authors, with increased emphasis on conceptual blending. In particular, the analysis of mathematical infinity, given in metaphorical form as the "Basic Metaphor of Infinity" (BMI) in Lakoff and Núñez (2000), is represented in blend form in Núñez (2005) as the "Basic Mapping of Infinity" (so, still "BMI").

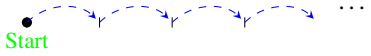
We show here how this blend works out in our setting. The BMI suggests that the notion of completed infinity, in particular the possibility of transfinite numbers in the sense of Cantor, comes from a blend of the notion of completed, finite process with that of a potentially infinite and endless process.

Thus take two corresponding input spaces, given by CASL specifications **FinEnd** and **Inf** corresponding to the following diagrams

**FinEnd:**




**Inf:**




- **FinEnd:** Completed Iterative Processes are those that from some initial state, terminate in a final state after a finite number of state transitions. One such case is chosen.
- **Inf:** Infinite Iterative Processes are those that continue indefinitely to change state.

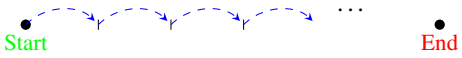
In both cases, the arrows indicate steps of the processes, and the process states are in a discrete linear order indicated by left-to-right order in the diagrams.

The generic space **Gen** simply identifies the start states, the notion of process step, and the linear ordering of states.

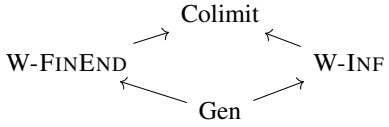
Now we can compute the blend of these spaces, which includes new features taken from both of the input spaces. This blend is *inconsistent*, for the following two reasons:

1. the number of states is finite (from **FinEnd**), and infinite (from **Inf**);
2. there both is an end state (from **FinEnd**) and is no end state (from **Inf**).

Search through the possibilities of weakening the input spaces by omitting as few axioms as possible among those involved in an inconsistency reveals the possibility of a structure with infinitely many states (from **Inf**) and an end state (from **FinEnd**). Computing the colimit from the weakened input spaces **W-FinEnd**, **W-Inf** gives a theory corresponding to this diagram:




Thus we have a blend as in the earlier examples:




## Prime Ideals as a blend

### Introduction

One of the most fundamental concepts of modern mathematics, which is the basis of commutative algebra and a seminal ingredient of the language of schemes in modern algebraic

geometry, is that of *prime ideal* (Grothendieck and Dieudonné, 1971; Eisenbud, 1995).

The terminology "prime ideal" relates to the older notion of "prime number". The initial aim of this work was to look for a blend between prime numbers (from the integers) and the ideals of a commutative ring, to see what would emerge. It turned out that *the blend process*, along with providing a definition for prime ideals, also *suggested an unexpected concept in the context of rings*, namely what will be called Containment Division Rings (CDR). In turn, this prompted questions and proofs about this concept – thus running the blend (space prevents description of this step in this paper).

We present a first blend involving weakening, followed by a second blend from fuller input spaces, where the emergent concept of CDR appears.

## The first conceptual space

Let $(R, +.*, 0, 1)$ be a commutative ring with unity (see the formal definition and examples in Eisenbud (1995)). Now, $R$ can be understood as the sort containing the elements of the corresponding commutative ring with unity. An ideal $I$ is a subset of $R$ satisfying the following axiom:

$$(\forall i, j \in I)(\forall r \in R)(i + (-j) \in I \wedge r * i \in I).$$

Let us define a unary relation (predicate) *isideal* on the set (sort) of subsets of $P(R)$ corresponding to this definition. Now, we define

$$\mathrm{Id}(R) = \{A \in P(R) : \mathit{isideal}(A)\}.$$

Ideals are "multiplied" using the following definition:

$$I \cdot_\iota J = \left\{ \sum_{k=1}^{n} i_k \cdot j_k : n \in \mathbb{N} \wedge i_1, \ldots, i_n \in I \wedge j_1, \ldots, j_n \in J \right\}.$$

In other words, $I \cdot_\iota J$ is the smallest ideal extending the set $\{i \cdot j : i \in I \wedge j \in J\}$.

The key property that we want to keep in the blend is the one saying that this operation $\cdot_\iota$ has a neutral element $1_\iota$, which can be seen as an additional notation for the ring. On the other hand, we want to see the containment relation $\subseteq$ as a binary relation over the sort $\mathrm{Id}(R)$.

Summarizing, our first conceptual space consists of sorts $R, \mathrm{Id}(R)$ and $P(R)$; operations $+, *, 0_R, 1_R, 1_\iota$ and $\cdot_\iota$; and the relations $\subseteq$ and *isideal*.

Let us denote this space by $\mathbb{I}$.

## The second conceptual space

Let $\mathbb{Z}$ be the set of the integer numbers. Here, we choose any partial axiomatization of them including at least the fact that $(\mathbb{Z}, *, 1)$ is a commutative monoid. We define also an upside-down divisibility relation $\lfloor$ defined as $e \lfloor g := g|e$, i.e. there exists an integer $c$ such that $e = c * g$. Let us define a unary relation *isprime* on $\mathbb{Z}$ as follows: for all $p \in \mathbb{Z}$, $\mathit{isprime}(p)$ holds if $p \neq 1$ and:

$$(\forall a, b \in \mathbb{Z})\ \big((ab \lfloor p) \rightarrow (a \lfloor p \vee b \lfloor p)\big).$$

Besides, we define the set (sort) of the prime numbers as

$$\mathit{Prime} = \{p \in \mathbb{Z} : \mathit{isprime}(p)\}$$

In the CASL language, we consider $\mathbb{Z}$ as the sort of the integer numbers, $*$ as a binary operation, *prime* as a predicate and $\lfloor$ as a binary relation, any of them defined over the sort $\mathbb{Z}$. We denote this conceptual space by $\mathbb{P}$.

## The Generic Space

The generic space $\mathbb{G}$ consists of a set (sort) $G$ with a binary operation $*_G$, a neutral element $S$ and a binary relation $\leq_G$.

## The Blending Morphisms

The morphism to $\mathbb{I}$ uses:
$\varphi(G) = \mathrm{Id}(R), \varphi(*_G) = *_\iota, \varphi(S) = 1_\iota$ and $\varphi(\leq_G) = \subseteq$;
the morphism to $\mathbb{G}$ uses:
$\delta(G) = \mathbb{Z}, \delta(*_G) = *, \delta(S) = 1$ and $\delta(\leq_G) = \lfloor$.

## The Axiomatization of the Blending

A straightforward colimit construction based on the input and generic spaces above yields a consistent space with properties inherited both from the prime elements into the integers and from the ideals of commutative rings; one of the concepts is a notion of prime ideals, another is that of CDR.[3] Here we describe briefly a weakening of the given spaces that makes the resultant blend more generally applicable.

From the properties defining the integers we transfer into the blend only the fact that $\mathbb{Z}$ is a set with a binary operation $*$ having 1 as neutral element and $\lfloor$ as a binary relation, without taking into account its formal definition.

Now after computing the colimit, we obtain that any element $P \in G$ (i.e., an ideal of $S$) satisfies the predicate *isprime* if and only if

$$P \neq S \wedge (\forall X, Y \in G = \mathrm{Id}(S))(X \cdot_\iota Y \subseteq P \rightarrow (X \subseteq P \vee Y \subseteq P)).$$

Thus, the predicate *isprime* turns out to be the predicate characterizing the primality of ideals of $S$ and the set (sort) $Prime$ turns out to be the set of prime ideals of $S$.

Using the weakened input spaces, the blending space consists of the axioms assuring that $S$ is a commutative ring with unity, $G$ is the set of ideals of $S$, *isprime* is the predicate specifying primality for ideals of $S$ and *Prime* is the collection of all prime ideals of $S$.

## Implementation for prime ideals over CDR-s as a blend

In this section we construct the concept of prime ideal over a CDR as a blend of the conceptual space of ideals of a commutative ring with unity and the conceptual space of the former second conceptual space where the axiom defining the upside-down divisibility relation is restored.

It is worth mentioning again that the definition of CDR-s was obtained after doing this implementation and therefore it could be seen as a form of "creative" result coming from the blending process.

After computing the corresponding colimit in HETS and interpreting "RingElt" as the sort containing the elements of the ring $S$, the theory defining the blend corresponds to the axioms defining a CDR (*S*), the set of all its ideals (*Generic*), the set all its prime ideals (*SimplePrime*) and a primality predicate (*IsPrime*). We present in Listing 5 just the

[3] A ring $R$ is a Containment Division Ring (CDR) if for all ideals $I$ and $J$ of $R$, $I \subseteq J$ if and only if $J$ divides $I$ (i.e. there exists an ideal $U$ such that $I = U \cdot_\iota J$).

theory corresponding to the colimit (omitting details of ring axioms and ideal generation).

```
spec SPEC =
    sorts Generic, RingElt, SimplePrime, SubSetOfRing
    sorts SimplePrime < Generic,IdGeneric < SubSetOfRing
    ops 0, 1, S : RingElt
    op __*__ : RingElt × RingElt → RingElt
    op __+__ : RingElt × RingElt → RingElt
    op __x__ : Generic × Generic → Generic
    pred IsIdeal : SubSetOfRing
    pred IsPrime : Generic
    pred __isIn__ : RingElt × SubSetOfRing
    pred gcont : Generic × Generic
    pred __generates__ : RingElt × Generic
    ∀ I : SubSetOfRing • I ∈ Generic ⇔ IsIdeal(I)
    ∀ x : Generic • x x S = x
    ∀ x : Generic • S x x = x
    ∀ A, B : Generic
    • gcont(A, B) ⇔ ∀ a : RingElt • a isIn A ⇒ a isIn B
    ∀ x, y : RingElt • x + y = y + x
    %% and further ring axioms ...
    ∀ I : SubSetOfRing
    • IsIdeal(I)
      ⇔ ∀ a, b, c : RingElt
          • ((a isIn I ⇒ a isIn S) ∧ 0 isIn I)
            ∧ (a isIn I ∧ c isIn S ⇒ c * a isIn I)
            ∧ (a isIn I ∧ b isIn I ∧ c isIn S ∧ b + c = 0
                ⇒ a + c isIn I)
    ∀ a : RingElt; A : Generic
    %% and axioms for generates and x ...
    ∀ x, y : Generic • gcont(x, y) ⇔ ∃ c : Generic • x = y x c
    ∀ p : Generic • p ∈ SimplePrime ⇔ IsPrime(p)
    ∀ p : Generic
    • IsPrime(p)
      ⇔ (∀ a, b : Generic
          • gcont(a x b, p) ⇒ gcont(a, p) ∨ gcont(b, p))
        ∧ ¬ p = S
end
```

Listing 5: Colimit for prime ideals over CDR-s

## A Challenge Example for Blending

### Computational Creativity via Blending

The examples shown thus far in the paper have been examples of blending in mathematics whose mechanisation has helped to identify some novel and unexpected results. The blending itself was a one-stage process where human input was required to identify the input concepts. A more ambitious aim of the approach of applying blending to the problem of computational creativity in mathematics, is to allow search to be done over multiple blends and for the *process* of blending to be controlled mechanically. In this section we describe very informally a mathematical domain that seems in some ways a natural candidate for a blending approach.

### Galois Theory

Galois theory develops a relationship between a polynomial $p(x)$ with coefficients in some field $F$, the extension of $K$ of $F$ (written "$K/F$") containing all of the roots of $p(x)$ in the algebraic closure of $F$, and the group $\mathbf{Gal}(K)$ of automorphisms of $K/F$ that fix the elements of $F$. The fundamental theorem of Galois theory states that there is a bijection between the subfields of $K/F$ and the subgroups of $\mathbf{Gal}(K)$; namely, subgroups correspond to their fixed fields. Using this correspondence, properties of polynomials can be derived, most famously the fact that quintic polynomials cannot be solved by algebraic operations and the extraction of roots.

We do not propose to reconstruct much of the theory here, but note that already in this basic account there are several steps that seem compellingly "blend-like."

In the first place, for field extension, $E$ is an extension of $F$ if $F$ is a subfield of $E$. We could derive the extension relationship from the input concepts $E$ and $F$ by "taking everything additional from $E$ and adding it to $F$." This is made specific in the process of *adjoining* elements, which simply means to augment the field with all fractions of formal finite sums and products of the adjoined elements with coefficients in the base field.

Second, the notion of the *splitting field* of a polynomial, namely the special extension $K/F$ containing all of the roots of $p(x)$. This could be formed conceptually by combining the concept "*the roots of a polynomial $p(x)$ with coefficients in a field $F$*" and the concept "*a field extension $E/F$ formed by adjoining certain elements to $F$*."

As above, we could then form the concept of $\mathbf{Gal}(K)$ by blending at the conceptual level. This time, there would be several constituent pieces: "*the roots of a polynomial $p(x)$ with coefficients in a field $F$*," "*the splitting field of $p(x)$*," "*the group of automorphisms of a field extension $E$*," "*the automorphisms that fix $F$*."

Finally, assuming that we have built $\mathbf{Gal}(K)$ in this fashion, we would like to know some of its properties. Consider the claim that *elements of $\mathbf{Gal}(K)$ permute the roots of $f$*. This time, instead of being purely conceptual, we want to work at the *process* level, and consider before-and-after descriptions of the result of applying $\varphi \in \mathbf{Gal}(K)$ to some $r$ with the property $p(r) = 0$. This is similar in some ways to the "Riddle of the Buddhist Monk", popularised by Koestler (1964), which is cited as an example of the power of blending.[4] However, this time the generic space is not a simple geometric machine, but rather an algebraic machine with several moving parts.

The proof of the claim is as follows. If $p(r) = 0$, then $\varphi p(r) = \varphi 0$. Since $\varphi$ is an automorphism, $\varphi 0 = 0$; and furthermore $\varphi$ distributes over the sums and products that make up the polynomial $p(x)$ and fixes its coefficients, therefore $\varphi p(r) = p(\varphi r)$. Chaining the equalities together, we have $p(\varphi r) = 0$.

[4] "A Buddhist monk begins at dawn one day walking up a mountain, reaches the top at sunset, meditates at the top for several days until one dawn when he begins to walk back to the foot of the mountain, which he reaches at sunset. Making no assumptions about his starting or stopping or about his pace during the trips, prove that there is a place on the path which he occupies at the same hour of the day on the two separate journeys."

In short, the proof is a fairly direct result of combining the definitions. Goguen (1992) suggests that "combination is colimit." Can we realise the proof through (one or several) colimit operations? And is there anything special about this proof? Apart from these more theoretical questions, the foregoing discussion raises the following technical issues:

**Field Extension** When reasoning about polynomials, it is useful to distinguish the three separate types – those of $E$, those of $F$ and those of $E/F$ as a supertype. Using blending machinery removes the distinction between these types.

**Splitting Field Extension Theorem** A challenging but creative step is to discover the theorem that extending $F$ *only* with the roots of $f(x)$ forms a field.

**Automorphisms** As mentioned in the background section, currently there is no way of computing colimits if automorphisms are characterised in higher-order logic. An alternative specification, or an implementation of colimit computation for higher-order logic is needed.

## Evaluation and Outlook

### Review of the current offering

(a) We began the paper with the reconstruction of certain mathematical objects, showing the technical feasibility of the approach.

(b) The more advanced example at the centre of the paper illustrates how this sort of reconstruction relates to mathematical practice.

(c) A future-oriented example exposes some technical challenges, while suggesting that blending could offer a novel approach to computer mathematics.

### Broader issues in evaluation

In addition to motivating a further investigation of the role blending can play in proofs, Galois theory, discussed above, is paradigmatic for other reasons. This discussion draws on the early 20th Century writings of Albert Lautman on the philosophy of mathematics and the subsequent interpretation of this work by Gilles Deleuze. It uses these ideas to propose an approach to embedding evaluation within the system itself.

Concerning the common features of Galois theory, class field theory, and the development of the universal covering surface in Riemann geometry, (Lautman, 2011, p. 126) writes:

> What is characteristic of the movement of the theories that will be considered here is the existence of an end conceived in advance as a term of the ascent.

This is reminiscent of our notion of internal evaluation that apply to the blend. To illustrate, let us briefly imagine how we would use blending techniques to move from porcupine+lion to the perfected *porculione*. Here, instead of field automorphisms that preserve mathematical structure and fix certain designated elements, we would look for mappings that preserve other properties that exist in the underlying domain. Porculiones would presumably have four feet, would be mammals, and would be omnivores; they should also be viable living creatures.

(Deleuze, 1994, pp. 227–228) follows Lautman in enthusiastically endorsing the Galoisian approach to mathematics:

> [T]he fact that an equation cannot be solved algebraically, for example, is no longer discovered as a result of empirical research or by trial and error, but as a result of the characteristics of the groups and partial resolvents which constitute the synthesis of the problem and its conditions (an equation is solveable only by algebraic means – in other words, by radicals, when the partial resolvents are binomial equations and the indices of the groups are prime numbers). The theory of problems is completely transformed and at last grounded, since we are no longer in the classic master-pupil situation where the pupil understands and follows a problem only to the extent that the master already knows the solution and provides the necessary adjunctions. For, as Georges Verriest remarks, the group of an equation does not characterise at a given moment what we know about its roots, but the objectivity of what we do not know about them. Conversely, this non-knowledge is no longer a negative or an insufficiency but a rule or *something to be learnt* which corresponds to a fundamental dimension of the object.

Although there is a commonality between blending and the Galoisian approach insofar as progressive refinement carries us toward a "perfected" conclusion, Deleuze's enthusiasm about the pedagogical situation would be significantly cooled here. It would seem, in many of our examples, that we only make progress "to the extent that the master already knows the solution and provides the necessary adjunctions."

However, this apparent infelicity may be less of a thick obstacle than it would initially appear. What seems to be most needed is a notion of a *question* inside the system. This would recover Lautman's basic thrust: "Scientific or not, every question has built in some assumptions about the form of the answer" (Larvor, 2011). In short, an experimental approach in which the system *asks* and *answers* questions would embed key aspects for evaluation in the system itself.

### Future work

The idea of using blending to carry out steps in a proof would provide a useful training ground for further development. The primary problem is: If blending is the realisation of "combinatorial creativity" how will we avoid being swamped by the combinatorial explosion of possible things to combine? The first challenge is thus fitting different mathematical components together in a sensible manner. A related challenge would apply when modifying the system to selectively experiment with the rules it uses. The objective in this case would be for the system to learn to associate different (useful) techniques with different types of problems.

## Conclusions and Remarks

The examples presented in this paper trace the development of the blending approach. The current paper begins with reconstructions, but also quickly shows how computed blends

can suggest new mathematical definitions and concepts of interest to practising mathematicians. The analysis offered here shows that this work is a building block that will be useful for future developments that are able to reason more flexibly about mathematical problems – and systematically find and propose new concepts and problems.

In future work, we will look more at the cognitive issues raised in this work. In particular, the use of *image schemas* can give a link between the computational and representational approach taken here, and the cognitive claims coming from authors such as Fauconnier and Turner, and Johnson. Here the work of Mandler and Canovás (2014) and Hedblom, Kutz, and Neuhaus (2014) gives an idea of how these underlying cognitive primitives can be expressed in logical form, and can thus play an explicit role in our modelling of creativity in mathematics.


## Acknowledgements

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# Using Argumentation to Evaluate Concept Blends in Combinatorial Creativity

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**Abstract**

This paper motivates the use of computational argumentation for evaluating 'concept blends' and other forms of combinatorial creativity. We exemplify our approach in the domain of computer icon design, where icons are understood as creative artefacts generated through concept blending. We present a semiotic system for representing icons, showing how they can be described in terms of interpretations and how they are related by sign patterns. The interpretation of a sign pattern conveys an intended meaning for an icon. This intended meaning is subjective, and depends on the way concept blending for creating the icon is realised. We show how the intended meaning of icons can be discussed in an explicit and social argumentation process modeled as a dialogue game, and show examples of these following the style of Lakatos (1976). In this way, we are able to evaluate concept blends through an open-ended and dynamic discussion in which concept blends can be improved and the reasons behind a specific evaluation are made explicit. In the closing section, we explore argumentation and the potential roles that can play at different stages of the concept blending process.

## Introduction

A proposal by (Fauconnier and Turner, 1998) called *concept blending* has reinvigorated studies trying to unravel the general cognitive principles operating during creative thought. According to (Fauconnier and Turner, 1998), concept blending is a cognitive process that serves a variety of cognitive purposes, including creativity. In this way of thinking, human creativity can be modeled as a blending process that takes different mental spaces as input and blends them into a new mental space called a *blend*. This is a form of *combinatorial creativity*, one of the three forms of creativity identified by Boden (2003). A blend is constructed by taking the existing commonalities among the input mental spaces (called the generic space) into account, and by projecting the structure of the input spaces in a selective way. In general the outcome can have an emergent structure arising from a non-trivial combination of the projected parts. Different projections lead to different blends and different generic spaces constrain the possible projections.

This poses challenges from a computational perspective: large number of possible combinations exhibiting vastly different properties can be constructed by choosing different input spaces, using different ways to compute the generic space, and selecting projections. Within the Concept Invention Theory project[1] (COINVENT), we are currently developing a computational account of concept blending based on insights from psychology, Artificial Intelligence (AI), and cognitive modelling (Schorlemmer et al., 2014). One of our goals is to address this combinatorial nature. One potential outcome of this work is a deeper understanding of the way combinatorial creativity works in general.

The formal and computational model for concept blending under development in COINVENT (Bou et al., 2014) is closely related to the notion of *amalgam* (Ontañón and Plaza, 2010). Amalgamation has its root in case-based reasoning and focuses on the issue of combining solutions coming from multiple cases. Assuming the solution space can be characterised as a generalisation space, the amalgam operation combines input solutions into a new solution that contains as much information from the two inputs solutions as possible. When input solutions cannot be combined, amalgamation generalises them by dropping some of their properties. This process of generalisation and combination can be expensive from a computational point of view, depending on the search space to be explored.

The amalgam-based approach for computing blends makes explicit the combinatorial nature of concept blending, which raises the issue of evaluating and selecting novel and valuable blends as opposed to those combinations that lack interest or significance. Although Fauconnier and Turner (1998) suggest a number of qualitative criteria that can be used for evaluating concept blends, it is not straightforward to chararacterise them in a computational model.

In this paper, we propose to explore an argumentative approach to understanding and evaluating the meaning, interest, and significance of concept blends. Specifically, we propose to view evaluating blends as a process of *argumentation*, in which the specifics of a blend are pinpointed and opened up as issues of discussion. Our intuition is that in the context of new ideas, proposals, or artworks, people use critical discussion and argumentation to understand, absorb and evaluate. We also consider the constructive roles that argumentation can play in concept blending.

Computational argumentation models have recently appeared in AI applications (Bench-Capon and Dunne, 2007;

[1]See `http://www.coinvent-project.eu` for details.

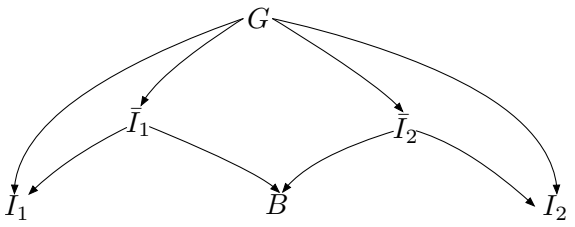



Figure 1: An amalgam diagram with inputs $I_1$ and $I_2$ and blend $B$ obtained by combining $\bar{I}_1$ and $\bar{I}_2$. The arrows indicate generalisation.

Rahwan and Simari, 2009), and we believe that incorporating argumentation can foster the development of a fuller computational account of combinatorial creativity. The current paper develops these themes at the level of (meta-) design; implementation is saved for future work.

### Roles of Argumentation in Concept Blending

Consider the *amalgam* diagram modeling the concept blending process (Figure 1): two input spaces $I_1$, $I_2$, two of their possible generalisations $\bar{I}_1$, $\bar{I}_2$, which have a generic space $G$ and blend $B$. When two input spaces cannot be combined because they do not satisfy certain criteria, the inputs have to be generalised for omitting some of their specifics. The combination of each specific pair $\bar{I}_1$, $\bar{I}_2$ yields a blend.

Informally, we can imagine argumentation taking place at various points in the amalgam diagram. In general this would happen in response to indeterminacy, that is, when some features of the diagram are underdetermined. We foresee that argumentation can be used:

**a.** to express opinions or points of view that can be used for guiding the selection/omission of specific parts of the input spaces; in particular, to select a specific pair of generalisation $\bar{I}_1$, $\bar{I}_2$ of the input spaces in the blending process;

**b.** to provide a computational setting for modeling discussions around the quality of a creative artefact, with the aim of evaluating and refining the generated blends.

In the first case, arguments would be about generalisation, i.e. which features should be preserved from $I_1$ and which features should be preserved from $I_2$. More complex inferences could be involved, for example in a case where $I_1$ is fixed, and constraints and various optimality criteria on the blend are imposed, which then yield various constraints on what the other input $I_2$ should be. We return to this point in the discussion section, and we focus for the most part on the second case.

In the second case, argumentation would be used to evaluate a range of blends, and the evaluation is carried out *post hoc*, by a variation of try-it-and-see. A range of blends are trialled, each one bringing out different (un)intended meanings. The evaluation is modeled as an argument, or dialogue in which the specifics of a blend are pinpointed and opened up as issues of discussion. This dialogue can be considered as an introspective evaluation, although it usually takes place among several parties as a means for the social development and understanding of creative artefacts. In this paper, we focus on this role.

### Our Approach

To exemplify our approach, we take the domain of computer icons into account. We assume that concept blending is the implicit process which governs the creative behavior of icon designers who *create* new icons by blending existing icons and signs. To this end, we propose a simple semiotic system for modeling computer icons. We consider computer icons as combinations of signs (e.g. document, magnifying glass, arrow etc.) that are described in terms of *interpretations*. Interpretations convey *actions-in-the-world* or *concepts* and are associated with shapes. Signs are related by sign-patterns modeled as qualitative spatial relations such as *above*, *behind*, etc. Since sign-patterns are used to combine signs, and each sign can have multiple interpretations, a sign-pattern used to generate a computer icon can convey multiple intended meanings to the icon. These are subjective interpretations of designers when they have to decide what is the best interpretation an icon can have in the real world. In this paper, we show how the intended meaning of new designed (blended) icons can be evaluated and refined by means of Lakatosian reasoning.

## Background

### Computational argumentation

Computational argumentation in AI aims at modeling the constitutive elements of argumentation, that are i) arguments, ii) attack relations modeling conflicts, and iii) acceptibility semantics for selecting valid arguments (Bench-Capon and Dunne, 2007; Rahwan and Simari, 2009).

The most well-known computational argumentation framework is due to Dung (1995). Dung defines an abstract framework to represent arguments and binary attack relations, modeling conflicts, by means of a graph. He defines different acceptibility semantics to decide which arguments are valid and, consequently, how conflicts can be resolved (Figure 2).

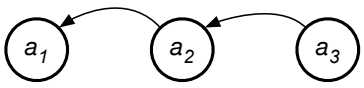



Figure 2: Dung framework example: Nodes represent arguments and edges (binary) attack relations. Argument $a_1$ is attacked by $a_2$ which is attacked by $a_3$. Thus, $a_2$ is defeated and $a_1$ can be accepted. $a_3$ is also accepted.

Abstract argumentation frameworks do not deal with how arguments are generated and exchanged. They merely focus on attack relations between arguments and acceptibility semantics. However, the intrinsic dialectical nature of argumentation is fully explored when an explicit argumentation process is considered. Then, the purpose of a dialogue becomes essential to determine how arguments should be generated and exchanged, and how a dialogue should be structured (Walton and Krabbe, 1995).

### Lakatosian argument and dialogue

Lakatos (1976) was a philosopher of mathematics who developed a model of argument, presented as a dialogue, to

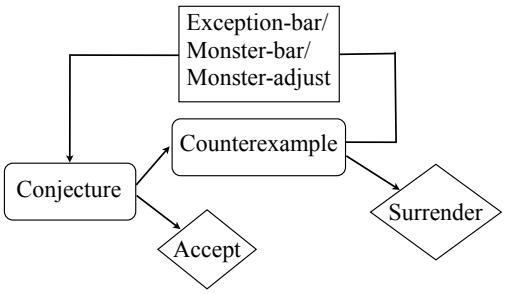



Figure 3: Our interpretation of Lakatos's game patterns.

describe ways in which mathematicians explore and develop new areas of mathematics. In particular, he looked at the role that conflict plays in such explorations, presenting a rational reconstruction of a dialogue in which claims are made and counterexamples are presented and responded to in various different ways. His resulting model describes conceptual continuity and change in the growth of knowledge, and contains dialogue moves, or methods, which suggest ways in which concepts, conjectures and proofs are fluid and open to negotiation, and gradually evolve via an organic process of interaction and argument between mathematicians. These dialogue moves are:

**Surrender** consists of abandoning a conjecture in the light of a counterexample.

**Piecemeal exclusion** is an exception-barring method that deals with exceptions by excluding a class of counterexamples, i.e., by generalising from a counterexample to a class of counterexamples which have certain properties.

**Strategic withdrawal** is an exception-barring method that uses positive examples of a conjecture and generalises from these to a class of object, and then limits the domain of the conjecture to this class.

**Monster-barring/monster-adjusting** is a way of excluding an unwanted counterexample. This method starts with the argument that a 'counterexample' can be ignored because it is *not* a counterexample, as it is not within the claimed concept definition. Rather, the object is seen as a monster which should not be allowed to disrupt a harmonious conjecture. Using this method, the original conjecture is unchanged, but the meaning of the terms in it may change. Monster-adjusting is similar, in that one reinterprets an object in such a way that it is no longer a counterexample, although in this case the object is still seen as belonging to the domain of the conjecture.

The moves above are not independent processes; much of Lakatos's work stressed the interdependence of creation and justification. These moves describe the evolution of both arguments and conclusions in mathematics, and as such constitute argument patterns, or schemes. However, they are a rational representation of exchanges between mathematicians and describe dynamic, rather than static arguments, presented as a dialogue. Thus, they also have temporal structure, and can be seen as a dialogue game, in which at any point various dialogue moves are applicable (see (Pease et al., 2014) for a description of Lakatos's methods in these terms). The fact that we include negotiations over definitions and changes in the conclusions being argued means that it is difficult to apply traditional abstract argumentation frameworks, which assume that such aspects are stable. However, we can see some of the moves in terms of Dung's framework: for instance if an initial argument for a conjecture forms $a_1$ in Figure 2, then $a_2$ might be a counterexample to the conjecture, and $a_3$ might be the monster-barring move.

The Lakatosian way of conceiving the reasoning as an open-ended discussion about a problem suggests that we can exploit Lakatos's moves for structuring dialogues for the evaluation of creative artefacts. Evaluation in creativity is not a static and rigid process, and the discussion should flow in a dynamic way. As such, in this paper, we propose to use Lakatosian reasoning to model the negotiation about the intended meaning of generated blends (icons). Figure 3 shows the dialogue game we will adopt to model these dialogues. For another formal framework of dialogue games for argumentation see (Prakken, 2005).

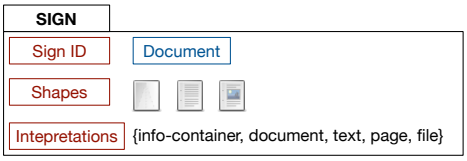
## Icons and Signs

We follow a semiotic approach to specify the intended meaning of computer icons. Semiotics is a transdisciplinary approach that studies meaning-making with signs and symbols (Chandler, 2004). Although it is clearly related to linguistics, semiotics also studies other forms of non-linguistic sign systems and how they may convey meaning; this includes not only designation, but also analogy, and metaphor. Although some people may regard Peirce's Sign Theory as the origin of semiotics, Saussure founded his semiotics (semiology) in the social sciences. Currently, cognitive semiotics and computational semiotics take their own perspectives on the relation between sign and meaning-making. In this paper, we take a semiotic approach to describe computer icons in the sense that icons, as a spatial pattern of shapes, are viewed as signs, and compositions of signs are interpreted to convey a meaning, as when we say 'this icon means the download is still active'.

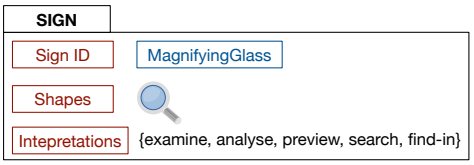
The shapes recurrently used in icons are interpreted as *signs*; screens, magnifying glasses and folders are examples of signs. A magnifying glass sign can be used in different icons in such a way that its meaning is *context-dependent*, that is, it depends on other signs related to it in different icons. We associate to each sign a set of *interpretations*, that encode the kinds of intended meaning associated to that sign as *actions-in-the-world* or *concepts*.

An icon is represented as a pattern defined by a collection of signs and qualitative spatial relations like *above*, *behind*, etc. We can find patterns of meaning that are shared among different icons by analysing recurrent patterns of signs and their spatial relation. We call them *sign patterns*. A sign pattern has an associated collection of *interpretations* that encode the intended meanings associated to that sign pattern.

Signs, sign patterns, and interpretations, which we will use in the paper, can be built by analysing and annotating existing libraries of computer icons. As we shall see, the inherent polysemy of signs, sign patterns and icons opens the way to use arguments for evaluating the quality or adequacy of new icons created by concept blending.

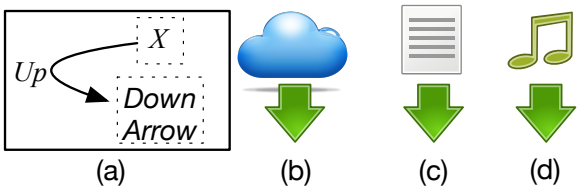



(I) The structure of the DOCUMENT sign, including associated shapes and interpretations.

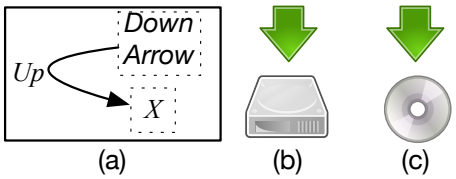



(II) The structure of the MAGNIFYINGGLASS sign, including associated shapes and interpretations.

Figure 4: Example of signs




(I) (a) the sign pattern FROM-DOWNARROW with three examples of the pattern where X is a sign for (b) cloud content, (c) document content, and (d) audio content.




(II) (a) the sign pattern DOWNARROW-TOWARD with two examples of the pattern where X is a sign for (b) a hard disk storage, and (c) an optical disk storage.

Figure 5: Example of different sign patterns used with the same sign DOWNARROW

## A semiotic system for icons

In this section, we will formalise the notions presented above. A *sign* $S$ is a tuple $\langle id, \mathcal{F}, \mathcal{A} \rangle$ where $id$ is a sign identifier, $\mathcal{F}$ is a set of shapes embodying the sign $S$ and $\mathcal{A}$ is a set of interpretations. We use $\mathcal{S}$ to denote the available set of signs. Figure 4 provides two examples. Figure 4I shows the structure of the DOCUMENT sign, with several shapes embodying the sign, and a list of interpretations that express how this sign is used in different ways to convey meanings such as *info-container, document, text, page, file*. Intuitively, this means that the shapes used in the icons are sometimes interpreted as a document and other times as a page, etc. Moreover, the specific shapes can be used interchangeably to embody a DOCUMENT, i.e. there is no clear distinction, regarding the shapes, between document vs. page vs. file. Another example of a sign is the MAGNIFYINGGLASS shown in Figure 4II, with interpretations *examine, analyse, preview, search*, and *find-in*.

We will also describe a library of annotated *icons* $\mathcal{I}$, where each icon $I \in \mathcal{I}$ consists of two parts: (1) a spatial configuration of signs and (2) the intended meaning of that icon. For instance, in Figure 5I, the icon (b) has the spatial configuration of a 'cloud on top of a downward-arrow' and its meaning is 'downloading content from the cloud'.

## Sign patterns

In our framework, sign patterns relate signs in icons using spatial qualitative relationships such as *above*, *behind*, *up*, *down*, *left*, etc. We assume that these relationships are represented as binary predicates, *Above*(X,Y), *Up*(X,Y), etc., where X and Y are variables ranging over signs in $\mathcal{S}$. For our current purposes, we use the qualitative spatial relationships defined in (Falomir et al., 2012).

Let us consider two examples of sign patterns that include the DOWNARROW sign. DOWNARROW has a vertical downward-pointing arrow shape and is associated with the interpretations {*down*, *downward*, *downloading*, *download-from* and *download-to*}. The sign pattern called FROM-DOWNARROW (shown in the schema labelled (a) in Figure 5I) uses the qualitative spatial relationship *up* between a variable X and the sign DOWNARROW. Examples (b), (c) and (d) in Figure 5I illustrate the intuitive meaning of the sign pattern FROM-DOWNARROW: 'downloading X'. Thus, example (b) refers to downloading cloud content, (c) document content, and (d) audio content.

The inherent asymmetry of arrows in general, and arrow signs particularly, can be appreciated when considering the opposite spatial relation, when the sign DOWNARROW is "up" from another sign (Figure 5II). Then, the sign pattern DOWNARROW-TOWARD is used to mean that X is the destination of the downloading. Example icons (b) and (c) are intended to mean that the data being downloaded (whose type or origin is now elided) is to be stored in a destination such as a hard disk or an optical disk.

## Evaluating Blends using Argumentation

As briefly described previously, the amalgam-based computation of concept blending amounts to combine different input spaces into a new space, called blend, by taking the commonalities of the inputs into account, by generalising some of their specifics and by projecting other elements. In the following, we describe how concept blending can account for modeling the creative process of a designer of computer icons.

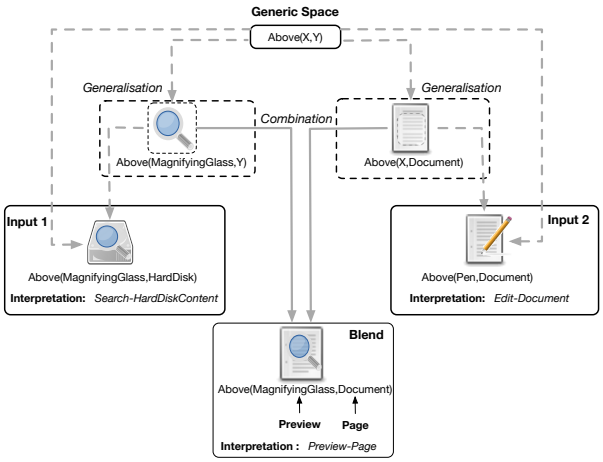



Figure 6: Generating an icon interpreted as *Preview-Page* through amalgam-based concept blending.

## A design scenario

Assume a designer is looking for creating a new icon with the intended meaning of previewing a document or a page. The creation of such icon can be achieved by the following amalgam-based concept blending process (Figure 6). In addition to the DOCUMENT and MAGNIFYINGGLASS signs, we assume we have available a HARDDISK sign and a PEN sign which have already been used to make icons.

**The input mental spaces.** The input mental spaces of the designer are an icon of a hard-disk with a magnifying glass hovering above it, whose meaning is *Search-HardDiskContent*, and an icon of a document with a pen above it, whose meaning is *Edit-Document*.

**The generic space.** The sign pattern *Above*(X,Y) is used in both icons. The first icon contains the relation *Above*(MAGNIFYINGGLASS, HARDDISK) between the MAGNIFYINGGLASS and the HARDDISK, and the second contains the relation *Above* (PEN,DOCUMENT) between the PEN and the DOCUMENT.

**Further generalisation.** Two generalisation steps are needed: *Above*(MAGNIFYINGGLASS,HARDDISK) $\rightarrow$ *Above(*MAGNIFYINGGLASS,Y*)*; correspondingly, *Above*(PEN,DOCUMENT) $\rightarrow$ *Above*(X,DOCUMENT).

**Combination via variable substitution.** We combine the schemas *Above*(MAGNIFYINGGLASS, Y) and *Above*(X, DOCUMENT) via [X/MAGNIFYINGGLASS, Y/DOCUMENT]. The icon of a page with a magnifying glass hovering above it is generated.

**The intended meaning.** The designer associates to the icon the intended meaning of *Preview-Page*, by selecting the interpretations (*Preview*, *Page*) for the MAGNIFYINGGLASS and DOCUMENT signs.

In this case, the designer decided that the intended meaning of *Above*(MAGNIFYINGGLASS, DOCUMENT) is *Preview-Page*, that is, a page can be examined without opening it. However, during the creative process, the designer could have generated other blends, not only by combining other signs, but also by selecting different interpretations associated to the MAGNIFYINGGLASS and DOCUMENT signs. For instance, the icon in Figure 7 still represents a page with a magnifying glass hovering above it, but it has been given a different intended meaning.

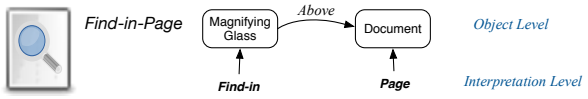



Figure 7: An example of interpreting the sign pattern of an icon as *Find-in-Page*.

The meaning of a blended icon cannot simply be considered right or wrong: interpretation depends on different points of view. Thus the evaluation of whether it is useful or valid for a specific purpose can be the object of a discussion.

## Arguments about intended meanings

In the icon domain, arguments may include a clear interpretation of any constituent signs in the icon if it is a composition of signs, or a good fit with other icons in the icon set.

For example, we can consider a counter-argument, i.e. an argument that *attacks* the interpretation $a_1$ "magnifying glass above document means *Preview-Page*" in Figure 6, to be phrased as follows:

$a_2$ : "However, the icon in Figure 6 can also be interpreted to mean *Find-in-Page*."

The rationale is that the MAGNIFYINGGLASS sign can often be understood as *finding* or *searching* for something. Thus, the icon can be also interpreted as *Find-in-Page* by associating the interpretation *find-in* instead of *preview* for the same sign MAGNIFYINGGLASS (Figure 7).

This attacking argument can be made at an abstract/conceptual level, for instance, by taking other possible blends of the DOCUMENT and MAGNIFYINGGLASS signs related by the sign pattern *Above*(X,Y) into account. Or, alternatively, if there is an icon library that contains an icon that 'satisfies' the argument above, then this attacking argument can be supported by a specific counterexample. Any of these two forms of attack evaluates negatively the icon in Figure 6. Therefore, if there are several alternative designs for a new icon, this attacking argument diminishes the degree of optimality/adequacy of that design with respect to alternative designs.

The original interpretation can be defended, as usually done in computational argumentation models, by a new argument that attacks the attacking argument $a_2$. For instance, the designer may say:

$a_3$ : "The icon in Figure 6 can only be interpreted differently if MAGNIFYINGGLASS is understood to mean *find-in* instead of *preview*. However, the other icons in

my library use MAGNIFYINGGLASS to mean *preview*, not *find-in*."

Argumentation semantics can then be used, once a network of arguments is built, to determine the outcome. For instance whether argument $a_1$, the original interpretation, is defeated or not can be determined as follows (Figure 2): in this example $a_3$ has no attack, so it is undefeated, which means it defeats $a_2$; since $a_2$ is defeated, the attack against $a_1$ is invalid and $a_1$ is undefeated (i.e. is accepted).

Arguments about the intended meanings of an icon can be embedded in a dialogue modeled in terms of Lakatos's moves and the dialogue pattern shown in Figure 3.

## Lakatosian reasoning for blend evaluation

Here we present a Lakatos-style dialogue between two players, a proponent $P$ and an opponent $O$. The goal of each player is to persuade the other player of a point of view, in this paper, the intended meaning of a new blended icon. In such a setting, we expect to see negotiations over the meaning of an icon take place between experts and novices, or between people designing icons and people using (interpreting) them, or various combinations.

To discuss a given icon using Lakatosian reasoning, we assume that an initial conjecture is about the interpretation of an icon usually being an *action-in-the-world* or a *concept*, together with an example of a particular icon and a particular interpretation. The conjecture could be constructed by inductive generalisation.

**Example 1.** In this example, Lakatosian reasoning is used for discussing the intended meaning of a new icon generated by concept blending:

**$P_1$:** "An icon with a magnifying glass over a page means *Preview-Page*" (Conjecture)
**$O_1$:** "I disagree, this icon (Figure 7) means *Find-in-page*." (Counterexample)
**$P_2$:** "No, this is a different case because the magnifying glass must be over pages with text on them to magnify (it shows what we're about to magnify)." (Monster-barring)

After this dialogue, it is agreed that the intended meaning of the icon is *Preview-Page* and the icon itself has been clarified. Alternatively, the proponent and the opponent could make a different evaluation by following different moves. For instance, if the proponent accepts the counterexample, then the intended meaning of the icon can be refined due to piecemeal exclusion:

**$P_1$:** "An icon with a magnifying glass over a page mean *Preview-Page*" (Conjecture)
**$O_1$:** "I disagree, this icon (Figure 7) means *Find-in-page*." (Counterexample)
**$P_3$:** "Ok, only icons with a magnifying glass over a page with text mean *Preview-Page*". (Piecemeal exclusion)

After this dialogue the intended meaning about the new icon has been changed by modifying the conjecture and taking the counterexample into account.

Sometimes players have different points of view due to the sign patterns they have used in their concept blending.

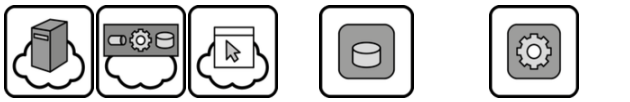



Figure 8: Interpreting the design of cloud icons[2]

**Example 2.** Let us imagine that the proponent has generated an intended meaning for an icon using the FROM-DOWNARROW sign pattern, whereas the opponent has used the DOWNARROW-TOWARD pattern (Figure 5 illustrates these cases). The two players can engage in the following dialogue:

**$P_1$:** "Look at icons in Figure 5I, icons with a DOWNARROW relate to content." (Initial Conjecture)
**$O_1$:** "The icon in Figure 5IIb has a DOWNARROW but doesn't relate to content." (Counterexample)
**$O_2$:** "The icon in Figure 5IIc also has a DOWNARROW but doesn't relate to content." (Counterexample)
**$P_2$:** "The conjecture is right because the two examples actually do relate to content as they are to do with storage and content is part of storage." (Monster-adjusting)

In this case, the proponent excludes the counterexamples using monster-adjusting, and reinterpreting them in a way that they are not counterexamples anymore.

A conjecture might even be at a higher level, for asserting that a particular metaphor is appropriate or inappropriate.

**Example 3.** For example, someone who is familiar with the 'gear means adjust setting' metaphor in one program may be comfortable with it in another program:

**$P_1$:** " An icon containing the 'gear' sign is a good one for Settings, because it invokes the idea of a gear change on a bicycle" (Initial Conjecture)
**$O_1$:** "The 'gear' sign does not invoke the idea of a gear change on a bicycle *from my point of view*." (Counterexample)
**$P_2$:** "Ok, you're right, it does not invoke the idea of a gear change on a bicycle, but it is often used for Settings." (Monster-adjusting)

**Example 4.** Argumentation may also consider the role a given abstract design plays within a given icon set.

**$P_1$:** "Even without knowing what the first or third icon in Figure 8I stands for, I can make a conjecture that it has to do with a server or a user interface accessed via the cloud. However, with the second icon, I'm not sure what it means. It is composed of various signs that I don't understand. It's probably badly designed." (Conjecture)
**$O_1$:** "Did you notice that icons in Figure 8II and Figure 8III are both defined as part of the same icon set? They mean 'Stateful component' and 'Processing component' respectively. Therefore, the second icon is actually well designed, because it uses signs appearing in other icons of the same icon set." (Counterexample)

[2]From `http://cloudcomputingpatterns.org`.

**$P_2$:** "But the second icon contains a pipe sign that is not used anywhere within the icon set, so I still don't know what the second icon means. If there were an icon with a pipe sign with a clear meaning, then I could understand the second icon better. " (Strategic withdrawal)

The main characteristic of employing Lakatosian reasoning is that it allows a dynamic and social development of the intended meaning of blended icons. This cannot be achieved by using only abstract argumentation frameworks, since they assume that the object of discussion does not evolve. Therefore, having an argumentation process of this kind has several advantages: it promotes not only open-discussions around the meaning of an icon, but also the construction of a discourse about how an intended meaning is obtained.

This is a desirable characteristic in computational creativity when evaluating creative outcomes such as concept blends. In this way, the evaluation evolves into a refinement process of an initial created concept, giving much more flexibility at the moment of deciding whether a blend is suitable.

## Discussion

We have illustrated the use of argumentation to evaluate completed blends. We alluded earlier to the role argumentation can play in the *generation* of blends, for instance by suggesting different ways to generalise the input spaces. Indeed, successive statements may serve to carry out the steps in the blending process iteratively, relaxing or refining as needed. These steps can be modelled using Lakatos's moves. From a conjectural candidate solution, to additional criteria that reveal this blend to be a 'monster' (i.e. which identify features of the candidate solution that cannot be allowed in the final solution for one reason or another), to adjustments that yield a more complete description of the problem and point the way toward a more satisfactory solution. An example of using argumentation for deciding which generalisations to use for creating a new icon is the following:

**A:** "We can create a different blend icon starting from the same icons of before." (see Figure 6)

**B:** "We could use the HARDDISK sign from the first icon and the DOCUMENT sign from the second icon."

**A:** "But putting the DOCUMENT sign above the HARDDISK does not make sense from my point of view."

**B:** "You're right, let's use the HARDDISK sign from the first icon and the PEN sign from the second icon."

**A:** "Sounds good, now we have a *Write-HardDisk* icon."

From this discussion and the previous sections, we think that it is feasible to bring the framework of argumentation inside the concept blending process. Moreover, this appears to work in a symmetric direction: the steps in an argumentation process can be carried out through blending. For instance, concept blending could be seen as the process behind the creation of rational arguments (Coulson and Pascual, 2006).

One area closely akin to the icon domain is the domain of *sentences* in a natural or artificial language. These can be evaluated for their coherence, succinctness, and fitness-to-purpose from a semantic standpoint (including relationship to other sentences), among other criteria; cf. (Abramsky and Sadrzadeh, 2014) for a category-theoretic view.

Since people have different standards for evaluation, they frequently disagree about what constitutes a satisfactory result, be it a final outcome or a design decision that is only a step the way to developing an artefact. They may also disagree at a more fundamental level about what can be considered a valid point of view or an appropriate manner of conducting an argument. For example, "Godwin's law" states that an online discussion ends when someone compares one of the discussants to Hitler and whoever made the comparison automatically loses the debate. Naturally, the validity of this principle is itself debatable. During the course of argumentation, the goalposts may shift, as new information is revealed about the domain under discussion, and about the discussants themselves. The relationship between argumentation and decision-making has been explored (Ouerdane, 2009), including the case of updating models of preferences (Ouerdane et al., 2014); the latter is quite similar to our previous work on Lakatos's games (Pease et al., 2014).

## Conclusion and Future Work

Computational models of combinatorial creativity faces the daunting issue of evaluating a large number of possible novel combinations. Particularly, Fauconnier and Turner (1998) propose a model that includes a collection of optimality principles to guide the construction of a 'well-formed integration network'. Our computational model, based on generalisations of input spaces and amalgams, makes this combinatorial nature more explicit. The heuristic criteria called 'optimality principles' are too underspecified to be used as computational measures to evaluate and select possible blends. Moreover, alternative numeric measures may be not enough to evaluate the quality or novelty of creative artefacts. Our intuition is that in the context of creative outcomes, people use argumentation to understand, criticise, modify and evaluate them, and that computational argumentation is a useful tool for computational creativity.

The domain of computer icons generated by blending, where the evaluation of new icons is focused on their intended meaning, shows that symbolic argumentation is a process that is adequate to distinguish well-formed icons from mix-and-match combinations, unambiguous and clear icons from ambiguous or incomprehensible icons. This domain supports our claim that numeric heuristic evaluation measures are insufficient to recognise good blends, and shows the usefulness of an argumentation-based process for identifying good blends, detecting their critical problems, and refining them in an evolving, open-ended process.

We have shown how Lakatosian reasoning can be used in evaluating concept blending for icon design. Our approach offers two main advantages. Firstly, the evaluation process can *improve* the blend, since the dialogue about it refines resulting blends. Secondly, the *reasons behind* a particular evaluation are made explicit. This is crucial given recent work on the importance of context in creativity judgments (Charnley, Pease, and Colton, 2012; Colton, Pease, and Charnley, 2011). Argumentation offers a framing story that

shows how and why a particular artefact was constructed, which can be presented alongside the artefact itself.

We envision several future works. First, we intend to specify an ontology for modelling the semiotic system presented and to build a library of icons. Having a domain knowledge will allow us to generate arguments by induction, for instance, by analysing icons cases. Moreover, it will also open the possibility to explore the use of value-based argumentation (Bench-Capon, Doutre, and Dunne, 2002) for selecting the input icons to be used in the concept blending process. This latter point is important, since usually the inputs of a blending process are assumed to be already provided. Second, as far as the interpretation of icons is concerned, we are thinking to take advantage of existing approaches to natural language processing and understanding, especially Construction Grammars (CxG). In CxG, the grammatical construction is a pairing of form and content. In our semiotic system, sign patterns seem equivalent to the form, while interpretations would be akin to the content. Working with a grammar would make evaluation more explicit, e.g. we could use quantitative measures of ambiguity; and this would open many other domains for application.

Finally, we plan to implement Lakatosian reasoning by employing existing computational tools for argumentation (Devereux and Reed, 2010; Wells and Reed, 2012). Our goal is to provide a computational argumentation framework and to integrate it into the framework for computational creativity we are developing in the COINVENT project.

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# Computational Poetry Workshop: Making Sense of Work in Progress

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**Abstract**

Creativity cannot exist in a vacuum; it develops through feedback, learning, reflection and social interaction with others. However, this perspective has been relatively under-investigated in computational creativity research, which typically examines systems that operate individually. We develop a thought experiment showing how structured dialogues can help develop the creative aspects of computer poetry. Centrally in this approach, we ask questions of a poem, inviting it to tell us in what way it may be considered a "creative making."

**Keywords**: computer poetry, social creativity, flowcharts, Writer's Workshops

'We *can* talk,' said the Tiger-lily: 'when there's anybody worth talking to.'

Through the Looking Glass, Lewis Carroll

## Introduction

We are writing in a large part to champion Alan Turing's proposal that intelligent machines should "be able to converse with each other to sharpen their wits" (Turing, 1951). The formalism that we propose builds on the notion of social cybernetics that flows from the following propositions of Heinz von Foerster's, which he uses to theorise systems in which participants can responsibly specify their own roles in relationship to other system participants:

"Anything said is said *by* an observer."
"Anything said is said *to* an observer."

(Von Foerster, 2003 [1979])

According to Jaako Seikkula and Tom Arnkil, who draw on the philosophical and literary analysis of Mikhail Bakhtin (Bakhtin, 2010 [1986], 1984 [1963]) in their approach to psychosocial work,

"Dialogues could be called 'the art of crossing boundaries'. Instead of trying to control others, the parties reach out towards each other to hear their views better, to generate shared languages and to join resources."

(Seikkula and Arnkil, 2014, p. 23)

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This paper outlines a study of social creativity with a dialogical emphasis, taking computer poetry as our working domain. It uses the Writer's Workshop model (Gabriel, 2002) as the virtual laboratory in which to conduct a thought experiment. The findings of our study are applied to the FloWr system (Charnley, Colton, and Llano, 2014). We focus on the following questions in turn:

- How has the social dimension of creativity been explored in CC to date?
- How can a created artefact tell us about its making, and what can this contribute to CC?
- How can computer poetry contribute to developing a process-based theory of poetics?
- What would have to change about the FloWr system to implement the computational poetry workshop approach?
- What are the pros and cons of the workshop approach?
- What might be the future role of dialogue in CC?

## Background

### Social creativity in CC

Minsky noted that computers need to be social if they are to deal with problems of any great complexity (Minsky, 1967, 1988). We believe that this is particularly true for challenges in computational creativity, since the essence of creativity lives in its appreciation by the creative entity itself and its audience. With creativity in 'the eye of the beholder' (Cardoso, Veale, and Wiggins, 2009), the ability to respond to evaluation during the creative process (Poincaré, 1929 [1908]; Csikszentmihalyi, 1988) becomes pivotal. Social creativity expands this paradigm by introducing co-creators to the process, and creating works that rely on dialogue, reflection, and multiple perspectives (e.g. the stages suggested by (Gervás and Leon, 2014)). 'Results' may be steeped in process and are not always based on consensus.

The Four Ps of creativity – the creative Person, Product, Process and Press (i.e. environment) (Rhodes, 1961) – have been emphasised in general creativity research. *Pluralising* these terms (Persons, Products, Processes) calls further attention to a social dimension of creativity, and would emphasise the way the "Press" accommodates multiple multidirectional perspectives akin to a social network in both the modern and original senses. The Pluralised Ps remind us

that in order to understand creativity it is not sufficient to model a lone creator or to generate an attractive artwork.

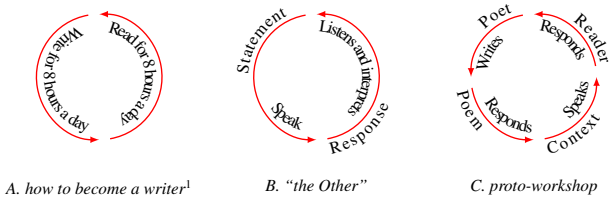
To date, computational creativity research has achieved many successes in computational generation of creative products, but the question of how these systems could adapt and learn from feedback to improve their creativity is less-explored in computational creativity (Jordanous, 2015). Evaluation has been advanced as a pivotal contributory part of the creative process, but researchers often give priority to generating artefacts that could be seen as creative over the task of incorporating feedback and evaluation within the processing of a creative system (Jordanous, 2011).

At the previous year's International Conference on Computational Creativity (ICCC 2014) the opening session had the theme "co-creation." However in the main proceedings of the conference, 36 out of 49 papers (approximately 3 in 4 papers) do not appear to mention social interaction or the ability to respond to feedback. Some notable exceptions highlight the usefulness of interaction and feedback for creative systems (McGraw and Hofstadter, 1993; Colton, Bundy, and Walsh, 2000; Sosa, Gero, and Jennings, 2009; Pérez y Pérez, Aguilar, and Negrete, 2010; Pease, Guhe, and Smaill, 2010; Saunders, 2012). Some of this work is influenced by the DIFI (Domain-Individual-Field-Interaction) framework (Csikszentmihalyi, 1988). However, social interaction between creative agents and their audience is often overlooked or relatively simplified: some examples in the domain of computer poetry presented below give the flavour. Increased development of the interactivity of creative systems, especially where this affects the way these systems works, has been highlighted as deserving more attention (Colton and Wiggins, 2012).

FloWr is a framework for implementing creative systems as scripts over processes that can be manipulated visually as flowcharts (Charnley et al., 2014). Its general approach consists of linking the inputs and outputs of code modules, called *ProcessNodes*, together to create a linear flow of data. The resulting Flowcharts can be constructed and executed visually through a GUI; however, they are ultimately represented as scripts, which are the main medium of FloWr. Experiments with automatic process generation in FloWr, reported in (Charnley et al., 2014), highlight the ability of the tool to do meta-programming and modify its own flowcharts. This suggests that FloWr has potential as an environment for modelling social creativity, where the observers are nodes and flowcharts, and their languages are, respectively, programming and meta-programming instructions.

### . . . and in computer poetry

In the domain of poetry-generation, there have already been several attempts to simulate social creativity by incorporating multi-agent systems. In WASP (Gervás, 2010), social behavior is simulated by incorporating a cooperative society of readers/critics/editors/writers consisting of specialized families of experts that cooperate during the poetry-generation process. The McGONAGALL system (Manurung, Ritchie, and Thompson, 2012) incorporates diverse modules as operators in evolutionary algorithms that produce poems fulfilling the constraints on grammar, meaning and poeticity. This approach facilitates the pursuit of several alternative solution paths in parallel, focusing on more promising results or coming back to former ideas. However McGONNAGALL does not provide any communication between modules. In the MASTER system for computer-aided poetry generation (Kirke and Miranda, 2013) a society of agents in various emotional states influences each other's moods with their pieces of poetry. The poetry-generation process is based on social learning as the agents interact by reciting their own pieces of poetry to each other. The generated poems are based on repeated words and sounds, and are closer in some ways to music than to typical language. Montfort, Pérez y Pérez, Harrell, and Campana (2013) and Misztal and Indurkhya (2014) use blackboard approaches to poetry-generation, in which independent specialized modules cooperate via a shared global workspace, *à la* (Baars, 1997). "Experts" exchange information using the blackboard, but without direct communication and without feedback about the reception of their created artifacts.




*A. how to become a writer*[1] *B. "the Other"* *C. proto-workshop*

Figure 1: (A) gives a simple recipe for the growth and development of a writer; (B) *response* always has dimensions that goes beyond the utterance that is overheard; (C) adds a *reader* who shares the context with the writer and responds.

In connection with our work in the current paper, we did a limited proof-of-concept reimplementation of some of the core methods of blackboard poetry system inside of FloWr; we include one of the generated poems and the corresponding flowchart.

## Methods

### "What are the proposed 'lab rats'?"

The generative side of the cycles in Figure 1 has been studied more than the reflective side. Our "lab rats" are, accordingly, not poems per se, but rather, *instances of reading and responding to poetry*. Naturally, such responses could be more or less "canned" (as with Michael Cook's humorously non-specific AppreciationBot[2]), so the question becomes: what constitutes an interesting and useful response, and how will these be developed? The idea of responses is useful at various levels. We focus here on staging an encounter between writer and reader.

### Writer's Workshops

Quoting (Gabriel, 2002, pp. 2–3):

> The original idea behind the writers' workshop was to do a *close reading* of a work... looking at the words on

[1] According to (King, 2000).

[2] https://twitter.com/appreciationbot

> the page rather than the intentions of the author or the historical and aesthetic context of the work. Under this philosophy, the workshop doesn't care much what the author feels about what he or she wrote, only what's on the page.

Framing and any other contextualisation of the work *as it is intended to be presented* is permitted, and receives critical attention. We define a *Workshop* closely following Gabriel's outline, to be an activity consisting of these steps: `presentation`, `listening`, `feedback`, `questions`, and `reflections`. The first and most important feature of `feedback` is for the listener to say what they heard; in other words, what, for them, is in the work. In some settings this is augmented with `suggestions`. After any `questions` from the author, the commentators may make `replies` to offer clarification. In related recent work, we have shown how the Workshop framework can help foster serendipitous discovery and invention (Corneli, Pease, Colton, Jordanous, and Guckelsberger, 2015; Corneli and Jordanous, 2015).

### Content as creative process

Giving agency to the poem rather than the poet's intentions, the poem illuminates its own creative process. This informs our approach to Workshop interactions, which are focusing on the poem observing its own construction. We're interested in context not in the literary or historical sense but in the micro-history of the poem's creative evolution. The originary and therefore unpredetermined nature of the creative process means that the outcome represents a more accurate and objective evidence of the process than the poet's attempt to explain the process. Moreover, to the extent that a creator knows what is expressed through the creative process, even he or she learns this only in the course of doing the work. Observers are only able to consider after the fact how a creator may have selected and rejected various possibilities. The *content* of the poem is no more and no less than how the poem was made.

> "In a poem, objective material becomes the content and the matter of the emotion and not just its evocative occasion." (Dewey, 1958 [1934], p. 69)

P. G. Whitehouse writing on Dewey's *Art as Experience* suggests that Dewey joins Collingwood in separating aesthetic emotion from any notion of inspiration that could be considered to be something like raw materials. An emotion is *aesthetic* when it "adheres to an object formed by an expressive act" (Whitehouse, 1978, pp. 149–156). However, "the art object does not have emotion for its significant content"; rather, the emotion "belongs to the self that is concerned in the movement of events toward an issue that is desired or disliked" (Dewey, 1958 [1934], p. 14).

### Aspects of the creative process

Doug Anderson and Carl Hausman take Collingwood's study further and map the creative process roughly as follows (Anderson and Hausman, 1992, pp. 299-305):

> Disturbance → aesthetic emotion → response → artist's decision on components of expression → feeling of easement plus a simultaneous emerging of a unique imaginative expression → alleviation → realising and converting prior psychical emotion → unique aesthetic experience including new conscious emotion

The poem is a *work of progress* before it is a *work in progress*. The purpose of a poetry workshop that attends to the content of the poem as process is to illuminate what the poet is exploring through his/her creative process and through the poem. The process of reading a poem is also a process of *poiesis* – and in the Workshop, the reader joins the writer in the process of creation. Asking questions like those listed in in Table 1 tells us what the constituent parts of the poem are doing.

### Relevance for CC research

From a CC standpoint, asking what the work tells us about the creative process gives an objective and critical focus on "creative evolution" (Bergson, 1911 [1907]) and provides an antidote to the seductions of mere generation. A poetry workshop gives participants the opportunity to read the drafts and final versions of poems by other Workshop participants, a shared culture of critique that can be applied to previously existing poems, and a structured way to gather feedback on one's own work in progress. These analyses, unbiased by the explanations of the (software) creator, will allow participants to explore and extend the conceptual space around poetry, or in practical terms, the toolbox the agents can access. "Extending" expresses both a refinement of the tools used and the introduction of entirely new tools. Moreover, reverse-engineering of the creative process from artefacts will help to teach agents participating in the workshop at which stage of *their* creative process these new tools or extensions could potentially be used. Dialogue in the workshop involves "respecting the voices of each of the participants" (Seikkula and Arnkil, 2014), be they agents, poems, or individual words – and suggests that we look at the "art of boundary crossing" that is to be found *inside* poems.

## Bridges between 'theory' and 'practice'

Our *ansatz* is that the Workshop could serve as a way to develop a process-based theory of poetics. There are certain prerequisites: in particular, an underlying context is required, shared (with respect to differing points of view) by the poet and the reader/listener (see Figure 1). Participants are assumed to have relatively stable, enduring but evolving, identities – either might be able to ask "Who am *I*?" and "Who are *you*?" (Bakhtin, 1984 [1963], p. 251). Answers would acknowledge a prepared mind with certain prior questions, abilities, involvements, and so on. However, within the Workshop dialogues, the discussion focuses solely on the work itself. Persistent identities allow participants to learn from these exchanges.

Table 1 contains a list of questions that a reasonably sophisticated poetry reader might ask about poems. This is complemented by a list of questions that could be addressed, in a

| Question | Examples |
|---|---|
| What are the register(s) of the poem? | cliched, instructive, imperative |
| Who is addressed? | reader, poet, friend, rival, confidante |
| What position(s) are present in the poem? | pleading, remonstrating, ephemeral |
| What is the poem doing with the reader? | accuse, bewilder, alienate |
| Who are the characters in the poem? | "the falconer", "you", narrator, "two men" |
| What is the role of image(s) in the poem? | "the sea", "a bicycle"; multiple meanings |
| What functions, mechanics, and paradigms are present for the reader to engage with? | communication, subverted cliche |
| What problems, discomforts, or diseasements are invoked in the poem? | horror, self-loathing, rejection, desire |
| How do these evolve? | E.g. an image may start to take over from a register |
| What is the world of the poem, and how does the poem distinguish between this and its perception of this? | "Surely", "must"; sacred vs mundane; perspectival vs surreal; tale vs telling |
| What are the overlaps, transitions, implicit dialogues? | "twinned" lines/ideas, juxtaposed parts of the poem |
| What role does the chronology of reading play, versus references to chronology and chronological positions within the poem? | flagged development, evolution, movement, stasis |
| How are lexical categories used? | solid nouns, tortuous adjectives, indistinct adverbs |
| Are there discernible allusive effects? | illustrating the literary apprenticeship of the author (or reader) |
| Where is the poet presented with respect to the poem? | Confidence, determination, exploration |

Table 1: Questions that we ask when reading a poem

straightforward programmatic manner (Table 2). Each of the examples listed in the right-hand column of Table 1 (and a plethora that are not listed) present a way of thinking about the poem. We can see these as roughly analogous to the agents in Table 2 (Minsky, 2006).

To illustrate, in response to a computer-generated poem:

*Oh dog the mysterious demon*
*Why do you feel startle of attention?*
*Oh demon the lonely encounter*
*ghostly elusive ruler*
*Oh encounter the horrible glimpse*
*helpless introspective consciousness*

A human critic might offer the following feedback:

1. The use of the word *mysterious* in the first line has no resolution, real or attempted, or quest to find one.
2. The use of the word *attention* is not being interrogated or acknowledged for its importance. Its qualifying word is *startle*, used here as an adjective; acknowledging the fact that the attention is noted, but is not yet part of the transformative of the poem.
3. This is repeated in the next references to the aesthetic experience as a *lonely encounter*, *exclusive ruler*, *horrible glimpse* and *introspective consciousness*.
4. The contact made between the poem and its own construction is qualified in negative terms attached to the words *demon*, *encounter* and *consciousness*.
5. This poem does not welcome the intimacy of bringing anything to aesthetic consciousness so that it might be expressed. Why do I say that? Because the words are generalised and *horribly* imprecise.
6. The poem does not move toward a better understanding of the ideas it alludes to. The vocabulary seems to associate exploration with fear and isolation and this is (paradoxically) quite an interesting acknowledgment of the poem's refusal to go anywhere i.e. to become a thing transformed by a creative process.

| Question | Agent concerned |
|---|---|
| ***Word level*** | |
| What is are dictionary definitions of this word? | WordNet expert |
| What are its etymological roots? | Etymology expert |
| Where did this word come from? | Provenance expert |
| What pronouns are used in the poem? | Pronoun expert |
| ***Phrase level*** | |
| What are the components? | Keywords expert |
| Do the components have a negative or positive connotation? | Association expert |
| What are the modifiers attached to the components? | Modifer expert |
| ***Sentence level*** | |
| What is the parse tree? | Grammar expert |
| ***Line level*** | |
| How long is the line? | Counting expert |
| Where does it break? | Breathing expert |
| Where is there white space? | Position expert |
| ***Poem level*** | |
| How are terms that exhibit emotion distributed within the poem? | Distribution expert |
| Where is there alliteration (rhyme, consonance) in the poem? | Phonics expert |
| Does the poem have a metrical structure? | Rhythm expert |
| How repetitive is the poem? | Repetition expert |
| Does the poem cohere? | Thematic expert |
| Does the poem have a progression? | Narrative expert |
| Where are the various elements of the poem concentrated? | Entropy expert |

Table 2: Questions we imagine a computer would currently be capable of answering when reading a poem

Each of these six points is *dual-voiced* in the sense that the critic is relaying the words of the poem with a new emphasis. Each such statement is one side of a micro-dialogue (Bakhtin, 1984 [1963], p. 73). The challenge is, of course, to bring the observations into the awareness of the computer poet, across the "analogue divide." Care should be taken not just to blythely program the computer with more rules, but rather to give attention to facilitating the process of learning new rules contextually. We continue with the example from this point of view in the following section.

First we will consider a reversal of roles, with the computer in the position of critic, looking at a passage from an historical piece of poetry. We have selected a passage from Robert Burns that might have – but in fact did not – serve as a model for the poem generated above.

*I'm truly sorry man's dominion*
*Has broken Nature's social union,*
*An' justifies that ill opinion*
*Which makes thee startle*
*At me, thy poor, earth born companion*
*An' fellow mortal!*

Naturally, the first problem is for the computer to *read* the poem. One of the approaches that is most appealing from our point of view is the automatic generation of a semantic network from the input text (Harrington and Clark, 2007). We could straightforwardly extend the methods of Harrington and Clark with notions drawn from Table 2.

1. The passage begins with *I*/*me*, locating the *poor, earth born* poet
2. *thee*/*thy* is another person, possibly the reader, who becomes *startled*
3. Singular *I* contrasts with the class *man*
4. *sorry* is a sad emotion
5. *truly* exaggerates sorry
6. *dominion* is large
7. *broken* and *union* are opposites
8. *sorry* and *justifies* are opposites
9. *union*, *companion*, and *fellow* are positive words about relationship and joining
10. *broken*, *ill*, *poor*, *startle* and *mortal* are related to frailty
11. *born* and *mortal* are related
12. There are a lot of rhymes in the poem, at the end of the lines, enjambed.

These comments are very different from the other reading above, and are differently interesting.

We've demonstrated that the computer is capable of asking objective questions of a poem. A similar semantic network approach would allow it to listen to feedback and take it on board, even when it doesn't understand the ways of thinking that generate this feedback. Again, this links the process of reading and writing poetry to a process of dialogue.

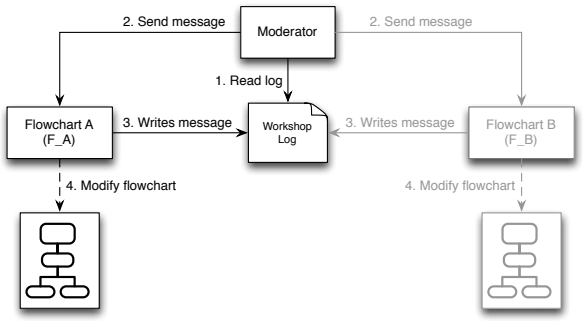



Figure 2: Schematic diagram for a workshop built in the FloWr system

## Seeds for a FloWr Garden

Keeping in mind the current limitations of FloWr – no looping or conditionals, only running one flowchart at a time and in one direction – a *conversation* between ProcessNodes or flowcharts is not immediately feasible. Figure 2 represents a hypothetical design in which a Workshop could take place with a minimally-altered version of FloWr. As shown in Figure 2, each participant in the Workshop would be represented by a single node. One of these nodes is a *moderator* in charge of dictating the interaction between the participants of the Workshop, while the rest represent *flowcharts* that have the ability to modify their own connections according to the discussions from the Workshop – this can be achieved by exploiting the scripting mechanism of FloWr and dynamically loading the new structure of the flowchart. Moreover, a shared *log* would contain the history of the messages exchanged during the Workshop and a queue of messages waiting to be delivered. We define four different types of messages that can be exchanged:

- ***comments*** about specific elements of a poem, or more general statements about how the poem affects this reader.
- ***questions*** to facilitate comprehension of this commentary; for instance, the questions can vary from simple requests of sources of information (e.g. files, input from another node, which resources a flowchart uses, etc.) to process-specific details (e.g. current conditions, purpose, other outstanding questions, etc.)
- ***answers*** would be associated to previous questions and may contain simple text such as an url for the source of information, or a piece of script representing a node used by a flowchart.
- ***suggestions*** are changes proposed by one participant to another. Similar to the answer, this can be as simple as suggesting the change of an information source, or more complex, such as suggesting the replacement of a node for an alternative node.

A Workshop session follows this communication protocol:

1. The moderator initialises an empty log and sends a message to the flowcharts to indicate that the session has started.
2. The flowcharts start writing messages in the log.
3. The moderator checks the current state of the workshop by reading the log.
4. The moderator selects the next message in the queue and passes it to the target flowchart.
5. The flowchart reads the message and acts accordingly, by either (*i*) modifying its connections or; (*ii*) sending a message back, i.e., writing to the log.
6. Step 3 is repeated until no further message are left in the queue.

**Example.** Figure 3 shows the poetry generator flowchart that generates the poem about the "demon dog" presented above. The flowchart uses two linguistic resources: ConceptNet (Liu and Singh, 2004), a semantic network of common knowledge, and Disco (Kolb, 2008), a semantic similarity words retrieval system. Let us assume the human critic **A** has access to the system through a "UI flowchart" like a Read-Eval-Print Loop (REPL), and the poetry generator **B** is mainly concerned with maintaining a generative flowchart like the one shown in the figure. The following exchange of messages can occur:

1. **Comment from participant A to participant B**: *The words "lonely encounter" and "elusive ruler" in lines 2 and 3 are generalised and imprecise.*
2. **Question from participant B to participant A**: *I identify the processes* Disco3 *and* Disco4 *as the source of the problem. Can you suggest an alternative to Disco?*
3. **Suggestions from participant A to participant B**: *Use WordNet or the Historical Thesaurus to find more expressive and specific terms for the core concepts in the poem; try to link the core concepts together by chaining together related concepts in ConceptNet or WordNet.*
4. **Action executed by participant B**: Receives suggestions, creates several alternative versions of the script, executes them and decides which one is most coherent and which conveys a sense of narrative.

From this exchange, the computer might learn (without ever being explicitly told) that expressive terms and narratives are related, and it might begin to discover a way to produce coherent poems with a narrative structure.

Since the computer has source code instead of a brain, we can use it to do control studies with process. However, in general source code does not uniquely determine process: contextual effects are what make an experiment an experiment. As described in (Cook and Colton, 2013), code may include hints about its expected operating context. This is related to our theme of embedding process within an artefact. In this connection, one extension to FloWr that would help to facilitate dialogue between flowcharts would be to add machine-readable *commentary* to ProcessNodes. Commentaries would label a node's inputs and outputs, describe its basic purpose, and provide information about procedure, conditional behaviour, mappings between processes and elements of a generated poem (like the mapping between Disco3 and "lonely encounter").

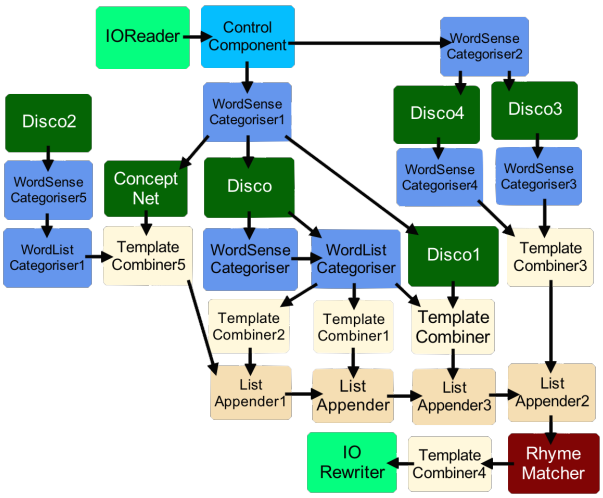



Figure 3: The flowchart that created the "demon dog" poem

Altered versions of a flowchart (Charnley et al., 2014) can be seen as parallel solutions that could be executed and compared on a population basis with respect to some prespecified metrics in order to make an informed decision on which suggestion(s) to follow, as hinted in the last step of the example. In (Colton, Pease, Corneli, Cook, and Llano, 2014) we explored the related idea of modelling system progress over time. Learning new rules contextually would offer one clear measure of progress. Caveat lector: considerable work would be required to realise the ideas we've described in FloWr or any other platform we're aware of.

## Discussion

**Potential applications.** The paradigm advanced in this paper would not remove the "generation" aspects of CC, but would pair them more closely with reflection. The same skills that support learning in a writers workshop may support a form of dialogue with the work itself, leading to richer creative artefacts that show us more about how creativity works. Focusing on social creativity does not suggest that we should devalue works from lone creatives, but it does suggest that we think about how we knit individuals together in the social fabric of the CC community. The current model at the International Conference for Computational Creativity (ICCC) is similar to many other academic conferences: we present *our* work to one another and build *our* sense of community in that way. But what about a track for computers to present their work? The idea of computers interacting in a workshop-like setting is not unprecendented. As Turing (1951) foresaw, computational software has become highly competent at Chess and reasonably competent at Go, partly through continuous practice pitting programs against each

other. Poetry could be approached in a similar way, reviving the *floral games* of the troubadours. Other creative arts may also be amenable to the same sort of approach. Gabriel mentions "brainstorms, critiques, charrettes, pair programming, open-source software projects, and even master classes" (Gabriel, 2002, p. 11). The sort of thinking we have developed here might be adapted to contexts like these.

**Potential criticisms.** It can be challenging and time-consuming to invite and process feedback, and the Workshop would often be seen as unnecessary for standardised production cycles that can already produce artefacts that are "good enough." Furthermore, since we often seem to *get* the computer to do just what we have in mind when we're programming, it might not make sense to treat it as a distinct other and invite it to participate in a dialogue. (Some REPL users may disagree, and may already think of programming as a dialogue.) From our read/write perspective on computational creativity, the most immediate problem is that appreciation of works of art is rather hard. Consider the difference between creating a video game (for example) and playing a video game. In the first case, the designer has full control over the rule-set, game mechanics, interaction devices and so forth. At least one computational video game designer can play its own games (Cook, Colton, Raad, and Gow, 2013), and an experiment shows that it is possible for an artificial game player to learn how to play classic video games using reinforcement learning, starting from raw pixels (Mnih et al., 2013) – but both are quite far from general-purpose game playing. This is itself a topic of contemporary research, and it serves to illustrate that coping with feedback is a major challenge for AI research. Finally, we are not in a position to make strong claims about the quality of workshopped artefacts, although our experience with poetry has shown us that high-quality poems are often exactly the ones which teach us about the creative process. We hope future research will explore this connection further.

## Conclusions

The ideas of social interaction, feedback, and evaluation have frequently been discussed in CC, but implementation and theorisation around these topics have been more limited. In the current paper, we suggest giving artefacts more agency, designing computer programs with more autonomy, and focusing research effort on creative evolution. We have shown that in principle computers can engage in dialogue about poems, which points to a theory of poetics rooted in the making of boundary-crossing objects and processes. In order to move from thought experiment to computational simulation, FloWr could be helpfully extended with further programmer facilities including loops, subroutines, and commentaries, along with the ability to generate-and-test in a population-based manner, and the ability to learn over time. Workshops and related approaches are suitable for autonomous learning and development of the creative process, but they face technical and also some theoretical limitations. Dialogue may offer a way to creatively push these limits, empowering both programs and programmers.

## Acknowledgements

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# The Search for Computational Intelligence

**Joseph Corneli** [1] and **Ewen Maclean** [2]

**Abstract.** We define and explore in simulation several rules for the local evolution of generative rules for 1D and 2D cellular automata. Our implementation uses strategies from conceptual blending. We discuss potential applications to modelling social dynamics.

## 1 Introduction

This paper takes a local approach to studying the evolution of cellular automata (CA), following on the global approach of "PICARD" [24].

> *Like a traditional one-dimensional CA, PICARD executions move from one iteration to another by some rule. However, whereas traditional CA's require the rule to be static and externally specified, PICARD infers the iteration rule from the current state of the CA itself.* [24, pp. 1–2]

PICARD's inferred rule is derived from the current state of the CA by a global characteristics, such as the number of 1's in the CA's current state (modulo 256), or the density $\rho$ of 1's (normalised as $\rho/256$). These global criteria are similar to Van Valen's theory of resource density as an "incompressible gel" [29].

In the current paper we introduce the notion of a MetaCA, in which CA rules are derived locally at each cell within the CA as it runs. Examples appear in Figure 1. Here, each colour represents one of the 256 standard one-dimensional CA rules. States evolve locally, according to globally-defined dynamics.

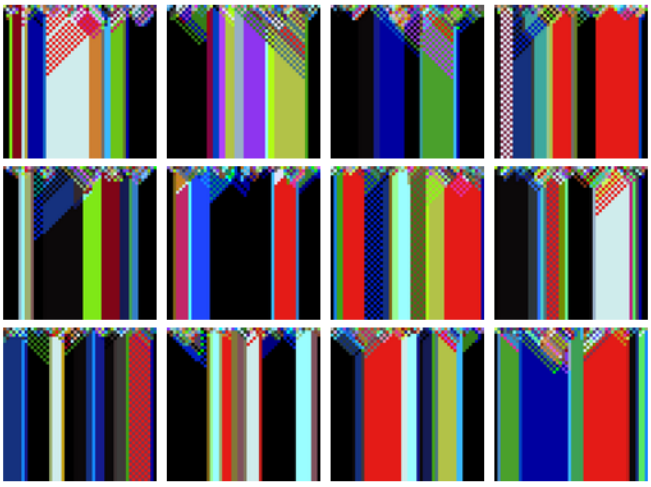


**Figure 1.** An illustration of MetaCA evolution

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## 2 Background

### 2.1 Cellular Automata

Each elementary 1D CA rule defines a mapping from all eight triples formed of 0's and 1's to the set $\{0,1\}$. Thus, for example the rule **01010100** is defined as the following operation:

$$\begin{array}{cccc} 0 & 0 & 0 & \mapsto \mathbf{0} \\ 0 & 0 & 1 & \mapsto \mathbf{1} \\ 0 & 1 & 0 & \mapsto \mathbf{0} \\ 0 & 1 & 1 & \mapsto \mathbf{1} \\ 1 & 0 & 0 & \mapsto \mathbf{0} \\ 1 & 0 & 1 & \mapsto \mathbf{1} \\ 1 & 1 & 0 & \mapsto \mathbf{0} \\ 1 & 1 & 1 & \mapsto \mathbf{0} \end{array}$$

The rules determine the next generation of a 1D CA locally, from three "parents". In the example, $\boxed{0}\boxed{0}\boxed{0} \mapsto \boxed{0}$ and $\boxed{0}\boxed{0}\boxed{1} \mapsto \boxed{1}$ and so on. There are 256 of these rule tables; the example above is Rule 84 in Wolfram's standard enumeration [31]. A crucial development in the history of CA research was the proof [5] that certain CA rules are Turing complete (in particular, Rule 110 enjoys this property).

Earlier classic works [14, 17, 23] exploring related "edge of chaos" effects. In [23, 17, 16], genetic algorithms are used to search the space of CA rules via crossover and mutation. This sort of evolution is global and is connected with the CA rule by a derived parameter, "Langton's $\lambda$" (cf. [14]). An overview of the "EvCA" programme is presented in [12]. CAs are also explored in two (and more) dimensions and with irregular topologies [7, 6]; in this paper, we develop both 1D and 2D examples. Closest to the work presented here is [26], which introduces the paradigm of *cellular programming*. As the name indicates, this approach is a fusion of ideas from cellular automata and genetic programming.

> *As opposed to the standard genetic algorithm, where a population of independent problem solutions globally evolves, our approach involves a grid of rules that coevolves locally.* [26, p. 74]

In cellular programming, local evolution of the CA rule makes use of a "fitness" metric ([26, pp. 79–81]), as the systems are evolved to perform certain global computational tasks. In the current effort system evolution is not directly guided by a specific fitness criterion. This paper defers any detailed *post hoc* analysis of MetaCA behaviour, although we hope to explore this further in a sequel, possibly following in the footsteps of the EvCA project [9, 10, 13].

### 2.2 Modelling social dynamics

Previous researchers have looked at CAs "as multi-agent systems based on locality with overlapping interaction structures" [6]. An

early application of cellular programming was to evolutionary game theory, a field with natural parallels (cf. [22]). We are inspired by recent work in this area on the evolution and failures of cooperation [1, 21, 27, 28] but we do not use a game theoretic approach. George Mead extends the term *social* to describe any scenario exhibiting emergent coevolution; this becomes central to our discussion.

> *What is peculiar to intelligence is that it is a change that involves a mutual reorganization, an adjustment in the organism and a reconstitution of the environment; for at its lowest terms any change in the organism carries with it a difference of sensitivity and response and a corresponding difference in the environment. . . . Now what we are accustomed to call social is only a so-called consciousness of such a process, but the process is not identical with the consciousness of it, for that is an awareness of the situation. The social situation must be there if there is to be consciousness of it.* [15, pp. 4, 48]

## 2.3 Conceptual Blending

One of our inspirations for working with cellular automata is that we are involved with a research project that studies computational blending [25], and cellular automata seem to offer a very simple example of blending behaviour. That is, they consider the value of neighbouring cells, and produce a result that "combines" these values (in some suitably abstract sense) in order to produce the next generation. We were also inspired by the idea of "blending" ordered and chaotic behaviour to produce edge-of-chaos effects. We propose to exploit existing formalisms of blending (in the style of Goguen [11]) in the context of cellular automata to investigate emergent and novel behaviours. The fundamental building blocks used in calculating concept or theory blends are:

**Input Concepts** are the concepts or theories which are understood have some degree of commonality (syntactic or semantic).

**Signature Morphism** is a definition of how symbols are mapped between theories or concepts.

**Generic Space** is the space which contains a theory which is common to both input theories.

**Blend** is the space computed by combining both theories. The computation is computed using a "pushout" from the underlying categorical semantics [18].

Once a blend has been computed, it may represent a concept which is in some way inconsistent. Equally it may represent a concept which is in some way incomplete. We can then either weaken an input theory, or refine the blend:

**Weakening** Given an inconsistent blend it is possible to weaken the input concept in order to produce a consistent blend. Weakening means removing symbols or axioms from the input concept.

**Refinement** Given a blend which represents a concept which is in some way incomplete, it is possible to refine the concept by adding symbols or axioms.

In this paper the primary examples have input concepts expressed in the same language, and indeed have the same specification. This means that the morphisms are not interesting and the calculated pushout could be computed without utilising the full machinery of category theory. Planned extensions will explore the idea of combining rules for cellular automata which may have entirely different techniques for expressing propagation (and we provide one example). For this reason, we target the Heterogeneous Tool Set (HETS) system [19] as an infrastructure for computing blends. We describe our current approach to blending in the context of cellular automata in Sections 3.2 and 3.3.

# 3 Implementation

## 3.1 Generating Genotypes

A MetaCA evolves a CA with 256 possible states – rather than the traditional $\boxed{0}$ and $\boxed{1}$ – where each state now corresponds to a "1D CA rule". By positioning three CA rules next to each other, we define a multiplication by applying the central rule bitwise across the alleles. For example, here is the result of "multiplying" $01101110 \times 01010100 \times 01010101$. In the context of such an operation, we refer to the central term as the "local rule." This example uses Rule 84 as the local rule, highlighted in bold.

| | | | | |
|---|---|---|---|---|
| 0 **0** 0 | | 0 | | *Apply local rule to "000"* |
| 1 **1** 1 | | 0 | | *Apply local rule to "111"* |
| 1 **0** 0 | | 0 | | *Apply local rule to "100"* |
| 0 **1** 1 | $\mapsto$ | 1 | | *Apply local rule to "011"* |
| 1 **0** 0 | | 0 | | *Apply local rule to "100"* |
| 1 **1** 1 | | 0 | | *Apply local rule to "111"* |
| 1 **0** 0 | | 0 | | *Apply local rule to "100"* |
| 0 **0** 1 | | 1 | | *Apply local rule to "001"* |

Realised in a simulation with random starting conditions, the results of this operation are not particularly impressive: they stabilise early and do not produce any interesting patterns (Figure 2).

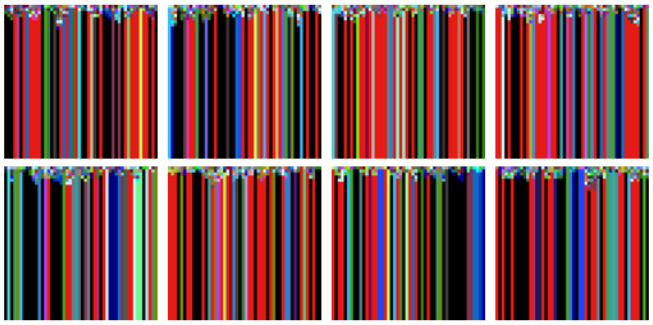


**Figure 2.** Under evolution according to the local rule without blending dynamics, a barcode-like stable pattern forms quickly

## 3.2 Introducing Blending

The blending variant says to first compute the "generic space" by noting the alleles where the two adjacent neighbours are the same, and where they differ. Only when the generic space retains some ambiguity (indicated by $\{0,1\}$) do we apply the local rule (again recorded on the centre cell at left and highlighted in bold) in a bitwise manner across each allele, to arrive at the final result.

| | | | | | |
|---|---|---|---|---|---|
| 0 **0** 0 | | 0 | | 0 | *Neighbours are both 0* |
| 1 **1** 1 | | 1 | | 1 | *Neighbours are both 1* |
| 1 **0** 0 | | $\{0,1\}$ | | $\boxed{0}$ | *Apply local rule to "100"* |
| 0 **1** 1 | $\mapsto$ | $\{0,1\}$ | $\mapsto$ | $\boxed{1}$ | *Apply local rule to "011"* |
| 1 **0** 0 | | $\{0,1\}$ | | $\boxed{0}$ | *Apply local rule to "100"* |
| 1 **1** 1 | | 1 | | 1 | *Neighbours are both 1* |
| 1 **0** 0 | | $\{0,1\}$ | | $\boxed{0}$ | *Apply local rule to "100"* |
| 0 **0** 1 | | $\{0,1\}$ | | $\boxed{1}$ | *Apply local rule to "001"* |

For illustrative purposes, this blend has been formalised in the HETS system by introducing CASL files to represent the 8 bit encodings (Listing 1, and corresponding development graph shown in Figure 3). In this example, the first computed blend is inconsistent as there is not a unique value representing the output value of each function. In order to resolve this, we weaken the input rules in CASL by removing the function values which cause conflict. Note that purposes of efficiency, we have implemented our 1D experiments in LISP rather than in HETS/CASL. We've put the working code on Github[3].

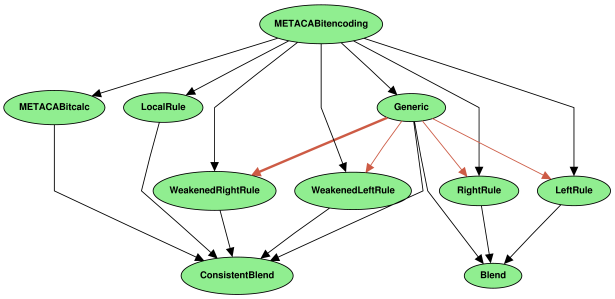
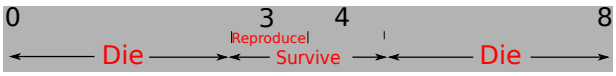



**Figure 3.** The development graph for calculating a blend of 8 bit encodings

## 3.3 2D Experiments

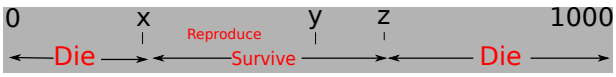
In order to extend the ideas presented so far in the 1D case, let us consider a variant of Conway's Game of Life [7], in which a global rule exists defining whether a square is alive or dead. We extend this by introducing the notion of a local rule at each square – a genotype, which governs the propagation of the phenotype.

In Conway's Game of life, one can view the rules for propagation as partitions on a finite interval $[0, 8]$.




The number on the line corresponds to the number of alive neighbours adjacent, in cardinal and inter-cardinal directions, to a given square. If the square is dead then it becomes alive (labelled reproduce) if the number of alive neighbours is exactly three. If there are five or more alive neighbours the square dies from overcrowding. If there are fewer than three alive neighbours the square dies from underpopulation. In all other cases the square maintains its status.

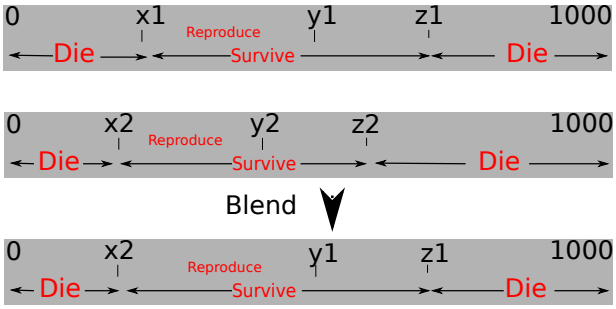
This can be generalised to partitions within a more finely grained line, for example from 0 to 1000, one creates a genotype $(x, y, z)$:




We introduce the corresponding notion of a *weight* for each cell. The *phenotype* of the cell is then a pair $(alive, weight)$ which denotes whether the cell is alive, and what weight is has. In this paper we always calculate a newly propagated weight as the average of the neighbours' weights.

[3] https://github.com/holtzermann17/metaca

The notion of local propagation is introduced by allowing the genotypes to be blended at each point where a cell remains or becomes alive. As we have represented the genotype as a partitioned line, we can, for example perform a blend where the partition is blended in such a way as the minimise the lowest bound and maximise the highest bound, and maximise the interval for reproduction. Given two genotypes $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$, the blend is $(\min\{x_1, x_2\}, \max\{y_1, y_2\}, \max\{z_1, z_2\})$:




Note that this is just one of several possible blending strategies, which we refer to as a *union* blend, since it maximises the partitions which pertain to survival. We consider alternative blends in our experiments.

# 4 Results

## 4.1 1D CAs

One of the first things we noticed was that even though the blending dynamic creates more interesting "CA-like" patterns than simple evolution according to the local rule (as illustrated in Figure 1), it also forms stable bands after this interesting initial period. In Figure 4, this is illustrated in a CA running with 500 cells over 500 generations. Figure 4 also includes a phenotype (in black and white) which is driven entirely by the genotype: that is, if the local genotype is $\boxed{\alpha}\boxed{\beta}\boxed{\gamma}$ where $\alpha, \beta, \gamma \in \{0, 1\}^8$ and the local phenotype is $\boxed{a}\boxed{b}\boxed{c}$ where $a, b, c \in \{0, 1\}$, then the genotype evolves locally according to the meta-rule $\alpha \times \beta \times \gamma$ (in the blending variant) while the phenotype evolves by applying the local rule $\beta$ to the data "$abc$."

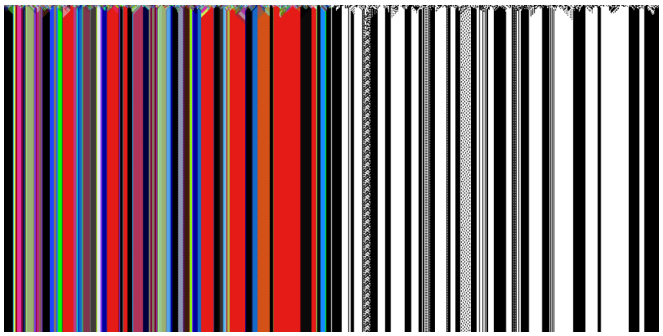


**Figure 4.** Phenotype with behaviour determined by genotype

In the phenotype layer, we see a few bands with interesting patterns, where the MetaCA at left has stabilised locally into one of the more interesting CA rules. However, at this scale we see that the long term evolution in the genotype layer is uninteresting: the structure observed in Figure 1 disappears quickly.

We therefore decided to introduce random mutations to the genotype, illustrated in Figures 5–7. With a high mutation rate, both genotype and phenotype are almost reduced to confetti. If we reduce the mutation rate sufficiently, some degree of stability is preserved, and the vertically striped bands are transformed into intermingling swaths of colour (Figure 6). We also see areas with more finely-grained structure in the phenotype layer.

In Figure 7, the colour-coded genotype layer has been replaced with a greyscale coding, and we see more clearly how the phenotype behaviour follows that of the genotype. That is, genotypes similar to Rule 0 (00000000) or Rule 256 (11111111) tend to produce 0 or 1, respectively, in the phenotype layer. Rules that output a blend of 0's and 1's are mapped to grey shades. Several interesting rules (Rule 110, Rule 30, Rule 90, Rule 184, and their reversals, bitwise inverses, and inverted-reversals) are highlighted in colour. In particular, Rule 110 variants are highlighted in red.

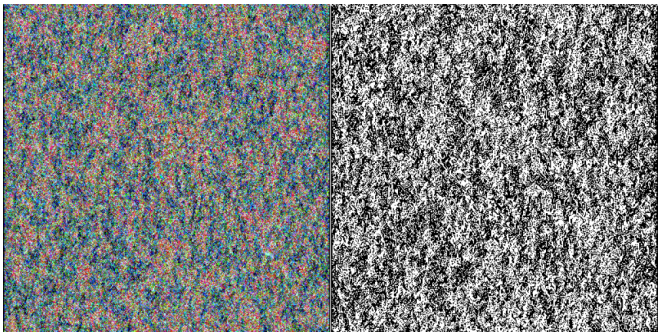


**Figure 5.** A high rate of mutation produces tantalising random structures

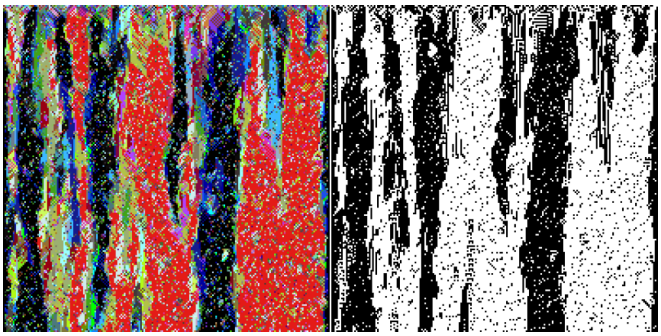


**Figure 6.** Throttling down the mutation rate preserves some of the large-scale stability while making room for variability

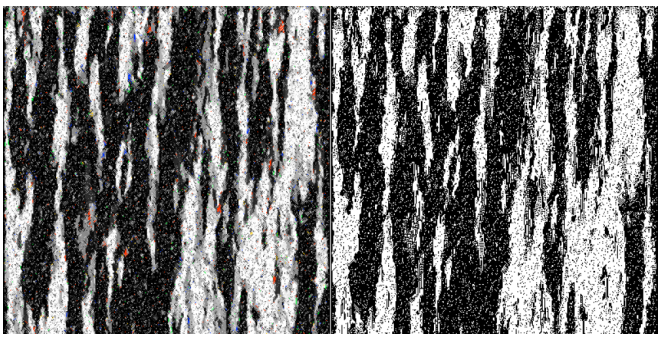


**Figure 7.** The search for intelligent life in the computational universe

We observe that Rule 0 and Rule 256 behaviour tends to predominate. Grey areas appear to be semi-stable. Red patches appear and disappear, as if independent planets evolve intelligent life and are then extinguished. With this physics, "intelligent life" seems inevitable, but also inevitably short-lived. One would have to look for another overall physics for intelligent behaviour to predominate.

A potential indication of the direction to look in is presented in Figure 8, which presents CAs generated by adjusting the typical blending evolution pattern by an (erroneously-programmed) mutation rule that only flips the first bit. We see that long-term behaviour in the genotype flutters randomly between Rule 0 (00000000) and Rule 128 (10000000). The short-term behaviour in the phenotype is nevertheless quite interesting, exhibiting many of the familiar lifelike edge-of-chaos patterns before ultimately succumbing to a version of Newton's First Law.

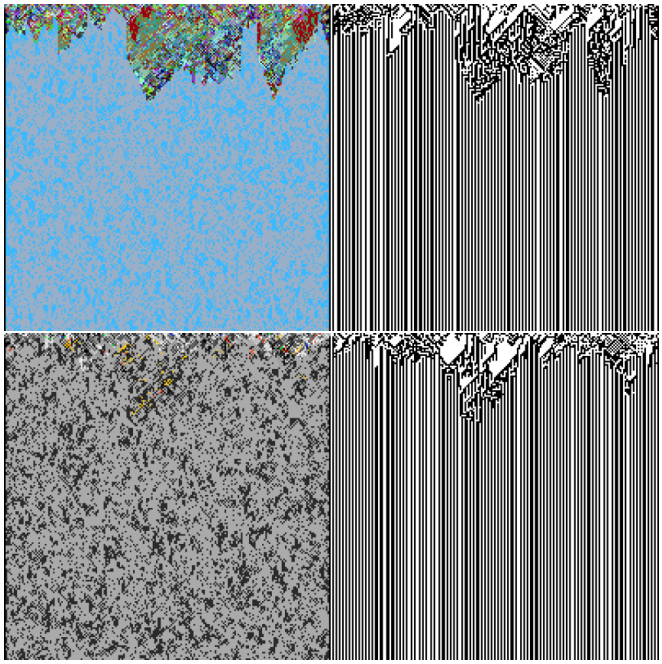
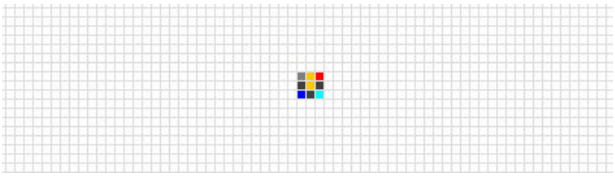


**Figure 8.** A skewed mutation pattern

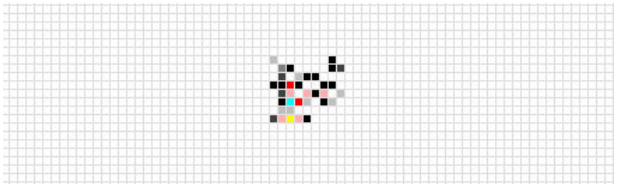
## 4.2 2D CAs

To see the behaviour of the union blend in action consider an initially populated grid, where colours represent the weights of alive cells:

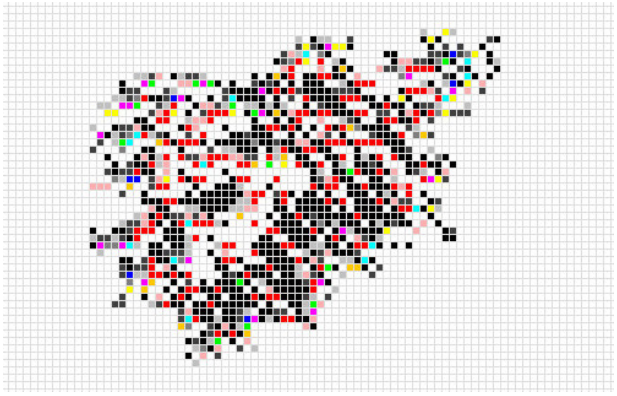


For this example, we initially restrict the computation of the blend for a particular cell to take place when the cell is alive in the next iteration. Also we compute the blend of genotype for all neighbours, whether dead or alive.

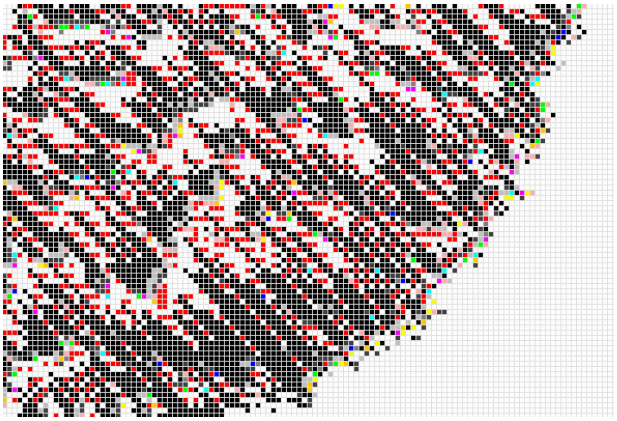
After 300 iterations the colony has grown a small amount:



Over time, the population continues to grow, with large patches of low-weight (black) cells:



Finally some structure starts to appear in the clustering:



The propagation that follows shows a population of cells which grows slowly over time. The majority of the members have low weight (represented by black squares), but interspersed within the population are chains of squares with high weight (represented by red squares) adjacent to dead cells (white).

### *4.2.1 Modified Blends*

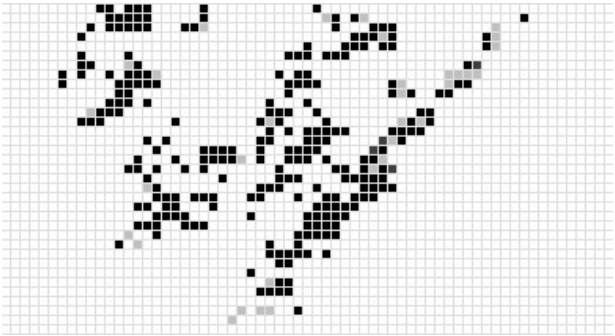
So far we have only showed the union blend working on the genotype. However, it is possible to use different blending techniques:

- Consider blending only the genotypes of alive neighbours, or all neighbours;
- Consider only blending genotypes for cells which are alive after propagation;
- Consider an *intersection* blend, where the partition sizes for survival are minimised;
- Consider an *average* blend, where the values of each genotype $(x_i, y_i, z_i)$ are summed and divided by either the number of alive neighbours, or the total number of neighbours.

As an example of different observed emergent behaviour consider a union blend where the blend is only computed from alive neighbours, and as before we compute only for cells which are alive at the next iteration. We start with an initial state:

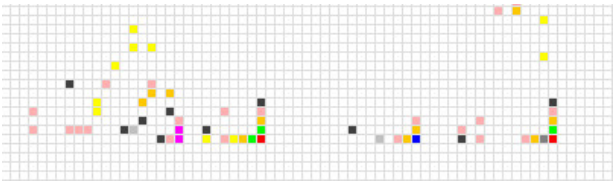


and observe a changing, but relatively steady pattern (resembling the motion of a flame) which does not grow in size using the union blend:

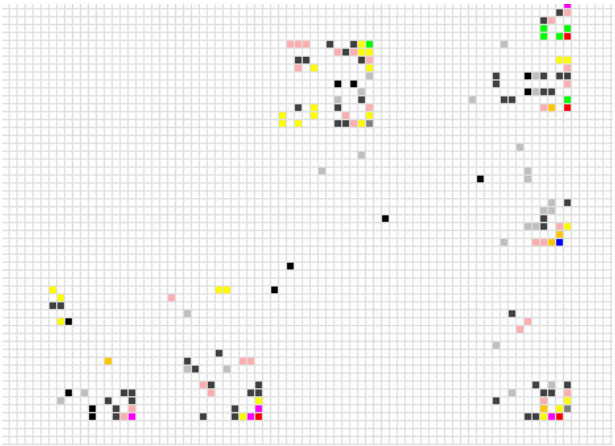


where the weight characteristic of the phenotype of each cell has fallen to very low.

Finally, consider applying instead an average blend under the same initial conditions:



Then we see a less steady but more active growth, with populations moving in triangular shapes away from population centres, leaving very small but steady and inactive populations behind:



The quickly-moving populations do not have a convergent weight characteristic in their phenotypes, as in the case with the union blend for the same initial conditions.

```
library metaca

logic CASL

spec METACABITENCODING =
    free type Bit ::= 0 | 1
    sort Triple
    ops t : Bit × Bit × Bit → Triple;
        bitop__ : Triple → Bit
end

spec METACABITCALC = % Calculate a blend given three 8-bit genotypes
    METACABITENCODING
then op blend____ : Triple × Triple → Bit
    ∀ t1, t2, t3 : Triple
    • bitop t1 = bitop t2 ⇒ blend t1 t2 = bitop t1
    • ¬ bitop t1 = bitop t2 ⇒ blend t1 t2 = bitop t3
end

spec LEFTRULE =
    METACABITENCODING
then • bitop t(0, 0, 0) = 0
    • bitop t(0, 0, 1) = 1
    • bitop t(0, 1, 0) = 1
    • bitop t(0, 1, 1) = 0
    • bitop t(1, 0, 0) = 1
    • bitop t(1, 0, 1) = 1
    • bitop t(1, 1, 0) = 1
    • bitop t(1, 1, 1) = 0
end

spec RIGHTRULE =
    METACABITENCODING
then • bitop t(0, 0, 0) = 0
    • bitop t(0, 0, 1) = 1
    • bitop t(0, 1, 0) = 0
    • bitop t(0, 1, 1) = 1
    • bitop t(1, 0, 0) = 0
    • bitop t(1, 0, 1) = 1
    • bitop t(1, 1, 0) = 0
    • bitop t(1, 1, 1) = 1
end

spec LOCALRULE =
    METACABITENCODING
then • bitop t(0, 0, 0) = 0
    • bitop t(0, 0, 1) = 1
    • bitop t(0, 1, 0) = 0
    • bitop t(0, 1, 1) = 1
    • bitop t(1, 0, 0) = 0
    • bitop t(1, 0, 1) = 1
    • bitop t(1, 1, 0) = 0
    • bitop t(1, 1, 1) = 0
end

spec GENERIC = % Common between left and right
    METACABITENCODING
then • bitop t(0, 0, 0) = 0
    • bitop t(0, 0, 1) = 1
    • bitop t(1, 0, 1) = 1
end

view LEFT : GENERIC to LEFTRULE % Morphism from Generic to Left
end

view RIGHT : GENERIC to RIGHTRULE % Morphism from Generic to Right
end

spec BLEND = % This will be inconsistent
    combine Left, Right
end

spec WEAKENEDLEFTRULE =
    METACABITENCODING
then • bitop t(0, 0, 0) = 0
    • bitop t(0, 0, 1) = 1
    • bitop t(0, 1, 0) = 1
    • bitop t(0, 1, 1) = 0
    • bitop t(1, 0, 0) = 1
    • bitop t(1, 0, 1) = 1
    • bitop t(1, 1, 0) = 1
end

spec WEAKENEDRIGHTRULE =
    METACABITENCODING
then • bitop t(0, 0, 0) = 0
    • bitop t(0, 0, 1) = 1
    • bitop t(1, 0, 1) = 1
    • bitop t(1, 1, 1) = 1
end

view WEAKENEDLEFT : GENERIC to WEAKENEDLEFTRULE
end

view WEAKENEDRIGHT : GENERIC to WEAKENEDRIGHTRULE
end

spec CONSISTENTBLEND = % A consistent blend as new 8 bit encoding
    combine WeakenedLeft, WeakenedRight
and METACABITCALC
and LOCALRULE
end
```

**Listing 1.** CASL source code listing calculating the running example 01101110 × 01010100 × 01010101 via the blending meta-rule

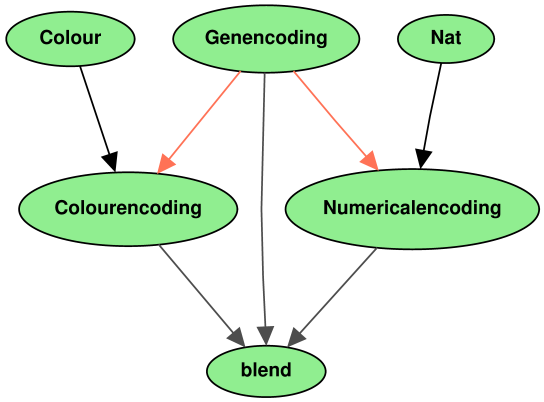



**Figure 9.** Blending different 2d genotypes

```
library metaca2d

logic CASL

spec NAT =
    sort Nat
    op max : Nat × Nat → Nat
    op min : Nat × Nat → Nat
end

spec COLOUR =
    sort Colour
    op maxhue : Colour × Colour → Colour
end

% a 2−d cellular automaton with numerical Genotype
spec NUMERICALENCODING =
    NAT
then sort NGenotype
    ops genotype : Nat × Nat × Nat → NGenotype;
        t : Nat × Nat × Nat → NGenotype;
        numblend : NGenotype × NGenotype → NGenotype
    ∀ g1, g2 : NGenotype; x1, y1, z1, x2, y2, z2, x3, y3, z3 : Nat
    • g1 = t(x1, y1, z1) ∧ g2 = t(x2, y2, z2)
      ⇒ numblend(g1, g2)
        = t(min(x1, x2), min(y1, y2), max(z1, z2))
end

% A colour CA Genotype
spec COLOURENCODING =
    COLOUR
then sort CGenotype = Colour
    op hueblend : CGenotype × CGenotype → CGenotype
    ∀ g1, g2 : CGenotype
    • hueblend(g1, g2) = maxhue(g1 as Colour, g2 as Colour)
end

% A generic space
spec GENENCODING =
    sort S
    sort Genotype
    op blend : Genotype × Genotype → Genotype
end

% A signature morphism from Generic to Numerical
view NUMERICALSM :
    GENENCODING to NUMERICALENCODING =
    S ↦ Nat, Genotype ↦ NGenotype, blend ↦ numblend
end

% A signature morphism from Generic to Colour
view COLOURSM :
    GENENCODING to COLOURENCODING =
    S ↦ Colour, Genotype ↦ CGenotype, blend ↦ hueblend
end

spec BLEND =
    combine NumericalSM, ColourSM
end
```

**Listing 2.** CASL source code using signature morphisms and pushout calculation to blend genotypes with different languages

## 5 Discussion

### 5.1 Research Contribution

The motivation for combining a notion of blending with cellular automata was to investigate ways in which cellular automata could be used to model processes, where propagation rules, or genotypes, were locally defined. The main research contributions in the field of two dimensional cellular automata are

- We built and implemented a framework where local propagation experiments can be performed;
- We used the HETS system to show that the notion of blending can be used to invent new propagation rules for different genotypes;
- We invented simply definable genotypes and blends of these genotypes to show proof of concept;
- Finally, we shared the results of simulations that illustrate qualitative behaviour in one and two dimensional MetaCAs.

The primary limitation of this work is that our results are purely observational at present. For example, the early experiments seemed to provide visual evidence that blending is useful: Figure 1 is more interesting than Figure 2. The robustness of our qualitative findings have been supported by developing a range of different experiments, for example, some analogy could be drawn between the "grey areas" observed in Figure 7 for the 1D case and the red-and-white chains that develop in the 2D case under union blending.

Our results confirm the basic finding of CA research: interesting global behaviour can arise from simple rules governing local interactions, with the added twist that these rules can also arise locally. The MetaCA setting seems to offer fertile ground for further computational research into evolutionary and co-evolutionary effects.

### 5.2 Social Interpretation

One can view the propagation of cells and patterns in a 1D or 2D MetaCA as a social process, and blending as a knowledge exchange. In the 2D case, we can think of the generated diagrams as illustrations of interactions between individuals with high knowledge, skill, or social impact (high weight), and those with less (low weight). The propagation in the "union" blend shows how large numbers of individuals with low social impact outnumber those with high social impact, but those with high social impact impose the emergent structure and determine the growth of the group of individuals.

In a fundamental respect our blending rules seem to embody a thought-provoking blend of two very different kinds of "ethics." Specifically, blending seems to introduce a dynamic similar to Carol Gilligan's *ethic of care* [8], which seeks to defend the relationships that obtain in a given situation. Here this is manifested by the question "Have my neighbours already formed a consensus?" This behaviour complements the local rule, which would correspond to Lawrence Kohlberg's *ethic of justice* (cf. [3]).

As we saw in Section 4.1, we would have to work harder to find meta-rules that give rise to an "intelligent universe" or in which life (considered as symbolic computation) plays an obvious negentropic role (*après* Bergson [4]).

One strategy that has not been developed here would be to make use of a "Baldwin effect" [2, 30], to use "learning" (considered as entropy) in the phenotype layer to drive (co)evolution. More specifically, $\boxed{0}\boxed{0}\boxed{0} \mapsto \boxed{0}$ and $\boxed{1}\boxed{1}\boxed{1} \mapsto \boxed{1}$ seem to be relatively uninteresting behaviours, but they are also hard to resist under the blending dynamics as we've defined them (compare Figures 4 and 7). Can we find ways to select against them?

### 5.3 Planned extensions

One observes that under our blending rule, the two non-entropic behaviours listed above are actually selected for, not against, because they are examples of the "neighbours match" condition. Indeed, reviewing the essential features of blending in the 1D case, we can use our basic principles:

> "*If neighbours match: use their shared value as the result.*
> *If neighbours don't match: use local logic to get the result.*"

to define a 1D CA rule, if we interpret "local logic" to mean "substitute my own value as the result." Here's how we would then define blending for triplets:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | $\mapsto$ | 0 | *Neighbours match* |
| 0 | 0 | 1 | $\mapsto$ | 0 | *Local logic* |
| 0 | 1 | 0 | $\mapsto$ | 0 | *Neighbours match* |
| 0 | 1 | 1 | $\mapsto$ | 1 | *Local logic* |
| 1 | 0 | 0 | $\mapsto$ | 0 | *Local logic* |
| 1 | 0 | 1 | $\mapsto$ | 1 | *Neighbours match* |
| 1 | 1 | 0 | $\mapsto$ | 1 | *Local logic* |
| 1 | 1 | 1 | $\mapsto$ | 1 | *Neighbours match* |

This is Wolfram's Rule 23: and as it happens, its evolutionary behaviour is not particularly interesting. Of course, for blending at the genotype level, "local logic" can be determined by any CA. Even so, when we use blending bitwise on alleles, we only ever run the local logic on half of the cases, and moreover it always the same half, determined by a "censored" version of Rule 23.

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | $\mapsto$ | 0 | *Neighbours match* |
| 0 | 0 | 1 | $\mapsto$ | $*$ | *Local logic* |
| 0 | 1 | 0 | $\mapsto$ | 0 | *Neighbours match* |
| 0 | 1 | 1 | $\mapsto$ | $*$ | *Local logic* |
| 1 | 0 | 0 | $\mapsto$ | $*$ | *Local logic* |
| 1 | 0 | 1 | $\mapsto$ | 1 | *Neighbours match* |
| 1 | 1 | 0 | $\mapsto$ | $*$ | *Local logic* |
| 1 | 1 | 1 | $\mapsto$ | 1 | *Neighbours match* |

Rather than using Censored Rule 23 as our template, we could instead have the template determined by phenotype data, thereby involving the phenotype as a "hidden layer" in the computation.

The standard template could be understood to be generated by locking in $\boxed{0}\boxed{0}\boxed{0} \mapsto \boxed{0}$ along with a "variation"[4] $\boxed{0}\boxed{1}\boxed{0} \mapsto \boxed{0}$ and the bitwise inverses of these. A wider class of templates could be calculated from arbitrary phenotype data by the same operations. What we would lose in abandoning the intuition associated with local blending, we may be repaid through a much more abstract but richer procedural blend, operating at the level of genotype+phenotype coevolution. At the very least, we can point to a generic space, namely the locked-in local rule which would be carried over (along with its variants) from the phenotype to the corresponding alleles.

As a simple example of cross-domain blending consider a genotype defined as in §3.3, and another which is defined by comparing the hue of just one neighbour. Their blend is a richer theory combining elements from both genotypes. CASL code expressing these concepts is given in Listing 2, and the resulting categorical diagram can be seen in Figure 9. Experimentation with more sophisticated genotypes and blends is ongoing.

---

[4] $\boxed{0}\boxed{1}\boxed{0} = \boxed{0}\boxed{0}\boxed{0} + \boxed{\ }\boxed{1}\boxed{\ }$

## 5.4 Future work

Coevolution has been understood to be relevant from both a philosophical [15] and empirical perspective [29]. Finding patterns that allow us to exploit Baldwin effects to drive the co-evolution of genotype and phenotype in the direction of intelligent behaviour is an interesting computational project. The MetaCA domain may help to show how to systematise some aspects of the search for the principles and techniques that underlie broader computational intelligence.

Expanding on the semantically simple domain of CAs, we would like to use HETS to formalise the mechanisms of social knowledge sharing and problem solving in fields like mathematics. It may be possible to encode mathematical problems in a MetaCA or cellular program and involve a group of agents in finding solutions to these problems as a society, in an emergent manner. This would be informed by ongoing empirical analysis of real problem-solving activities [20] developed in parallel to the simulation work presented here.

# 6 Conclusion

This research was inspired by the aim to build an example of computational blending that matched, to some extent, the way blending might work in social settings. One person suggests an idea, and another offers a variant of that, a third brings in another idea from elsewhere and some combination is made. The next day, things head in another direction completely. Our progress in this research project has followed this sort of trajectory: from an initial critique of blending theory ("it's not dynamic enough to be social!") to some tentative examples showing how large-scale system dynamics can be driven by local behaviour in an emergent manner. Perhaps the most interesting aspect of this research is the relationship between these emergent dynamics and the meta-rules. Whereas previous CA research has shown that complex global behaviour can be generated from a set of simple, local rules, this project gives an enticing glimpse of a future research programme that carries out a computational search for those very rules (out of the many possible) that lead to system behaviour we would recognise as "intelligent."

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# Chapter 1
# Stakeholder Groups in Computational Creativity Research and Practice

**Simon Colton, Alison Pease, Joseph Corneli, Michael Cook, Rose Hepworth and Dan Ventura**

**Abstract** The notion that software could be independently and usefully creative is becoming more commonplace in scientific, cultural, business and public circles. It is not fanciful to imagine creative software embedded in society in the short to medium term, acting as collaborators and autonomous creative agents for much societal benefit. Technologically, there is still some way to go to enable Artificial Intelligence methods to create artefacts and ideas of value, and to get software to do so in interesting and engaging ways. There are also a number of sociological hurdles to overcome in getting society to accept software as being truly creative, and we concentrate on those here. We discuss the various communities that can be considered stakeholders in the perception of computers being creative or not. In particular, we look in detail at three sets of stakeholders, namely the general public, Computational Creativity researchers and fellow creatives. We put forward various philosophical points which we argue will shape the way in which society accepts creative software. We make various claims along the way about how people perceive software as being creative or not, which we believe should be addressed with scientific experimentation, and we call on the Computational Creativity research community to do just that.

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## 1.1 Introduction

It seems uncontroversial to state that one of the long-term goals of research into Computational Creativity is to see creative software embedded in society: Apple's iTunes will one day compose new music for us, rather than just recommending it; Microsoft's PowerPoint will suggest jokes for a speech we're writing; videogames will be constructed on the fly to fit our preferences and mood; software will routinely make scientific discoveries; and household appliances will be endowed with creative abilities, like a refrigerator able to concoct a recipe to fit its contents. It is also uncontroversial to point out that another long-term goal of the field is to further our understanding of human creativity, both individually and in societies, through computer simulation.

Computational Creativity researchers have made steady progress towards software which creates, by employing, advancing and inventing novel Artificial Intelligence, natural language processing, graphics, audio and other techniques for creative purposes. There is, of course, much progress still to be made technically, so that software can be creative and be seen to be creative, in order for consumers to be provided with valuable artefacts and enjoyable creative experiences. In addition to the technological hurdles faced, it is clear that certain sociological issues stand in the way of progress. That is, people naturally tend towards thinking that nuts-and-bolts, bits-and-bytes machines will never have a creative spark, and different sets of people instantiate this tendency in different ways. Through much engagement and outreach, we have come to the conclusion that understanding people's conceptions of software being creative is an important tool to be used towards the long-term goal of understanding human creativity, and that favourably guiding these conceptions will be essential in bringing about the long-term goal of embedding creative software in society.

In largely separate tracks of research, we have examined how creative software is perceived by three different types of *creativity stakeholders*—people who may have something to gain or lose from software which is creative—from a practical and a philosophical perspective. We address the different types of creativity stakeholders in general in Sect. 1.3, and concentrate in the rest of the chapter on three particular types. In particular, in Sect. 1.4, we address members of the general public exposed to creative software. Following this, in Sect. 1.5, we address observer issues within Computational Creativity research itself. Finally, in Sect. 1.6 we address videogame designers, as an exemplar of a focused community of creative individuals within which creative software has begun to make an impact. We posit that, because of the different issues that each stakeholder community raises with creative software, it currently helps to study them independently, and suggest approaches to altering the perception that people have of software in these groups in different ways. However, by bringing together these strands for the first time here, we can begin to discuss more unified approaches to the presentation of software written to be autonomously creative.

Throughout this chapter, we propose hypotheses about how each set of stakeholders perceive software as being creative or not, based on practical experiences,

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philosophical studies and theoretical advances. We believe that our arguments in favour of these claims are sufficiently strong for them to be taken to the next level and tested scientifically through observer-based experimentation—and that the hypotheses provide an agenda the Computational Creativity research community cannot ignore. To conclude in Sect. 1.7, we suggest some practical ways in which these claims (which are presented as numbered hypotheses) could be investigated. In order to explain and support the claims we make, in the next section, we first present a philosophical perspective on the notion of creativity, which will introduce ideas that underpin the material in the rest of the chapter.

## 1.2 A Perspective on Creativity

We hold that creativity is a *secondary* and *essentially contested* quality of a person, and that linguistic usage of terms related to creativity can often be declarative illocutionary speech acts. We unpack these assertions below. Firstly, we believe that attributions of creativity are contextualist, having no truth value which is independent of context, perception and interpretation. In this way we see creativity attributions as analogous to the Lockean notion of a secondary quality [1]. Locke distinguished *primary* and *secondary qualities*, where the former are taken to be intrinsic to an object, for example, its mass, and the latter are understood to be perception-dependent, for example, colour. While these Lockean qualities are directly tied to sensory perception, as opposed to the aesthetic and social category of creativity, the distinction is still a useful one here, since it highlights different types of properties. Dennett's intentional stance [2] is also of interest here: we may adopt a "creativity stance" towards a person and interpret their work *as though they were being creative*, in order to better understand (rather than predict) their behaviour. Likewise, we may find that the "creativity stance" provides a new way of understanding the behaviour of a piece of software which goes beyond the physical details of the program.

Gallie introduced *essentially contested concepts* as those for which "the proper use . . . inevitably involves endless disputes about their proper uses on the part of their users" [3, pp. 169], to which Gray added that the disputes ". . . cannot be settled by appeal to empirical evidence, linguistic usage, or the canons of logic alone" [4, pp. 344], and Smith noted that ". . . all argue that the concept is being *used inappropriately* by others" [5, pp. 332]. In the *Cambridge Handbook of Creativity*, Plucker and Mabel assert that:

> . . . despite the abundance of definitions for creativity and related terms, few are widely used and many researchers simply avoid defining the relevant terms at all. [6, p 48]

Clearly, certain notions such as *art* are essentially contested concepts, looking at the multitude of articles written each year in the popular and cultural press asking: "But is it Art?" Indeed, Gallie points out that the statement: "This picture is painted in oils" can be disputed whilst the disputants nevertheless agree on the proper usage

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of the terms involved, whereas the assertion "This picture is a work of art" is likely to be contested

> . . . because of an evident disagreement as to—and the consequent need for philosophical elucidation—of the proper general use of the term "work of art" [3, pp. 167].

As a recent example, the question of whether videogames should be classed as art was raised by a Guardian art critic [7], to which the Guardian games editor responded:

> Here is a good way to tell if a critic is having a moment of madness: they will attempt to define art. The greatest philosophers in history have floundered on the question, many simply avoided it altogether, preferring to grapple with more straightforward questions—like . . . the existence of God. Art is ethereal, boundless, its meaning as transient as the seasons. When you think you have grasped it, it slips through your fingers [8].

While this is only one example, it serves as an exemplar of the kinds of debates that occur daily about the nature of art. While it is true that the preoccupation with expressing creativity is a relatively modern aspect of the visual arts, if the notion of *art* is indeed essentially contested within our culture, then the notion of the *creativity* that went into producing a given artwork should be seen accordingly. In particular, a selection of criteria for what counts as creativity is required in any coherent scheme for understanding and evaluating creativity in art. This is the perspective advanced by Jordanous [9], with which we agree—although we also agree with her point that there is unlikely to be broad and lasting agreement about just what the precise criteria of creativity actually are. We can further justify the idea that proper usage of the term *creativity* involves endless debate about its proper usage by reference to the multitude of volumes written about improving, managing and assessing creativity in people, organisations and society. Indeed, as a society, we are better off if we do not agree about what creativity means—in the sense that the disputes we have about this are an engine for change and progress, and it would surely be stultifying if we all suddenly agreed on this most important of concepts. While it is problematic for various areas of study—not least Computational Creativity—that creativity is an essentially contested quality of any person, it is something we need to embrace and even celebrate. For more in-depth discussion of these issues, see Jordanous [9, Chap. 3]. We may ask, in practice, what does it mean to *say* someone or something is "creative"? Austin informally introduced the notion of an *illocutionary act* as a locution that also serves to perform another action [10]. Searle further categorised such speech acts into: *assertives*, *directives*, *commissives*, *expressives* and *declarations* [11]. *Declarations* in particular are understood to change reality in accordance with the proposition stated. An example of such a speech act is: "I pronounce you husband and wife." We believe that—in certain circumstances—people can bestow the reality of a person being creative simply by stating it. To see this, we recall the contested nature of creativity, and the assumption that there is no general consensus about what makes someone creative. It follows that people who are not particularly invested in the creativity (or lack thereof) of someone else may be swayed by the declarative speech act of a third party in a position of authority. When Nicholas Serota, long time director of the Tate art museums and galleries, says that a piece is a great work of art, that work

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becomes (at least temporarily) a great work. When he states that a particular artist is unusually creative, who are we to argue? Given that the sentence 'X is creative' is often shorthand for: 'Most people agree that they perceive X to be creative', such authorities can essentially bring into being the creativity of X, regardless of whether X perceives him/herself as creative or not.

## 1.3 Communities of Creativity Stakeholders

In order to understand the different groups of creativity stakeholders, the relationships between them, and the ways in which meaning is continually being created, negotiated and re-created, we can look to sociology. In Latour's *Actor Network Theory* [12], he describes such stakeholders and diverse social groups as actors in a network. Meaning is created socially via actors who cluster into diverse stakeholder groups. These groups are in constant flux, as relationships, actors and ideas within the groups change and come into conflict with each other. Latour holds that understanding such dynamics in the network is essential to understanding processes of innovation and knowledge-creation in science and technology. The process by which a network is formed and comes to be represented as a single entity is called *translation*, and is a key concept in the Actor Network Theory. Translation consists of various phases: the initial formation of a programme and identification of actors in a new network with a novel, shared goal (*problematisation*); the strengthening of the network via formal and informal means (*interessement*); ways of evolving the network and providing structures for new members to join (*enrolment*); and acquiring the resources and power to build an effective institution which can achieve its goal (*mobilisation*).

In the case of Computational Creativity, relevant creativity stakeholders include researchers, the wider AI community, funding bodies, experts in the psychology of human creativity, neuroscientists, artists, art critics, journalists, philosophers, educators, the public, and so on. Each group has accompanying visions, beliefs and goals, in which they have, to a varying degree, invested (and which, to a varying degree, define them as a group). We hold that understanding such different perspectives and their interactions is essential if software is ever to be deemed creative by mainstream consumers of cultural artefacts. In this section, we consider these stakeholder groups and in particular use Latour's notion of translation to look at how Computational Creativity researchers have evolved into a community. We also look at some of the relationships between the groups, both in the context of Computational Creativity and the wider scientific arena.

### *1.3.1 The Computational Creativity Stakeholders*

Members of the Computational Creativity community are largely people with a background in Artificial Intelligence or computer science and an interest in creativity.

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They are usually professional academics with the infrastructure of a university supporting them. AI is itself a young field—originating in the 1950s—and, since initial attempts to build general intelligence machines, has fragmented into many different specialisations and subdisciplines: once established, these then form the *internal environment* for any new area, in terms of providing ideas, methods and concepts, and at times, competition. Academic measures of the health of such subdisciplines include the amount of funding awarded, the number of lectureships or professorships in the field, the existence of a journal and an international conference series, and other scientifically respectable incentivisation schemes and recognition.

It was against this backdrop that AI researchers with an interest in creativity found themselves in the late 1990s. Given their background, they were not only accustomed to the idea that machines can be intelligent, but their very livelihood depended on that premise. So it was not, perhaps, such a huge leap to the idea that machines can be creative. However, since there was no infrastructure supporting research into Computational Creativity, early researchers largely had to establish their reputation in different (possibly related) areas of AI and build up the Computational Creativity community almost on their own time, sometimes taking considerable career risks to do so.

Latour's notion of *translation* can help us to understand how the community formed. *Problematisation* occurred when a few core people identified the goal of building creative software as a subdiscipline of AI. Between them, they had the influence and organisational power to make *Creativity in AI and Cognitive Science* the theme of the AISB'99 Convention (co-chaired by Geraint Wiggins, Helen Pain and Andrew Patrizio). This featured a keynote address by Margaret Boden, a cognitive scientist known for her popular writing on creativity in people and in machines [13, 14]. The initial symposium was followed up by four further events[1] held at AISB'00 - AISB'03, and a series of workshops on creative systems at major AI conferences. We present an extract from the editors' introduction to the Proceedings of the Symposium on Creative and Cultural Aspects of AI and Cognitive Science, held at AISB in 2000 in Fig. 1.1. This was the *interessement* phase. These were further consolidated with the International Joint Workshops on Computational Creativity (IJWCC), held 2004–08, during which time the community grew from twenty, or so, to double that (*enrolment*). Finally, the community was considered healthy enough, strong enough and large enough to launch the first International Conference on Computational Creativity in 2010. For a history of the field up to this stage, see [15] in a special issue of the AI magazine on Computational Creativity.

The community continues to evolve and grow, with the series having recently held its Fifth Annual International Conference (2014), with around 90 delegates. In order to organise and guide the international series, a Steering Committee was set up consisting of anyone who had chaired an IJWCC event, and they formed the Association for Computational Creativity (ACC) in 2010 and set out rules which enabled new members to join and old members to leave the Association (*mobilisation*). Landmark

[1] *Creative and Cultural Aspects of AI and Cognitive Science* (2000) and then simply *AI and Creativity in Arts and Science* (2001–2003).

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> Following on from the successful AISB'99 Convention, whose theme was the study of creativity in AI and Cognitive Science, the purpose of this symposium is to bring together researchers interested in all AI and cognitive aspects of creativity and cultural enterprise. The aim of holding one unified meeting, instead of several simultaneous smaller ones, is to promote communication between those studying different aspects of creativity, and this has been mostly achieved: the papers (and the corresponding presentations) are for the most part grouped by their relation to creativity in general, rather than to a particular domain. The exceptions to this are two papers on important low-level aspects of modelling musical creativity.
>
> The four sections which have naturally arisen, then, are headed Exploratory Creativity, Modelling & Supporting Creative Processes, Techniques for Modelling Musical Creativity, and Methodology & Evaluation.

**Fig. 1.1** An excerpt from the preface of the Proceedings of the Symposium on Creative and Cultural Aspects of AI and Cognitive Science, held at AISB in 2000, written by Geraint Wiggins. Note the 'natural' emergence of themes within the field, although of course these are very much subject to the Call for Papers, the communities who received the call, the instructions given to the reviewers, the reviewers themselves and the editor's vision

events during this time included the first ever award of a Chair in Computational Creativity (to Geraint Wiggins, in 2004, by Goldsmiths, University of London) (only one of two—the other being held by Simon Colton also at Goldsmiths, University of London, awarded in 2013); the first PhD with Computational Creativity in its title (Anna Jordanous, University of Sussex, 2012 [9]) and the first NSF and EU calls for proposals in Computational Creativity (*CreativeIT, NSF Program Solicitation 09–572* [16] and *Objective ICT-2013.8.1, Technologies and scientific foundations in the field of creativity* [17, p. 81]). The process has been carefully managed throughout, with an eye on political as well as intellectual developments. Social factors have also played a key role, being inextricably linked to internal development of scientific knowledge [18].

### *1.3.2 Other Creativity Stakeholders*

Each of the other stakeholder groups will have a similarly fascinating history. Some, such as the EU funding body, are tightly bound and have a formal definition of themselves and their goals. Others, such as the general public—for whom the concept of *translation* is meaningless—are much more loosely defined. Of note is who the decision makers are in each of these groups. In the Computational Creativity community, it is clear that a few people have had a huge influence, and it is likely that this is also the case for other groups of stakeholders. It may be worth considering these in detail, especially from a point of view of motivation and power. For instance, Boden's way of seeing creativity dominated the first decade of the community growth. Likewise, a few core individuals working for the EU had the influence to prioritise research into Computational Creativity, and to fund around €10 m worth of projects.

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- **The general public**

When describing what they do, to a layman, most researchers into Computational Creativity will probably have experienced reactions such as: "A computer that is creative might be dangerous—it might kill us"; "Creativity is a celebration of humanity, and the very idea of Computational Creativity cheapens that"; "I read a poem or listen to music to communicate with another human being. I don't want to communicate with a computer, I want a live human connection", and so on. It is important to determine where these ideas come from, whether they are grounded in anything, whether we should try to counter them, and if so, how? While such emotional responses are not necessarily negative, it might be the case that they hinder reasoned debate. Public perception of Computational Creativity derives from multiple sources, including journalistic coverage (or lack of it), science fiction narratives, opportunities to consume computationally created artefacts and so on. We look further at observer issues in the general public in Sect. 1.4.

- **Fellow creatives**

Creative people sometimes voice the worry that "Computers are going to put us out of a job". This group is similar to the general public in terms of influences and attitudes. It seems that artists might be being encouraged to worry about software replacing them, because such sensationalist stories sell newspapers. We study a particular community of creative people, namely videogame designers in Sect. 1.6.

### *1.3.3 Relationships Between the Different Stakeholder Groups*

There have been several interactions between the Computational Creativity community and members of the public and fellow creatives. For instance, Colton and Ventura hosted a festival of Computational Creativity in 2013, *You Can't Know my Mind* [19], and other events have followed on from this. Historical relationships between scientists and the public can also elucidate current interactions. In other fields, there have been some explicit campaigns to manufacture doubt, by parties who are threatened by specific scientific advances. For instance, the tobacco industry tried to discredit and discourage the notion that smoking is bad for our health; likewise the fossil fuel industry did the same in the case of global warming. Here we see that a few powerful actors can sometimes bring an entire body of established scientific knowledge into question.

Ravetz argues that scientific ignorance may in some ways be as prone to social construction as scientific knowledge [20, 21], cited in [22, p. 37]. Stocking and Holstein [22] explore different perceptions that journalists have of their roles, concluding tentatively that journalists construct scientific ignorance consistent with their own interests. Even without such dark agendas, there are other examples from the history of science in which public perceptions conflict with scientific thinking and have been managed, or controlled, in order to bring them into line with current scientific results. Famous examples in which scientific advances have challenged our image of ourselves and our universe include Copernicus's heliocentric model,

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which challenged our view that the earth is the centre of the universe; Darwin's theory of evolution, which challenged concepts of what it means to be human, to be distinct from other animals, and the notion that our existence has a higher purpose; and Lemaître et al.'s Big Bang theory, which challenged the view that the universe is a stable, stationary entity. In all of these cases the scientists faced their own challenges of reconciling their findings with their religious or world views, and then a process of outreach was necessary in order to gain wider social acceptance. Thus, we see Thomas Huxley—"Darwin's bulldog"—promoting Darwin's theory in the face of many varied and negative responses to it (some of which are recorded in [23]) and helping it to gain wider acceptance, transitioning from scientific to social fact. Today, people in the fields of genetically modified food and stem cell research endeavour to gain wider social acceptance in the form of media coverage and well-funded outreach programmes aimed at educating both school children and the wider community.

Computational Creativity is in a particularly difficult position, since its main research question concerns an essentially contested concept. On certain understandings, the question "can machines be creative?" may be answered negatively, without further elaboration or debate. Thus, we see part of the job of the Computational Creativity community consisting in the delivery of outreach programmes, in which creative software is demonstrated and explained, and the artefacts it has produced exhibited in a setting in which consumers of creative artefacts might begin to appreciate them. In [24], Franzen et al. explore the impact that such dissemination activities can have on scientific progress, and argue that the right name, image or metaphor has the power to make or break relations between a scientific discipline and the public. For instance, consider Dolly the sheep from the Roslin Institute in Edinburgh and Ida the primate fossil from the Messel Pit in Germany. These names make it easier for the discoveries to be visualised and discussed. Arbib and Hesse go further, stating that "scientific revolutions are, in fact, metaphoric revolutions" [25, p. 156], cited in [26, p. 5].

In addition, then, to sociological narratives, it is important to consider language use by each stakeholder group. The role of spin doctors is well-known in the political arena, in which those who bestow power are influenced in their thinking by vocabulary, metaphors and frames. In our case, the public have the power to bestow or withhold the word "creative" when describing software. Thus, we need to consider the language that we use. Lakoff [27] argues that we fit new information into pre-existing frames, which are built up slowly over time, and if we don't have appropriate frames, then we might misunderstand the information. Using the wrong frame, which is triggered by specific vocabulary, even to deny a message, only reinforces the frame. Thus, rather than trying to argue that "creative software is not scary", we should build up our own vocabulary, frames and metaphors for thinking about it.

**Hypothesis 1** Different stakeholder groups (including Computational Creativity researchers, the general public, domain creatives, psychologists, philosophers, educators, critics, journalists, bureaucrats, etc.) assess creativity in software differently, and there is no one-size-fits-all approach to presenting what software does and what it produces in the best way to increase perception of creativity.

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Given this, we believe it is currently appropriate to study stakeholder groups separately, as we do in the following sections.

## 1.4 Observer Issues with the General Public

We introduce here three notions, namely *essential behaviours*, *the humanity gap* and *software accounting for its actions*. We believe these are important in understanding how people generally react to the idea of software being creative, and thus are important in managing and shaping those reactions. To end the section, we present a case study in handling public perception of creativity in software, and we introduce another notion, namely that of *accountable unpredictability*.

A working definition of the field of Computational Creativity research as a subfield of Artificial Intelligence research given in [28] is as follows:

> The philosophy, science and engineering of computational systems which, by taking on particular responsibilities, exhibit behaviours that unbiased observers would deem to be creative.

While this definition is not universally accepted (with a challenge to focus on system-level creativity rather than individual responsibilities given in [29]), variations of it have been used to describe the field for many years.

The usage of the word 'unbiased' in the above definition hints at a problem encountered in evaluating projects where generative software produces *artefacts* (poems, paintings, sonatas, recipes, theorems, etc.) for human consumption. In particular, people generally have natural biases against, but also occasionally in favour of, artefacts produced by computers over those produced by people. In particular, negative, so called 'silicon', biases have been observed under experimental conditions [30, 31]. Hence, in stipulating that observers must be unbiased, the definition above emphasises a scientific approach to evaluating progress in the building of creative systems, whereby experimental conditions are imposed to rule out, or otherwise cater for, such biases. One such experimental setup is the *Turing-style comparison test*, where computer-generated and human-produced artefacts are mixed and audience members make choices between them with zero context given about the processes involved in their production. It is seen as a milestone moment if audiences cannot tell the difference between the artefacts produced by people and those produced by a computer. We believe there are many problems in the application of such tests in the general context of presenting the processing and products of creative software, as expanded in the subsections below.

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### 1.4.1 Essential Behaviours

We suggest not asking people if they believe software is behaving creatively, but rather concentrating on whether they perceive the software to be acting *uncreatively*. Using our standpoint above that the notion of creativity is essentially contested [3], we expect that no matter how sophisticated our software gets, we will not see consensus on such matters. However, we have found that people agree much more on notions of uncreativity: if a program doesn't exhibit certain behaviours onto which certain words can be projected, then it is easy to condemn it as being uncreative. Building on the foundational arguments given in [32], we propose that audience members can too easily label software as uncreative if they are unable to project any of the following words onto the behaviours they perceive software to be exhibiting:

> skill, appreciation, imagination, learning, intentionality, accountability, innovation, subjectivity and reflection

We have found that assessing the level of projection of these words onto the behaviours of software can help us to gauge people's opinions about (the lack of) important higher-level aspects of software behaviour, such as autonomy, adaptability and self-awareness. Note that we make no claim about the above behaviours being sufficient for a perception of creativity: a necessary set of behaviour types for avoiding the uncreativity label is not the same as a sufficient set of behaviour types for gaining the creativity label. This mis-interpretation of our aims for highlighting the above essential behaviours has propagated somewhat, for instance in [33].

**Hypothesis 2** Creativity in people and software is essentially contested and secondary, and hence it might be advantageous to work on people's perception of *uncreativity* in software, as this is easier to predict/manage. Software exhibiting the essential behaviour types highlighted above is necessary for it to avoid being labelled as uncreative. Eventually, when there are no good reasons to label software as uncreative, people may choose to label it as creative.

### 1.4.2 The Humanity Gap

One could argue that, given the particularly human-centric nature of creativity, and that a human connection is paramount in much of the arts, it is simply inappropriate to use the term 'creative' to describe software. The status quo is that we currently haphazardly apply human terminology related to creativity to software, which often requires the projection of other human qualities onto software, such as it being juvenile, which is inherently error prone, given that computers are patently not people. Another option is to ignore the non-human nature of software and concentrate on what it produces, rather than on what it is, or what it does. To begin to address the kind of silicon biases described above, researchers often compare the interpretation of computer-generated and human-produced artefacts in a rather extreme "blind

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experiment" situation in which knowledge about the personality of the artist and their practice is entirely missing. The philosophical grounding of such an approach [34, 35] matches the motivation of several art movements [36, 37] and many individual artists who have expressed a desire for their work to be taken at face value (see [38] for examples and further discussion).

We argue that in modern culture, a curious thing can happen when artists attempt to remove all reference to themselves and their process from discussions about the artistic (and commercial) value of their work. That is, in the absence of such information, people may tend to fill in the gaps about personality and process, and may do so in ways which bolster the credibility of an artist and increase the perceived value of his/her works. Indeed, one could argue that—in the same way that artists invite people to interpret the imagery in artworks in their own way by not prescribing what people should see/read/hear, in refusing to provide meta-level details about personality and process, artists, writers and musicians are actually (purposefully or not) inviting art lovers to invent interesting and engaging back-stories about who they are and what they do.

In such a context of non-disclosure, the comparison of the situation for computer-generated artefacts with the situation for human-produced artefacts is not particularly favourable. The vast majority of people have little or no idea about programming or programs, and may even harbour a desire not to find out about these things. Thus, when invited to assess a computer generated painting or poem, say, without background knowledge, they are denied any opportunity to invent a back-story, as they cannot project personality traits or romantic situations onto the computer, and cannot enter into any dialogues. More importantly, this situation can lead to people realising how much they value the human connection, whether actual or imagined, in such situations. We posit that there is a *humanity gap* that must be faced by Computational Creativity researchers who want their software to enhance society by being creative for artistic and utilitarian purposes.

Turing-style experiments, which epitomise the practice of non-disclosure, are intended to reduce variables so that a scientific study of the value of computer generated artefacts can be undertaken. One could argue that these contexts are intended to help people realise how much they value the aesthetic appeal of art, literature and music, regardless of other factors. This may be true, but we believe that such tests can actually help people realise how little they can relate to the computational origin of artefacts. In [39], we raise other issues with Turing-style comparison studies: in particular, we suggest that they encourage naïvety and pastiche generation in creative software. As a final point, it is clear that such experimental conditions are not sustainable if we are to enhance society with creative software. In the long term, biases about machine creation need to be embraced and managed, rather than factored out through experimental setups.

**Hypothesis 3** Turing-style comparison tests serve to highlight the humanity gap, and while they might serve short-term scientific gain, they are damaging to the long-term goal of embedding creative software in society.

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### 1.4.3 Software Accounting for Its Actions

We argue in [32, 40] that people take into account how a person or software operates when they assess the value of the output it produces. To address this issue, we advocate a development path to follow when building creative software: (i) the software is given the ability to provide additional, meta-level, information about its process and output, e.g., giving a painting or poem a title (ii) the software is given the ability to write commentaries about its process and its products (iii) the software is given the ability to write stories—which may involve fictions—about its processes and products, and (iv) the software is given the ability to engage in dialogues with people about what it has produced, how and why. Indeed, giving software the ability to discuss its creative works would mirror Turing's original proposal for an intelligence test [34] to a greater extent than tests focusing only on consumer perception of artefacts. As a preliminary example, in [41], we demonstrated a poetry generation system which is able to provide commentaries about its poetry, and how and why it produced a particular poem.

As we discuss in [42], in a computational setting, there are advantages to software being immersed in environments where serendipity might occur. However, accounting for lucky events that trigger creative acts may actually lessen the celebration and hence the impact that the acts have. It is important to note that people tend not to describe their processes and products in the explicit way we advocate for software, preferring to maintain some level of mystery. Nevertheless, we believe that, at this stage in the development of computationally creative systems, it is important to address the humanity gap—without aspiring to eliminate it. *Framing* [40] serves to highlight that intelligent processing was used to produce artefacts, which is an important first step. Given that audience members will typically not be able to come up with an interesting backstory without some scaffolding, positive acts of framing are likely to have more fruitful impact than an overall air of mystery.

Another possible way to address the humanity gap is to manage people's expectations about the level of humanity they will encounter through a computationally produced artefact. In the same way that when people buy an *e-book* they know they are not going to get a physical object, we advocate telling audiences that they are reading a *c-poem*, and hence—in the knowledge that it was produced computationally—they will get a reduced human connection. We can go further in *re-imagining* traditional artefacts, for instance in suggesting that a c-poem is actually a doublet of texts, one which resembles a traditional poem and another which provides a commentary about the motivations, actions and results of the software's processing. We believe this will highlight the humanity gap, but that it will do so in such a way as to help people to engage with and appreciate the creative process, and better enjoy the artefacts produced by software.

**Hypothesis 4** The humanity gap can be addressed by re-imagining the nature of creative artefacts, to manage expectations of humanity. In particular, it is advantageous for software to account for its processes and products through additional material such as a commentary.

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### 1.4.4 A Case Study in Automated Portraiture

As part of an exhibition with The Painting Fool[2] system [43] in 2013, we enabled the software to produce portraits for people live in a gallery, as described in [19]. Managing the expectations and perceptions of the observers was a key aspect of this project. To this end, we hung posters describing the behaviour of the software as exhibiting aspects of intentionality, imagination, skill, appreciation, reflection and learning (six of the essential behaviours described above). Moreover, the software's actions and output were tailored to support the perception of these behaviours and an impression of creativity in the software by observers present in the exhibition, especially those sitting for a portrait.

Portraits were painted with people sitting in front of a laptop. It was immediately made clear that (i) the software was modelling a 'mood' to direct its painting, and (ii) the sitter was very much a tool for the software, not the other way around. This was achieved by opening remarks from the software such as: "Thank you for being my model. I'm in a negative mood right now, so I would like you to express a sad emotion." This was followed by The Painting Fool explicitly directing the sitter, while video recording them. A still image was then extracted where the sitter was expressing an emotion. Machine vision techniques were applied to remove the background, into which was substituted one of 1,000 abstract art images, to which one of 1,000 image filters was applied. The filter was chosen to increase the chances that the resulting image might reflect a changing simulated mood gained through reading newspaper articles, as described in [19]. The same filter was applied to the face of the sitter placed in the foreground, producing in a few seconds an image conception, or sketch for the portrait, such as the first image of Fig. 1.2.

Following this, a canvas appeared on screen, and a hand holding either a pencil, paint brush or pastel stick made virtual marks on the canvas leading to a non-photorealistic rendering of the background and foreground of the portrait, taking between 2 and 10 minutes, depending on the style. An example portrait is given at the bottom of Fig. 1.2, which was printed and given to the sitter, along with the commentary (the whole of Fig. 1.2). The most important aspect of the commentary is the expression of intention, by first showing a conception of the type of portrait the software aimed to produced, then showing what it produced and finally analysing and criticising—using machine vision techniques described in [44]—its results with respect to its aims.

The purpose of the exhibition was cultural, not scientific, and no experimentation was undertaken. From our experience, however, we contend that the behaviours exhibited by the software and explained in poster form enabled people to be surprised by the resulting portrait (and many of the 100 or so sitters in the exhibition were very surprised), while still projecting creativity onto the software. This upheld the aim of the *You Can't Know my Mind* exhibition: as it used some intelligence, and could explain its actions, it was somewhat appropriate to employ the word 'mind' with reference to The Painting Fool. However, as the process was unpredictable due to

[2] Online presence: www.thepaintingfool.com.

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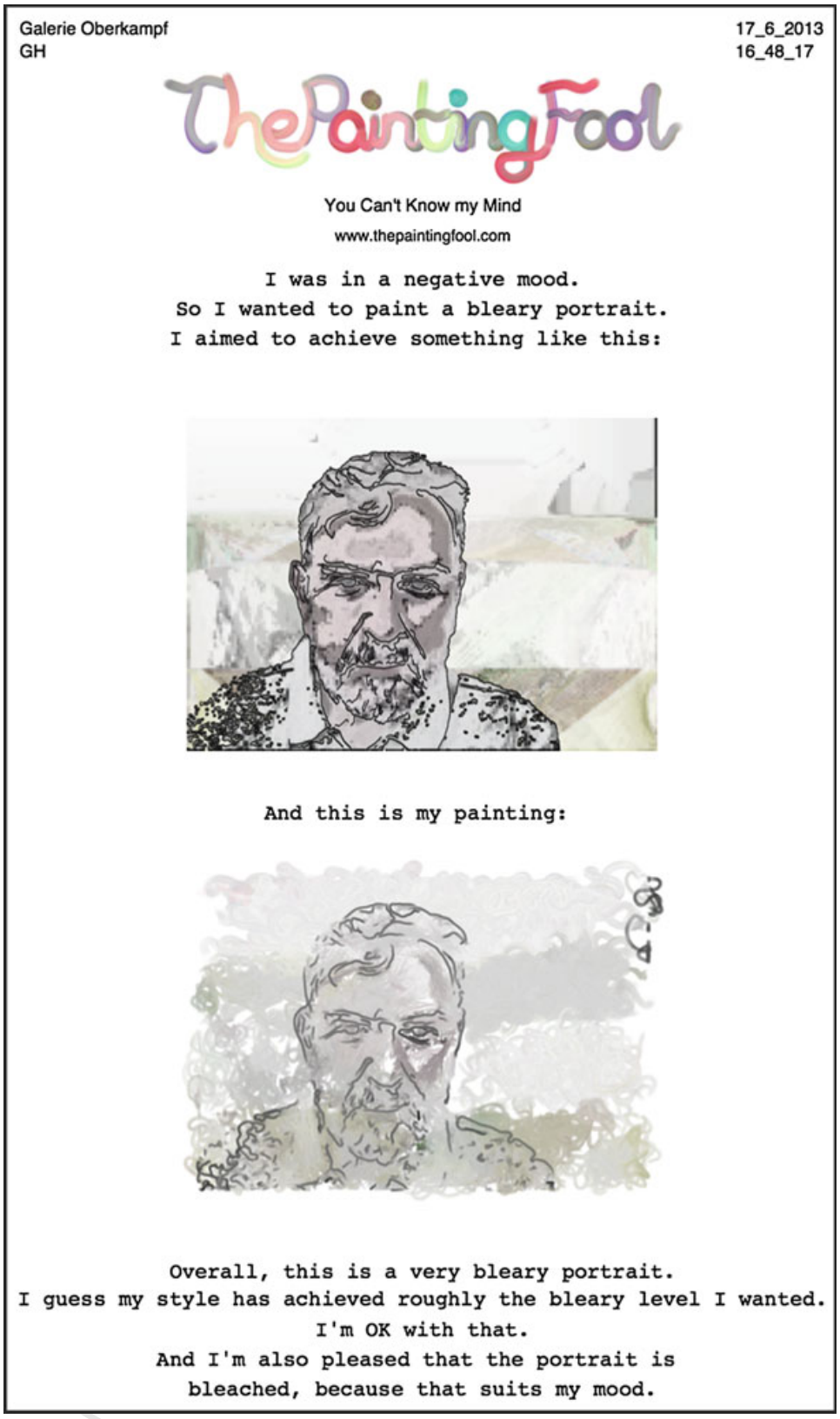



**Fig. 1.2** Example commentary by The Painting Fool, from the *You Can't Know my Mind* exhibition, Paris, June 2013.

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the dynamic nature of the software's changing mood, it was impossible to know this mind, and people realised that some software is written not to be a tool, but to be a creative individual. In fact, when in the most negative of moods, The Painting Fool refused to paint a portrait and sent the (often shocked) sitter away, citing a particularly depressing keyphrase in a particularly distressing newspaper article that it had recently read.

In these cases, The Painting Fool pointed out explicitly: "No random numbers were used in coming to this decision". This is because we feel that *accountable unpredictability* is important for creative systems. That is, we have found that when people realise that a certain important event has happened or an important artefact has been produced because of a random act, any dialogue (perceived or real) comes to an abrupt halt, and detracts from the creative experience. In contrast, unpredictability through accountable actions such as reading newspaper articles can add a great deal to a creative experience, at the very least by providing additional talking points.

**Hypothesis 5** Accountable unpredictability enhances the experience people have when told about software creating an artefact, whereas random number based unpredictability detracts from the experience.

## 1.5 Formally Capturing Progress in Creative Systems

Naturally, another major set of stakeholders in the notion of software being creative are the Computational Creativity researchers who aim to write such systems, and use them to study creativity in people and machines. As they are familiar with the issues of simplistic arguments for and against creativity in software, these stakeholders require more formalism in any argumentation put forward to support the hypothesis of increased creativity in software.

We have focused on formalising the general notion of *progress* in Computational Creativity research. To do this, we first introduced the FACE and IDEA descriptive models in [45, 46]. The FACE model categorises generative acts by software into those at (g)round level, during which base objects are produced, and (p)rocess level, during which methods for generating base objects are produced. These levels are sub-divided by the types of objects/processes they produce: $F_g$ denotes a generative act producing some framing information, $A_g$ denotes an act producing an aesthetic measure, $C_g$ denotes an act producing a concept and $E_g$ denotes an act producing an example of a concept. Generative acts producing new processes are defined accordingly as $F_p$, $A_p$, $C_p$ and $E_p$. Tuples of generative acts are collated as *creative acts*, and various calculations and recommendations are suggested in the model with which to compare creative systems. We developed the IDEA model so that creative acts and any impact they might have could be properly separated. We defined various stages of software development and used an ideal audience notion, where people are able to quantify changes in well-being and the cognitive work required to appreciate a creative act and the resulting artefact and/or process.

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The majority of researchers develop software using only themselves as an evaluator, because observer-based models are too time-consuming to use on a day-to-day basis. These informal in-house evaluation techniques generally do not capture the global aims of the research project, or of the field (e.g., producing culturally important artefacts and/or convincing people that software is acting in a creative fashion). This can lead to situations where systems are presented as feats of engineering, with little or no evaluation at all [9]. In [47], we argue that assessing *progress* is inherently a process-based problem, and hence we concentrate our formalism on processes, tempered with aspects of artefact evaluation. In the subsections below, we present this formalism with worked examples, followed by a case study describing the development of an evolutionary art system.

### 1.5.1 Formal Assessment of Progress

We combine the most useful aspects of the IDEA and FACE models, the list of essential behaviours described in Sect. 1.4.1, and certain aspects of assessing artefact value in a diagrammatic formalism for evaluating progress in the building of creative systems. We focus on the creative acts that software performs, the artefacts it produces and the way in which audiences perceive it and consume its output. We simplify by assuming a development model where a single person or team develops the software, with various major points where the program is sufficiently different for comparisons with previous versions. We aim for the formalism to be used on a daily basis without audience evaluations, to determine short term progress, but for it also to enable fuller audience-level evaluations at the major development points. We also aim for the formalism to help determine progress in projects where there are both *weak* and *strong* objectives, focused, respectively, on the production of increasingly higher valued artefacts, and on increasing the perception of creativity people have of the system. We found that the original FACE model didn't enable us to properly express the process of building and executing generative software. Hence another consideration for our formalism is that it can capture various timelines both in the development and the running of software in such a way that it is fairly obvious where the programmer contributed creatively and where the software did likewise.

### 1.5.2 Diagrammatic Capture of Timelines

Taking a realistic but abstracted view of generative software development and deployment, we identify four types of timeline. Firstly, generative programs are developed in **system epochs**, with new versions being regularly signed off. Secondly, each process a program undertakes will have been implemented during a **development period** where creative acts by programmer and program have interplayed. Thirdly, at run-time, data will be passed from process to process in a series of creative and

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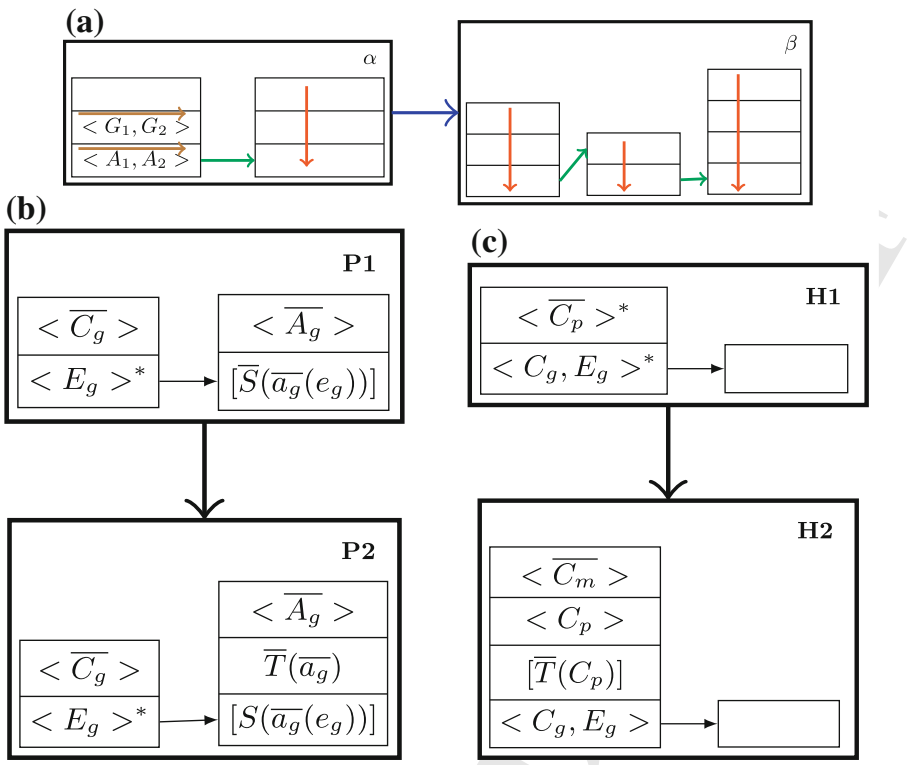



**Fig. 1.3** **a** Key showing four types of timelines **b** progression of a poetry system **c** progression of the HR system

administrative **subprocesses** performed by software and programmer. Finally, each subprocess will comprise a sequence of generative or administrative **acts**.

We capture these timelines diagrammatically: the four different kinds of transitions are highlighted with coloured arrows in Fig. 1.3a. The blue arrow from box $\alpha$ to $\beta$ represents a change in epoch at system level. The red arrows overlapping a *process stack* represent causal development periods. The green arrows represent data being passed from one subprocess to another at run-time. The brown arrows represent a series of generative/administrative acts which occur within a subprocess. Inside each subprocess box is either a ⟨creative act⟩ from the FACE model (i.e., a sequence of generative acts), or an [administrative act] which doesn't introduce any new concept, example, aesthetic or framing information/method. Administrative acts were not originally described in the FACE model, but we needed them to describe certain progressions during software development. For our purposes here, we use only $T$ to describe a translation administrative act often involving programming, and $S$ to describe when an aesthetic measure is used to select the best from a set of artefacts. We employ the FACE model usage of lower-case letters to denote the output from the corresponding upper-case generative acts. Furthermore, we extend the FACE notion of (g)round and (p)rocess level generative acts with (m)eta level acts during which process generation methods are invented. As in the original description of the FACE model, we use a bar notation to indicate that a particular act was undertaken by the programmer. We use a superscripted asterisk (*) to point out repetition.

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As a simple example diagram, Fig. 1.3b shows the progression from poetry generator version P1 to P2. In the first version, there are two process stacks, hence the system works in two stages. In the first, the software produces some example poems, and in the second the user chooses one of the poems (to print out, say). The first stack represents two timesteps in development, namely that (a) the programmer had a creative act $\langle \overline{C_g} \rangle$ whereby he/she came up with a concept in the form of some code to generate poems, and (b) the software was run to produce poems in creative acts of the form $\langle E_g \rangle^*$. The second stack represents the user coming up with an idea for an aesthetic, e.g., preferring lots of rhyming, in creative act $\langle \overline{A_g} \rangle$, and then applying that aesthetic $\overline{a_g}$ him/herself to the examples produced by the software, in the *selection* administrative act $[\overline{S}(\overline{a_g}(e_g))]$, which maps the aesthetic $\overline{a_g} : \{e_g\} \rightarrow [0, 1]$ over the generated examples, and picks the best one. In the $P2$ version of the software, the programmer undertakes the *translation* act $[\overline{T}(\overline{a_g})]$, writing code that allows the program to apply the rhyming aesthetic itself, which it does at the bottom of the second stack in box $P2$.

Figure 1.3c shows a progression in the HR automated theory formation system [48] which took the software to a meta-level, as described in [49]. HR operates by applying production rules which invent concepts that categorise and describe input data. Each production rule was invented by the programmer during creative acts of the type $\langle \overline{C_p} \rangle$, then at run-time, HR uses the production rules to invent concepts and examples of them in $\langle C_g, E_g \rangle^*$ acts. In the meta-HR version, during the $\langle \overline{C_m} \rangle$ creative act, the programmer had the idea of getting HR to form theories about theories, and in doing so, generate concept-invention processes (production rules) in acts of the form $\langle C_p \rangle$. The programmer took meta-HR's output and translated it via $[\overline{T}(C_p)]$ into an implemented production rule that HR could use, which it does at the bottom of the stack in box H2.

### 1.5.3 Comparing Diagrams and Output

Examining the transition from one epoch-level diagram to another should provide some shortcuts to estimate audience reactions, especially when these are linked to strong objectives. As with the original FACE model, the diagrams make it obvious where creative or administrative responsibility has been handed over to software, namely where an act which used to be barred has become unbarred, i.e., the same type of generative act still occurs, but it is now performed by software rather than programmer. For instance, this happened when the $\overline{S}$ became an $S$ in Fig. 1.3b and when the $\overline{C_p}$ became a $C_p$ in Fig. 1.3c. At the very least in these cases, an unbiased observer would be expected to project more autonomy onto the software, and so progress in the strong sense has likely happened. In addition, the diagrams make it obvious when software is doing more processing in the sense of having more stacks, bigger stacks or larger tuples of acts in the stack entries. Moreover, the diagrams make it clear when more varied or higher-level creative acts are being performed by

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the software. All of these features have the potential to convince audience members that software is being more sophisticated, and can be taken as a preliminary indicator of progress.

When dealing with actual external evaluation, where people don't know what the software does, we suggest that the diagrams above (or verbalisations/simplifications of them) can be used to describe the software to audience members, to explain what the software does, and what the programmer has done in the project. Audience members can then be asked whether they would project any of the essential behaviours from Sect. 1.4.1 onto any of the creative acts undertaken by the programmer or by the system. Thus, one method for estimating progress from version $v1$ of a creative system to version $v2$ that takes into account features of both processing features and artefact quality would be:

- show audience members the diagrams for $v1$ and $v2$ as above, and explain the acts undertaken by the software, then
- show audience members the output from $v1$ and $v2$, and,
- ask each person to compare the pair of product and process for $v1$ with that of $v2$.

A statistical analysis could then be used to see whether the audience as a whole evaluates the output as being better, worse or the same, and whether they think that the processing is better, worse or the same in terms of the software seeming less uncreative.

### *1.5.4 A Case Study in Evolutionary Art*

Evolutionary art—where software is evolved which can generate abstract art—has been much studied within Computational Creativity circles [50]. Based on actual projects which we reference, we hypothesise here the various timelines of progress that could lead from a system with barely any autonomy to one with nearly full autonomy. Figure 1.4 uses our diagrammatic approach to capture three major lines of development, with the final (hypothetical) system in box 8 representing finality, in the strong sense that the software can do very little more creatively in generating abstract art. Since features from earlier system epochs are often present in later ones, we have colour-coded individual creative acts as they are introduced, so the reader can follow their usage through the systems. If an element repeats with a slight variation (such as the removal of a bar), this is highlighted. Table 1.1 is a key to the figure, which describes the most important creative and administrative acts in the systems. Elements in the key are indexed with a dot notation: *system.process-stack.subprocess* (by number, from left to right, and top to bottom, respectively). System diagrams have repetitive elements, so that the timelines leading to its construction and what it does at run-time can be read in a stand-alone fashion.

Following the first line of development, system 1 of Fig. 1.4 represents an entry point for many evolutionary art systems: the programmer invents ($\overline{C_p}$) (or borrows) the concept formation process of crossing over sets of mathematical functions to

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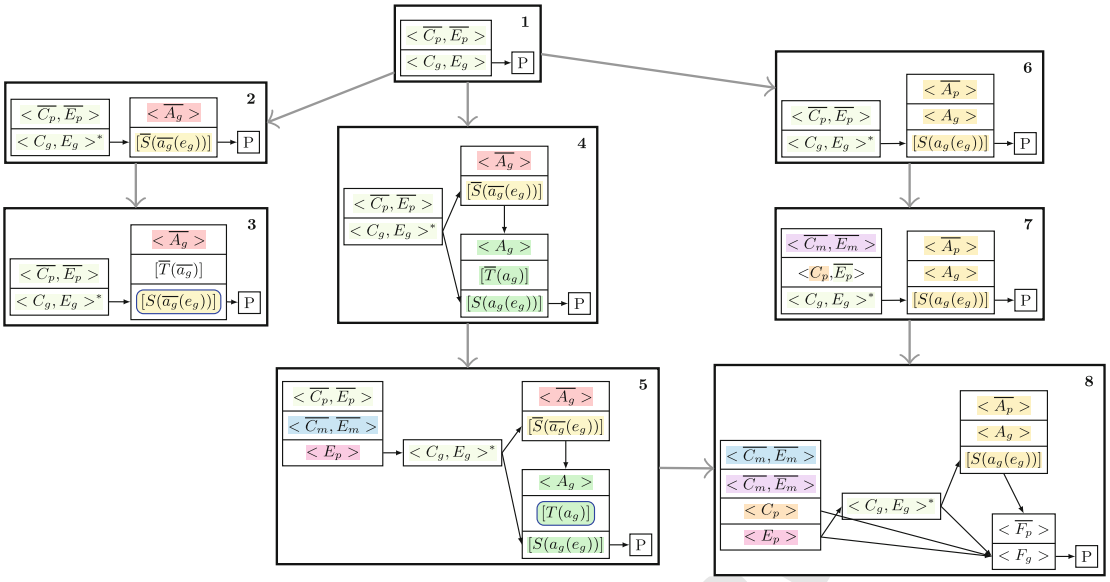


**Fig. 1.4** The progression of an evolutionary art program through eight system epochs, taken from [47]

produce offspring sets. He/she also has an idea ($\overline{E_p}$) for a *wrapper* routine which can use such a set of functions to produce images. He/she then uses the program to generate ($C_g$) a set of functions and employ the wrapper to produce ($E_g$) an image which is sent to the (P)rinter. The crossover and subsequent image generation is repeated multiple times in system 2, and then the programmer—who has invented ($\overline{A_g}$) their own aesthetic—chooses a single image to print. In system 3, as in the poetry example above, the programmer translates their aesthetic into code so the program can select images. This is a development similar to that for the NEvAr system [51].

Following the second line of development, in system 4, the programmer selects multiple images using his/her own aesthetic preferences, and these become the positives for a machine learning exercise as in [52]. This enables the automatic invention ($A_g$) of an aesthetic function, which the programmer translates by hand $\overline{T}(a_g)$ from the machine learning system into the software, as in [53], so the program can employ the aesthetic without user intervention. In system 5, more automation is added, with the programmer implementing their idea ($\overline{C_m}$) of getting the software to search for wrappers, then implementing this ($\overline{E_m}$), so that the software can invent ($E_p$) new example generation processes for the system.

Following the final line of development, in system 6, we return to aesthetic generation. Here the programmer has the idea ($\overline{A_p}$) of getting software to mathematically invent fitness functions, as we did in [54] for scene generation, using the HR system [48] together with The Painting Fool [43]. In system 7, the programmer realises ($\overline{C_m}$) that crossover is just one way to combine sets of functions, and gives ($\overline{E_m}$) the software the ability to search a space of combination methods ($C_p$). The software does this, and uses the existing wrapper to turn the functions into images. System 8 is the end of the line for the development of the software,

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**Table 1.1** Key to Fig. 1.4

| ID | Event | Explanation |
|---|---|---|
| 1.1.1 | $\overline{C_p}$ | The programmer invents the idea of crossing over two sets of mathematical functions to produce a new set of mathematical functions |
| 1.1.1 | $\overline{E_p}$ | The programmer implements a wrapper method that takes a set of mathematical functions and applies them to each $(x, y)$ co-ordinate in an image to produce an RGB colour |
| 1.1.2 | $C_g$ | The software generates a new set of functions by crossing over two pairs of functions |
| 1.1.2 | $E_g$ | The software applies these functions to the $(x, y)$ co-ordinates of an image, to produce a piece of abstract art |
| 2.2.1 | $\overline{A_g}$ | The programmer had in mind a particular aesthetic (symmetry) for the images |
| 2.2.2 | $\overline{S}(\overline{a_g}(e_g))$ | The programmer uses his/her aesthetic to select an image for printing |
| 3.2.2 | $\overline{T}(\overline{a_g})$ | The programmer took their aesthetic and turned it into code that can calculate a value for images |
| 3.2.3 | $S(\overline{a_g}(e_g))$ | The software applies the aesthetic to select one of a set of images produced by the software |
| 4.3.1 | $A_g$ | The software uses machine learning techniques to approximate the programmer's aesthetic |
| 4.3.2 | $\overline{T}(a_g)$ | The programmer hand-translates the learned aesthetic into code |
| 4.3.3 | $S(a_g(e_g))$ | The software applies the new aesthetic to choosing the best image from those produced |
| 5.1.2 | $\overline{C_m}$ | The programmer has the idea of getting the software to search through a space of wrapper routines |
| 5.1.2 | $\overline{E_m}$ | The programmer implements this idea |
| 5.1.3 | $E_p$ | The software invents a new wrapper |
| 5.4.2 | $T(a_g)$ | The software translates the machine-learned aesthetic itself into code |
| 6.2.1 | $\overline{A_p}$ | The programmer has the idea of getting the software to invent a mathematical fitness function |
| 6.2.2 | $A_g$ | The software invents a novel aesthetic function |
| 6.2.3 | $S(a_g(e_g))$ | The software selects the best image according this aesthetic |
| 7.1.1 | $\overline{C_m}$ | The programmer has the idea of getting the software to invent and utilise new function combination techniques, generalising crossover |
| 7.1.1 | $\overline{E_m}$ | The programmer implements this idea so that the software can invent new combination techniques |
| 7.1.2 | $C_p$ | The software invents a novel combination technique |
| 8.4.1 | $\overline{F_p}$ | The programmer has the idea of getting the software to produce a commentary on its processes and the images it produces |
| 8.4.2 | $F_g$ | The software produces a commentary about its process and product |

as it brings together all the innovations of previous systems. The software invents aesthetic functions, innovates with new concept formation methods that combine mathematical functions, and generates new wrappers which turn the functions into

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images. Finally, the programmer has the idea ($\overline{F_p}$) of getting the software to write commentaries, as in [41], about its processing and its results, which it does in generative act $F_g$.

Tracking how the system diagrams change can be used to estimate how audiences might evaluate the change in processing of the software, in terms of the extended creativity tripod described above. Intuitively, each system represents progress from the one preceding it, justified as follows:

**1** $\rightarrow$ **2**: $\langle C_g, E_g \rangle \rightarrow \langle C_g, E_g \rangle^*$

Simple repetition means that the software has more *skill*, and the introduction of independent user selection shouldn't change perceptions about *autonomy*.

**2** $\rightarrow$ **3**: $\overline{S} \rightarrow S$

By reducing user intervention in choosing images, the software should appear to have more *skill* and *autonomy*.

**1** $\rightarrow$ **4**: Introduction of $A_g$ and $S(a_g(e_g))$ acts

Machine learning enables the generation of novel aesthetics (albeit derived from human choices), which should increase perception of *innovation*, *appreciation* and *learning*, involving more varied creative acts.

**4** $\rightarrow$ **5**: Introduction of an $E_p$ act, $\overline{T} \rightarrow T$

Wrapper generation increases the variety of creative acts, and may increase perception of *skill* and *imagination*.

**1** $\rightarrow$ **6**: Introduction of $A_g$ and $S(a_g(e_g))$ acts

The software has more variety of creative acts, and the invention and deployment of its own aesthetic—this time, without any programmer intervention—should increase perception of *intentionality* in the software.

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**6** $\rightarrow$ **7**: Introduction of a $C_p$ act

Changes in the evolutionary processes should increase perceptions of *innovation* and *autonomy*.

**5, 7** $\rightarrow$ **8**: Introduction of an $F_g$ act

Framing its work should increase perceptions of *accountability* and *reflection*.

With all strands brought together, the programmer does nothing at run-time and can contribute little more at design time. The software exhibits behaviours onto which we can meaningfully project words like skill, appreciation, innovation, intentionality, reflection, accountability and learning, which should raise impressions of autonomy, and make it difficult to project uncreativity onto the software.

**Hypothesis 6** The diagrammatic formalism given above—or some extension of it—is sufficient to capture the creative acts performed in building and running any kind of generative software. Moreover, when this is used alongside audience evaluation of the artefacts produced, a formal assessment of progress in creative software development can be achieved.

## 1.6 Software as Part of a Creative Community

For each domain in which creative software operates, there is a community of people who have a stake in the notion of whether software working in that domain is perceived as creative. As described in this section, we have recently started to embed our software in such a community, for various reasons, including the study of how people react to it and to the work that it produces. These experiments will form part of a larger study of how people accept (or not) creative technologies that undertake activities which used to be the purview of people only.

### *1.6.1 Accountable Subjectivity*

Applying aesthetic judgements and expressing preferences are important kinds of activity that contribute to the perception of a person or piece of software as being creative. Aesthetics and preferences allow a creative entity, be it a person or software, to express founded judgement (even if we regard the judgement as worthless, or subjectively 'wrong') on creative artefacts, both those created by the entity itself

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and those created by others. It can also serve as a driving force behind future creation, allowing someone to work towards goals that they have set themselves and strengthening claims of intentionality.

Despite this, little work has been done to build systems which can generate aesthetic preferences of their own and apply them intelligently. One reason for this may be the uncomfortable clash between the subjective and the objective that so often affects research in Computational Creativity. The notion of 'optimality' in many creative domains, particularly those associated with the arts, is a contentious one and leads to much criticism of systems which attempt to quantify the quality of an artefact. The idea of having a system quantify the quality of an *opinion* on creative artefacts is equally controversial, if not more so. Similarly, in the past, the question of how to quantify the degree to which a system is creative was also a subjective and controversial task. In this case, researchers such as Ritchie found it useful to use metrics which dealt with abstract notions of creativity without directly laying out objective measures of quality for any particular artefact or medium. Ritchie's criteria are described in [55], and have been used in many evaluations of creative systems in a variety of different fields and media.

We propose here a similar set of criteria which apply to aesthetics or preferences rather than creative systems. By using abstract metrics, we can avoid talking about aesthetic measures in objective ways, while retaining a meaningful vocabulary with which to describe different kinds of aesthetics. These metrics can be used to evaluate *aesthetic comparator functions*, namely binary functions which take two examples of a type of object, and then return $-1$, 0 or 1 depending on whether the first object is preferred less, the same as, or more than the second object. Assuming we have an aesthetic function $f$, and a set of objects the function expresses a preference over, $O$, we define the following criteria which can be used to differentiate aesthetic functions from one another. Note that these metrics do not necessarily represent a linear gradient of quality—different types of aesthetic function may be desirable in different scenarios.

The first metric is *specificity*. Specificity captures the degree to which the aesthetic represents a *total order* over the set of objects $O$. If an aesthetic can offer a definite preference (that is, a nonzero result) for many of the objects, it will have a high specificity, and vice versa. High-specificity aesthetics might suggest the aesthetic is experienced or well-developed in some way, if it is able to make clear distinctions between many different artefacts.

The second metric is *transitive consistency*. This captures how self contradictory the aesthetic function is. Suppose we have three artefacts: $A$, $B$ and $C$, and our function $f$. We can write $A < B$ to indicate that $B$ is preferred to $A$. We might expect that if $A < B$ and $B < C$ then $A < C$. Transitive consistency measures what proportion of $O$ this holds for. In some scenarios, we might want a high transitive consistency, as this indicates a lack of contradictions in the preferences being expressed. However, in some scenarios, preferences can be complex and multi-objective, and it might be the case that transitivity does not hold for highly subjective opinions about artefacts produced by creative acts.

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The third metric is *agreement*. Instead of being expressed in terms of a single aesthetic function, agreement is expressed about two different aesthetics, which we can call $f$ and $g$. Agreement measures the proportion of the object set $O$ that $f$ and $g$ agree on. This can be *strict*, in which case $f$ and $g$ must return exactly the same value for two objects to be said to agree. Alternatively, agreement can be *non-strict*, in which case $f$ and $g$ can either return the same value, or one of the functions can return zero (no preference) to be said to agree. Informally, agreement lets us assess how closely two aesthetic functions are aligned with each other. Of course, they may be in close agreement for very different reasons—this metric simply establishes similarity in the result of the subjective judgements.

**Hypothesis 7** The perception of creativity in software which produces artefacts within a creative community will be increased if the software can exhibit subjective judgements about its own work and that of others, and defend those judgements in an accountable way. This can be seen as part of a bigger picture of software exhibiting a personality, in order to be accepted into a creative community.

### *1.6.2 A Case Study in Automatically Designed Videogames*

A *game jam* is a contest where entrants attempt to make a videogame from scratch in a short period of time, normally with the added restriction of a theme which developers must incorporate into their game somehow. Ludum Dare is one of the largest regularly occurring game jams in the game development community, taking place three times a year and garnering over 2,000 entries in December 2013, where developers were given the theme 'You Only Get One'. The ANGELINA system is an automated videogame designer developed to investigate issues surrounding Computational Creativity in a ludic and interactive context [56]. Many different versions of ANGELINA have been developed, working with various different kinds of game, technologies and user guidance. The most recent iteration, ANGELINA-5, was designed to enter game jams, by allowing it to be given just a theme in plain text as a starting point. This theme is then interpreted by ANGELINA-5 and used to influence the design of the game.

ANGELINA-5 entered Ludum Dare for the first time in December 2013, the 28th edition of the event. One of the objectives was to investigate the reactions of various groups of people to a piece of creative software entering such a contest. To gain more insight into these groups, we entered two games designed by ANGELINA-5 to Ludum Dare 28. In the first submission, *To That Sect*,[3] we included a commentary generated by ANGELINA-5 to illustrate the actions of the system, as well as multiple paragraphs describing the research behind ANGELINA-5 and identifying the game as the creation of a piece of software. In the second submission,[4] we anonymised

[3] *To That Sect* game: www.tinyurl.com/tothatsect.

[4] *Stretch Bouquet Point* game: www.tinyurl.com/stretchpoint.

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**Table 1.2** Percentile rankings for ANGELINA-5's two games entered into Ludum Dare 28, and its single entry to Ludum Dare 29 *(Jet Force Gemini)*

| | To That Sect | Stretch Bouquet Point | Jet Force Gemini |
|---|---|---|---|
| Overall | 36 | 29 | 23 |
| Fun | 34 | 30 | 26 |
| Audio | 73 | 43 | 74 |
| Graphics | 43 | 33 | 36 |
| Mood | 77 | 39 | 80 |
| Innovation | 64 | 33 | 59 |
| Theme | 32 | 30 | 26 |
| Humour | 48 | 59 | 51 |

Note that higher percentile rankings indicate higher achievements. There were 780 submissions in the LD28 track, and 1,004 entries in the LD29 track

ANGELINA-5's commentary to remove references to it being software-based, edited it for grammar, and added no supplementary explanation about the software, the origin of the game, or anything to connect the game with a digital author. The ratings process for Ludum Dare takes place in the 22 days following the contest, and is conducted as a peer review system, where each entrant is asked to rate and review games by other entrants. Ratings are given as marks out of five for eight categories: Audio, Graphics, Mood, Theme, Humour, Fun, Innovation and Overall. AQ2

The results for the two entries by ANGELINA-5 can be seen in Table 1.2. While we were unable to get specific vote data, we do know that 70 people rated *To That Sect*, the non-anonymised submission, while 26 people rated *Stretch Bouquet Point*.[5] While it is impossible to calculate confidence intervals for these ratings without the vote data, we can see that they differ by hundreds of positions for some categories such as Mood and Audio. We can also see a noticeable difference in the comments left by some of the reviewers underneath both submissions, in terms of their tone and attitude when dealing with each game. Many commentators indirectly criticise the anonymised game, with comments such as "You made me feel something there. Don't make me put it into words though". Other commentators made more obvious statements of criticism or praise, such as "This was a rather annoying experience" or "This game feels dreamy. The audio is intense." Only one comment included both praise and criticism. We attribute the indirect or sarcastic comments to an unwillingness to potentially criticise a person for performing poorly, even though other reviewers were less tactful. Ludum Dare is often used as a learning experience for amateur developers, and many children enter using simple game creation tools. We believe many reviewers felt uncomfortable with direct criticism for this reason.

By contrast, comments on *To That Sect* were more balanced in nature, often offering both praise and criticism in equal amounts, e.g., "Angelina seems really good at creating an atmosphere with both sound and visuals. But the game part of it seems

[5] This is due in part to ANGELINA-5's small following on the internet, which promoted the non-anonymised submission more than normal.

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a bit lacking still". In the description of the game, we asked people to rate it as they would any other Ludum Dare entry, hoping to dissuade people from reviewing the *concept* of ANGELINA-5 rather than the game itself. Nevertheless, many reviewers suggest that their scores were influenced by their appraisal of ANGELINA-5 as a novel system, rather than what it was capable of creating, e.g., "creating a program to create your game . . . [is] certainly not something you see every day. On that front alone, this gets a lot of points for innovation". These results suggest that reviewers were unable to separate the creator from the artefact, and were incapable of reviewing the game as if created by a person. For instance, *To That Sect* rated 282nd of 780 for Innovation. These ratings are subjective, and it is hard for us to objectively assess them. However, we do not believe there is anything particularly innovative about *To That Sect*. As such, we must attribute this high ranking to reviewers assessing the game as a product of ANGELINA-5. It seems that reviewers projected (human) innovation in the ANGELINA project onto the game it produced.

We can compare the results of ANGELINA-5's debut in Ludum Dare with the results garnered from a second entry to the game jam in April 2014, Ludum Dare 29. This time ANGELINA-5 was only entered into the game jam once, with the game *Jet Force Gemini*, created in response to the theme *Beneath The Surface*. As with the non-anonymised entry in Ludum Dare 28, *Jet Force Gemini* was entered with a commentary describing some of the decisions contributing to the design process. Table 1.2's rightmost column shows the results for *Jet Force Gemini* in contrast to the entries in Ludum Dare 28. The number of entries in Ludum Dare 29 was nearly 30 % larger than Ludum Dare 28; ANGELINA's percentile scores drop for four of seven specialised categories, and fall dramatically in the *Overall* rating.

We believe this is evidence of the relationship between the observers and ANGELINA shifting over time. While some of the comments underneath *Jet Force Gemini* indicate that the reviewer is encountering ANGELINA-5 for the first time (which is unsurprising, since the number of reviewers account for less than 1 % of total Ludum Dare entrants) others explicitly note that they are reviewing ANGELINA-5's games for a second time. One states that 'I'm sorry to say that I can't really see improvements from last time', indicating that there is either an expectation of growth on the part of the software, or an expectation that the software's author will grow the software over time. Despite many of the other comments being generally positive, the drop in ratings suggests that people perhaps feel less compelled to rate ANGELINA-5 highly for novelty value alone. Given that Ludum Dare is a community built on the idea of improving creative skills through regular practice, it is interesting to note the expectation of growth shown by some reviewers. We hypothesise that this may be a factor which is particularly important for creative individuals in assessing creativity, as opposed to other types of observer.

We can also examine reactions to particular elements of ANGELINA-5's work and compare it to critiques of similar games. One comment on *To That Sect* states "If it [had] added shooting at the statues that you must avoid and a goal how much ships you have to collect, it would have been better. It felt like playing [an] 'art-message'

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type of game". *LITH*[6] is a game entered into the competition by a human designer, where the player navigates a maze and collects bags of gold coins, while avoiding patrolling robots. They can escape to an exit at any stage, with their score being the amount of gold collected. While not an exact duplicate, the rules of *LITH* are very similar to those of *To That Sect*, i.e., search for as many objects of a certain type as possible, while avoiding another object, then exit. *LITH* was entered in the same track as ANGELINA-5's games, and ranked 95th Overall, 125th for Fun, and 274th for Theme. None of the comments on *LITH* reference the game's rulesets in a critical way. Notably, *LITH* ranks 259 places above *To That Sect* for Theme. This is significant, as the *LITH* designer justifies its theme in a fairly thin way, by saying simply that the player only has one opportunity to save their score (which they do by ending the game, as in *To That Sect*). The games are by no means identical: *LITH*'s level is more closed in to accentuate a feeling of claustrophobia, but the similarities are many. This analysis suggests a fundamental difference in how people evaluate a game when they have knowledge and when they have no knowledge of its designer and design process.

**Hypothesis 8** There can be both positive and negative biases at work when people consume artefacts in the knowledge that computers created them. By managing both cases in a creative community context, we can increase perception of software as being creative and enjoyment of the artefacts produced. This increase will be further fuelled if the software shows clear growth in sophistication in the field, and expresses this through its processes and products.

## 1.7 Conclusions and Future Work

Simply stated, one of the main aims of research into Computational Creativity is to one day see creative software properly embedded into society. To achieve this aim, larger sectors of society need to join the effort, including creative communities within the arts and sciences, the creative industries, technology firms, and the next generation of Artificial Intelligence researchers. Hence, we need to convince certain sets of stakeholders that creative software is no fantasy, but a potential reality that will bring benefits to society. As described above, we have studied three sets of stakeholders, namely the general public, fellow Computational Creativity researchers, and a specific community of creative people, namely videogame designers. These studies have enabled us to make concrete hypotheses related to how stakeholder communities perceive creativity in software, and how best to manage that perception in the future. Based on our immersion in the stakeholder communities mentioned, we have argued above in favour of the truth of the hypotheses, with extended discourse and argumentation given in [38, 47] amongst other papers. We believe it is now time to turn the hypotheses into experiments designed to see whether the ways in which sets of stakeholders perceive and react to creative software fit our beliefs.

[6] *LITH* game: www.tinyurl.com/lith-ludum.

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Our first hypothesis is pitched somewhat at a meta-level, in that it proposes that different stakeholder groups see creative systems differently and their perception of software behaviour could and should be managed in a bespoke manner. We can therefore imagine an experiment where we present the processes and products from creative software to different stakeholder groups and assess their reaction to see if there is indeed a difference in how different groups react, learning from analyses of the results. Hypothesis 2 encompasses much of our philosophical position on the notion of creativity being essentially contested and secondary in nature. One can imagine restricting participants in an experiment to fairly constrained groups, and testing whether there is general (healthy) disagreement about the nature of creativity in people and software or not, and further testing whether there is more consensus about software being uncreative. To properly test Hypothesis 2, we would need to ask participants about the essential behaviours—such as intentionality, learning and reflection—they perceive to be taking place in software and see how it affects their perception of uncreativity in the system.

Our third hypothesis makes a bold statement: that blind comparison tests damage the long-term goal of embedding creative software in society, by emphasising the evident humanity gap. If this effect is true, it would be borne out by a Turing-style test where, when people are told that it was software that produced an artefact that they particularly liked, they were also asked about whether their perception of the creative act and/or the artefact had changed in light of the new knowledge. More pointed questions about the nature of any change in perception could lead to insights about how to manage the humanity gap in future projects. This would lead into an experiment to address Hypothesis 4, where computer generated artefacts were presented as re-imagined pieces with specific management of the relative lack of humanity in the generation of the artefacts. The re-imagining would specifically include commentaries and other framing information produced by the creative system. If Hypothesis 4 is correct, people would appreciate the re-imagined versions of artefacts more than those presented merely as computer-generated versions from the human oeuvre.

By proposing that random number generation detracts from an experience of a creative act, whereas more accountable unpredictability can benefit the experience, Hypothesis 5 is more specific than those preceding it. We can imagine an experiment where one set of participants are told that a particularly impressive creative act (in terms of the processing performed and/or the resulting artefacts) was because of a random event, and another set are given interesting framing information about what led—in a non-random way—to the same unpredictably good creative act. If the latter group appreciated the creative act and its results more than the former group, the truth of the hypothesis would be upheld.

We have already started work on testing Hypothesis 6, i.e., that the formalism presented in [47], can capture notions of progress when building creative systems. That is, we have used the formalism to capture abstracted timelines leading to the building of certain creative systems, and timelines where that software operates and produces artefacts of value. However, to convince the Computational Creativity researcher stakeholders of the value of the formalism, we need to work with them to capture the essence of their approaches to implementing and operating creative

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software. Moreover, our audience evaluation model is far from complete. We plan to employ the criteria specified in [55], for more fine-grained evaluations of the quality, novelty and typicality of artefacts. We will also import audience reflection evaluation schemes from the IDEA descriptive model, e.g., change in well-being, cognitive effort and emotional responses such as surprise and amusement.

The final two hypotheses we present above relate to communities of creative people into which creative software is implanted. To address Hypothesis 7, we will need to implement software behaviours which can meaningfully be described as subjective, and we plan to do so with the ANGELINA videogame generation system, and others such as The Painting Fool automated artist. With such systems, we can experiment to see whether members of the creative community are more impressed by subjective software or not. Such an experiment could be simultaneously used to address the final hypothesis, with knowledge of the computational origins of artefacts systematically withheld in order to see whether positive or negative biases hold in different creative communities. Similarly, experiments where participants are told about the intellectual growth of a system could be carried out, to see if this influences their impression of the software. An analysis of the findings from such experiments could help pave the way for software to be full members of these kinds of communities.

Looking at the three stakeholder groups studied here, we see some emerging generalities. In particular, looking at behaviours where systems exhibit subjectivity and intentionality, it seems clear that in all three groups, *personality modelling* in software has the potential to increase the impression that people have of what software does and, in turn, what it produces. This is part of a new understanding of creative acts as being potentially interesting, even dramatic, episodes of activity which can amuse and engage people, rather than a means to the end of producing an artefact of value. This is in contrast with the traditional idea that the value of the output from software can increase people's appreciation of the creativity it exhibits. While the traditional view is often correct, it is not the only model of managing perceptions of creativity in software.

The hypotheses presented here are only a subset of those which should be proposed and addressed in the future of Computational Creativity research. Not addressing such issues would be a mistake, as stakeholder perception of creativity in software will in part dictate the number of researchers and businesses coming into the field. Done badly, handling of stakeholder perceptions could stall the forward progress achieved towards embedding creative software in society. As a recent controversial example, online retailer Amazon briefly sold T-shirts with slogans such as *"Keep Calm and Rape a Lot"* [57]. The T-shirt company responsible posted an apology on its website, and insisted that the offending articles were "automatically generated using a scripted computer process running against hundreds of thousands of dictionary words". This may be the first example of computer generated artefacts causing such offence and a company—while taking responsibility—blaming generative software for poor quality artefacts, while tacitly acknowledging that the software had taken on unsupervised creative responsibilities in their workplace.

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Situations where software is employed independently for creative purposes in commerce and elsewhere are likely to become more commonplace in the future. As a more positive example, IBM researchers have recently undertaken research to explore the commercial potential of Computational Creativity [58], with particular emphasis on culinary creativity [59, 60]. Creative software will make great inventions and make terrible mistakes in the future, and this will lead to a re-evaluation of humanity as being the centre of the creativity universe. Managing stakeholder perceptions of creativity in software will be paramount in making this transition as smooth and as fruitful for society as possible.

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# Building the Peeragogy Accelerator

Joseph Corneli, Dorota Marciniak, Charles Jeffrey Danoff, Charlotte Pierce, Paola Ricaurte, Jan Herder, Suz Burroughs, George Brett, John Graves

**Abstract**

The Peeragogy Project facilitates the reflective practice of peer learning and peer production. We are building living projects that continuously evolve, and we have come together with several of our own projects to build a space for mutual aid. In short, we are building a peer produced Peeragogy Accelerator. We present the background and operating principles of the Accelerator, and describe some of the Accelerated projects: SOLE out of the Box, which aims to build a self-sufficient distributed makerspace; PlanetMath.org, an online mathematics community that will soon run its first peeragogical Calculus course; and *Managing Innovation and Change*, a semester-long academic course on building startups that is using peeragogy concepts. The Peeragogy Accelerator works "transversally" across these and other projects, supporting both peer-to-peer and Project-to-Project collaboration. We are actively seeking to involve others from the OER world.

**Keywords**: peer learning, peer production, peeragogy

## Introduction

The Peeragogy Project combines reflection and practice in support of peer learning and peer production. An emphasis on both *doing* and *understanding how* gives us a distinct perspective. The project synthesizes communication and contributions from globally distributed members. We are teachers, students, and researchers; editors, publishers, and writers; activists, makers, and (co)designers; but inside the project, there are no "teachers" or "students." Each contributor brings their own vision and values, and we all learn. When we come temporally together, the "Aha!" moments come thick and fast. In the process, differing viewpoints emerge, but we find agreement to be less important than co-creation of a space for learning and productive work. We emphasize the central importance of an environment of mutual respect and enjoyment. This approach has profound implications for the way we do things, and what we choose to work on.

We have always preferred to *drink our own champagne* – but new wine needs new bottles. Having co-produced the *Peeragogy Handbook* (Rheingold *et al.*, 2014) and learned a lot in the process, we are now building a peer produced Peeragogy Accelerator – a living resource for those seeking assistance with applying the reflective practice of peer learning and peer production in their own projects. A brief "Who, What, When, Where, Why, and How" exercise from the *Peeragogy Handbook* plays the role of *S. cerevisiae* for our new brew.

1. **Who: Fluxuating roles.** Some of the people involved in the Accelerator were involved as co-authors of the *Handbook*. Others joined the Accelerator driven by interests in topics such as open source, open science, open education, and peer-to-peer, informal education. We have different visions: it's *not about consensus.*

2. **What: Nature of the project**. The Peeragogy Project has built peeragogy as "something for us" rather than "something for them." The Accelerator will help make the *Handbook* useful for new projects: we improve our methods as we go.

3. **When: Time management.** We hold regular weekly meetings on Mondays at 6PM UTC, with break-out meetings and asynchronous collaboration as needed. We publish the *Handbook* annually on January 1st (Public Domain Day). The Accelerator will formally induct the first "class" in June.

4. **Where: Journey vs destination.** We value the process more than the product. The Accelerator aims to be a meta project, *a place for people to bring their own projects* and receive support from each other.

5. **Why: Putting peeragogy into practice.** "Success" in our own projects will provide case study material to *improve* the Handbook.

6. **How: Linearity vs Messiness.** Our "How to Get Involved" page contains a list of tools and activities. Keeping in mind the importance of polycentric leadership in the Peeragogy Project, people should feel free to innovate. Our general operating principles are below.

We are different from other incubator and accelerator projects in that we are entirely peer produced. Instead of signing away a percentage of future profits or some of us paying tuition to others for mentoring services, contributors "p(l)ay" together, with volunteer effort in the present.

## Operating principles for the Peeragogy Accelerator

### A living project

1. We build *living projects* that can continuously evolve.

2. We measure our progress by things *actually done*, and the process by which they are done.

3. Although we can, and do, adopt concrete goals (for example, to write a book or collaborate on this paper), we do not expect our work to be *final*. Every contribution leads to improvement.

4. In the Accelerator, people can *branch, fork and merge* other projects, as they might with source code in a Git repository. Each version or thread has its own life.

**Peeragogy Labs: try new things, and discover what works!**

5. Everyone chooses their own meaningful contributions – and each may (optionally) have their own business models in mind.

6. We use non-restrictive licenses, and each member can re-package Peeragogy in their own way if they want to. For example, a fee-based peeragogy workshop and pay-to-download Peeragogy notebooks are entirely compatible with what we're doing with the all-volunteer Accelerator and the co-created CC-Zero *Peeragogy Handbook*.

7. Although we released our *Handbook* into the Public Domain, we do like to know who made things with it, how they made it, and why. We manage this information in an extra-legal way. Indeed, we cannot enforce informed re-use, since there is no uniform way to express this sort of information. We strive for good ways to express and implement sharing at this level, building towards a *creativity commons,* above and beyond the legal instruments of the Creative Commons.

**Measure of success → values**

8. Instead of measuring "success" via downloads or sales, we ask:

   *What have we actually done? How was it done? What values did we evolve in ourselves? What did we learn to support? What values did we share with the world? How did we experientially support something for someone? What did we change in the world in the process? How did we improve our own well-being? How did we evolve as a group?*

9. Accordingly, in general we measure in a qualitative way, rather than a quantitative way. We can certainly do quantitative studies in order to help understand how things work, but this understanding does not come from measuring isolated and arbitrary scalar values like "participation".

**Creating contexts**

10. Ask "What's the next step?" rather than "What's the ultimate goal?" If we are going in a given direction, it does not mean we need to go forever in that direction. An important part of peeragogy is making it clear how people can become meaningfully involved with what you are doing.

11. For example, when the Peeragogy Project was just about writing the *Handbook*, that was the one thing people could get involved with. In the Accelerator, you do not need to be a co-author to become involved – you need only share an idea for a peer learning or peer production project. And we can always strive to go further in this way.

# Background

Peer production in education is not new. Booker T. Washington's students and faculty learned agriculture, architecture and construction as they built the Tuskegee Institute, producing an educational institution that thrives today, more than a century after Washington (1901) wrote these words:

> *I was determined to have the students do not only the agricultural and domestic work, but to have them erect their own buildings.* […] *Not a few times, when a new student has been led into the temptation of marring the looks of some building by lead pencil marks or by the cuts of a jack-knife, I have heard an old student remind him: "Don't do that. That is our building. I helped put it up."*

Information technology amplifies opportunities for peer-to-peer collaboration. Following Benkler's (2002) seminal work, Shaw & Hill (2014) describe an important new way of working that has developed considerably over the last decade:

> *Commons-based peer production – the distributed creation of freely accessible information resources through the mass aggregation of many small contributions – represents a model of collective action and public goods production that integrates the use of digital communication networks and information technologies.*

These authors make it clear that peer production is not a panacea, but it is also not the only relevant recent innovation. Design patterns, invented by Christopher Alexander in the field of architecture in the 1970s (Alexander, Ishikawa, and Silverstein, 1977), and spreading from there to to software (Gamma *et al.*, 1994; Gabriel, 1996), and later, to other fields, provide a way to communicate and share effective practice. "Pedagogical Patterns" is a network that came together to develop and provide easily transferable strategies showing how best to bestow knowledge. The *Peeragogy Handbook* includes our own catalog of design patterns that explore how peers can produce knowledge together, while building a shared "how-to".

When we think of *open practice*, following Asger Jorn (1994 [1960]), we think of an exploration of the "polyvalence of the unique". Some of the groundwork for peeragogy was developed when two of us did this at P2PU, signing up for each other's courses and critically examining the institution as a whole as the term progressed (Corneli & Danoff, 2011).

Peer learning is not something to "provide" (Boud & Lee, 2005), but something to do – and, of course, something to study – but we strongly prefer to do this in the mode of studying *with*. As with the Las Endredadas network of Latin American activists, the Infusion Project built by creative professionals, and the *Open Advice* book created by some of the leading figures in open source software, we are building our own meta-project to pursue this goal.

# Works in Progress in the Accelerator

The current list of Accelerated projects is as follows:

- SOLE Out of the Box,
- PlanetMath.org,
- Mr. Danoff's Teaching Laboratory,
- "PeerPubU", a project by the Independent Publishers of New England (IPNE),
- Managing Innovation and Change, a course at the Bill Greehey School of Business, St Mary's University in San Antonio,
- The Peeragogy Accelerator itself
- … and enrollment for this round is open until June 1st, 2014.

In the limited space, we will not be able to describe all of these projects in depth, so we focus on a few examples. More information can be found on the landing page for the Peeragogy Accelerator group on the Commons Abundance Network.

**SOLE out of the Box**

This project is building a strategy to create a self-sufficient, Self Organizing Learning Environment (SOLE) (Mitra & Dangwal, 2010), in the form of a distributed 'makerspace.' We want to tackle two interrelated global problems: access to education and access to meaningful employment. The innovation would be to build a system that sustains the development of self-directed learners, through a network of living centers, which have correct complexity to evolve sustainably and abundantly. These "Learner Makerspaces" are a distributed network of micro universities. They are considered social businesses, because their purpose is to provide employment while each participant learns. A core group of organizers initiate the makerspace, building initial working models and an "Owners Manual". Investors contribute to capital and infrastructure. Wages and operating expenses are generated from the activity of the makerspace. The project's initial income plan is to abundantly produce food through the science of aquaponics, using Arduino monitors. This leads to a STEAM-focused project-based makerspace and will provide the foundation for a 'Cyber Cafe' to initiate the SOLE.

**PlanetMath.org**

Following on Joseph Corneli's (2014) thesis work, in which PlanetMath users helped develop new design patterns that could be used to improve PlanetMath as a learning platform, there were many high-level and low-level programming changes to make, including development work to improve PlanetMath's presentation of books. We plan to import mathematics books from Project Gutenberg and other sources, and apply optical character recognition (OCR) techniques to retrodigitize more math books ourselves. One of the books we have already imported is an early 20th Century Calculus textbook (Davis & Brenke, 1912), which includes some rather quaint examples, but which otherwise looks quite contemporary when rendered in MathML. We are planning an online peeragogical Calculus course, whose syllabus will feature this text and other resources from around the 'net. This course is built around design patterns – including patterns to deal with the common points of confusion associated with Calculus.

**Managing Innovation and Change**

This course on *Managing Innovation and Change* is targeted for Junior and Senior high performing undergraduates at the Bill Greehey School of Business at St. Mary's University in San Antonio. The course integrates peeragogical concepts in an "amortized" manner. Students have increasing autonomy as the course progresses.

The course is structured with a flipped classroom format and labs or workshops in the first half of the semester and a real world practicum project graded on execution and impact in the second half. Assessment is based on a student-generated peer assessment rubric. Accelerator members have offered peeragogical support, discussing the course concepts and design in a weekly online PreK-16 practitioners forum.

Our objective in this course is to fill the current educational model gaps, by teaching collaboration skills and integrated systems thinking. We are frequently reminded by students that we address skills and concepts not covered by any other courses in the school such as teaming, social media marketing, cloud collaboration, web site editing, and strategies for coping with high performance contexts.

The change in expectations and context has been disorienting for students, as is the online learning portion for the course. The technology presents one learning curve, and the human aspects of the digital context are another. We opted to strictly require online participation. We frequently read in weekly journals that we are opening up experiences of things that students feel they should be getting throughout their degree program.

**Auto-Acceleration of the Peeragogy Accelerator**

Peeragogy is large and diffuse, but in order to facilitate progress at a rapid pace, we have convened a small working group of persons with relatively concrete goals. The Accelerator is supported by peeragogy mavens, but, critically, it is also open to newcomers who are trying to understand and integrate peeragogy processes and metaphors into their context. Working together, we all learn.

## Discussion

Building on our peer-to-peer experiences working together on the *Peeragogy Handbook* we have re-organized ourselves for Project-to-Project collaboration. Our methods include "Hive editing" document meetings (working as a group using recorded Hangouts on Air together with Google Docs or Etherpads), and "Wrapping" our works-in-progress with summaries for a wider contributing public. It is worth emphasizing, again, that we all have our own working methods and preferences, and that we do not require everyone to do things the same way. Our attention options for our meetings include: participating in the live hangout, watching the video stream and joining asynchronously later via the Peeragogy in Action Google+ community. When writing collaboratively, some like to edit, others to comment. We learn from each other's approaches and values, and create new values together, e.g., observing that it is very useful, when one person is talking during a meeting, for another to type a transcript of what that person is saying. We encourage contributing projects to carefully specify their own operations and the qualitative outcomes that they are seeking through Acceleration.

**Limitations and Future Work**

We cannot currently point to a case study that demonstrates “X% passing without peeragogy, and Y% passing with peeragogy,” or anything of this sort. We may be experiencing a “Peeragogy Island” effect: we are quite happy with what we are doing, but it does not always resonate with outsiders. Even so, examples like *Managing Innovation and Change* show that peeragogy can be not only compatible, but helpful at numerous points within a (relatively) mainstream course.

How can we escape from Peeragogy Island? Involving others from the OER world, as we are doing in our workshop for OER14, may be very helpful for this. We have improved on our previous outreach activities by providing more concrete ways for people to get involved. Many Future Work ideas are recorded as “What’s Next” sections in the patterns in our pattern catalog – a “living” design for a living project.

**Summary of our contributions**

Peer learning and peer production benefit from participatory engagement at every level. Building novel systems for collaboration that are themselves peer produced offers users rich opportunities for meaningful engagement. The *Peeragogy Handbook*, including the latest version of our design pattern catalog, is a peer produced Open Educational Resource, available under the terms of the Creative Commons Zero license: it can be freely adapted. We continue to improve and refine the *Handbook* through our work in the Peeragogy Accelerator.

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# NNexus Reloaded

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**Abstract.** Interlinking knowledge is one of the cornerstones of online collaboration. While wiki systems typically rely on links supplied by authors, in the early 2000s the mathematics encyclopedia at PlanetMath.org introduced a feature that provides automatic linking for previously defined concepts. The NNexus software suite was developed to support the necessary subtasks of concept indexing, concept discovery and link-annotation. In this paper, we describe our recent reimplementation and revisioning of the NNexus system.

## 1 NNexus 1.0 – Introduction

PlanetMath.org is a mathematics digital library, built "the commons-based peer production way" [Kro03]. Like Wikipedia, which launched the same year, PlanetMath has been created by volunteer contributors from around the world. However, unlike Wikipedia, PlanetMath focuses solely on mathematics. Since its launch, it has used custom software both to support the display of mathematical expressions, and to facilitate the integration of new user-contributed content. One of the features designed to assist in content integration was an *autolinking service*. This service allowed authors to write without concerning themselves with wiki-style links to technical concepts that had already been added to the corpus. Instead, these links would be added automatically – and links would be recalculated and adjusted automatically as the encyclopedia grew, using a sophisticated caching and expiry system. The system provided an example of named entity recognition [NS07], where the entities to identify in submitted text are article titles, the names of terms defined in the articles, and any known synonyms. The process of adding links to named entities in text has come to be known as "wikification" [Rat+11].

In 2006, NNexus 1.0 began the process of decoupling autolinking from PlanetMath, and provided integration with other corpora (Wikipedia, Mathworld) on a demonstration basis [GKX06], an effort that has matured with the current release.

## 2 NNexus 2.0 – Reload, Refresh, Refactor

The primary goal of our rebuild was to decouple fully from the old Noosphere system on PlanetMath.org. A strong contributing motivation was that Noosphere was in the process of being deprecated on PlanetMath and replaced by

the new Planetary system [CD12]. The new NNexus works with Planetary, but also functions in a stand-alone fashion, and is published as a software library on the Comprehensive Perl Archive Network (CPAN).[3] It has been refactored to operate either as a web service,[4] or programmatically via an API. NNexus accepts arbitrary HTML input and performs concept discovery against its concept index, followed by a serialization of the mined data, either as stand-off metadata or by in-place embedding.

Concept indexing is performed by NNexus' built-in web crawler. It is based on a plugin architecture. Every indexed web resource requires its own indexer class, which contains the custom rules for detecting the concept definitions in the page. For example, PlanetMath's key terms are found in RDFa metadata that has been deposited in the encyclopedia pages, whereas the Digital Library of Mathematical Functions lists its defined concepts in its index as bold-anchored elements.

The current NNexus release ships with a database that integrates the concepts from seven web resources for mathematical concepts. These include the three best-known web resources for mathematics – Wolfram's MathWorld; PlanetMath.org; and Wikipedia – as well as Springer's Encyclopedia of Mathematics; the Digital Library of Mathematical Functions (DLMF); the nLab (which focuses on category theory); and the recently created MathHub.info.

At the time of writing, the NNexus index contains just under 50,000 unique concepts in its index. With the introduction of client-side tools for embedding NNexus [Gin13], we can also report successful auto-linking in third-party platforms such as arXiv.org and Zentralblatt MATH.

## 3 Concept Discovery

The NNexus implementations to date have only scratched the surface of the knowledge discovery problem. NNexus performs longest-token matching, aided by classic preprocessing techniques (stopword lists, morphological normalization) to discover all possible concept candidates. Concepts are considered discovered if there is an exact match between the linguistically normalized input document and the identically normalized concept index. When a concept $A$ is a substring of a concept $B$, since they both match at the same starting point of the input, preference is given to the longer string $B$. To demonstrate, take $A$ to be "fundamental groupoid" and $B$ to be "fundamental groupoid functor".

This simplistic approach leads to false positive hits, for instance, in words that have multiple part-of-speech uses or words that have both technical and everyday meanings. Accordingly, each of the following examples becomes a candidate for linking, even though the words in the right-hand column are not being used in a technical sense.

[3] Run **cpan NNexus** to install the software locally.

[4] A demonstration instance is available at `http://nnexus.mathweb.org`. Sending an HTML snippet as the body of a POST request will return the link-annotated snippet back, embedded in a thin JSON wrapper.

"*Let G be a* ***group***" "***group*** *the numbers in rows*"
"***chain*** *in a graph*" "***chain*** *made of steel*"
"***permanent*** *of a matrix*" "*using a* ***permanent*** *marker*"

This particular phenomenon is less observable as the length of the concept grows, both because longer words are less frequently overloaded and because multi-word concepts rarely have non-technical meanings. Inversely, the problem is particularly challenging in concepts with short single-word names.

Another challenge is disambiguating between overloaded concept names, used differently in different scientific areas. To address that, NNexus does not immediately return all of the named entities it discovers. Instead, it first uses a clustering algorithm, based on a distance metric between the classes of the MSC [Ame09] categorization scheme, which determines a kernel of closely related concepts. We have observed our distance metric is effective in separating concepts from typically disjoint subfields of science, but less successful in making fine-grained distinctions in subfields that tend to have a lot of mutual connections. For example, "entanglement" is a concept both in quantum mechanics and graph theory. NNexus is able to tell which meaning is intended by contextually clustering with the rest of the discovered concepts. However, generic concepts such as "equivalence" tend to be redefined in closely related subfields, and NNexus cannot tell these apart.

In addition to these technical limitations, NNexus is limited by the quality of the metadata provided by its indexed sources. For example, as pointed out by one of the reviewers of this paper, PlanetMath currently has no article on "classical logic", and links to this term are currently being directed to PlanetMath's article on quantum logic. This looked like a rather strange error until we realized that the quantum logic article includes a definition of the term "classical logic", in contravention of the "one main concept per article" norm.

## 4 NNexus 3.0 Revolution – an Outlook

Auto-linking continues to be a useful tool around PlanetMath. For instance, Planetary added support for contributing problems and problem sets, and technical terms in problems are linked to definitions drawn from the encyclopedia. We plan to add a PlanetMath feature where, given a contributed piece of text, a small "course packet" of preliminaries would be built on the fly, created out of auto-linked encyclopedia articles. Thanks to the metadata in the links provided by NNexus, we will be able to consider both "incoming" and "outgoing" links – this means we can discover applications of a concept as well as simpler concepts.

Some other efforts that we plan to explore include autolinking in math blogs, such as the blogs indexed on `http://mathblogging.org/`. NNexus could build a "term cloud" of technical terms from across the math blogosphere, providing a useful access method that parallels the familiar tag cloud.

The main challenge ahead is to solve the problem of reliable concept discovery. The immediate goal is to achieve reliable disambiguation of overloaded concept words (such as "set" or "group"), possibly by employing the help of a

part-of-speech tagger. A complementary idea is to improve precision by augmenting longest-token matching with weights derived by statistical term-likelihood analysis. As statistical term-likelihood methods do not depend on an a priori fixed lexicon, they could also be used to detect concepts that are not yet included in the index. That would allow us to enable another desirable feature – the automatic creation of dangling links (similar to Wikipedia's "red links").

Deeper scrutiny of mathematical formulas and terms will allow us to link occurrences of math constants in MathML, both globally, for symbols like the reduced Planck constant $\hbar$, and locally, following the annotation of the corresponding natural language term, such as "Assume a cyclic group $\mathbb{Z}_{mn}$ ...".

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# Assessing Progress in Building Autonomously Creative Systems

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**Abstract**

Determining conclusively whether a new version of software creatively exceeds a previous version or a third party system is difficult, yet very important for scientific approaches in Computational Creativity research. We argue that software product and process need to be assessed simultaneously in assessing progress, and we introduce a diagrammatic formalism which exposes various timelines of creative acts in the construction and execution of successive versions of artefact-generating software. The formalism enables estimations of progress or regress from system to system by comparing their diagrams and assessing changes in quality, quantity and variety of creative acts undertaken; audience perception of behaviours; and the quality of artefacts produced. We present a case study in the building of evolutionary art systems, and we use the formalism to highlight various issues in measuring progress in the building of creative systems.

## Introduction

Creativity, we believe, relates to a perception that others have of certain behaviours exhibited by some person or system, rather than an inherent property of people or software: in this sense it is a secondary quality. Moreover, we believe that, just as the endless debates about "is it art?" fuel innovation in the arts, the endless debates about "is it creative?" are a force for good: they drive forward creative practices and Computational Creativity research. A longer discussion of this philosophical position is given in (Colton et al. 2014), and an exposition of creativity as being *essentially contested* (Gallie 1956) is given in (Jordanous 2012).

In such a context of energetic and subjective debate about creativity, it has been difficult to derive systematic approaches to assessing progress in the building of software for creative purposes. One main issue has been the cross-purposes of the creativity project(s) for which software is developed. A useful analogy with the notions of weak and strong AI has arisen recently in Computational Creativity research. Focusing on software which generates artefacts such as poems, paintings or games, we can say that *weak Computational Creativity* objectives emphasise the production of increasingly higher valued artefacts, whereas *strong Computational Creativity* objectives emphasise increasing the perception of creativity people have of the system. This is similar to the distinction put forward in (al-Rifaie and Bishop 2012). In many projects, there are both strong and weak objectives, and often they are not complementary. For instance, increasing autonomy in software may lead simultaneously to higher perception of creativity and lower value artefacts being produced. This is described as the *latent heat* problem in (Colton and Wiggins 2012), and is analogous to U-shaped learning, where to get better, we first have to get worse.

The objectives for a project usually influence the assessment methods employed. In particular, to assess progress with respect to weak objectives, it makes sense to evaluate the quality of the artefacts produced. In contrast, for strong objectives, it makes more sense to assess what software actually does and how and why people perceive it as creative or not. To this end, in (Colton, Pease, and Charnley 2011) we introduced the FACE descriptive model to formalise descriptions of the creative acts undertaken by software, and the IDEA model to formalise the impact those creative acts might have on people. Subsequent attempts to use these models to describe particular systems have highlighted another major issue: the assignment of programmer/software ownership of creative acts. Along with other issues in applying it to describe systems, we have found the FACE model to be inadequate for fully capturing the interplay of creative acts between programmer and program in this respect.

We describe here the next stage of our formalism for capturing notions of progress in building creative systems. We first provide a potted history of how progress has been evaluated in Computational Creativity research, and lay out some intuitive notions of progress. Given our philosophical and practical standpoints, we place less emphasis on asking whether artefacts are 'better' than previously. We also avoid direct questions about 'creativity' in computational systems. Instead, we integrate (i) aspects of the FACE and IDEA models (ii) objective measures of quality, quantity and variety of creative acts and (iii) audience perceptions of software behaviour and quality of output. We present a two-stage method for estimating whether obvious or potential progress or regress has occurred when building a new system. This involves diagrammatically capturing various timelines in the building and execution of a system, then comparing diagrams. We use the method to describe progress of an evolutionary art system, leading to a general discussion about how the approach could be used in practice. We conclude by describing future directions for this formalism.

## Background in Assessing Creative Progress

The assessment of progress in building creative systems has been a bespoke and multi-faceted endeavour, driven by various, often opposing objectives ranging from understanding human creativity to practical generation of artefacts to the raising of philosophical questions. The majority of practical researchers who engineer and test software joined the Computational Creativity field with objectives in the weak sense of getting software to produce quality artefacts. Hence the first way in which progress was assessed was Boolean: if software reliably produces artefacts of a particular type, then this is progress over software which was unreliable or unable to produce artefacts of the required form.

In such a context, Turing-style discrimination tests indicated a particularly strong milestone: if certain artefacts – usually hand-selected – looked/sounded so like human-authored counterparts that observers couldn't tell the difference, progress had certainly been made. This approach was pioneered by (Pearce and Wiggins 2001) who were one of the first to emphasise the importance and role of evaluation in Computational Creativity, and to propose a concrete way of applying Popperian falsificationism. However, despite them urging caution at depending on the discrimination test to evaluate creativity, direct comparison of human produced and computer generated artefacts has frequently been used to assess progress. We further criticised such Turing-style tests in Computational Creativity, for, among other reasons, encouraging naïvety in software and the generation of pastiches (Pease and Colton 2012). Moreover, we question whether this methodology, while beneficial for short-term scientific progress, is actually detrimental to the longer-term goal of embedding creative software in society (Colton et al. 2014). The work of (Ritchie 2007) was an important step away from simplistic discrimination tests, establishing an approach to assessing the value of artefacts according to their *novelty*, *typicality*, and *quality* within a genre. A number of practitioners have used this approach to compare and contrast their systems, e.g., (Pereira et al. 2005).

As the field matured, attention moved from *mere generation* to programs able to assess, critique and select from their output. Often searching large spaces, software was required to find the best artefacts using mathematically derived or machine-learned aesthetic/utilitarian calculations (Wiggins 2006). If a later version of software – with more sophisticated internal assessment techniques – was able to produce higher yields of higher quality artefacts when assessed externally, then clear progress had been made.

Audience perceptions of software became a focus, as the field further matured. Jordanous used methods from linguistics to determine how people are using the word 'creativity', and which other concepts are associated with it, and then used crowdsourcing techniques to evaluate a creative system in terms of the associated concepts (Jordanous 2012). As a complement to Jordanous's work in which she tried to *capture* society's perception of creativity, researchers began investigating ways to *influence* people's perception of creativity in software. Software assessing its own work made it *appear* more intelligent, and *seem* more creative. This led to the engineering of software that *framed* its processes and outputs by producing titles, commentaries and other material. (Charnley, Pease, and Colton 2012) propose that this may increase perception of creativity, and audiences would possibly appreciate the artefacts produced more. Studying audience perceptions of creativity in software opened many research avenues, but raised an important problem: that the original product-based assessment methods no longer capture all intuitions of what constitutes progress in the field.

From a strong perspective, some researchers, including ourselves, are not content to accept the underlying assumption of product-based evaluation methods: if better artefacts are produced, the software must have been improved, hence people will project higher perceptions of creativity onto the software and progress will have been made. As mentioned previously, the main problem here is that increasing autonomy – which must happen if strong objectives are to be met – can decrease artefact value. Conversely, when the objectives of a project are weak, it is perfectly natural to decrease software autonomy to produce artefacts of presentation quality, especially when a concert/exhibition is looming, but this is unlikely to increase any perceptions of creativity.

Concentrating on understanding perceptions of software creativity by the general public, we introduced the *creativity tripod* in (Colton 2008b) as three types of behaviours which were necessary (but not necessarily sufficient) for software to avoid being labelled as 'uncreative'. We proposed that people are influenced by their understanding of what software does when assessing its output. We argue that it is easy to ascribe uncreativity to software which is not simultaneously seen as *skillful*, *appreciative* and *imaginative*.

Focusing on assessment of progress by peers, we introduced the FACE and IDEA descriptive models in (Colton, Pease, and Charnley 2011) and (Pease and Colton 2011). The FACE model categorises generative acts by software into those at (g)round level, during which base objects are produced, and (p)rocess level, during which methods for generating base objects are produced. These levels are subdivided by the types of objects/processes they produce: $F_g$ denotes a generative act producing some framing information, $A_g$ denotes an act producing an aesthetic measure, $C_g$ denotes an act producing a concept and $E_g$ denotes an act producing an example of a concept. Generative acts producing new processes are defined accordingly as $F_p$, $A_p$, $C_p$ and $E_p$. Tuples of generative acts are compiled as *creative acts*, and various calculations and recommendations are suggested in the model with which to compare creative systems. We developed the IDEA model so that creative acts and any impact they might have could be properly separated. We defined various stages of software development and used an ideal audience notion, where people are able to quantify changes in well-being and the cognitive work required to appreciate a creative act and the resulting artefact/process.

We have arrived at a very observer-centric situation in the assessment of progress towards creative systems, in which progress can only be measured using feedback from independent observers about both the quality of artefacts produced and their perceptions of creativity in the software. Unfortunately, the majority of researchers develop software using only themselves as an evaluator, because observer-

based models are too time-consuming to use on a day-to-day progress. These informal in-house evaluation techniques generally do not capture the global aims of the research project, or of the field (e.g. producing culturally important artefacts and/or convincing people that software is acting in a creative fashion). In many cases, systems are presented as feats of engineering, with little or no evaluation at all (Jordanous 2012). We argue that assessing *progress* is inherently a process-based problem. We focus here on modeling diachronic change across multiple levels.

## A Formal Assessment of Progress

We combine the most useful aspects of the IDEA and FACE models, an enhanced creativity tripod, and aspects of assessing artefact value into a diagrammatic formalism for evaluating progress in the building of creative systems. We focus on the creative acts that software performs, the artefacts it produces and the way in which audiences perceive it and consume its output. We simplify by assuming a development model where a single person or team develops the software, with various major points where the program is sufficiently different for comparisons with previous versions. We aim for the new formalism to be used on a daily basis without audience evaluations, to determine short term progress, but for it also to enable fuller audience-level evaluations at the major development points. We also aim for the formalism to help determine progress in projects where there are both weak and strong objectives. We found that the original FACE model didn't enable us to properly express the process of building and executing generative software. Hence another consideration for our new model is that it can capture various timelines both in the development and the running of software in such a way that it is obvious where the programmer contributed creatively and where the software did likewise.

With the above aims in mind, we envisage a scenario where we are comparing two versions of creative software $v1$ and $v2$. At the highest level, we split the assessment method into a two stage process as follows:

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1. Diagrams are drawn for both $v1$ and $v2$ which capture the interplay of programmer and program behaviours as timelines during both the development phase and the runtime execution of both versions of the software.

2. The diagrams for $v1$ and $v2$ are compared by an audience to determine if the second system represents progress over the first in terms of process. Similarly, the output from $v1$ and $v2$ is compared, to see if progress has been made.

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### Stage 1: Diagrammatic Capture of Timelines

Taking a realistic but abstracted view of generative software development and deployment, we identify four types of timeline. Firstly, generative programs are developed in **system epochs**, with new versions being regularly signed off. Secondly, each process a program undertakes will have been implemented during a **development period** where creative acts by programmer and program have interplayed.

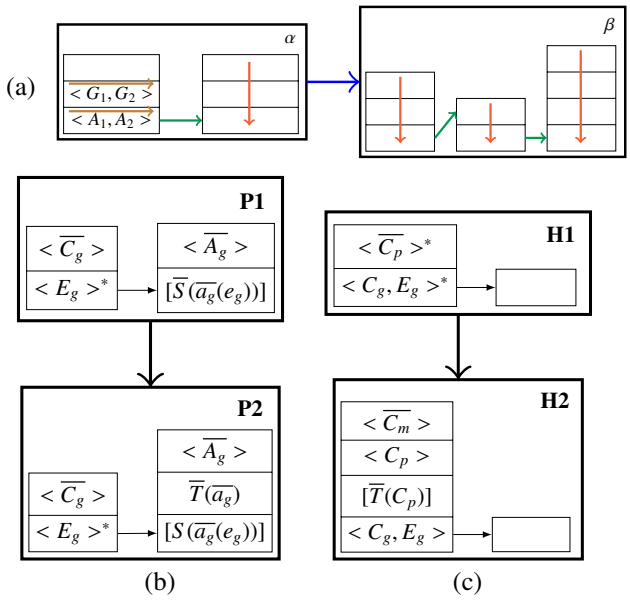



**Figure 1:** *(a) Key showing four types of timelines (b) Progression of a poetry system (c) Progression of the HR system.*

Thirdly, at run-time, data will be passed from process to process in series of creative and administrative **subprocesses** performed by software and programmer. Finally, each subprocess will comprise a sequence of generative or administrative **acts**. We capture these timelines diagrammatically, highlighted with coloured arrows in Figure 1(a). The blue arrow from box $\alpha$ to $\beta$ represents a change in epoch at system level. The red arrows overlapping a *process stack* represent causal development periods. The green arrows represent data being passed from one subprocess to another at run-time. The brown arrows represent a series of generative/administrative acts which occur within a subprocess.

Inside each subprocess box is either a < creative act > from the FACE model (i.e., a sequence of generative acts), or an [ administrative act ] which doesn't introduce any new concept, example, aesthetic or framing information/method. Administrative acts were not originally described in the FACE model, but we needed them to describe certain progressions of software. For our purposes here, we use only $T$ to describe a translation administrative act often involving programming, and $S$ to describe when an aesthetic measure is used to select the best from a set of artefacts. To add precision, we indicate the output from which generative act the administrative routine is applied, and to which examples a ground aesthetic is applied. To enable this, we employ the FACE model usage of lower-case letters to denote the output from the corresponding upper-case generative acts. We extend the FACE notion of (g)round and (p)rocess level generative acts with (m)eta level acts during which process generation methods are invented. As in the original description of the FACE model, we use bar notation to indicate that a particular act was undertaken by the programmer. We use a superscripted asterisk ($^*$) to point out repetition.

As a simple example diagram, Figure 1(b) shows the progression from poetry generator version P1 to P2. In the

first version, there are two process stacks, hence the system works in two stages. In the first, the software produces some example poems, and in the second the user chooses one of the poems (to print out, say). The first stack represents two timesteps in development, namely that (a) the programmer had a creative act $< \overline{C_g} >$ whereby he/she came up with a concept in the form of some code to generate poems, and (b) the programmer ran the software to produce poems in creative acts of the form $< E_g >^*$. The second stack represents the user coming up with an idea for an aesthetic, e.g., much rhyming, in creative act $< \overline{A_g} >$, and then applying that aesthetic $\overline{a_g}$ him/herself to the examples, $e_g$, produced by the software, in the *selection* administrative act $[\overline{S}(\overline{a_g}(e_g))]$, which maps the aesthetic $\overline{a_g} : \{e_g\} \rightarrow [0, 1]$ over the generated examples, and picks the best one. In the $P2$ version of the software, the programmer undertakes *translation* act $[\overline{T}(\overline{a_g})]$, writing code that allows the program to apply the rhyming aesthetic itself, which it does at the bottom of the second stack in box $P2$.

Figure 1(c) shows a progression in the HR automated theory formation system (Colton 2002) which took the software to a meta-level, as described in (Colton 2001). HR operates by applying production rules which invent concepts that categorise and describe input data. Each production rule was invented by the programmer during creative acts of the type $< \overline{C_p} >$, then at run-time, HR uses the production rules to invent concepts and examples of them in $< C_g, E_g >^*$ acts. In the meta-HR version, during the $< \overline{C_m} >$ creative act, the programmer had the idea of getting HR to form theories about theories, and in doing so, generate concept-invention processes (production rules) in acts of the form $< C_p >$. The programmer took meta-HR's output and translated $[\overline{T}(C_p)]$ it into an implemented production rule that HR could use, which it does at the bottom of the stack in box H2.

## Stage 2: Comparing Diagrams and Output

In both simple cases of Figure 1, it is clear that progress has been made in the strong sense, but not clear in the weak sense, as the output could easily be degraded by the more sophisticated processing of the systems. The diagrams help us to capture the creative interplay between software and programmer at design time and run time. However, given that the ultimate aim of both strong and weak projects is to impress audiences with process and product, any assessment of progress must be done in a context of audience evaluation. However, as mentioned previously, audience evaluation is too expensive to help assess progress on a day to day basis. Hence, it seems sensible for the programmer to step in and act as a proxy for a perceived audience: we advocate the programmer putting themselves in the position of the type of person they would expect to form their audience, and answer questions about the products and processes accordingly.

Examining the transition from one diagram to another should provide some shortcuts to estimate audience reactions, especially when there are strong project objectives. In particular, as with the original FACE model, the diagrams make it obvious where creative or administrative responsibility has been handed over to software, namely where an act which used to be barred has become unbarred, i.e., the same type of generative act still occurs, but it is now performed by software rather than programmer. This happened when the $\overline{S}$ became an $S$ in Figure 1(b) and when the $\overline{C_p}$ became a $C_p$ in Figure 1(c). At the very least in these cases, an unbiased observer would be expected to project more autonomy onto the software, and so progress in the strong sense has likely happened. In addition, the diagrams make it obvious when software is doing more processing in the sense of having more stacks, bigger stacks or larger tuples of acts in the stack entries. Moreover, the diagrams make it clear that more varied or higher-level creative acts are being performed by the software – again, this was one of the benefits of the original FACE model. Both of these have the potential to convince audience members that software is being more sophisticated with respect to various behaviours described below, and hence can be a shorthand for progress.

When dealing with actual external evaluation, where people don't know what software does, we suggest that the diagrams above (and verbalisations/simplifications of them) can be used to describe to audiences what the software and what the programmer have done in a project. In this way, using also their judgements about the artefacts produced, people can make fully informed decisions in evaluation studies. As a general philosophical standpoint, we suggest not asking people if they believe software is behaving creatively, but rather concentrating on whether they perceive the software as acting *uncreatively*. Our argument for this is that the concept of creativity is essentially contested (Gallie 1956), hence, no matter how sophisticated our software gets, we should not expect consensus on such matters. However, we have found that people agree much more on notions of uncreativity: if a program doesn't exhibit behaviours onto which certain words like *intentionality* can be projected, then it is very easy to condemn it as being uncreative.

Hence, we advocate not asking a set of questions from which we can conclude that an audience member thinks that software is creative, but rather asking questions from which we can determine whether they think that software is acting uncreatively. It may seem like rather a negative admission, but we believe that the best way to get people to accept software as being creative is for them to eventually realise that there is no good reason to call it uncreative. Even then, people would be perfectly at liberty to say that while software is not uncreative, it is not creative either: creativity and uncreativity do not appear to be exact opposites. With this in mind, we have boiled down audience evaluation of behaviour to asking people whether they would project certain words onto software in reaction to understanding what it did in the context of a particular project. We then tentatively conclude that they believe the software is uncreative if they don't project onto it some or all of these words, as originally intended in the creativity tripod proposition (Colton 2008b).

In the five years since the introduction of the creativity tripod, we have slowly added additional behaviours which we have found to be important in the perception of creativity in software. That is, for people to take seriously software as being not uncreative, we believe it needs to exhibit behaviours onto which people can meaningfully project (at least) these

| Product change | Process change | Weak | Strong |
|---|---|---|---|
| Up | Up | OP | OP |
| Up | Down | PP | PR |
| Up | Same | OP | PP |
| Down | Up | PR | PP |
| Down | Down | OR | OR |
| Down | Same | OR | PR |
| Same | Up | PP | OP |
| Same | Down | PR | OR |
| Same | Same | PP | PP |

**Table 1:** Guidelines for using change in evaluation of product and process in gauging (O)bvious or (P)otential (P)rogress or (R)egress, in both weak and strong agendas.

eight words: skill, appreciation, imagination, learning, intentionality, accountability, innovation, subjectivity and reflection. We have found that assessing the level of projection of these words onto the behaviours of software can help us to gauge people's opinions about (the lack of) important higher-level aspects of software behaviour, such as autonomy, adaptability and self-awareness.

The method we suggest for estimating progress from version $v1$ of a creative system to version $v2$ is to: (a) show audience members the diagrams for $v1$ and $v2$ as above, and explain the acts undertaken by the software, then (b) show audience members the output from $v1$ and $v2$, and (c) ask each person to compare the pair of product and process for $v1$ with that of $v2$. A statistical analysis could then be used to see whether the audience as a whole evaluates the output as being better, worse or the same, and whether they think that the processing is better, worse or the same in terms of the software seeming less uncreative. This takes into account the phenomenon described in (Colton 2008b) whereby the process can influence value judgements for artefacts.

To use this analysis to estimate progress, it's important to first prioritise objectives for the project locally in terms of strong and weak agendas. Then, taking the audience evaluation of change in output and in process, we suggest using the guidelines in Table 1. Here, we have stipulated that certain evaluation pairs indicate obvious progression (OP) or obvious regression (OR). For instance, in the weak sense, when the evaluation of output goes up and the evaluation of process increases or stays the same, it seems clear to indicate obvious progress. Other cases are not so clear-cut, for instance when evaluation of artefacts goes up, but evaluation of process goes down. In this case, we suggest that this is potential progress (PP) in a weak agenda, and potential regress (PR) in a strong agenda. In such cases, we give our judgements for whether it is likely, after more development, that $v2$ will be viewed retrospectively as a progressive success or a step backwards. Note that we have tended to be optimistic, e.g., when evaluation of output and process stay the same, we say that this is potential progress in both weak and strong agendas. Note also that this table is meant to be used flexibly, possibly in a context of more fine grained analysis. For instance, the focus of a subproject might be to increase audience perception of intentionality, and if this increases while audience perception of the value of the process as a whole reduces, it should still be seen as progress.

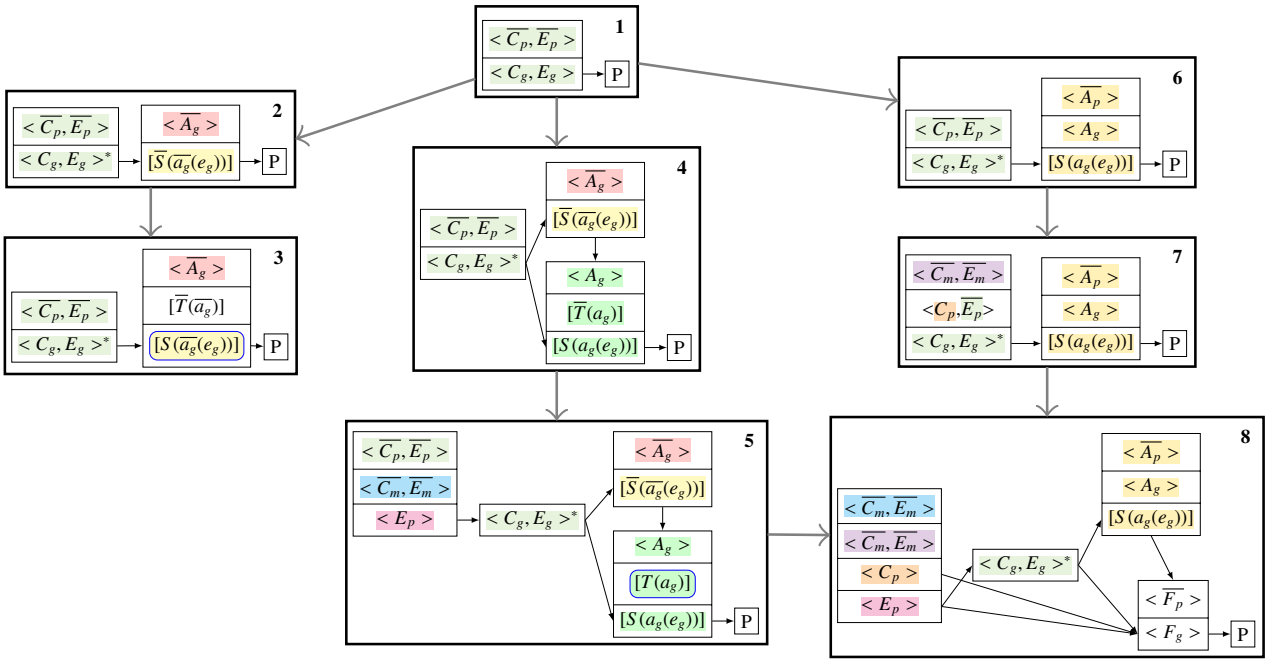
## A Case Study in Evolutionary Art

Evolutionary art – where software is evolved which can generate abstract art – has been much studied within Computational Creativity circles (Romero and Machado 2007). Based on actual projects which we reference, we hypothesise here the various timelines of progress that could lead from a system with barely any autonomy to one with nearly full autonomy. Figure 2 uses our diagrammatic approach to capture three major lines of development, with the final (hypothetical) system in box 8 representing finality, in the strong sense that the software can do very little more creatively in generating abstract art. Since features from earlier system epochs are often present in later ones, we have colour-coded individual creative acts as they are introduced, so the reader can follow their usage through the systems. If an element repeats with a slight variation (such as the removal of a bar), this is highlighted. The figure includes a key, which describes the most important creative and administrative acts in the systems. Elements in the key are indexed with a dot notation: *system.process-stack.subprocess* (by number, from left to right, and top to bottom, respectively). System diagrams have repetitive elements, so that the timelines leading to its construction and what it does at run-time can be read in a stand-alone fashion.

Following the first line of development, system 1 of Figure 2 represents an entry point for many evolutionary art systems: the programmer invents ($\overline{C_p}$) (or borrows) the concept formation process of crossing over sets of mathematical functions to produce offspring sets. He/she also has an idea ($\overline{E_p}$) for a *wrapper* routine which can use such a set of functions to produce images. He/she then uses the program to generate ($C_g$) a set of functions and employ the wrapper to produce ($E_g$) an image which is sent to the (P)rinter. The crossover and subsequent image generation is repeated multiple times in system 2, and then the programmer – who has invented ($\overline{A_g}$) their own aesthetic – chooses a single image to print. In system 3, as in the poetry example above, the programmer translates their aesthetic into code so the program can select images. This is a development similar to that for the NEvAr system (Machado and Cardoso 2002).

Following the second line of development, in system 4, the programmer selects multiple images using his/her own aesthetic preferences, and these become the positives for a machine learning exercise as in (Li et al. 2013). This enables the automatic invention ($A_g$) of an aesthetic function, which the programmer translates by hand $\overline{T}(a_g)$ from the machine learning system into the software, as in (Colton 2012), so the program can employ the aesthetic without user intervention. In system 5, more automation is added, with the programmer implementing their idea ($\overline{C_m}$) of getting the software to search for wrappers, then implementing this ($\overline{E_m}$), so that the software can invent ($E_p$) new example generation processes for the system.

Following the final line of development, in system 6, we return to aesthetic generation. Here the programmer has the idea ($\overline{A_p}$) of getting software to mathematically invent fitness functions, as we did in (Colton 2008a) for scene generation, using the HR system (Colton 2002) together with The Paint-




| ID | Event | Explanation |
|---|---|---|
| 1.1.1 | $\overline{C_p}$ | The programmer invents the idea of crossing over two sets of mathematical functions to produce a new set of mathematical functions. |
| 1.1.1 | $\overline{E_p}$ | The programmer implements a wrapper method that takes a set of mathematical functions and applies them to each $(x, y)$ co-ordinate in an image to produce an RGB colour. |
| 1.1.2 | $C_g$ | The software generates a new set of functions by crossing over two pairs of functions. |
| 1.1.2 | $E_g$ | The software applies these functions to the $(x, y)$ co-ordinates of an image, to produce a piece of abstract art. |
| 2.2.1 | $\overline{A_g}$ | The programmer had in mind a particular aesthetic (symmetry) for the images. |
| 2.2.2 | $\overline{S}(\overline{a_g}(e_g))$ | The programmer uses his/her aesthetic to select a preferred image for printing. |
| 3.2.2 | $\overline{T}(\overline{a_g})$ | The programmer took their aesthetic and turned it into code that can calculate a value for images. |
| 3.2.3 | $S(\overline{a_g}(e_g))$ | The software applies the aesthetic to select one of a set of images produced by crossover and the wrapper. |
| 4.3.1 | $A_g$ | The software uses machine learning techniques to approximage the programmer's aesthetic. |
| 4.3.2 | $\overline{T}(a_g)$ | The programmer hand-translates the machine learned aesthetic into code. |
| 4.3.3 | $S(a_g(e_g))$ | The software applies the new aesthetic to choosing the best image from those produced. |
| 5.1.2 | $\overline{C_m}$ | The programmer has the idea of getting the software to search through a space of wrapper routines. |
| 5.1.2 | $\overline{E_m}$ | The programmer implements this idea. |
| 5.1.3 | $E_p$ | The software invents a new wrapper. |
| 5.4.2 | $T(a_g)$ | The software translates the machine-learned aesthetic itself into code. |
| 6.2.1 | $\overline{A_p}$ | The programmer has the idea of getting the software to invent a mathematical fitness function. |
| 6.2.2 | $A_g$ | The software invents a novel aesthetic function. |
| 6.2.3 | $S(a_g(e_g))$ | The software selects the best artefact according to its aesthetic function. |
| 7.1.1 | $\overline{C_m}$ | The programmer has the idea of getting the software to invent and utilise novel combination techniques for sets of functions, generalising crossover. |
| 7.1.1 | $\overline{E_m}$ | The programmer implements this idea so that the software can invent new combination techniques. |
| 7.1.2 | $C_p$ | The software invents a novel combination technique. |
| 8.4.1 | $\overline{F_p}$ | The programmer has the idea of getting the software to produce a commentary on its process and artwork by describing its invention of a new aesthetic, combination method and wrapper. |
| 8.4.2 | $F_g$ | The software produces a commentary about its process and product. |

**Figure 2:** *The progression of an evolutionary art program through eight system epochs.*

ing Fool (Colton 2012b). In system 7, the programmer realises ($\overline{C_m}$) that crossover is just one way to combine sets of functions, and gives ($\overline{E_m}$) the software the ability to search a space of combination methods ($C_p$). The software does this, and uses the existing wrapper to turn the functions into images. System 8 is the end of the line for the development of the software, as it brings together all the innovations of previous systems. The software invents aesthetic functions, innovates with new concept formation methods that combine mathematical functions, and generates new wrappers which turn the functions into images. Finally, the programmer has the idea ($\overline{F_p}$) of getting the software to write commentaries, as in (Colton, Goodwin, and Veale 2012), about its processing and its results, which it does in generative act $F_g$.

Tracking how the system diagrams change can be used to estimate how audiences might evaluate the change in processing of the software, in terms of the extended creativity tripod described above. Intuitively, each system represents progress from the one preceding it, justified as follows:

**1 → 2**: $< C_g, E_g > \rightarrow < C_g, E_g >^*$

Simple repetition means that the software has more *skill*, and the introduction of independent user selection shouldn't change perceptions about *autonomy*.

**2 → 3**: $\overline{S} \rightarrow S$

By reducing user intervention in choosing images, the software should appear to have more *skill* and *autonomy*.

**1 → 4**: Introduction of $A_g$ and $S(a_g(e_g))$ acts

Machine learning enables the generation of novel aesthetics (albeit derived from human choices), which should increase perception of *innovation*, *appreciation* and *learning*, involving more varied creative acts.

**4 → 5**: Introduction of an $E_p$ act, $\overline{T} \rightarrow T$

Wrapper generation increases variety of creative acts, and may increase perception of *skill* and *imagination*.

**1 → 6**: Introduction of $A_g$ and $S(a_g(e_g))$ acts

The software has more variety of creative acts, and the invention and deployment of its own aesthetic – this time, without any programmer intervention – should increase perception of *intentionality* in the software.

**6 → 7**: Introduction of a $C_p$ act

Changes in the evolutionary processes should increase perceptions of *innovation* and *autonomy*.

**5, 7 → 8**: Introduction of an $F_g$ act

Framing its work should increase perceptions of *accountability* and *reflection*.

With all strands brought together, the programmer does nothing at run-time and can contribute little more at design time. The software exhibits behaviours onto which we can meaningfully project words like skill, appreciation, innovation, intentionality, reflection, accountability and learning, which should raise impressions of autonomy, and make it difficult to project uncreativity onto the software.

## Discussion

Capturing what programmers and software do creatively over long periods and during complicated program executions is difficult and open to variability. The systems in the above case study could easily have been interpreted and presented differently. In essence, we have provided some tools for presenting software development in terms of creative acts, and suggested a mechanism for turning audience perceptions into estimates of progress. We advise flexible application in both cases. In particular, the difference between potential progress and potential regress is quite subtle. Both mean that it is too early to determine whether progress or regress has been made, and the programmer should proceed with caution: the former suggesting cautious optimism and the latter, cautious pessimism. Practically speaking, the programmer may want to review longer term goals, archive previous versions, and/or clarify research directions.

Our approach is currently more tailored to capturing progress in software behaviour than its output. We would understand some resistance to the approach, particularly from researchers with agendas for Computational Creativity in the weak sense. For example, if product evaluations remain the same, yet processing evaluations go up, this is presumably because the software is performing more sophisticated routines. From a weak perspective, the simpler version of the software clearly has advantages, as it produces the same results in a more understandable way. In certain application domains, for instance mathematical discovery, where aesthetics like truth are of paramount importance, a simpler method for finding a result is usually preferred. While reducing complexity of processing normally requires considerable invention or intervention, unless such invention is done by the software itself, the resulting simplicity would tend to increase perceptions of uncreativity in software, regardless (or, indeed, because of) how easy it is to understand what it has done.

Our approach is also more tailored towards capturing progress from version to version of the same software than to comparing different programs. However, we have used the formalism to compare systems in the same application domains, such as mathematical discovery systems AM (Lenat 1976) and HR (Colton 2002), and various poetry and art generators. The comparative approach works somewhat here, because it was possible to compare diagrams meaningfully to suggest where one system would likely be perceived as an improvement over the other. However, full application of the approach may be difficult as the context for evaluating artefacts (and the processes producing them) can change greatly with small changes in artefact composition. For instance, we recently attempted to compare one-line "What if...?" ideas produced textually by three systems. We found that it was not possible to conceive a fair approach involving an audience to determine which system's artefacts or processes were the best. Fields like Machine Learning have largely homogenised the testing of their systems in a problem-solving paradigm. Given the tacit requirements for software to surprise us through its output and processing, and to innovate on many levels, it seems unlikely that such standardisation could apply in Computational Creativity research.

## Related Work

Diagrammatic approaches to software modelling have been extensively studied in the last two decades. The best known example is the Unified Modelling Language (UML), managed by the Object Management Group (OMG), a standard that is widely used to visualize the design of systems (`www.omg.org/spec`). The main objectives of modelling with UML are to represent the architecture of a system, including use cases, deployment, information flow diagrams, etc., and to model system behaviour and data flow via activity diagrams, state machines, sequence diagrams, etc.

Progress at the process level can be modelled with UML by diagramming the steps used to complete a task within the system. However, UML is not typically applied to model progress at the level of system epochs, although two UML diagrams can of course be compared on the basis of the functionality they describe. Some diagrams created using the UML model, such as use case diagrams, enable designers to specify the agents that participate in the development of a system: people, external processes, other systems and the system itself can all be modelled as agents. However, there is no formal notation to distinguish between the different agents, rather, they are simply assigned a label which is meaningful for the system designer. The OMG have also developed other graphical notations specialised for other aspects of systems modelling. For instance, the Business Process Model and Notation (BPMN) is used to model business processes by extending the original activity diagrams of UML. The specific objective of BPMN is to provide a high-level overview of business systems, rather than detailed information about how the system works.

UML diagrams have also been used in the context of formal methods. In particular, the UML-B language (Said, Butler and Snook 2009) enables the modelling of Event-B specifications as UML-like diagrams. Event-B is a formalism based on set theory for the modelling and verification of systems (Abrial 2010). One of the main aspects of Event-B is the use of refinement to handle the complexity of systems at different levels of abstraction. UML-B can be used to diagrammatically model a system at increasing levels of refinement, and system consistency can then be verified through mathematical proof. However, UML-B considers one system at a time, so it is not possible to use this formalism to model creative change as system development progresses.

Using the Event-B formalism, it is possible to model aspects of the environment, such as external systems that affect the behaviour of the modelled system. The aim is to ensure that the designed system will work in harmony with its operating environment. However, there is no clear way to delimit the aspects of the model that are related to the environment and those that are part of the final system. Again, the environment is simply identified by the designer assigning meaningful names to the state representing it. Other related approaches include Z-notation (Spivey 1992), the Vienna Development Method (Jones 1990) and the B-method (Abrial 1996). The objective of these approaches is to verify properties of systems. Progress would be meaningful at the modelling level, i.e., by building models that offer increasing detail (and assurance) about how a given system works.

Petri nets provide a graphical notation used primarily to model systems with concurrency (Girault and Valk 2003). With petri nets, progress at the process level is modelled in the form of state transitions, and data is represented by abstract tokens, with no data values assigned. An extension, called coloured petri nets (Jensen and Kristensen 2009), allow data values to be assigned to tokens. Neither type of petri net is used for modelling changes through versions of a system. Petri nets are an event-based modelling language and representations of agents (such as the programmer or the system) are not included in the formalism.

## Conclusions and Future Work

We have presented a new diagrammatic formalism for assessing progress in building creative systems. Our aims were to enable more precise understanding of progress in Computational Creativity in general, and in mapping the progress of particular systems. In doing so, we aimed to bring closer together public/peer appreciation of progress, strong/weak agendas, and day-to-day/milestone progress assessments. The new approach involves producing diagrams of systems that depict creative acts in timelines, which are compared in a context of audience evaluation of process and product. When applied, the formalism captures some intuitive notions, including: quality of artefacts; quantity, level and variety of creative acts performed; and audience perception of software behaviour. To enable better understanding of process, and more informed audience judgements about (un)creativity, the diagrams explicitly separate creative acts coming from the programmer and the program. Even in the absence of audience participation, the diagrams themselves can be used in combination with straightforward assumptions about audience reactions to system design features to perform low-cost estimates of progress in a strong agenda.

We motivated the approach throughout with various philosophical standpoints, as per (Colton et al. 2014), supported by a critical review of the ways in which progress in building creative systems has been measured historically. To highlight the potential for the formalism, we presented a case study where the progress through eight versions of evolutionary art software was mapped and justified.

Our audience evaluation model is far from complete. We plan to employ the criteria specified in (Ritchie 2007), for more fine-grained evaluations of the quality, novelty and typicality of artefacts. We will also import audience reflection evaluation schemes from the IDEA descriptive model, e.g., change in well-being, cognitive effort and emotional responses such as surprise and amusement. We have so far used the diagrammatic approach to fully depict timelines in the building of generative software producing mathematics, visual art, poetry and video games, including dozens of system diagrams (omitted for space reasons). This has worked well, but there are still some subtle improvements required to capture better the functioning of the software at run-time.

(Gabriel and Goldman 2000) describe system development environments with many contributing programmers, and multiple interacting, self-programming, and self-updating distributed systems (Gabriel and Goldman 2006). It would be straightforward to modify our formalism to deal

with multiple agents, for example by turning bars into superscripts. However, this does complicate the notion of progress: if system $\mu$ chooses to hand off creative control to system $\nu$, this would amount to changing a superscript – but it's not immediately clear that this should count as progress in the same way that removing bars does. If the agents are considered to be full partners in the creative process, $\mu$ and $\nu$ may well have their own perspectives on what counts as progress, and this needs to be formalized.

Broadly speaking, we expect that the distinction between strong and weak agendas will eventually disappear: in order to produce higher quality artefacts, more sophisticated systems involving behaviours perceived as creative will be required, and audiences will expect to project notions of creativity onto software to fully appreciate its output. In such a context, assessing processes and products simultaneously will be important, and we hope versions of this diagrammatic approach will enable this. In (Colton, Goodwin, and Veale 2012), we used the FACE model as a driving force for poetry generation software, rather than as a descriptive tool. We hope that system developers will similarly begin to think about their software in the above diagrammatic terms, in order to suggest interesting new avenues for implementation.

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# Peer produced peer learning :

## A mathematics case study

*by*

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A thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy.

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Submitted July 31, 2013

# Abstract

This research project develops around a technological intervention intended to transform a peer produced reference resource into a peer produced learning environment. Through the work described in this thesis, PlanetMath.org, an early online community devoted to mathematics, has now become a mathematical practicum, and a laboratory for learning science.

A new theory that describes the nexus of peer production and peer learning is foundational for the research programme. The candidate theory was initially developed during a pilot study based on online field work at the Peer-2-Peer University. The new theory – which is given the name "paragogy" – has implications for designers, researchers, educators, and others whose work relies on peer learning and peer production. Further research and development work in the PlanetMath context helped to refine the theory, and applied it along with a range of mixed methods to develop an anthropologically-inspired study of modern mathematics.

A quantitative approach was employed to detect the factors of interaction that influence learning outcomes, using legacy data from PlanetMath. A qualitative, interview-based approach was employed, to understand the desiderata potential users of a new system emphasizing peer learning. The new software system was implemented, informed by paragogy and these stakeholder perspectives, using Drupal and other open source components. Field work with PlanetMath users after the launch of the new system employed an emergent design process to elaborate the theory and develop a viable approach to ongoing development and codesign.

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> *There is a pool of blood somewhere, a place you came from. You will find this blood petrified into stone and it is red. It comes from a sacred spot common to all people, where even enemies are turned into friends and relatives.*
>
> *Lame Deer, Seeker of visions*, 1972

Among all my friends and relations I particularly wish to thank my father, Steve Corneli. I remember when I asked him for help with a geometry problem, and he said to me: "Do you just want the answer, or do you want to really understand how it works?" I was far too ashamed to admit that what I wanted was the answer, so I think I said, "Both." I thought about it some more afterwards, and realized, what I really wanted was a computer program that would help me understand how to get the answer. Well, I don't have the computer program yet. But I know a lot more about how it works.

Figure 4 is © 2013 Gerry Stahl and is used here with permission.

Figure 5 is © 2013 Sean Goggins and is used here with permission.

Tables 11, 12, and parts of Chapter 9 appear in altered form in the *Peeragogy Handbook*.

Appendix B draws extensively on notes from a public lecture on "Modelling the mathematical discovery process" presented by Sir Timothy Gowers in his Maxwell Institute Lecture at the University of Edinburgh, November 2, 2012.

# Publications

Eleven recent peer reviewed publications helped to explicate the main ideas in the thesis in the context of academic dialog. These papers were published in venues devoted to wikis, open knowledge, learning design, learning science, computer mathematics, and commons governance. The thesis draws on and complements recent experiments with research-through-writing (Tomlinson et al., 2012), and joint work on the *Peeragogy Handbook* (Rheingold et al., 2014). The benefits of collaboration in research are particularly clear for research on collaboration, as this allows theoretical ideas about the topic of study to be developed and grounded in practice. The associated publication track record reflects the interdisciplinary and mediatedly-social nature of this project. That said, in conformity with academic norms, the text presented here remains the responsibility of a single author.

Chapter 2 (Historical Background):

- Joseph Corneli (2011), The PlanetMath Encyclopedia, in *Workshop on Mathematical Wikis (MathWikis-2011) at ITP (2nd International Conference on Interactive Theorem Proving) 2011, Nijmegen, NL, August 27th, 2011.*

  This paper provides a retrospective look at PlanetMath's first decade. This material is used to to develop a dispassionate perspective on PlanetMath's strengths and weaknesses.

Chapter 3 (Theoretical Background):

- Joseph Corneli and Alexander Mikroyannidis (2012), Crowdsourcing Education on the Web: A Role-based Analysis of Online Learning Communities, in Alexandra Okada, Teresa Conolly, and Peter Scott (eds.), *Collaborative Learning 2.0: Open Educational Resources*, IGI Global.

  This book chapter provides an analysis of learning in terms of its constituent roles. This is reassessed here, to better emphasize the importance of context.

- Joseph Corneli and Charles Jeffrey Danoff (2011), Paragogy, in *OKCon 2011: Proceedings of the 6th Open Knowledge Conference.*

  This was the first published paper on the paragogy framework, developed in a pilot study at the Peer-2-Peer University. Paragogy is the central idea that is developed in the thesis.

– Joseph Corneli and Alexander Mikroyannidis (2011). Personalised and Peer-Supported Learning: The Peer-to-Peer Learning Environment (P2PLE), *Digital Education Review*, 20.

  This paper reinterprets paragogy as a set of design principles. These are drawn upon to present a understand stakeholder perspectives on the PlanetMath re-build.

– Joseph Corneli, Paragogical Praxis, *E-Learning and Digital Media*, 9(3), 2012.

  This paper looks at the methods that people can use to "do" paragogy. Together with the design principles mentioned above, this work is used to develop a consistent framework for analysis of peer produced peer learning in terms of system change.

Chapter 5 (Methodology):

– Joseph Corneli and Marisa Ponti (2012), Detecting mathematics learning online, in Hodgson V., Jones C., de Laat M., McConnell D., Ryberg T. & Sloep P. (eds.), *Proceedings of the 8th International Conference on Networked Learning 2012*, 2-4 April, 2012, Maastrict, NL.

  This paper considers the prospects for modeling the causes of learning in an online context. The paper presented foundational designs for both a quantitative analytic approach, and a qualitative heuristic approach. These ideas are sharpened in Chapter 5 to describe the reseach methods that will be used in subsequent chapters.

Chapter 8 (Implementation):

– Joseph Corneli, Constantin Jucovschi, and Alexander Mikroyannidis (2011). PlanetMath Redux: Web 2.0 infrastructure for mathematical problem solving, in *Technology-Enhanced Learning for Mathematics and Science: Landmark Research and New Contributions (TELMAS@ECTEL'2011), September 21, 2011, Palermo, Italy.*

  This paper presented a preliminary outline of what it would mean to convert PlanetMath into a "learning environment." This is expanded here with a requirements analysis informed by research with stakeholders.

– Joseph Corneli and Mircea Alexandru Dumitru (2012). PlanetMath/Planetary, in *Proceedings of Works in Progress track, MathUI and OpenMath workshops at the Conferences on Intelligent Computer Mathematics 2012.*

  This paper presented a technical overview of the Drupal 7 based rebuild of PlanetMath's infrastructure. Chapter 8 presents an analysis of the deployed system, including screenshots.

Chapter 9 (User evaluation):

– Joseph Corneli, Dorota Marciniak, Charles Jeffrey Danoff, Charlotte Pierce, Paola Ricaurte, Jan Herder, Suz Burroughs, George Brett and John Graves (2014). Building the Peeragogy Accelerator, in Megan Quentin-Baxter, ed., *Proceedings of OER14: building communities of open practice*,

  This paper describes work-in-progress on project-to-project collaboration in the Peeragogy project, supported by a design pattern methodology. The approach to collaboration described in this paper informed my interpretation of user feedback on PlanetMath/Planetary.

Chapter 10 (Discussion):

– Joseph Corneli, Anna Keune, Amanda Lyons, and Charles Jeffrey Danoff (2013). Peeragogy in Action, in Kaitlyn Braybrooke, Jussi Nissilä and Timo Vuorikivi, eds., *The Open Book*, #3 in the Reaktio Series, published by The Finnish Institute in London, UK. ISBN 978-0-9570776-3-8.

  One key question in my discussion chapter is whether PlanetMath is useful for instructors, in practice. This book chapter a presents a general-purpose design for peer learning activities; it is instantiated here with a concrete example.

– Joseph Corneli (2014), Towards a Creativity Commons, *Fifth "Workshop on the Ostrom Workshop" (WOW5)*. Working group: "The Commons as Transformative Paradigm".

  This paper helped to clarify the philosophical issues underlying the design pattern methodology. I draw on this work to help explain the broader implications of the research presented here.

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# Part 1
# Introduction

# 1 Studying Mathematics the Commons-Based Peer Production Way

**Mathematics is a socially shared, symbolically – and technologically – mediated *cultural* activity. This suggests that one way to understand mathematics better is to look at how people *do* mathematics.**

## 1.1 What this is and is not

This does not mean watching over individuals who are solving mathematics problems and taking notes on how often they look out the window and whether or not they ultimately get the correct answer. Nor will this study rely, for the most part, on the even more typical (and obvious) idea: being that person oneself. As Eugene Matusov argues:

> *Focusing on the individual as the unit of analysis blinds researchers and educators from seeing the systemic complexity of psychological phenomena and from considering systemic solutions to complex psychological and social problems.* (Matusov, 2007)

Rather than studying mathematics in the traditional way, by solving mathematics problems – or focusing exclusively on studying discrete instances of others' problem solving – the proposal is, to the extent possible, to study mathematics as a cultural system. In itself this perspective is not new. Ludwig Wittgenstein wrote that "*mathematics is after all an anthropological phenomenon*" (Wittgenstein, 1956, §V.26). Wittgenstein looked at the foundations of mathematics primarily through the lens of language. For example: "*A definition signifies a change in technique*" (Wittgenstein, 1976, p. 33).

In 1997–1998, as a beginning student of cultural anthropology and multivariable calculus, it occurred to me that mathematics was perhaps the ideal example of "culture",

when this is understood to mean:

> [S]*ocially acquired, symbolically mediated systems of techno-ecological adaptive strategy, organization* [...] *and cosmology*[.]
>
> Luther P. Gerlach, quoted in (Brandt, 2005)

At the time, my ambition to study mathematics as a participant observer meant doing the obvious thing and becoming a mathematics student. It was only later, when I became acquainted with current computational and computer-aided approaches to mathematics, that the foundations for the current study would begin to develop in earnest.

What I have more recently observed and participated actively in, and what will be described here, is the collaborative design and development of an online learning space for mathematics. This work features mathematical culture at its own most "cultural" level, and evokes "soft" cultural patterns – for instance, values pertaining to enculturation. To talk about enculturation rather than learning here is "*emphasizing not just a conditioning process but an adaptation process*" (Brandt et al., 2006). Indeed, at least in the current setting, this is a dual adaptation: not only of people to culture, but of the culture itself.

This study will explore the ways people think about mathematics and ramify their "native" theories and practices of mathematical culture. It will develop a theory of *peer produced peer learning*, inspired by and as a reflection on mathematical practice, but with broader implications. The work draws inspiration from anthropology and philosophy, but the methods and primary outcomes pertain to humanistic computing and to computer science. The following sections describe the research setting and approach, indicate the main scientific and practical contributions, and outline the argument of the thesis.

## 1.2 Setting and audience

> *PlanetMath is a virtual community which aims to help make mathematical knowledge more accessible. PlanetMath's content is created collaboratively: the main feature is the mathematics encyclopedia with entries written and reviewed by members.*
>
> Aaron Krowne, c. 2001, on HTTP://PLANETMATH.ORG

The PlanetMath encyclopedia currently defines over 15000 mathematical terms, covering both basic and advanced mathematical topics. Each page in the encyclopedia is accompanied by a threaded forum discussion. The development of the site and its highly custom software was the central topic in Aaron Krowne's Masters thesis, *An Architecture for Collaborative Math and Science Digital Libraries* (Krowne, 2003A).

PlanetMath's byline is "*math for the people, by the people.*" It is an example of a resource created through "*non proprietary production by peers who do not interact either through a firm or through a market*" (Benkler, 2002); in short, PlanetMath is *peer produced.* As Chapter 2 explains, a particular sort of wholly virtual ownership underlies PlanetMath's original design, but the site is nevertheless a "*community-managed commons*" (Gardner & Stern, 1996).

Although the encyclopedia is historically PlanetMath's central feature, it is not the project's sole commitment. PlanetMath is better described as a mathematics digital library built "*the commons-based peer production way*" (Krowne, 2003в). In a joint paper exploring potential new directions for the project, we wrote:

> *We find it compelling that, in the context of a digital library, marginal conversations within a text provide a chance for readers to interact with primary authors and with each other, and to become primary authors themselves, all at once. While marginalia are considered to be vandalism in physical library books, in a digital library, there is no reason to fear them – they can easily be hidden away.* (Corneli & Krowne, 2005)

The research question that I will be concerned with here is:

> What is needed to transform this peer produced digital library and mathematics reference resource into a peer produced peer learning environment?

The historical audience of the PlanetMath website was anyone who might consult an online encyclopedic dictionary of mathematics, or who would use a forum to discuss questions about mathematics. This population includes advanced undergraduates, graduate students, mathematics professors, and professionals. The historical co-creators of PlanetMath formed a very particular subset of this population. These were people who were interested in discussing mathematics over the internet, asking questions and sharing existing knowledge, and often learning in the process. Certainly, this was peer produced peer learning, but learning was not the focus – creating an encyclopedia was. Chapter 2 traces the history of the PlanetMath project, and sets the stage for its potential evolution.

My hope is that in the future, PlanetMath's audience – or better still, its public – will expand to include anyone who might consult a mathematics textbook or Schuams outline, or who might take or design an online mathematics course, tutorial, or seminar. Users will be able to collectively inquire into the techniques and approaches that work best, using PlanetMath as a laboratory. Much as GNU/Linux offers a complete free operating system for computers, PlanetMath or its eventual successor may expand to be a go-to place for mathematicians, mathematics teachers, and students at all levels.

For now, however, there are a range of technical issues at stake which limit the scope of the current work to a far more modest scale, and which correspondingly narrows the range of users served. The technical focus of the current work is on building infrastructure that supports sharing, discussing, and solving textbook-style problems at the university or beginning graduate level. A prototypical user would be anyone doing independent study of mathematics at this level – whether or not they are formally enrolled as a student. The particular audience that the research focuses on are those socially-minded independent learners who would be interested in benefiting from or contributing to peer support using this technology.

A concrete use case can be described, building on an essay by computer scientist Clayton Lewis, who writes about the new potential of "DIY Education". In order to explore this theme, Lewis imagines himself caught in a time warp. Whereas $\text{Clayton}_{1963}$ "*lived in the dormitories of the college he attended*," due to the rising costs of education, $\text{Clayton}_{2010}$ "*has no such easy choice*" (Lewis, 2010). He looks around for free or low-cost options, and discovers the Peer 2 Peer University.*

> *Following his computer interests, Clayton explores the course offerings at P2P U. He's attracted by the Python programming course, and also by a cooperative program in Web technology offered jointly by P2P U and the Mozilla Foundation, World of Webcraft. He also thinks about Differential Equations for Engineers, but is put off by the fact that it isn't pure math, something about which he has a childish prejudice. He does sign up for DIY Math, facilitated by a doctoral student at the Open University, because it is open as to topic. He has a vague idea that there should be some interesting mathematics around games, not what is called "game theory", which he has read enough about to know isn't what he's interested in, but something that allows one to analyze games like penning the pig. (He doesn't know that this is called "combinatorial game theory".) He's hoping that the people in DIY Math can help him find something like that. P2P U has no requirements structure, so he doesn't sign up for anything outside his areas of interest.* (Lewis, 2010)

However, as Chapter 3 will demonstrate, *DIY Math* would actually be a singularly bad choice. One problem is that in following his own interests, $\text{Clayton}_{2010}$ "*doesn't have the guiding hand of curriculum* [...] *because P2P U doesn't have a curriculum.*" Nevertheless, Lewis points out that:

> *Anyone could create* [...] *a roadmap for learning in a particular field, or a roadmap for becoming "an educated person" in any more general sense they fancy, and share this via P2P U.* (Lewis, 2010)

One of the key aims behind the changes to PlanetMath is to support the creation, sharing, and critique of such learning roadmaps for the mathematics domain. In addition to peer learning, there is a complementary use case related to peer teaching.

*HTTP://P2PU.ORG

Folded inside of this point there is an important question. If both peer teaching and peer learning are important here, what about other roles? In particular, what about the person or persons who will be involved with creating the website, which offers suitable conditions for others to teach and learn? This points to a very important secondary audience, both for this thesis and the accompanying new software system, which has been been developed and released as free software. These aim to reach the people whose contributions will continue to drive the system forward.

## 1.3 The relevance of an anthropologically informed approach

The studies developed in this thesis are not ethnographic, but I view them as anthropological in spirit. As Tim Ingold explains, "*anthropology is not ethnography*" (Ingold, 2008). He offers this perspective on what anthropology can be: "*Quite simply, it would be an inquiry into the conditions and possibilities of social life, at all times and everywhere*" – in contrast to the more limited "*objective of ethnography*," which is "*to describe the lives of people other than ourselves.*" A lot hinges on Ingold's understanding of "*social life.*" He writes: "*Societies are not entities analogous to organisms, let alone to machines.*" Indeed, he takes an opposite view: "*Organic life is social.*" Ingold's socially-inspired and process-oriented ontology refers to the "*essential interpenetrability or commingling of mind and world*" and to "*the unfolding of a continuous and ever-evolving field of relations within which beings of all kinds are generated and held in place.*"

Ingold's philosophical views are connected to an approach to inquiry that has significant bearing on the approach that will be taken here.

> *What truly distinguishes anthropology, I believe, is that it is not a study **of** at all, but a study **with**. Anthropologists work and study **with** people. Immersed with them in an environment of joint activity, they learn to see things (or hear them, or touch them) in the ways their teachers and companions do.* (Ingold, 2008), original emphasis

A system like PlanetMath is an approximate description, in an active rather than static form, of the way mathematics is done. At its best, such a system simultaneously creates new ways to do mathematics. In the last analysis, the interpretation of mathematical practice at PlanetMath is an interpretation made together with the user community. Notably, if the system is entirely incongruous with what mathematical practice *is*, then people who are interested in mathematics will have very little use for it!

In expanding the theme of "*studying with*," Ingold emphasizes that he is ultimately far less interested in "*description*" than in "*a practice of correspondence.*" In a recent book bringing anthropology together with archaeology, art, and architecture, he writes:

> *To practise this method is not to describe the world, or to represent it, but to open up our perception to what is going on there so that we, in turn, can respond to it.*
> (Ingold, 2013, p. 7)

For Ingold, after Lave (1990), learning is "*understanding in practice*," and after Gibson (1979), it is based upon "*an education of attention*," which is, drawing on his own fieldwork experiences, "*a process of active following*" (Ingold, 2013, pp. 9, 13, 2, 1). These points are are not discipline-specific: they apply equally to hunting and pottery-making, computer programming and mathematics.

The key feature of Ingold's anthropology that brings it closer to art than to ethnography is that it aims to be "*transformational*" rather than "*documentary*" – here noting that "*no genuine transformation in ways of thinking and feeling is possible that is not grounded in close and attentive observation*" (Ingold, 2013, pp. 3-4).

The fundamental place that Ingold reserves for the social in his thinking bears a similarity to George Herbert Mead's conception, which again emphasizes transformation. For Mead (1932), sociality is the "*stage betwixt and between the old system and the new one*," which occurs whenever a new form of organization is being established, but when neither the new form nor the broader environment has stabilized; *cf.* (Aboulafia, 2012). A view in which the environment is not taken as fixed, but as co-emergent with the people who dwell in it is reflected in the PlanetMath context, in the sense that both content and software are expected to grow and develop along with use.

One further quote can be brought forward to illustrate the essential idea of "*studying with people*" that underlies Ingold's participatory anthropology. This quote describes peer learning in negative form – in other words, it says how things are usually done in mainstream education.

> *Students are told that anthropology is what we do with our colleagues, and with other people in other places, but not with them. Locked out of the power-house of anthropological knowledge construction, all they can do is peer through the windows that our texts and teachings offer them.* (Ingold, 2008)

In the parallel setting of mathematics, there are some exemplary exceptions to this norm. Educator Alan Schoenfeld, who describes *metacognition* as the key to working effectively in mathematics (Schoenfeld, 1987), builds his pedagogical practice around asking his students reflective questions like these:

> *What (exactly) are you doing? Can you describe it precisely? Why are you doing it? How does it fit into the solution? How does it help you? What will you do with the outcome when you obtain it?* (Schoenfeld, 1987, p. 206)

He argues that

> [T]*he interactions among mathematicians, and the sense of community they support, are part of what sustains mathematics – that the practice of mathematics is a human endeavor and very much a cultural one.* [...] [E]*ntry into that culture (or some culture that supports the same values) may be necessary to understand and appreciate mathematics.* (Schoenfeld, 1987, p. 214)

Schoenfeld asserts that, for practicing mathematicians, the classroom "*is an essentially alien culture*," and that "[w]*hat we need is a program of 'cultural design' for schooling.*"

## 1.4 What (exactly) is to be done

Having described a basic outlook in which mathematics is seen as a cultural and indeed as an already-anthropological affair, it remains to say what concept of culture is most relevant to the study of mathematics, and what will be done here using this concept. The "*socially acquired, symbolically mediated systems*" definition from Gerlach, quoted above, provides a useful big picture, but in its breadth, it does not exactly give a notion of culture that we can easily grasp. It is, however, compatible with a pointed cybernetic notion of culture, developed by Clifford Geertz. First, a bracketing admonition:

> *Culture is not a power, something to which social events, behaviors, institutions, or processes can be causally attributed; it is a context, something within which they can be intelligibly – that is, thickly – described.* (Geertz, 1973, p. 14)

Without yet worrying about the issues attached to these ideas of context and thick description, the following quote provides a working definition of what *culture* itself can be taken to mean:

> *I want to propose two ideas. The first of these is that culture is best seen not as complexes of concrete behavior patterns – customs, usages, traditions, habit clusters – as has, by and large, been the case up to now, but as a set of control mechanisms – plans, recipes, rules, instructions (what computer engineers call "programs") – for the governing of behavior. The second idea is that man is precisely the animal most desperately dependent upon such extragenic, outside-the-skin control mechanisms, such cultural programs, for ordering his behavior.* (Geertz, 1973, p. 44)

So, for Geertz, culture is already-programming![*] Of course it has, by and large, up to now, been programming for humans, not for computers. It is also important to point out that Geertz is explicitly referring to *cybernetic* programming. In other words, his

[*]Geertz's notion of culture derives from that of Talcott Parsons – "*patterned or ordered systems of symbols which are objects of the orientation of action, internalized components of the personalities of individual actors and institutionalized patterns of social systems*" (Parsons, 1951) – "*suitably emended*"; *cf.* (Geertz, 1973, pp. 254), (Peacock, 1981).

concept of cultural control systems "*presupposes that control is partial and not complete*" (Burroughs, 1978).

In light of Ingold's emphasis on trained attention and in-the-world experience, it is not surprising that he is suspicious of the idea of culture as an external control system. In a survey that treats Geertz and other prominent anthropological figures, he writes:

> [O]*ne learns to perceive in the manner appropriate to a culture, not by acquiring programmes or conceptual schemata for organising sensory data into higher-order representations, but by 'hands-on' tasks whose successful fulfilment requires a practised ability to notice and to respond fluently to salient aspects of the environment.*
>
> (Ingold, 2000, p. 167)

And yet, cybernetic control systems are not acquired in the way we might learn a second (or first) language; culture is primarily context, not content; *cf.* (Ingold, 2000, pp. 163–166). Rather, one might say that these systems are present, as more or less salient features of an environment, on the understanding that these features are emergent along with those who dwell in the environment. Whereas Geertz calls "*any object, act, event, quality, or relation which serves as a vehicle for conception*" a "*symbol*" (Geertz, 1973, p. 91), Ingold seems to be more cautious: "*every feature* [of the landscape] *is a potential clue, a key to meaning rather than a vehicle for carrying it*" (Ingold, 2000, p. 208).

Independent of the precise weight we assign to symbol systems as opposed to meaning making, it would be hard to deny that one the greatest cultural shifts in history is connected with the fact that, in recent decades, certain classes of symbolically-mediated programs can be processed with computers. Granted, people continue to perform "*actions that are far beyond the capabilities of any machine yet devised*" (Ingold, 2000, p. 171). What's of interest here is what happens when humans and computers work together.

> *We have been programming universal computers for about 50 years. Programming provides us with new tools to express ourselves. We now have intellectual tools to express "how to" as well as "what is." This is a profound transformation: it is a revolution in the way we think and in the way we express what we think. For example, one often hears a student or teacher complain that the student knows the "theory" of some subject but cannot effectively solve problems. We should not be surprised: the student has no formal way to learn technique. We expect the student to learn to solve problems by an inefficient process: the student watches the teacher solve a few problems, hoping to abstract the general procedures from the teacher's behavior on particular examples. The student is never given any instructions on how to abstract from examples, nor is the student given any language for expressing what has been learned. It is hard to learn what one cannot express. But now we can express it!* (Sussman, 2005)

Not only can we express "how to" with computers – we can change it! Or at least we're in a better position to try.

As a thought experiment: imagine the distillation of the world's mathematical knowledge into a computer program that could coach students through difficult learning tasks in mathematics in an efficient manner. With a few adaptations, this program might also serve as a creative partner to professional mathematicians. With further adaptations, it may be able to come up with and prove interesting theorems on its own. The development of such a system would certainly constitute a change in mathematical culture. It would also have significant impact on in the way mathematical thinking is applied, in fields ranging from logic to logistics.

In order to move in this direction, one approach is to set aside the AI aspects at first, and focus instead on building a useful collection of *content*. The computer would be used primarily as a communication tool and database. As time goes by, additional sophistication can be added, expressing the user community's collective wisdom about what works and what doesn't. A commons-based peer production system is a suitable *context* for exploring cultural adaptation of this sort. The idea is that a software system created together with its users can be used to build not just a description of mathematical culture, but an actual working model that can serve as an engine for change.

**Theory.** A 6-month pilot study as a participant observer at the Peer-2-Peer University helped develop a general picture of the way learning and adaptation works (and sometimes fails to work) in peer production communities. For example, people often learn computer programming concepts and techniques while building free/open source software and participating in discussions on a developer mailing list. *Paragogy* aims to be a comprehensive theory of peer produced peer learning, and a qualitative complement to Benkler's theory of peer production.

**Tool.** In collaboration with colleagues at in the Knowledge Adaptation and Reasoning for Content (KWARC) group at Jacobs University, PlanetMath's custom code from 2001 was replaced with a new platform built using contemporary technologies. The aim was not just to modernize PlanetMath, but to build a reusable and extensible platform, which we called "Planetary" (Kohlhase et al., 2011). The particular instantiation of this system on PlanetMath is designed to support peer learning according to the paragogical understanding.

**Test.** Launching the Planetary system on PlanetMath constitutes a major change for users. The real-world deployment of the system gave the opportunity to study the response to this change, supported by focus groups and interviews. PlanetMath users provided concrete suggestions on how to extend the software model, which helps build a picture of the way they think about mathematics, and which also contributes to validating the paragogical approach to design, learning, and co-production.

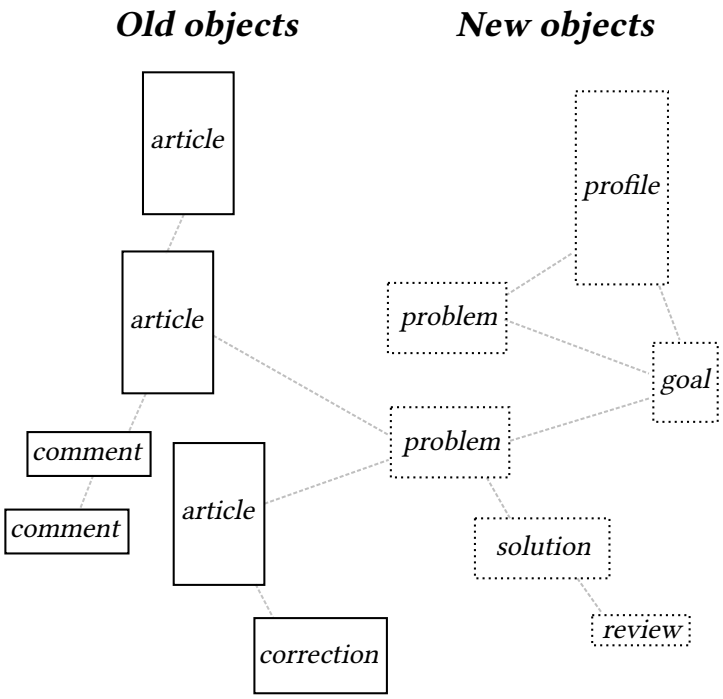



**Figure 1:** *This is the sort of extension we'd like to develop*

## 1.5 Trajectories

**Advances in method and theory.** Using a commons-based peer production system as a way to study and express culture it is an example of the kind of self-conscious experiment with representation found to be lacking by Geertz (1973, fn. p. 19). This approach may be of interest to contemporary cultural anthropologists, who are increasingly involved in participatory research, and fieldwork in networked settings. Paragogy contributes to a broader social science literature, and develops a perspective in which peer production and peer learning are mutually constituting. Although the methods are inspired in part by anthropology and philosophy, the outcome is a new computational account of mathematical collaboration that should prove useful in the design and evaluation of agent-based simulations, as well as online learning environments. The understanding of mathematical social behavior that is developed may be of interest to philosophers of mathematics. These contributions to method and theory are supported by practical advances in learning design and assessment, and provide a framework within which to assess claims about online mediated peer learning, as detailed below. These aspects may be of immediate interest to educators, as well as to education theorists.

**New designs for peer learning.** Gráinne Conole's (2008) perspective on "*learning design*" is that it is best understood as a way to create and represent practice. The design we are using (Figure 1) will include all of the authoring and editorial practices associated with building and maintaining PlanetMath's encyclopedia – and a new layer for problem

solving. The first layer features legacy activities like writing and editing articles, forum discussions, corrections and requests for articles on new topics. The second layer is new, and includes support for setting and solving problems, connecting problems and encyclopedia articles, discussing and reviewing problems, and creating course packets that combine problems with expository texts. From the user's perspective, the environment should offer some significant advantages over a standard textbook or problem archive. For example: nothing is more daunting than being faced with a problem to solve and not knowing what the terms in that problem mean. On PlanetMath, we will be able to provide automatically-generated links to the definitions of technical terms in problem statements. In the event that the linked material is not enough, and the user will be able to make annotations asking help. Answers to these questions will help improve the quality and relevance of the encyclopedia. Over time, we expect a large archive of worked-out examples, heuristics, hints, and strategies to accumulate, helping future users overcome the cold-start problem. Rather than having textbook problems serve as a daunting obstacle to application or research, they should, in this context, serve as stepping stones to relevant and meaningful engagement in mathematical practice.

**A new technique for measuring learning efficacy.** How can we validate claims about learning quality? John Anderson, Lynne Reder, and Herbert Simon advanced a consumer protection perspective on this question.

> *We should make a larger place for responsible experimentation that draws on the available knowledge–it deserves at least as large a place as we now provide for faddish, unsystematic and unassessed informal "experiments" or educational "reforms." We would advocate the creation of a "FEA" on analogy to the FDA which would require well designed clinical trials for every educational "drug" that is introduced into the market place.* (Anderson, Reder, & Simon, 2000)

Techniques for evaluating the efficacy of learning strategies and interventions can also be immediately relevant to users and system designers, in real time. Feedback on learning relevance could help users improve their problem solving skills. Chapter 6 draws on a recently developed approach from computational statistics to address these issues.

**New frontiers for peer produced peer learning.** Partly inspired by the idea of paragogy, Howard Rheingold convened the Peeragogy project,[*,†] which I subsequently joined as a contributing author, and later, editor. Rheingold positions the *Peeragogy Handbook* as "*a resource for any group of people, who want to learn any subject.*"[‡] This thesis will help to refer these claims to sound theoretical and empirical foundations.

[*] http://dmlcentral.net/blog/howard-rheingold/toward-peeragogy

[†] http://peeragogy.org

[‡] http://www.youtube.com/watch?v=SDuSpOUtyJE

## 1.6 Terminology

In cases where it is necessary to distinguish between the "original" PlanetMath built by Aaron Krowne and the "new" PlanetMath developed in the course of this thesis work, I will refer to them as PlanetMath/Noösphere and PlanetMath/Planetary, respectively.

Paragogy is the name for the theory of peer produced peer learning that is developed in this thesis, drawing on earlier collaborative writings with Charles Jeffrey Danoff, and, perhaps confusingly, on contemporary collaborative work in the Peeragogy project. Peeragogy is Howard Rheingold's neologism, adapting my earlier usage, chosen because this term is relatively self-explanatory. What "paragogy" might lack in directness, it makes up in nuance. A term that causes people to stop and think helps problematize the idea of peer learning. The term evokes but also challenges the idiom of the "guide on the side." It is a preexisting word in both English and Greek, so it brings additional associations through its etymology.* There are models of peer learning that remain "*provisionist*" rather than peer produced (Boud & Lee, 2005): in other words, the opportunity for peer learning is provided by some group, and taken up by others. Paragogy comes with a built-in immunity to provisionist thinking. As Chapter 3 explains, the emphasis in paragogy is on co-creating the context in which learning and production take place. The term *peeragogy* is most easily read as a synonym for peer learning. In this document, *Peeragogy* will always refer to the Peeragogy project, and its creations and namesakes (the *Peeragogy Handbook*, Peeragogy patterns, the Peeragogy Accelerator, *etc.*). I use *paragogy* as a shorthand for "the theory and practice of peer produced peer learning."

## 1.7 Related work

The concept of *emergent design* was applied by David Cavallo to create learning environments together with their users (Cavallo, 2000A, 2000B, 2004). Cavallo (2000B) describes his work with Thai villagers to help design local applications for the Logo computer system as "*applied epistemological anthropology*." More recently, this theme was taken up by Cavallo's PhD student, Arnan Sipitakiat (2007), in work with children learning robotics. This practice aims to learn along with research subjects as they build on their existing knowledge and skills, typically in *constructionist* scenarios – in which "*everything* [is to] *be understood by being constructed*" – as opposed to "*instructionist*" scenarios that emphasize teaching (Papert & Harel, 1991).

The idea of emergent design will be connected, in the current work, to the possibility of a "*problematic*" understanding of reality, after Deleuze (1968). Deleuze calls this the philosophy of difference, and it is grounded in his thinking about mathematical dy-

*Greek: *παραγωγή*, meaning "production", "generation"; English: "coaptation", *i.e.* "the adaptation or adjustment of parts to each other", and also "letters added for emphasis or to change the sense of a word."

namics. Building on the associated nexus of philosophical and scientific ideas, Manuel DeLanda (2011) makes a survey of the history of life through a series of computer simulations of dynamical systems, starting with water and ending with pyriamids.

The array of "*man-made information sources for the directive ordering of human conduct*" (Geertz, 1973, p. 251) that are relevant to mathematical practice can be treated with a similar multi-layered computational approach. Much more so than Geertz, the associated idea of *cybernetics* is associated with the work of the anthropologist Gregory Bateson, who insisted on the importance of not separating the "*internal*" and "*external mind*" (Bateson, 1972, p. 470). Ingold continues this line of thought and treats "*organism plus environment*" as a "*developmental system*" (Ingold, 2000, p. 19). Ingold does not talk about development using the language of difference from Bateson and Deleuze but emphasizes *growth* and, especially, *correspondence* (Ingold, 2013, pp. 21, 107). Chapter 4 will draw on Bateson's perspectives on learning.

Economics and organizational theory are almost entirely in the background in the current work: the focus is on phenomena that economists tend to treat as *externalities*. Briefly, one way to deal with externalities this is to create new markets, and another is restructure the firm; both scenarios were examined by Coase (1960, 1937), who emphasized the costs associated with exchange or transformation. Benkler's (2002) idea of commons-based peer production is a third model built around positive externalities that coalesce into new shared resources. Ostrom's (1990) design principles for the governance of common pool resources can be related to Beer's (1979, 1981) Viable System Model, which describes a cybernetic approach to organization.* Paragogy is developed as a parallel framework, focused on the dynamics of learning and adaptation.

Mathematics is usually managed through professional societies, firms, and the guild-like structures of academia. The possibility of managing it in a more "*open*" fashion has been explored by various authors; some, like (Buswell et al., 2004), considering technical issues, others focusing on social engineering (Pitman & Lynch, 2014). To my knowledge there is no prior study on considerations related to learning mathematics the commons-based peer production way. Within the field of mathematics education, most research focuses on classroom settings (Cole, 1998; Cobb, Wood, & Yackel, 1993; Cobb, 1989).

There has been broadly related research on learning in Wikipedia and Wikiversity (Lawler, 2005, 2011), and on learning in free/libre/open source software (Meiszner, 2010). I will draw on the latter in Chapter 4. It is also worth mentioning the traditions of *collaborative learning* (Bruffee, 1984), and *networked learning*.† The latter takes inspiration from Ivan Illich and Christopher Alexander. I draw on Alexander's design pattern methods, connecting them to the idea of emergent design and Deleuze's philosophy.

*http://p2pfoundation.net/Cybernetics_of_the_Commons

†http://en.wikipedia.org/wiki/Networked_learning

## 1.8 Summary of the argument

**Part 2 of the thesis describes the research domain.**

I make a case for using a peer production model to build learning resources for mathematics. I also make it clear that PlanetMath needs a new direction if it is to have contemporary relevance. I introduce paragogy through a description of the case study at P2PU in which it developed. The key theme in paragogy is to combine learning at the individual level with contextual change, that is, change in organization or environment. It is a theory of learning and adaptation. I develop paragogy as a framework, which is then used to organize the research literature that is relevant to peer produced peer learning in mathematics. This yields a preliminary set of criteria that can help address the main research question.

**Part 3 describes the research methodology and preliminary studies.**

A key philosophical idea in the thesis is the idea of emergence: the idea that the whole is always more than the sum of its parts. My methodology is rooted in Manuel DeLanda's ideas about emergent dynamics that develop within and across different domains and different levels of analysis (DeLanda, 2006A, 2011). At the micro-level, I develop a statistically-based study of the social antecedents to learning in PlanetMath/Noösphere. This shows how a system like PlanetMath could be used as a research instrument to study patterns of change from a quantitative perspective. I gather ideas about macro-level systemic change from representatives of three important stakeholder groups: students, mathematicians, and researchers who study mathematical practice.

**Part 4 presents the primary results and discussion.**

I describe the implementation of the new Planetary system, which has been built taking the ideas from stakeholders on board. After the system was deployed on PlanetMath, a user evaluation study was conducted. Based on material drawn from interviews with users about their experience with PlanetMath/Planetary, I develop a revised design that better reflects the purposes that users see the site as serving. Finally, I conclude that a paragogical approach to emergent design can be effectively applied within mathematical culture as a way "*to pose the problems set within it and to it by the differential relations it incarnates*" (Deleuze, 1968, p. 235).

# Part 2
# The research domain

# 2 Historical Background

**Could learning mathematics become more like the learning that happens in the course of developing free/open source software?**

## 2.1 The PlanetMath Encyclopedia

### Beginnings

The pre-history of PlanetMath was tied to the fate of a similarly-named website, MathWorld.* Eric Weisstein began collecting the material now found in MathWorld as a high school student, and continued the project as a college student in the late 1980s. "Eric's Treasure Trove of Mathematics," went online in 1995, when Weisstein was a graduate student in astronomy at the California Institute of Technology.†

In November 1998, Weisstein made a deal with the CRC Press to publish his encyclopedia in book format, as the *CRC Concise Encyclopedia of Mathematics* (Weisstein, 2003). One year later, Weisstein accepted the position of Encyclopedist at Wolfram Research, Inc., and the newly-rebranded "MathWorld" site was unveiled in December 1999.‡ In March 2000, CRC Press sued Weisstein and Wolfram Research for copyright violation, forcing MathWorld off of the internet (US Patents Quarterly, 2000). Eric Weisstein was understandably dismayed: "[I]*f you ever assemble a body of knowledge that you want to share with others, you don't want to go through what I have just gone through.*"§

*http://mathworld.wolfram.com

†http://www.echarcha.com/forum/archive/index.php/t-19516.html

‡http://en.wikipedia.org/wiki/Eric_W._Weisstein

§http://www.echarcha.com/forum/archive/index.php/t-19516.html

MathWorld users Nathan Egge and Aaron Krowne, then undergraduates at the Virginia Polytechnic Institute and State University, were also dismayed – and in Autumn, 2000, they devised the idea for PlanetMath: it would be a collaboratively created mathematics reference work that would have resistance to copyright threats built-in, in the form of an open content license.

By Summer 2001, the basic infrastructure for creating an encyclopedia was complete, and a fledgling community had grown up around PlanetMath. It was not alone: Wikipedia also launched that year. Mayo Fuster Morell describes 2001 as a watershed year in the history of online creation communities – the year when the focus spread "*from free software to free culture*" (Morell, 2011).

The CRC lawsuit was settled for an undisclosed sum in late 2001, and on November 6, 2001, MathWorld returned to the internet.* But in the mean time, a new online community had been born – with some very different principles and practices. Whereas MathWorld's terms of use disallow archival copies, PlanetMath regularly published snapshots of the content for download. Moreover, under the terms of the site's license (originally the GNU Free Documentation License, later the Creative Commons Attribution-ShareAlike license)†,‡ users are permitted – and, indeed, encouraged – to copy, mirror, redistribute, print, remix, and reuse PlanetMath content for commercial or other purposes – so long as all such works are published under the same license as PlanetMath, granting downstream users the same rights.

## Stabilization

The key reference on the founding and foundation of PlanetMath is Aaron Krowne's (2003a) Master's thesis, *An Architecture for Collaborative Math and Science Digital Libraries*, written at Virginia Tech under the supervision of Ed Fox. In this thesis, Krowne describes how the early design and development of the site benefited from continuous feedback in the `#math` IRC channel on Undernet.§ He also details the key technical and community features of the site as they developed in this period:

- A state-of-the-art system for displaying mathematical notation on the web, starting from LaTeX sources.
- A flexible authority model that can support both wiki-style articles (that anyone can edit), and a more academic style, where articles are owned by one person, who may, if they wish, grant co-authorship permissions to chosen others, and who

*http://mathworld.wolfram.com/about/faq.html#history
†http://www.gnu.org/licenses/fdl-1.3-faq.html
‡http://creativecommons.org/licenses/by-sa/3.0/us/
§irc://irc.undernet.org/math

must respond to separate commentary from peer reviewers in order to maintain ownership (Krowne & Bazaz, 2004).

- A discussion forum attached to every encyclopedia article, which helps give the resource its "*pedagogical slant*."*
- An autolinking service that helps integrate content into the site, by enabling authors to focus on the contents of one article at a time.
- The Mathematics Subject Classification (MSC) devised by the American Mathematical Society and Zentralblatt MATH provides a hierarchical browsing mechanism.
- A peer review workflow built around a content-level bug reporting feature called "corrections", and user-messaging feature called "watches." The workflow includes a feature whereby articles are "orphaned" if a correction is not responded to after a given period of time. A system for bug reports is particularly important in light of the authority model mentioned above: the historical PlanetMath is *not* an "*encyclopedia anyone can edit*"† in the typical wiki sense.
- "Attachments" expand individual articles with detailed proofs and related expositions.
- A scoring feature provides a rough estimate of how much value each user has contributed to the site.
- Downstream use and modification of PlanetMath's content is permitted, under the terms of the Creative Commons Attribution-Share Alike license (CC-By-SA), a copyleft free content license.

One can readily spot ways in which the core ideas of commons-based peer production – *modularity*, *granularity*, *integration* (Benkler, 2002) – had an influence on the design of the site. Articles are modular and relatively self-contained. Comments and corrections are even more fine-grained than articles. The Mathematics Subject Classification, attachment feature, and autolinking service connect related articles, assisting with integration. Article ownership modularizes control, and provides a protectionary measure for authors that is not given by the content license. Raymondite "homesteading" was built in to PlanetMath's custom software, Noösphere, from the beginning – and the system takes its name from Eric Raymond's (1998) essay.

In 2003, PlanetMath incorporated, and in 2005, obtained non-profit status, so that it could accept tax-deductible donations in the US. Together with a small stream of advertising revenue, donations covered hosting and other maintenance costs since that time.

*HTTP://PLANETMATH.ORG/HISTORY.HTML

†HTTP://EN.WIKIPEDIA.ORG

## Mathematics problems

This section describes my own early involvement with the PlanetMath project.

From 2002-2004, I was enrolled as a graduate student in mathematics at the University of Texas in Austin, and in possession of a large and growing personal collection of tersely-written definitions and proofs related to the department's preliminary exams. For example:

(lebesgue outer measure: fact: lebesgue outer measure is infimum of lebesgue outer measures of open supersets)

1: $X \subset \mathbf{R}^n$

2: $L = \{O \subseteq \mathbf{R}^n : O \supset X\}$

3: $|X|_e = \inf_{O \in L}(\{|O|_e\})$

My name for this collection was the "Austin Problems in Mathematics – Cross-Index", named following the pattern established by *Berkeley Problems in Mathematics* (De Souza & Silva, 2004), and styled APM-Ξ.* In fact, this work had as much to do with the tradition of computer mathematics in the air in Austin as it had to do with exams.

This tradition was embodied in systems and projects like QED (Boyer et al., 1994), Maxima (Joyner, 2006), ACL2 (Kaufmann & Strother Moore, 1996), AM (Lenat, 1976; Lenat & Brown, 1984), Cyc (Lenat, 1995), ISAAC (Novak, 1976; Novak & Bulko, 1993), and KM (Clark & Porter, 1999) among others. Although I was in the mathematics department and not the computer science department, I was quite drawn to this body of work. My notion was to express all of the definitions and theorems relevant to the prelim exams in a symbolic form that could be relatively easily translated into computer code.†

I was convinced that writing down all of the relevant definitions, theorems, and proofs in meticulous detail would effectively guarantee that I would be able pass the exams – even if I wasn't able to build a mathematical artificial intelligence that would pass them on my behalf (or tutor me) in time. "Relevance," to my mind, was determined by

*HTTP://METAMESO.ORG/~JOE/MATH/XI.PDF

†Per HTTP://METAMESO.ORG/~JOE/MATH/100.PDF, the translation of the above example into a more computer-friendly syntax would have been:

```
(defthm leb-outer-measure-inf-over-open-supersets (X)
  (subset X Rn)
  (let ((L (setof (open-subset O Rn) :st (supset O X))))
    (eq (leb-outer-measure X)
        (inf :over O :in L :of (leb-outer-measure O)))))
```

inclusion among the problems and ideas found in old exams – which were available from the department's filing cabinets (without solutions).

I purposefully neglected the fact that the required first year courses were expressly designed to prepare students for *this* year's preliminary exams, which would be written by the instructors for these courses. Accordingly, course homework and in-class exams would have been a better source of material for my cross-index. But here I found myself caught in something of a perfect storm. Compared with my earlier experiences with more tutorial-style instruction as an undergraduate, the almost exclusively lecture-oriented classroom culture was stifling. I had been further spoiled by early participation in collaborative mathematics research. I thought that the instructors were supposed to help me learn mathematics, and indeed, help me to get to the research frontier as quickly as possible; instead, it seemed as if they were putting in effort to make the entire experience as *difficult* as possible. Finally, I was aware that other students would often study for the courses together, but I wasn't sure what to think about this. Was it encouraged? Was it cheating? As it turned out, I wasn't invited, and I didn't invite myself.

Instead, I spent most of my time building my cross-index of old exam material, reading philosophy books, and learning how to program in LISP with support from the contributors to mailing lists related to GNU Emacs, "*the extensible, customizable self-documenting display editor*" (Stallman, 1981). And I thought: why stop with the graduate curriculum? And so the goal shifted: let's build a copyright-free computer simulation of all known mathematics!

It seemed to me that the mathematics community was entirely uninterested in any potentially liberatory aspects of computers. A new project focused on making computer mathematics relevant to human mathematicians – here I was inspired by GNU and Deep Blue – seemed to be the only suitable response. I called this the *Hyperreal Dictionary of Mathematics* project; *cf.* (Baudrillard, 2002).[*,†]

In 2003, I discovered the PlanetMath website and initiated a correspondence with Aaron Krowne about the potential for overlap between PlanetMath and HDM. I admired the fact that he had put together a real, working, system, with actual users. Considering the magnitude of HDM project, I knew it would eventually need a crowdsourcing component. PlanetMath wasn't doing quite what I had in mind, but it seemed like a good place to start. In Autumn of that year, we uploaded the contents of the APM-Ξ into the PlanetMath encyclopedia as world-editable "seed entries".

This was not received particularly favorably by PlanetMath users, which generated a range of useful reflections.[‡] The primary concerns raised by community members were:

[*] HTTP://WIKI.PLANETMATH.ORG/CGI-BIN/WIKI.PL/HDM

[†] Kellner (2007) offers a useful overview of Jean Baudrillard's writing.

[‡] HTTP://WIKI.PLANETMATH.ORG/CGI-BIN/WIKI.PL/ONE_WEEK_IN_OCTOBER

(1) The entries could not be understood without reading an accompanying FAQ, and were not "human readable";

(2) A casual visitor to the PlanetMath website might get the wrong impression about the nature of the encyclopedia when looking at the owned by the user apmxi; and,

(3) Nearly 600 entries had been introduced into PlanetMath by the site's administrator in one big batch, circumventing, at least in outward appearances, the site's usual model of careful review and collaborative editing of entries.

Subsequent to a poll, it was decided that the entries owned by apmxi would be "orphaned," and any that were not adopted by community members after a week would be deleted from the encyclopedia. This was the fate that befell most.

It was clear in hindsight that the APM-Ξ material – independent of its potential value for computer mathematics – was *not* something that had a positive value for most people in the PlanetMath user community. Although it was something of a personal embarassment, the apmxi incident was a testament to the strength of PlanetMath's norms, and it illustrated the role that shared norms play in maintaining an integrated common resource. It also showed that the specific affordances of computer technology, for instance, for bulk processing, or for using hypertextual methods to present alternate treatments of a given topic, would need to be used and developed carefully, so as to work well for the *people* involved.

As could have been expected, over the next year, things ground slowly to a halt for me in Texas. With no prelim exam success, I was transferred into the Master's program, and given the opportunity to work with the mathematician and computer scientist Robert Boyer, but he and I couldn't see eye to eye about a thesis topic. At the end of Autumn in 2004, I was formally asked to leave.

Although I had been an errant and irresponsible mathematics graduate student, I had assembled another identity along the way: that of a necessarily interdisciplinary researcher working on understanding and transforming mathematical culture. Although PlanetMath was in many respects "marginal" to mainstream mathematics, and my project was on the fringes even of PlanetMath, this was a digital library – where the margin was perhaps not such a bad place to be.*

*In connection with the idea of "*math for the people, by the people*", is worth pointing out that Arendtian *constituent moments* "*enact felicitous claims to speak in the people's name, even though those claims explicitly break from the authorized procedures or norms for representing the popular voice*", *cf.* (Frank, 2009, p. 210).

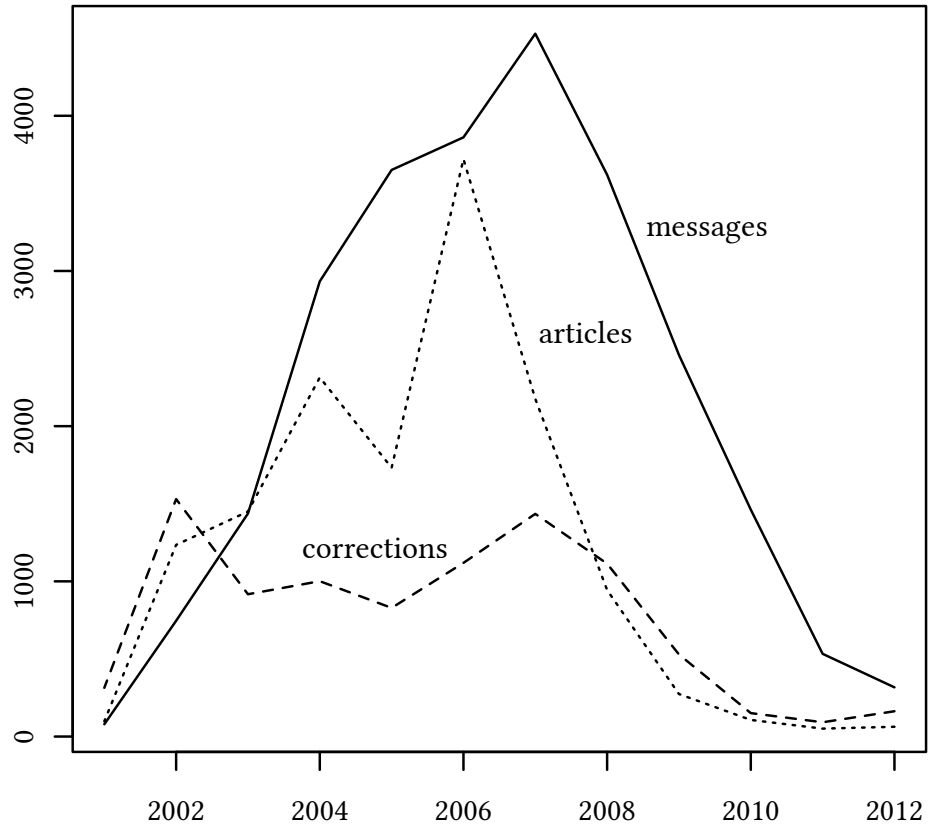



**Figure 2:** *Plot showing the number of messages, articles, and corrections added to PlanetMath over the last decade*

## Reach and scope

Since PlanetMath describes itself with the by-line "by the people, for the people," one way to assess its quality, success, and vitality is through data on user participation. As an initial foray, Figure 2 shows the contributions per year in various sectors of the site. As is typical for online communities, contributions are uneven, and roughly follow a "power law" distribution, *i.e.* $f(x) = ax^k$. The relevant exponent is approximately $-4$, which represents a much steeper dropoff than popularity on the internet ($\approx -3$), energy efficiency of scale for living organisms ($\approx -2$), or word frequencies in natural language texts ($\approx -1$); *cf.* (Corneli, 2011).

PlanetMath can also be evaluated as an encyclopedia. A brief summary, following the framework used by Emma Previato (2004) in her review of the *CRC Concise Encyclopedia of Mathematics* (Weisstein, 2003), gives a look at how PlanetMath measures up (Table 1).

| | |
|---|---|
| **Coverage** | The median entry would be an advanced undergraduate or beginning graduate topic. PlanetMath is generally considered to have more in-depth treatment of technical issues, e.g. of proofs, than that found in Wikipedia. Over 15000 terms are defined in over 9000 articles. |
| **References** | Present in many articles, although there is not yet a unified database of references or style of presenting them (this is planned). |
| **History** | 600 items, mostly 20$^{th}$ Century or later (HTTP://PLANETMATH.ORG/MSC_BROWSER/01Axx) |
| **Audience** | Consistent with the coverage, there have been 4127 posts in the "Graduate/Advanced" forum, 4261 posts in the "University/Tertiary" forum, and 1199 posts in the "High School/Secondary" forum. |
| **Clarity** | There is no hard and fast rule. Some articles will tend to be minimalistic, but precise (particularly when the authors have English as a second language). Other articles even from native speakers may be verbose and vague. In any case, debates over clarity of presentation are intense, and a high standard is maintained. |
| **Pictures** | There are over 600 images, but since this is only about 7% of the number of articles, the prevalence could be improved. A unified database/gallery of pictures would help. |
| **Accuracy** | At the time of this writing there are 48 outstanding corrections; more than 14080 corrections have been filed since the site began, though 2337 are classified as "addenda", meaning that no mistake is implied, and 9182 are classified as "meta/minor", which suggests that in the entire history of PlanetMath some 2561 real errors have been found through the peer review process (and all but a few fixed!). Note that these numbers do not take into account changes initiated by authors without prompting, and would tend to under-represent error fixing in world-editable articles. |
| **Unusual** | PlanetMath provides "math for the people, by the people". |
| **Weight** | PlanetMath can be comfortably edited from a lightweight laptop weighing about 1kg (and browsed from a mobile device weighing considerably less). |

**Table 1:** *Succinct review of the PlanetMath Encyclopedia*

The dropoff in participation in recent years is presumably due to a mixture of factors: saturation of the encyclopedia, technical problems with the site's software, and the rise of successful competition. Most notable among the "competitiors" are two other sites, Wikipedia and StackExchange, both using the same license as PlanetMath.*,† It is clear from Figure 2 that PlanetMath will need a new approach if it is going to continue to be a productive community.‡

As Sophia Angeletou et al. write:

> *Health of online communities is a relatively new and complex concept that is codependent on the emergence and evolution of user behaviour in those communities.*
> (Angeletou, Rowe, & Alani, 2011)

## 2.2 The broader context: Other candidate solutions

There are already several *gratis* (and some *libre*), well-established players in the open/online mathematics learning space. In addition to Wikipedia and StackExchange, mentioned above, other relevant service providers include MITx, Coursera, OCW, edX, Stanford, Khan Academy, PurpleMath.com, OpenStudy.com, and functionspace.org. On offer are articles, Q&A forums, courses and course materials, problem sets and videos, as well as peer support in various forms.

For now, through it's economy of scale, Wikipedia is dominating in the encyclopedist prosumer market; *cf.* (Jankowski, 2013). After more than a decade of work, PlanetMath is valuable as a highly curated topic-specific reference resource, but Wikipedia is more active. It also seems to define many more mathematics terms than PlanetMath, although a structured comparison of coverage is still being charted.§ Perhaps surprisingly, even Wikipedia is less active than it once was (Simonite, 2013), and there appear to be bigger patterns at work, with social media on the rise, while centralized peer production sites are less used.

Could PlanetMath have any technical, or ethical, advantage (Landini, 2012) over existing technical innovations like Wikipedia, Massive Open Online Courses (MOOCs), Q&A sites like StackExchange, and videos collections and problem sets like the ones produced by Salman Khan and his colleagues? Is there already an adequate answer to the demand

*"*There are approximately 29,280 mathematics articles in Wikipedia,*" from HTTP://EN.WIKIPEDIA.ORG/WIKI/PORTAL:MATHEMATICS

†As of June 16, 2013, math.stackexchange.com and MathOverflow contain 147768 and 42735 questions, respectively.

‡Note that hundreds of PlanetMath articles have been used as seed material for world-editable articles in Wikipedia, under the auspices of the (somewhat inaccurately named) "PlanetMath Exchange". HTTP://EN.WIKIPEDIA.ORG/WIKI/WIKIPEDIA:WIKIPROJECT_MATHEMATICS/PLANETMATH_EXCHANGE

§Deyan Ginev, personal communication

for high-quality, low-cost, globally accessible, mathematics instruction? This brief discussion focuses particularly on comparisons with Khan Academy.

Khan Academy,[*] founded by Salman Khan in 2006, is built around a collection of instructional videos hosted on YouTube, which are supplemented with work-book style exercises, mostly for school-level topics.[†] It was initially best known for mathematics content, although more recently, it has added content in a range of other fields.

Khan Academy has helped to popularize the "flipped classroom" model, in which lectures are viewed out of class, and homework is assigned to be done in class (Khan, 2012). The key thought behind this model is that teachers can help students who are struggling, or use class time to invite students to help one another. Adopting a flipped classroom model for graduate-level mathematics would have helped address my discontents at UT Austin, since in-class discussion would have replaced note taking as the norm.

However, Khan Academy and most current MOOCs still have a highly-centralized production model, whereas PlanetMath's content is entirely user-built. Khan Academy's application of the non-free Creative Commons Attribution-NonCommercial-ShareAlike license (CC-By-NC-SA) reflects their centralized stance.[‡,§]

PlanetMath can continue to grow in a comparatively demand-driven and participatory manner, which facilitates the development of advanced material and detailed expositions. To use Landini's (2012) term, PlanetMath is considerably more "*malleable*" than Khan Academy and most MOOCs. Public wikis provide another relatively malleable alternative – but they are most malleable at the content level. In fields like sociology, it makes good sense to invite students to author or edit encyclopedia articles as part of the learning process (Konieczny, 2007). It is hard to imagine a mathematical curriculum that did not focus on problem solving. A customized version of a mainstream content management system, together with robust support for LaTeX, makes Planetary at least as flexible a typical wiki, and can help build additional workflows.

While a centralized production model makes a great deal of sense for a resource with highly integrated content (like a course), a distributed production model makes more sense when building a resource that is relatively disaggregated (like a Q&A collection). PlanetMath/Planetary allows for both disaggregated one-off questions and answers, and more cohesive textbook or course-like exposition – and it seeks to link these two models together. As with Khan Academy, the new PlanetMath content should be useful for self study, which may facilitate more efficient use of in-person interactions. And like MOOCs, PlanetMath should also simultaneously open up new avenues for peer support

[*] http://www.khanacademy.org/

[†] https://www.khanacademy.org/exercisedashboard

[‡] http://creativecommons.org/licenses/by-nc-sa/3.0/

[§] http://freedomdefined.org/Licenses/NC

online. Unlike these more centralized models, PlanetMath is oriented around end-user co-production of structure, preferring an "emergentist" approach to what Boud and Lee (2005) would call the "*provisionist*" approach.

In the first place the question is not whether this approach is "better," but what it may possibly be better for. I would argue that an emergentist strategy is better for working with advanced content. Only relatively well-understood content can be developed locally, for any given locale. This suggests a deeper point, which does indeed take on an ethical dimension. In short, we should eschew provisionist approaches to mathematics on the basis of Euclid's assertion that *there is no royal road to geometry*. Even if we managed to build an computer tutor that delivered superb personalized instruction, and which made learning what we currently think of as difficult mathematics fairly easy, this too should ultimately be surpassed. That would be, as ever, a lot of work. The most important insights would surely come from the system's users.

The fact that the courses and lectures provided as part of mainstream mathematics are good enough "for most people" is not a real counterargument, since, in the first place, most people have never set foot in an advanced mathematics classroom, and secondly, we don't yet know how effective an alternative emergentist approach would be. Existing educational research suggests that if they were available, personalized tutorials would be more effective than classrooms (Bloom, 1984). Unless there was compelling evidence to show that an emergent peer learning approach was either ineffective, or otherwise impractical, it seems vital to give it a serious try, at least on an experimental, demonstation, basis.* Emergence, here, does not mean adding more and more elements to a loose network of associations and expecting meaning to appear, and neither does it at entail giving up on training as such. Rather, it is a question of establishing a suitable *ethos*, as a higher-order constraint on behavior; *cf.* (Sloterdijk, 2013, pp. 162–163), (Deacon, 2003). Once this is done, the corresponding ethical argument is no longer needed: the point will have become, as mathematicians like to say, obvious (Cooksey, 2005, p. 3).

## 2.3 The road ahead

Although it has experienced both a rise and a fall, PlanetMath/Noösphere was reasonably successful as an online community: the software stabilized early on and has required little upkeep, while the site continued to grow. The downside of early stabilization is that relatively few new features were developed since the early 2000s. The sign that there might be a problem – and some unrealized potential – was that feature

*Research mathematics already has many of the features typical of peer production, and the standards for openness in modern mathematics tend to go beyond those in many other academic fields. For instance, Article 2 of the WIPO Copyright Treaty holds that mathematical concepts, per se, are not copyrightable. Mathematical expressions may be further excepted by the legal notion of *scènes à faire*.

requests kept coming in, while very little new code was being written.* Despite these concerns, there is still an active interest in the project nearly a decade and a half after its creation.

Figure 3 presents a use case for PlanetMath that inspires the technical work carried out in support of this thesis. The diagnostic features described in the use case were not realized, although in light of the new semantic features that are included in Planetary, they may not be far off. Certainly they would not require a mathematical "Deep Blue".

As with flipped classrooms, had a system like this existed in 2003, I would have been a very happy user, and contributor. My basic concern at the time was: couldn't learning mathematics be done the same way I was learning LISP? There was detailed documentation available in Emacs, together with the full source code, a Read-Eval-Print Loop (REPL) for trying things out, and an integrated debugger. If I ran into questions that I couldn't resolve on my own using these resources, there were public mailing lists with participants who were always glad to help me, and to suggest improvements to any new packages I might upload. Although much remains to be done, PlanetMath/Planetary should eventually support this style of learning in mathematics.

Indeed, the renewed activity in software development is based on a redesign, not just a rebuild. PlanetMath/Planetary puts more emphasis on content reuse, and, being using a mainstream content management system, makes the system better suited than Noösphere was to support ongoing development. In a philosophical sense, PlanetMath represents a new and distinct type of encyclopedia, and it is now poised to begin its final ascent to the literal classical meaning of *ἐγκύκλιος παιδεία* ("recurrent instruction"). Because the technology that supports the site is special-purpose (and is not constrained to be "a wiki"), it can helped zero in on the specific set of features that support commons-based peer production – and now peer learning – in mathematics.

## 2.4 Summary

From a development standpoint, this is an critical moment, since there are mature, extensible, web frameworks available, together with a relevant collection of content, together with a clear demand for a new approach. As a researcher, this is interesting, because it affords the opportunity to examine technological change as an insider in the process. At the same time, PlanetMath occupies a novel position: although it is host to various examples of mathematical practice, it is still outside of the mathematical mainstream. It is useful as vantage point from which to survey and reflect on mathematics as a whole.

*http://wiki.planetmath.org/cgi-bin/wiki.pl/Feature_Requests

***Primary Actor***: A student, Madeleine, who is trying to learn multivariable calculus.

***Main Success Scenario***:

1. Madeleine is enrolled in an advanced calculus course at university. She learns about PlanetMath from her instructor who recommends it as a place for extra practice with homework problems. Madeleine creates an account, fills in basic profile information, and starts solving problems that the system supplies based on the information she added.

2. The problems that the system supplies are automatically linked to reference resources in PlanetMath's encyclopedia. This expository material gives Madeleine easy access to the relevant mathematical concepts, examples, and hints needed for solving the increasingly difficult practice problems. However, she eventually runs into a problem where neither the automatically supplied information, nor her current knowledge of the subject, is sufficient. She's completely stuck on a problem having to do with water flow in a pipe! Madeleine attaches a help request to the problem: "I understand that I have to use the two variables $x$ and $y$ to solve for water flow, but I don't understand what the boundary limits of the equations would be: do I have to convert it to polar coordinates?"

3. This request is noticed by Natalie, a mathematics graduate student who regularly looks at the feed showing "recent requests for help with advanced calculus." She sees that the reference resources linked to Madeleine's problem are probably not sufficient, and that Madeleine's idea about using polar coordinates would work. Natalie makes some changes to the encyclopedia indicating that converting to converting to polar coordinates can be necessary in pipe flow problems, and sketches an example. Natalie then checks that this information links to Madeleine's problem correctly, and alerts Madeleine to the changes. With this new information, Madeleine is not only able to solve her problem, but can proceed with confidence: she had the right idea after all!

**Figure 3:** *An example use case for problem solving support on PlanetMath*

Although there are other platforms that people can use to discuss problems, none of them clearly present the opportunity to make something that is better than a textbook from the point of view of learning, cohesion, and exposition. Considering the difficulty of managing advanced material and personalizing the learning experience, I make the case for using a somewhat distributed peer production model, rather than a centralized provisionist model. Although there are many great textbooks, and great teachers, we don't yet know whether the computer, or computer mediated communication, can deliver comparable (or complementary) results. The kind of research that is developed in thesis is needed in order to make that research possible.

This project builds on a decade of experience with the PlanetMath community. In this chapter, I have highlighted some of the problematic aspects that experience. I came to the project with the view that mathematics itself is already a kind of simulation, even if mostly still carried out on paper and with chalk – and, indeed, already a kind of virtual community, even if most of its activities take place offline and in person. My philosophy at the time was "*To simulation we reply by simulation*" (Baudrillard, 1983, p. 87); these days, it's closer to "*What it is is up to us*" (Rheingold, 2000, p. 39). This applies to PlanetMath, certainly, and may apply to mathematics more broadly, if we're willing to tackle some problems that aren't drawn from filing cabinets (Deleuze, 1966, p. 15). To supplement my participant observer perspective on these matters, I've presented some figures on participation drawn from the PlanetMath database, which confirm the need for a change in direction if the PlanetMath project is to continue in an active form.

# 3 Theoretical Background

**Theories of pedagogy can help us "*get to the root of conditioning through practising repetitions*" (Sloterdijk, 2013, p. 199) – but they fail to model the productive, explicating, way in which learning unfolds in online peer production communities. These features are better described with the new-old word, *paragogy* (from *παραγωγή*, production).**

## 3.1 Paragogy

The term *paragogy* (literally, "para-" alongside, "-gogy" leading) is used here to characterize the critical study and practice of peer produced peer learning, adapting the classical concept of *pedagogy* and the relatively recent notion of *andragogy* (Knowles, 1968) to a peer learning context. The need for a theory of this nature was articulated in dialog and collaborative research with Charles Jeffrey Danoff (Corneli & Danoff, 2011a, 2011b). Having first met in person at Wikimania 2010, and then again online at P2PU where we were preparing to become course facilitators, we agreed to enroll in one another's first round of courses and share feedback. P2PU aims to provide a social layer that is not supplied by default with Open Educational Resources,* and our agreement presented us with another small scale social context within P2PU. Paragogy was developed in Danoff's *Collaborative Lesson Planning* course, as we tried to understand why my *DIY Math* course had gone so badly (see Appendix A for the course description and post mortem). We used paragogy to examine a range of courses and support experiences at P2PU. After describing this analysis, the chapter concludes with a discussion of paragogy as a unified perspective on individual and contextual change.

*http://www.shuttleworthfoundation.org/projects/p2pu/

## DIY Math

During Autumn of 2010, I made a brief and fairly naive effort to run a discussion-based online mathematics course at P2PU called "*DIY Math*", with the hope that participants would self-organize a suitable peer-learning structure. My motivation for running the course was partly to experiment with peer learning and self-organizing structures in mathematics. At the same time, I was also interested in exploring possible organizational connections with P2PU, where I hoped PlanetMath might eventually come to play the role of a mathematics department. Table 2, drawing on a broader discussion of roles in education from (Corneli & Mikroyannidis, 2012), applies a framework from Ken Wilber (1997) to summarize the overall picture.

When *DIY Math* actually ran, participants quickly lost interest, and all participation stopped after two weeks.* Although this was clearly a failed experiment from a teaching perspective, it is actually quite useful as a baseline reading for the broader project of understanding peer produced peer learning. The experiment shows what happens in a learning project where there is essentially no structure provided at the outset.

The rapid failure of *DIY Math* provoked a certain amount of self-criticism, but it also led me to ask the following critical questions about the learning context at P2PU:

- *What if P2PU was taking an ongoing survey of the "wished for" course topics?* Then a future course organizer would be able to create courses specifically tailored to the interests of self-selected participants.
- *What if every participant specified their level of commitment in advance?* Building a "contract" with the facilitator – and with the community – would help ensure that people would be make appropriate commitments (and keep them).
- *What if facilitators themselves got more coaching and peer support?* Ongoing discussions in the P2PU community mailing list helped a little, but not enough. Quality control was sacrificed to free experimentation. However, it was not entirely clear how a shared knowledge base about "what works" would develop out of the individual experiments.

More than the bare-minimum structure will be needed for peer learning in mathematics to succeed. The next term, after polling people about their interests, I ran a second more traditionally structured and focused course ("*Math for Game Designers*"), with a detailed syllabus prepared in advance, and this course did not go much better in terms of student participation than *DIY Math*. Again, this suggests that it is not simply more structure but the *right* structure that is needed.

*https://groups.google.com/d/msg/diy-math/X8uCWYRFzVQ/ccLCATWEYvoJ

| | |
|---|---|
| | ***An "ideal" participant in the DIY Math course:*** |
| **I** | say what I want to study |
| **We** | talk about difficulties and successes |
| **Its** | discussions on a shared mailing list |
| **It** | helps me learn mathematics (and improve my skills at being a self-directed and peer-to-peer learner) |
| | ***A P2PU course facilitator:*** |
| **I** | come up with a course I'd like to facilitate, and then facilitate it |
| **We** | discuss ideas about how our courses might work and what "facilitation" means (e.g. as opposed to "teaching") |
| **Its** | discussions on a community mailing list and other settings (including discussions with participants in the course as it runs) |
| **It** | helps me improve my skills at a course designer and facilitator (and it's fun talking about and practicing this stuff!) |
| | ***An a posteriori picture of a DIY Math participant:*** |
| **I** | try to figure out what to study |
| **We** | sometimes give or get advice that isn't always so helpful (and most of it isn't for me, anyway) |
| **Its** | a bunch of good intentions that lead nowhere |
| **It** | confirms my sense of the difficulty of learning anything in a self-directed fashion (and the difficulty of mathematics in particular) |

**Table 2:** *Roles surrounding the* DIY Math *course at P2PU*

## Paragogy principles

I devised and shared five "paragogical principles" with Danoff's *Collaborative Lesson Planning* course, where they were jointly revised. I chose andragogy (Knowles, 1968) as a starting point for adaptation, knowing that since andragogy is a theory of *adult education*, and the aim was a new theory of *peer learning*, serious alterations would need to be made. I made use of Laurie Blondy's (2007) critique, which shows that the following major assumptions of andragogy often fail in online mediated learning applications.

1. **That adult learners are self-directed.**
2. **That they bring a wealth of experience to the educational setting.**
3. **That they enter educational settings ready to learn.**
4. **That they are problem-centered in their learning.**
5. **That they are best motivated by internal factors.**

A paragogical view suggests that everyone has a mixture of self-directed, other-directed, dependent, and independent, behaviors, without saying that any of this is good or bad. It suggests that a particular sort of self-concept may be less important than a robust concept of "*shared context in motion*," which constrains but is also shaped by behavior, as per Kitaro Nishida; *cf.* (Abe, 1988). It emphasizes social intelligence over so-called "mental age," and focuses on social dynamics rather than individuals.*

A paragogical view says that, whatever our past experiences may be, we have a lot to learn about learning. In order to show why, it is again helpful to translate the key term: "*Μάθησις means learning; μαθήματα, what is learnable*" (Heidegger, 1977). On the strong etymological view, mathematics is the study of things insofar as they are learnable, and what makes something learnable is exactly what makes it mathematical. For Heidegger, this is the axiomatic. Deleuze uses a related but less logistic conception: the problematic. "*Learning is the appropriate name for the subjective acts carried out when one is confronted with the objecticity of a problem*" (Deleuze, 1968, p. 204); problems are themselves constituted in a supple and always "*meta-mathematical*" dialectic, understood to be "*a system of differential relations between genetic elements*" (Deleuze, 1968, pp. 203, 229).

At the same time, from a paragogical perspective, learning is not an automatic default; it is a specific sort of highly conditioned side-effect. One may be stuck in a rut. A *peer* should not be conceptualized as someone who is stuck in the same rut. Rather, people who learn together or from one another, who help one another get a better grasp of the world, are understood to be "peers."

The paragogue often has an application in mind, but learning "*always takes place in and through the unconscious*" (Deleuze, 1968, p. 204), and one problem leads to others. "*Learning to swim or learning a foreign language* [...] *tears us apart but also propels us into a hitherto unknown and unheard-of world of problems*" (Deleuze, 1968, p. 241).

Finally, realizing a motivation seems at least as important as its existence in the first place. Merely indicating that a given motivation is "internal" or "external" is not particularly incisive. More can be learned by realizing motivations where this is possible, then going on to the next thing.

*Après* andragogy, then, these are the five paragogical principles:

1. **Changing context as a decentered center.**
2. **Meta-learning as a font of knowledge.**
3. **Peers provide feedback that wouldn't be there otherwise.**
4. **Learning is distributed and nonlinear.**
5. **Realize the dream if you can, then wake up!**

* *Cf.* Eugene Matusov [§1.1, p. 3].

It should be elucidating to mention straight away that the new five-point framework can be understood as an extension of "I/We/Its/It" from the previous section. In these terms, the framework amounts to "Context/I/We/Its/It". Whereas andragogy deals with what Manuel DeLanda would call "*tendencies*," paragogy deals with – necessarily contextual – "*capacities*."

> *While tendencies make a list of essential properties look falsely permanent capacities explode it since for a given whole it would be impossible to list all the different ways in which it can affect and be affected by innumerable other wholes.*
>
> (DeLanda, 2011, p. 186)

Nonaka and Takeuchi (1995) drew on Nishida's thinking to build the SECI model of knowledge production.* This model has close parallels with Ken Wilber's (1997) "I/We/Its/It". It is curious that neither of these frameworks makes an explicit place for context, although Nonaka and Takeuchi would likely agree with Gregory Bateson that the key point is that "[i]*t is the context which evolves*" (Bateson, 1972, p. 164). In a more detailed interpretation, "Context" could be decomposed into nested, overlapping, or adjacent contexts. Context, thought of as an "*environment*," plays an important role in *constructivist* thinking and learning design:

> *Thinking of instruction as an environment gives emphasis to the 'place' or 'space' where learning occurs. At a minimum, a learning environment contains: (1) the learner; (2) a 'setting' or a 'space' wherein the learner acts, using tools and devices, collecting and interpreting information, interacting perhaps with others, etc.*
>
> (Wilson, 1996, p. 4)

Again, in the paragogical view, the environment is not thought of as simply "given."

> *Interactions observed in complex ecosystems need not be regarded as expressing self-organizing properties of the systems themselves; instead they can be understood as the consequences of the various and variable adaptive strategies of individual organisms living together in restricted spaces.* (Vayda & McCay, 1975)

Andrew Vayda and Bonnie McCay position adaptivity within a "*neo-Darwinian selection theory*," in which life is subject to various "*hazards*."

> *Important here is the notion of* ***process*** *of response, including processes whereby the unit of action may shift from individuals to various forms (and degrees of inclusiveness) of groups and perhaps back to individuals, in accord with the magnitude, persistence, and other characteristics of the hazards in question.*
>
> (Vayda & McCay, 1975), original emphasis

*SECI stands for *Socialization/Externalization/Combination/Internalization.*

The notion of *process* underscores to the critical importance of a context in which that process can unfold.*

Within the field of education, Marlene Scardamalia's notion of "*collective cognitive responsibility*" (Scardamalia, 2002) is sympathetic, although it remains basically provisionist. Her claim is that in the standard classroom model, "*all the higher-level control of the discourse is exercised by the teacher.*" Students are "*reactive*" and "*receptive*," their work focused on "*tasks and activities.*" Small groups and decentralized, constructive, computer mediated communication are seen as two possible alternatives that "*turn more responsibility over to the students.*"

## 3.2 P2PU: An "After Action Review"

The paragogy principles are intended to serve as guidelines for building successful peer learning experiences. Corneli and Danoff (2011A) looked at how each of principles was (or was not) implemented at P2PU. A summary of this analysis is presented below. In addition to the paragogy principles, the US Army's (2002) After Action Review (AAR) provided the basic technology for *doing* paragogy. The AAR's intended purpose is to "*identify strengths and shortcomings in unit planning, preparation, and execution, and guide leaders to accept responsibility for shortcomings and produce a fix.*" It places much more emphasis on learning at the group level than the corresponding basic technology of andragogy described in (Carlson, 1989). Although the context of application is different, there are parallels between the AAR and the Action Research cycle; *cf.* (Carr & Kemmis, 2003; Lawler, 2008). The four steps in the AAR are:

1. *Review what was supposed to happen (training plans).*
2. *Establish what happened.*
3. *Determine what was right or wrong with what happened.*
4. *Determine how the task should be done differently the next time.*

(US Army, 2002)

**Changing context as a decentered center: Mapping system dynamics and semantics** We organized multiple courses where participants were supposed to interact and learn about the subject matter: *Collaborative Lesson Planning* Fall 2010 and Winter 2011 (co-organized by Charles Jeffrey Danoff and Dr. Majorie King); *DIY Math* (Fall 2010), and *Math for Game Designers* (Winter 2011) by Joseph Corneli; *Open Governance and Learning* (co-organized by Corneli and Marisa Ponti); and, in Spring 2011, *Shaping P2PU*,† which was an "intervention" based on a preliminary version of the paragogical evaluation presented here. Due to critically low participation, the mathematics courses did

* *Cf.* Eugene Matusov [§1.1, p. 3] and Tim Ingold [§1.3, p. 7].

† HTTP://NEW.P2PU.ORG/EN/GROUPS/P2PU-THE-COURSE/

not run to completion. Participation in *Collaborative Lesson Planning* and in *Open Governance and Learning* was minimal, but sufficient for a conversation to be sustained for the entire 6 week session. The theory of paragogy was born in these discussions. More than 30 people signed up for *Shaping P2PU*, but participation was very low. These experiences were typical: participation within courses was globally uneven, and falling over time. Our best experiences as course organizers happened when we were committed to working through the material ourselves. Combining facilitator dedication with gentle prompts to participants to follow through on their commitments could go a long way towards keeping engagement at a reasonable level, but this only works when commitments are somewhat clear in the first place. The case of *Shaping P2PU* shows that organizer commitment is not enough. The P2PU ecology contains an implicit rubric for learning and engagement: from the time a member signs up for a course, to its completion, peers go through a cycle.* P2PU could implement more formal check points throughout the cycle, requiring participants to specify, reaffirm, or adapt their commitments in relationship to judgments about quality. The AAR could be useful in this regard.

**Meta-learning as a font of knowledge: Transparency, accountability, and tone** Support for facilitators was offered as a P2PU course (*Course Design Orientation*), in mailing lists, via weekly phone calls, in a Q&A issue tracker, and via other informal channels. Participants in courses were presumed to be ready and willing to contribute in a useful fashion. The core members of P2PU are doing a lot of work, and the project is moving forward, with grant funding, incorporation, and several new staff positions. However, apart from contractual agreements within the nonprofit and basic standards for etiquette, participants have little or no accountability to one another. The P2PU governance model was said to be based on "*rough consensus*," after MIT professor David Clark's description of the Internet Engineering Task Force at the July 1992 IETF conference: "*We reject: kings, presidents and voting. We believe in: rough consensus and running code.*"† Pursuant to this, it would be helpful if a concise discussion of the current consensus norms and "best practices" for organizers and participants could be made available. The current *Course Design Handbook* provides one starting point, but it would represent a better consensus if it was improved by capturing insights from successful and less successful courses.‡ This sort of resource would be particularly useful for newcomers and to people who cannot regularly attend the community telephone calls.

**Peers provide feedback that wouldn't be there otherwise: Dealing with problems in a respectful way** Discussions about P2PU happen in the community mailing list and other places mentioned above. At the time when we were doing this research,

*https://wiki.mozilla.org/Drumbeat/SoW-engagement-ladder

†http://www.wired.com/wired/archive/3.10/ietf_pr.html

‡http://wiki.p2pu.org/w/page/27905271/Course-Design-Handbook

bug reports and feature requests related to the software were supposed to go into an issue tracker hosted on Lighthouse.* Discussions about P2PU also happen in courses, where they may be viewed as distractions from the content of the course. Even within the mailing list, it can be difficult to keep track of the full range of ideas circulating at any given time. And, apart from development work, it can often be hard to tell what's happening around P2PU. Presumably participants who have identified critical and unsolvable problems simply leave. The Q&A tracker† and mailing list both provide ways to build factual knowledge, but seem less effective for building *strategic* knowledge. Broadly, we would like to have had more clarity about how to contribute to the process of shaping P2PU.

**Learning is distributed and nonlinear: Design considerations** Facilitators are supposed to choose and assemble suitable learning resources for their courses (blogs, OER, etc.), in which everyone is supposed to learn something. In our experience, this is essentially what happened, but it is hard to measure when, whether, and how much knowledge was gained. Dealing effectively with complexity is hard, and more attention to modeling the learning process could help. However, it is not clear how or whether the data model used by P2PU's current platform – based on shared activity streams drawn from around the web – relates in a significant way to *learning*. It seems that P2PU would have to work hard in order to be able to use anything but "participation" as a proxy measure for learning. In terms of broader issues of quality control, one suggestion would be for P2PU core members – including staff – to start using the P2PU platform to organize their activities in the open, sharing what they are learning in the process of working on the site.

**Realize the dream if you can, then wake up: High level roadmap** At one time, the high-level vision at P2PU was arguably a Declaration of Independence from Formal Education.‡ But arguably each participant has their own vision, and their aims are often much less radical.§ When we were building this analysis, P2PU had recently had its first board meeting, but, so far, documentation about the organization's vision and roadmap have not been presented to or affirmed by the user community – nor has the user community presented any stipulations to the organization. P2PU has made considerable progress (e.g. in the form of successful grant applications), but without more transparency about these efforts, the ability of non-core members to learn from organizational successes is limited. In order to maximize the possibilities for others to contribute, P2PU should work on a public roadmap that leads from now up to the point where the vision is achieved. Both vision and roadmap should be revised as needed.

*http://p2pu.lighthouseapp.com/dashboard

†http://help.p2pu.org/kb

‡http://www.youtube.com/watch?v=t8wxUbU1W_0#t=12m11s

§http://www.youtube.com/watch?v=t8wxUbU1W_0#t=13m12s

## 3.3 Paragogical praxis

The study at P2PU helped develop a notion of "principled paragogy" – an essentially normative theory. These principles can be re-evaluated to set the stage for "emergent paragogy" – a descriptive theory. The paragogy principles point to five specific kinds of change that actions within a learning context can accomplish (Corneli & Mikroyannidis, 2011). These five points express the paragogical principles in a more uniform way.

1. **Changing the nature of the space** : $\Delta$ *context*
2. **Changing what I know about myself** : $\Delta$ *metalearning*
3. **Changing my perspective** : $\Delta$ *feedback*
4. **Changing content or connectivity** : $\Delta$ *distributivity*
5. **Changing objectives** : $\Delta$ *achievement*

If change is a "*quintessential element*" in learning (Burger & Starbird, 2012), it makes sense to give it a central role in learning design. The question "what changed?" provides a unified lens through which to examine and evaluate problem solving heuristics, like Pólya's (1945), as well as more open-ended problem solving processes, like those outlined by Schoenfeld [§1.3, p. 9]. Chapter 7 will use these five dimensions of change to understand user requirements for new problem solving support in PlanetMath/Planetary. Paragogy is a theory of peer learning in which the learner is not a separate species. With this in mind, the five paragogy principles can also be translated into practical suggestions on how to be an effective paragogue (Corneli, 2012). The following points outline a method for studying how the five factors of change mentioned above *themselves* change. With care, this engine can be run in reverse – not just to study change, but to bring it about.

1. **Develop empirical studies and a critical apparatus** : $\nabla$ *context*
2. **Find companions for the journey** : $\nabla$ *metalearning*
3. **Work with real users** : $\nabla$ *feedback*
4. **Study and build nonlinear interfaces** : $\nabla$ *distributivity*
5. **Limit philosophizing** : $\nabla$ *achievement*

Chapter 5 describes the connections between these five dimensions and the design pattern methodology that is applied in Chapter 9. Together, the two levels of change are similar to what Gregory Bateson calls *proto-learning* and *deutero-learning*, or, more plainly, Learning I and II (Bateson, 1972, pp. 174, 292–306). Bateson's perspective on learning is discussed in Chapter 4, where it is used to illustrate the connections and differences between paragogy and other contemporary learning theories.

## 3.4 Summary

This chapter presents paragogy, a new theory of peer produced peer learning. Paragogy is different from earlier theories of learning, in the sense that broader contextual change is considered along with change in the persons who are learning.

Although paragogy initially developed in the mode of constructive critique, it is not tied to a project of "*fixing*" education; *cf.* (Kernohan, 2013). Indeed, paragogy is not presented primarily as an educational approach, but as a description of the way learning works in peer production communities. Paragogy has been amplified and extended within the Peeragogy project (Rheingold et al., 2014), which has helped make it useful to educators (Purser, Towndrow, & Aranguiz, 2013; Baker & Surry, 2013), and others (Corneli et al., 2014).

In order to give a foundation to the key ideas in paragogy, the presentation in this chapter has drawn on course design, participant observation, and on subsequent papers that explored the relevance of paragogy for system design and research (Corneli & Mikroyannidis, 2011; Corneli, 2012).

# 4 Literature Review

**While there is no significant prior literature on "peer produced peer learning," the paragogy framework suggests a way to organize and survey prior work in adjacent fields.**

## 4.1 Prelude: Learning and adaptation

This section will explore *learning*, focusing on Gregory Bateson's (1972) perspective, which was introduced briefly in the previous chapter [§3.3, p. 41], connecting it to the notion of *adaptation* that appears frequently in anthropological literature. The recent work of Andreas Meiszner (2010), who studied learning in the setting of free software and open courses, helps to show how these ideas relate to practical application and to contemporary learning theories. In the later sections of this chapter, this material will be useful in analyzing the literature related to peer produced peer learning in mathematics.

Bateson's hierarchy *Learning I*, *Learning II*, *Learning III*, etc., begins with *Zero Learning*, which denotes no change in the subject, only a stimulus and a reaction that is already "*soldered in*" (Bateson, 1972, p. 288). As with paragogy, Bateson understands learning to be fundamentally related to change, but in his model, he imposes the requirement that the environment should *not* change while learning is happening. His assertion is that "[t]*he notion of repeatable context is a necessary premise for any theory which defines 'learning' as change*" (Bateson, 1972, p. 296). Bateson does allow a hedge, which is that the context is only "*somehow*" or "*theoretically*" the same. The problem is that, here, Bateson is understanding context as "*a metamessage which classifies the elementary signal.*" Rather than talking about a repeatable context, it would be clearer to simply say that the stable classification of signals is what is important. Even in the case where a

bell sometimes means dinner and sometimes means no dinner, there is something to be learned, which is that the bell is capricious. All classification systems must "*fit innumerable, more or less similar cases – which means, strictly speaking, never equal*" (Nietzsche, 1873). Indeed, in some sense, what one is learning is precisely the classification system, with whatever built-in uncertainty pertains to it. This is compatible with Bateson's more formal definition: "*Learning I is change in the specificity of response by correction of errors of choice within a set of alternatives*" (Bateson, 1972, p. 298). The definition of Learning II, naturally enough, builds on the definition of Learning I:

> *Learning II is a change in the process of Learning I, e.g. a corrective change in the set of alternatives from which choice is made, or* [...] *a change in how the sequence of experience is punctuated.* (Bateson, 1972, p. 298)

Getting better at solving problems of a given type over time is an associated phenomenon, which related to the notion of *transfer of learning* whereby a problem in a new context is treated as if it were the same as a problem encountered in an earlier training context. Again, context change is perfectly compatible with learning at this level.

As an important class of examples, day-to-day communication works largely due to the fact that "*the stream of events is commonly punctuated into contexts of learning by a tacit agreement between the persons regarding the nature of their relationship*" (Bateson, 1972, p. 304). The pattern by which we should define Learning III and so forth is now relatively clear. Since "*what is learned in Learning II is a way of punctuating events*" (Bateson, 1972, p. 305) then what is learned in Learning III is a way of punctuating the punctuation of events – and so on. In other words, Learning II is about creating the boundary conditions in which a given pattern of control will be exercised, subject to the standard caveat that this control is partial. In a paragogical view, punctuation of this sort could also be called *context creation* – although often more is involved in context creation than just bracketing.

Bateson introduces several additional ideas that expand on this hierarchy, and which will be useful in what follows. The first is the distinction between analog and digital signals. Often the two are combined, as in language learning (Iverson & Thelen, 1999); indeed, "*there is a continuous gradation from the ostensive through the iconic to the purely digital*" (Bateson, 1972, p. 296).

For example:

> [A]*n English-speaking mathematician confronted with a paper by a Japanese colleague* [...] *gazes uncomprehendingly at the Japanese ideographs, but he is able to partly understand the Cartesian graphs in the Japanese publication.* (Bateson, 1972, p. 378)

Digital signals generally have conventional interpretations (*e.g.* the bell), whereas analog signals are more direct (*e.g.* the dinner), and need not stand for anything else. Clearly any conventions must be learned in order to be used. "*Verbal language*," in particular, is "*almost (but not quite) purely digital*" – and furthermore, "*primarily oriented toward things*," rather than "*the patterns and contingencies of relationship*" (Bateson, 1972, pp. 378, 377).*

Here, *relationship* especially means a context for communication, which as we've seen, means a context for learning, and, more broadly, a context in which cybernetic control is exercised. Bateson shows through many examples that humans and other mammals typically communicate about their most meaningful relationships with one another in nonverbal – *i.e.*, analog – ways. He introduces the term *μ functions* to describe communication acts, whether digital or analog, spoken or unspoken, which are ways of "voicing" relationship.

> *We shall say that "A dominates B" if A and B show by their behavior that they see their relationship as characterized by sequences of the type $a_1 b_1 a_2$ where $a_1$ is seen (by A and B) as a signal defining conditions of instrumental reward or punishment; $b_1$ as a signal or act obeying these conditions; and $a_2$ as a signal reinforcing $b_1$. Similarly we shall say that "A is dependant on B" if their relationship is characterized by sequences $a_1 b_1 a_2$, where $a_1$ is seen as a signal of weakness; $b_1$ as a helping act; and $a_2$ as an acknowledgement of $b_1$.* (Bateson, 1972, p. 305)

Some examples may help clarify these concepts. Wittgenstein's builders operate at the level of Learning I and execute a digital program – "*so that A can shout 'red slab!', and B will pass slabs of a certain colour*" (Bloor, 1983, p. 22). Given the thin description, we don't know for sure that *A* dominates *B*, which is certainly what it sounds like, or is perhaps dependent on *B*. In any case, the language of the builders is entirely about things, and the relationship is "voiced" through the embodied pattern of compliant acts.

In a real-world example, Richard Stallman solved a problem with printer jams at the MIT AI lab by installing custom code "*to notify every user with a waiting print job that the printer was jammed.*"

> *"If you got that message, you couldn't assume somebody else would fix it," says Stallman, recalling the logic. "You had to go to the printer. A minute or two after the printer got in trouble, the two or three people who got messages arrive to fix the machine. Of those two or three people, one of them, at least, would usually know how to fix the problem."* (Williams & Stallman, 2010, p. 3)

*Note that Geertz does not precisely say that a *symbol* is digital, but his list of exemplars – "*any object, act, event, quality, or relation which serves as a vehicle for conception*" – is comprised of discrete terms, rather than processes; in particular, a *relation* rather than a *relationship*. *Cf.* [§1.4, p. 9].

This message had the $\mu$ function of expressing a relationship between particular individuals and a printer, but it also served as a collective expression of the mutual interdependence of the members of the lab as a whole. This episode was an example of Learning II, enacted in such a way as to change the punctuation of experiences for the group. Furthermore, it served as a motivating example for an episode of Learning III, leading to the creation of the GNU project – based on the observation that custom messaging was only possible because the source code that controlled the printer's was accessible and could be changed.

It is now possible to make connections from this theoretical view to some of the standard learning theories. Here, I will largely follow in the footsteps of Andreas Meiszner (2010), whose PhD thesis asks whether the kind of learning that happens in free software communities can be applied to build open courses. Chapter 2 was framed around a similar motivation: "Can we make learning mathematics more like learning in free software?"

Meiszner draws on the description of self-directed learning from Gerald Fischer and Eric Scharff (1998), who write about the potential of "*Economies of Educational Knowledge*" that can be used by self-directed learners. According to Fischer and Scharff:

1. **Learning requires environments in which learners can be active designers and contributors.**
2. **Learning is highly tuned to the situation in which it takes place.**
3. **Learning is knowledge-dependent.**
4. **Learning needs to account for distributed cognition.**
5. **Learning is affected as much by motivational issues as by cognitive issues.**

These points bear a manifest similarity to the paragogy principles [§3.1, p. 36], however Fischer and Scharff retain a basically provisionist orientation. Their primary interest is in "*support for*" self-directed learners who act "*primarily as designers of some artifacts*." This is, indeed, the typical case in peer production, and the basic form in *constructionism* (Papert & Harel, 1991). However, as there is little emphasis on co-creating the learning context, this is a kind of "proto-paragogy."*

By and large, Meiszner finds the above-listed characteristics "*well presented within FLOSS*." He documents further specific features of the way learning works in free software communities, for instance, he notes the "*fragmentation and self-selection of learning materials*." He extracts sensible-sounding recommendations for designers of open courses, for example: "*Enable learners to engage* [with] *activities that are not restricted to*

*Fischer and Scharff's primary sources are Lauren Resnick (1989) for the first three points, who grounds them in the cognitive science literature; the contemporary work of Mitchell Resnick (1996) on distributed constructionism for the fourth point; and Mihály Csíkszentmihályi (1990) for the fifth.

*an artificial university setting*" and "*Implement peer support mechanisms - a large support network provided voluntarily by peers in a collaborative manner nearly 24/7.*" The problem then becomes, who will implement these proposals?

*Connectivism*, introduced by George Siemens, emphasizes the role that "*self-organizing networks*" and "*actionable knowledge*" play for learning in the "*Digital Age*" (Siemens, 2005). Massive Open Online Courses (MOOCs) designed with a connectivist orientation have many features matching Meiszner's recommendations. However, there seem to be some important differences.

> [O]*ne of the cornerstones of learning in FLOSS, is to 'enable re-experience', to 'learn from what others did',* [...] *Most of the discourse, debates or discussions of the CCK08 course are very close to the FLOSS case, a significant difference however is that those activities do not culminate within a 'solution', 'product', 'guide', 'report' or other type of 'concrete outcome' that would allow future cohorts to take this outcome as a starting point.* (Meiszner, 2010)

As the *Connectivism and Connective Knowledge* course was re-run a few years later as CCK11, this assertion must be questioned.* Nevertheless, the emphasis for most participants in connectivist MOOCs has been on "*dialogical*" features (Ravenscroft, 2011) – crafting an experience rather than a product. Without a doubt, such conversations are peer produced – but, still, without any aspersions, this is the kind of learning that takes place on "*Facebook, Twitter, G+, blogs, and other media*" (Purser et al., 2013), rather than the kind of learning that takes place in free software projects. In framing the central tenets of connectivism, George Siemens remarks that "*The pipe is more important than the content within the pipe*" (Siemens, 2005). The broad success of technologies for communication like StackExchange in learning applications notwithstanding, the paragogue would rather side with René Magritte here – "*Ceci n'est pas une pipe.*" Rather, after Benkler,

> *There is simply the question of governance in the relations among users of a class of software platforms that have certain degrees of freedom in their design, resulting in a variety of social affordances, and therefore facilitating a variety of social and economic interactions.* (Benkler, 2006b)

In this respect, paragogy is closer to *constructivism*, inasmuch as the latter, at least in its "radical" form, places an emphasis on *adaptation* rather than on knowledge transfer. As per Ernst von Glasersfeld, "[t]*o be adapted* [...] *means no more and no less than to be viable*" (von Glasersfeld, 1996). However, while von Glasersfeld intends for radical constructivism to be compatible with "*an ontological 'reality'*," he asserts that his framework "*denies the human experiencer the possibility of acquiring a true representation of*

*http://cck11.mooc.ca/

*it*" (von Glasersfeld, 1996). Bateson, for his part, would reject the separation of mind from body that this entails (Bateson, 1972, p. 470). Von Glasersfeld's focus on the construction of conceptual schemes and models leaves out broader contextual and relational features that are present, for example, in Vayda and McCay's perspectives on environmental change and group-level action and selection, which consider the adaptivity of the entire "*process of response*" [§3.1, p. 37].

Paragogy not only "*eschews pipeline models of transmitting knowledge*" (Papert & Harel, 1991) but treats communication as an assemblage with emergent properties, co-constituted by its participants, and subject to the vagaries of existence in the real world. In this, respect, we find paragogy at once in both its (em)*phatic* and *productive* senses.* Following Malinowski (1936), Julia Elyachar explains her concept of *phatic labor*:

> *Malinowski shows how language such as gossip and chatting can be a means of establishing ties for their own sake, rather than for the purpose of conveying any information in particular.* [...] *I argue that this labor produces communicative channels that can potentially transmit not only language but also all kinds of semiotic meaning and economic value.* (Elyachar, 2010)

The synthesis of a channel relies on the key feature that makes context creation different from the mere punctuation of experience, and different from standard models of communication: "*There is always more in the effect than there was in the cause*" (Harman, 2008). But these emergent effects must be *channeled*, or else they dissipate (Deacon, 2014). In short: paragogy is concerned with generating and voicing the $\mu$ functions that sustain relationships in which learning can take place, and through which meaning can be made. This process is "*morphogenetic*," not "*hylomorphic*" (Ingold, 2013, pp. 21–22) – but moreover it is associated with problems which possess an objective reality, as when "*a population of interacting physical entities, such as the molecules in a thin layer of soap* [...] *adopt a form which minimizes free energy*" (DeLanda, 1998). In the case of PlanetMath, with reference to the framework from Chapter 3,

1. **The co-created context is a mathematics website,**
2. **in which learning is part of a social knowledge building process,**
3. **carried by mediated peer feedback,**
4. **which unfolds in an architecture devised to support collaboration;**
5. **wherein the "dream" is one of shared community.**

Bringing together and contextualizing accounts from key figures associated with these themes, the chapter arrives at a picture of mathematical sociality that can be used to reframe the research question as an array of related research problems (Chapter 5).

* *Cf.* [§1.6, p. 14, Footnote ∗].

## 4.2 Mathematical thinking

How do people understand mathematics? This is the key issue in William P. Thurston's paper, "On Proof and Progress in Mathematics". Thurston had this admonishment for the mathematics community:

> *We need to focus far more energy on understanding and explaining the basic mental infrastructure of mathematics – with consequently less energy on the most recent results.* (Thurston, 1994)

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| human language |
| sensory perception |
| logic and deduction |
| intuition, association, and metaphor |
| stimulus/response |
| process and time |
| socialization |

**Table 3:** *Criteria from William Thurston*

Note the emphasis here is on the *human experience* of mathematics. Thurston particularly emphasizes the importance of a social dimension for building a successful mathematics research programme. He begins with a point drawn from his own experience and very similar to Eric Raymond's insight that "*some very successful projects become 'category killers'*" (Raymond, 1998). Rather than clearing out mathematical topics on one's own, he observes that the long run it is much better to involve other people.

Thurston points out that people are able to make use of a wide, embodied channel to get involved. The way people *think* about mathematics (Table 3) is vastly different from what we see in the relatively dry and disembodied channel that typifies mathematics *papers*. Thurston describes mathematics as "*the theory of formal patterns*."* Nevertheless, he argues that

> [R]*eliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas.* (Thurston, 1994)

**The heart of mathematics** *Problem solving* is considered by Paul Halmos to be "*the heart of mathematics*" (Halmos, 1980). Together with Thurston's definition, this speaks to a dual role for problem solving activities: they cross back and forth between the informal, experiential, "human" ways of thinking about a problem, and more "formal" static results (solution, proof, often "correct", etc.). "[T]*he mathematician's main reason for existence is to solve problems*" (Halmos, 1980). Adding problem solving interactions is a key "ergonomic" step in PlanetMath/Planetary, since it brings it into closer alignment

*He remarks, "*to discuss this would be a whole essay in itself.*" Thurston may be thinking here of formal structural organizations, not only in terms of logic; *cf.* (Thompson, 1917), (Alexander, 1964, p. 134). Thurston's former student Benson Farb indicates: "He changed our idea of what it means to 'encounter' and 'interact with' a geometric object. The geometry that came before almost looks like pure symbol pushing in comparison." HTTP://BLOGS.SCIENTIFICAMERICAN.COM/OBSERVATIONS/2012/08/23/THE-MATHEMATICAL-LEGACY-OF-WILLIAM-THURSTON-1946-2012/

with what people *do* when they do mathematics. Michael Atiyah argues that the "*core*" of mathematics is has as least much to do with theory building as it has to do with problem solving:

> [W]*hat is the good of a theory if it does not solve problems and what is the good of an infinite collection of disjoint problems, however interesting each individual one may be.* (Atiyah, 1974)

Timothy Gowers emphasizes that both problem solving and theory building can achieve the important end of organizing and communicating mathematical thought (Gowers, 2000). It would seem that communication lies near the heart of mathematics – and if that is so, then surely the choice of media for mathematical communication matters.

**Studying mathematics online** Khan Academy was discussed briefly in Chapter 2. If we look beyond the video format and the associated flipped-classroom pedagogy (Bergmann & Sams, 2012; Khan, 2012) we can say a bit more. Remarking on an inherently collaborative online mathematics learning environment, the Virtual Math Teams project, Gerry Stahl suggests that people studying online collaboration are in an uniquely good position, since they have access to the same information that participants in these environments have (Stahl, 2010). This is not even the case for persons researching *offline* interaction, who typically need to apply instruments like think-aloud protocols to build an explicit trace of what people are thinking (Nielsen, Clemmensen, & Yssing, 2002). (Of course, in practice, the users of an online system may have scratch work or side-conversations that are in no way indexed by the system.) One exemplary piece of recent research in this genre is a project on collaborative online peer supported proof-generation exercise that was conducted by Alison Pease and Ursula Martin. "Seventy four minutes of mathematics" (Pease & Martin, 2012) analyzes the activities that took place during *Mini-Polymath III*, a project in which Terrence Tao invited students to collaborate on his blog to solve an old Mathematical Olympiad problem. Having access to a trace of the entire conversation allowed Pease and Martin to categorize contributed comments as *concepts*, *examples*, *conjectures*, or *proof-related* – a framework inspired by Imre Lakatos's (1976) theory of informal mathematics.

**Questions of procedure** The Polymath project, which served as the inspiration for the Mini-Polymath series, aimed to answer the question "*is massively collaborative mathematics possible?*"* This project was convened by Timothy Gowers, who developed 15 rules to address the "*questions of procedure*" that he anticipated would arise.†

*http://gowers.wordpress.com/2009/01/27/is-massively-collaborative-mathematics-possible/

†http://gowers.wordpress.com/2009/02/01/questions-of-procedure/

In particular, the rules emphasize that a good contribution represents a "*quantum of progress*." Gowers progressively and carefully explicated the imperative to share comments that are "*not fully thought out.*" As Ursula Martin had commented ten years earlier with regard to the emerging field of *experimental mathematics*, that the growth of experimentation requires the "*development of community standards as to experimental methodology*," emphasizing not only technical matters, but also "*the early sharing of insights*" (Martin, 1999). Community standards are essentially synonymous to community norms, like those we would have liked to see more clearly elucidated at P2PU (Chapter 3 [§3.2, p. 39]). Community standards are part of any technology for shared communication; without them, communication is more difficult, or impossible; *cf.* Bateson on the tacit agreements that create contexts for learning [§4.1, p. 44].

Gowers's rules provide one starting point for community standards in *social* experiments in mathematics. These rules suggest a fairly precise understanding of mathematical socialization. It is not simply a matter of chatting things over, rather, it is a slow process of big ideas, and a faster intermediate process of breaking these ideas down into smaller parts. This recovers the *modularity* and *granularity* aspects the Commons-Based Peer Production model. Ultimately, participation in the Polymath project was not as "massive" as had been hoped (Gowers & Nielsen, 2009). This points to a potential expanded role for *integration*: not only to connect ideas into finished products, but also to connect the workflows of more persons. At the same time, the question about massive collaboration in research needs to be made more precise. Reasonably large scale collaboration in research mathematics had been going on much earlier, after a fashion: consider Mersenne's 17th century system (Bartle, 1995); and for a review focusing on Bourbaki, see Leo Corry on "Writing the Ultimate Mathematical Textbook" (Corry, 2009).

**Making mathematical meanings** There is a correlate to the quantum of progress in research mathematics to be found in a paper by Constant Leung about the process of learning elementary mathematics (Leung, 2005). Leung mentions three key points about learning mathematical language, and *mathematical vocabulary* in particular. Leung points out that when you learn mathematics, you have to learn both "*formal*" and "*semantic*" features of the concepts involved*; you have to "*think through*" the concepts involved; and that learning it is "*incremental*," in that meanings expand and develop by building on one another. This tells us what mathematical vocabulary does. Introducing new terminology is a way to represent new ideas – a quantum of progress in a learning or discovery process. It would not be entirely unreasonable to take learning vocabulary as a model for learning mathematics.

*Thus, for instance, a square is "*a figure with 4 straight equal sides forming 4 right angles*" (*formal/core*), and "*it has four lines of symmetry, and rotational symmetry of order 4*" (*semantic/non-core*).

> [W]*ords don't just represent what we claim to know already, they also allow us to make observations and to formulate novel meanings within a negotiated range of acceptable/accepted possibilities or limits.*" (Leung, 2005)

**Mathematical semiotics** Michael Halliday's view that a language contains terminology *and* a theory of human experience *and* an enactment of interpersonal relationships (Halliday, 2006, p. 50) is consistent with Leung's view that learning words is actually a matter of learning and enacting process, and consonant with Thurston's perspectives on mathematical experience. Halliday makes it clear that learning scientific or mathematical language is not just a matter of learning new vocabulary words. Writing about students learning science, Halliday said that the difficulty is not just with "*technical terms*" but with the "*the total patterns of the discourse*" (Halliday, 2006, pp. 200–201). The salience of these comments is amplified when we consider Halliday's language-based theory of learning, which says that, for humans, learning how to relate to others, and indeed learning how to think, are intrinsically linguistic processes. To illustrate these claims, Halliday presents a detailed study of the development of thought through 21 distinct features of childhood language development (Halliday, 1993). In this work, all human learning activities are seen as "*semiotic*": the system of language is concomitant with processes of meaningful acts. Halliday's thinking about system and process is reminiscent of Louis Hjemslev's thesis:

> [F]*or every* ***process*** *there is a corresponding* ***system****, by which the process can be analyzed and described.* (Hjelmslev, 1961, p. 9), original emphasis

In this way, we move back and forth between words/symbols, system/language, and process/meaning.

**Learning mathematics from examples (and by doing)** But does a language-based theory adequately describe the way people learn mathematics? A paper by Xinming Zhu and Herbert A. Simon demonstrates the feasibility of an alternative learning-by-example approach (Zhu & Simon, 1987). In this case, the examples were spelled out in advance in textual form. The underlying model is simple, and best suited for rote learning (Waterman, 1975). This approach to learning from worked-out examples contrasts strong with the "*Moore method*" (Jones, 1977; Cohen, 1982), a variation on the practice known as "*discovery learning*" (Bruner, 1961). Here, the syllabus consists of a selected list of theorems to prove. Clearly, in order to prove non-trivial theorems, one needs "*a reasonable degree of sophistication concerning the logic of verification and concept formation*" (Strike, 1975). The Moore method generally relies on the rediscovery of a known system behind the revealed process. On the other hand, the Zhu/Simon model is an example of a system with only a minimal process: it provides a limited language that only

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| | If a problem seems familiar, try *reasoning by analogy.* If you solved a similar one in the past, and can adapt to the differences, you may be able to re-use that solution. |
| | If the problem still seems too hard, *divide it into several parts.* Every difference you recognize may suggest a separate subproblem to solve. |
| | If it seems unfamiliar, *change how you're describing it.* Find a different description that highlights more relevant information. |
| ▽ | If you get too many ideas, then *focus on a more specific example* – but if you don't get enough ideas, make the description more general. |
| | If a problem is too complex, *make a simpler version* of it. Solving a simpler instance may suggest how to solve the original problem. |
| | Asking "*what makes a problem hard?*", may suggest another approach – or a better way to spend your time. |
| | When your ideas seem inadequate, remember someone more expert at this, and *imagine what the expert would do.* |
| ⊗ | Whenever you find yourself totally stuck, *stop what you're doing* now and let the rest of your mind find alternatives. |
| | The best way to solve a problem is to already *know how to solve it* – if you can manage to retrieve that knowledge. |
| | If none of these methods work, you can always *ask another person for help.* |

**Table 4:** *Problem-solving heuristics suggested by Minsky, with new mnemonics*

makes it possible to reexpress the guiding text. When compared with Halliday's detailed developmental model of learning as a linguistic/semiotic/meaning making process, both of these models appear quite flawed.

**A linguistic desert** In a series of memos written for the One Laptop per Child (OLPC) project,* Marvin Minsky points out that young students are not given many mathematical words or ideas. "*It is hard to think about something until one learns enough terms to express the important ideas in that area*" (Minsky, 2008–2009).† However, Minsky's concern is not solely with increasing the rate of vocabulary acquisition, but with learning new ways to think and problem solve. He proposes several example heuristics (*i.e.*, ways of thinking about problems) that could be taught to students (Table 4). In the OLPC memos, he suggests that programming may be a better way to teach heuristic reasoning than mathematics. Elsewhere, Minsky emphasizes that to build real understanding, it is necessary to be able to "*represent something in several ways*" (Minsky, 2007, p. 6).

*HTTP://ONE.LAPTOP.ORG/

†*Cf.* Gerry Sussman: it's "*hard to learn what one cannot express*" [§3, p. 10].

## 4.3 Collaborative knowledge building

The model of learning from discovery and examples, where both are embedded in a social process, is similar to Marlene Scardamalia's (2002) conception of knowledge building, mentioned in Chapter 3. These ideas build upon earlier work that was developed jointly with Carl Bereiter (Bereiter, 1994; Bereiter & Scardamalia, 1993; Bereiter, Scardamalia, Cassells, & Hewitt, 1997). These works emphasize heuristic design guidelines such as: "*There is a way into the central knowledge space for all participants*" (Scardamalia, 2002).

| Tacit preunderstanding |
|---|
| Personal attention (⋆) |
| Personal comprehension (⋆⋆) |

| (⋆) Personal attention |
|---|
| Public statements |
| Argumentation and rationale |
| Shared understanding |
| Collaborative knowledge |
| Cultural artifacts |
| (⋆⋆) Personal comprehension |

**Table 5:** *Criteria from Gerry Stahl*

"A model of collaborative knowledge building" is a short paper by Gerry Stahl (2000), in which he talks about "*learning as a social process of knowledge building*." The aim of Stahl's paper is to add a cognitive dimension to the early, more informal, theories of knowledge building. In particular, the paper aims to mesh "*personal understanding*" (considered as a cycle) with "*social knowledge building*" (considered as another cycle) – see Figure 4.* Table 5 presents a minimal rendition of the diagram's main generative cycles and mutually-satisfying boundary conditions.

Stahl's model does an admirable job of organizing the key ideas of social knowledge building, while simultaneously offering a potentially descriptive theory. However, as a cognitive model, it remains fairly schematic.

**Another candidate structuration** In a recent paper by Sean Goggins and co-authors, Stahl's work on collaborative knowledge building inspires another graphical interpretation (Figure 5), where *context*, *people*, *artifacts*, and *interactions* are organized in *contextualized interaction* (Goggins, Mascaro, & Valetto, 2013). This model also considers people, artifacts, and a shared context. There is less of a focus on the particular *teloi* of belief, understanding, and comprehension. However, there is more of an emphasis on small-scale feedback loops, which are connected to a large-scale research cycle that aims to identify the social connections that are created within the groups being studied.

**Modeling the process of learning** "Modeling the process, not the product of learning" (Akhras & Self, 1999, pp. 3–28) was published around the same time as Stahl's short paper. Here, Fabio Akhras and John Self look for ways to model learning within a

*In this revision to Stahl's (2000) diagram, "*social knowledge building*" splits into "*small group interaction*" and "*commmunity knowledge building*", and what was formerly "*personal belief*" is now labeled "*personal attention*" (Stahl, 2013).

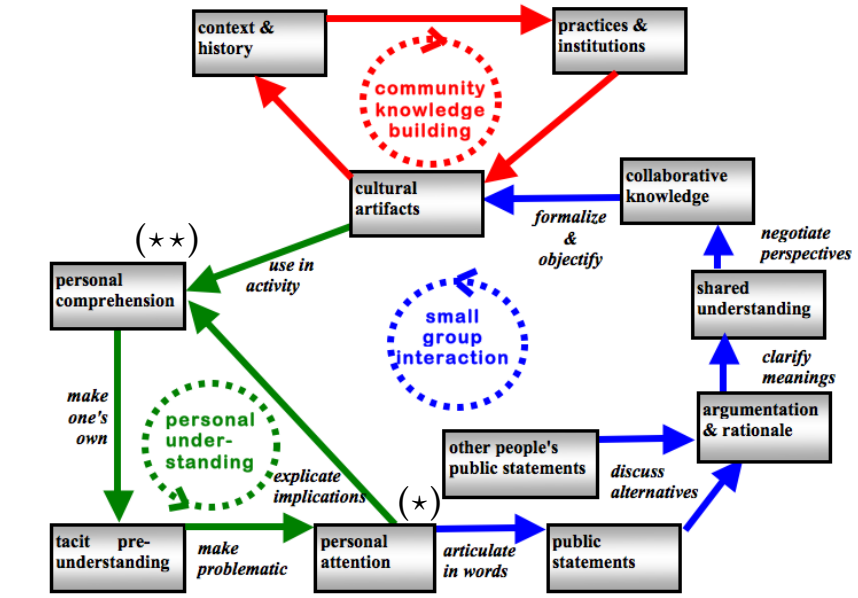



**Figure 4:** *Stahl's diagram of factors involved in collaborative knowledge building, Figure 8-4 from* Translating Euclid: Creating a human-centered mathematics, *© 2013 Gerry Stahl (used with permission; annotations* $(\star)$ *and* $(\star\star)$ *added to indicate the points of departure and arrival for the two main cycles)*

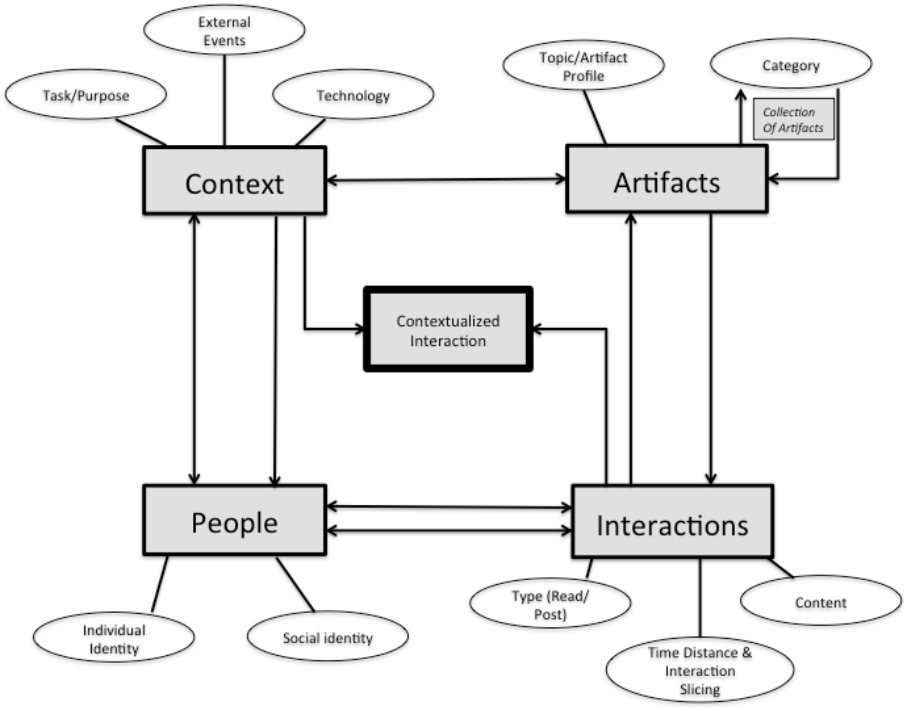



**Figure 5:** *Overview of the Group Informatics Model, Figure 1 from "Group informatics: A methodological approach and ontology for sociotechnical group research", © 2013 Sean Goggins (used with permission)*

"*constructivist mindset*." The classic approach to student modeling in intelligent tutoring systems seeks to capture domain knowledge in certain objective representations, which the student is then supposed to internalize – they cite Wenger (1987) as an example. According to Akhras and Self, knowledge is an inherently process-based phenomenon that "*cannot be objectively defined and statically represented.*" Akhras and Self therefor restrict themselves to modeling the "*context*," "*activity*," and "*temporal factors*" of learning. In a subsequent work, they go on to argue that "*if you want to take a constructivist approach the ITS framework will not help because of the different issues addressed*" (Akhras & Self, 2002). Commentators Michael Young et al. (2002) use Akhras and Self as a launch point to call for the consideration of the multidimensional nonlinear dynamics associated with learning interactions – a perspective that is quite sympathetic to the paragogical view.

**The (multidimensional) Zone of Proximal Development** As certain basic problems get easier to solve, people move on to more difficult ones. This principle is perhaps best studied within the field of child psychology, where it was formalized by Lev Vygotsky as the *Zone of Proximal Development*,

> *a sphere formed by the aggregate of vectors that pass through a 'point' of difficulty and that delineate a child's diverse possible areas of development.* (Zaretskii, 2009)

When the learner moves within this multi-dimensional "*zone*," he or she may overcome particular difficulties with the help of competent others. Once a given challenge can be satisfactorily met with help, it can often later be managed independently. Of course, we do not assume that social interactions, *per se*, will always have a positive effect on development or learning (Hogan & Tudge, 1999, p. 61). In particular, we cannot expect that a given peer will have the skills or sensitivities needed to help navigate and maintain the ZPD of another (*ibid.*, p. 57). The multidimensional Zone of Proximal Development provides a satisfying generalization of the basic form described by Stahl (see Figure 4 and Table 5, above). Both systems are reminiscent of the *microsphere* defined by Peter Sloterdijk, whose key feature is its bi-centered nature and whose basic paradigm is life in the womb (Sloterdijk, 2011, p. 540).

**The emergence of mathematical meaning** Cobb and Bauersfeld (1995) present an edited volume containing several detailed analyses of classroom-based teaching and learning practices. The focus is on examining what happens when the teacher shifts from a pedagogy revolving around teaching what is seen as the correct way to think about things, to pedagogy where the teacher is, or tries to be, genuinely interested in the way students think as well. For example, students are encouraged to come up with their own ways to add numbers like 56 and 67. In addition to the traditional strategy of lining up the numbers in columns and carrying digits, one might round down to 55 plus

65, see that this is the same as $60 \times 2$ and add back the 3 points from rounding at the end. Or one might work the sum as $(50 + 6) + (60 + 7)$. If not limitless, the varieties of mathematical experience are at least highly diverse, and it seems the teacher must quickly muster some patience to deal with the enthusiastic creativity of her pupils, while also remembering to encourage them to reflect on one another's ideas. This appears to be an effective way to teach mathematical creativity. However, one wonders how much interest the student answers really have for the teacher, especially after the curriculum has been run several times. Ilana Horn underscores that "[t]*he focus on student thinking requires a genuine curiosity about young people and their ideas*" (Horn, 2012). Relating recent pedagogical experiments in the JUMP Math project, John Mighton writes:

> *Research has shown that many elementary teachers* [...] *are mathphobic or have very rudimentary knowledge of math.* [...] *In following the online lesson plans* [from JUMP Math] *teachers learn the math as they teach. Many have become excited about their new understanding of the subject and have formed volunteer networks to support and mentor other teachers.* (Mighton, 2014)

These examples illustrate a view of mathematics education in which peer pedagogy and paragogy are potentially compatible, featuring ongoing, active, and creative learning for all involved. The emergence of mathematical meaning goes along with the formation of new practices, for example, effective practices of metacognition, as described by Alan Schoenfeld [§1.3, p. 9] – and may be accompanied by new social arrangements, as described by Mighton. Indeed, the social in Mead's sense of an unstable "*betwixt and between*" [§1.3, p. 8] tends to be well exemplified in learning; *cf.* (Deleuze, 1968, p. 204).

The following section emphasizes the role emergence can play in the co-creation of an environment that meets the proximally-developmental needs of the peer learner, whose not-just-(en)cyclical learning style has been traced above.

## 4.4 The ecological approach in learning design

"The Ecological Approach to the Design of E-Learning Environment: Purpose-based Capture and Use of Information About Learners" is a paper by Gordon McCalla that talks about how to use interaction data to help build a learning environment (McCalla, 2004). McCalla does not use the word "stigmergy" in this paper, but his view of mediated interaction matches that definition well, if we understand *stigmergy* to mean:

> [T]*he use of environmental conditions as instigators of action and the overall ability of the group to perform problem-solving activity that exceeds the knowledge and the computational scope of each individual member.* (Clark, 1998, p. 234)

A key part of McCalla's view is that the system adapts as its external environment

changes, where its "*external environment includes learners, teachers, the subject matter being learned, and the technology that implements the e-learning system.*" Such a perspective is in line with the broader *emergentist* approach in cognitive science (and robotics), in which systems are programmed directly by the environment, and can make use of environmental features for purposes of self-programming (Clark, 1998).

The idea in McCalla's model is to engineer systems that can capture information about users and individual use events. Data describing use events has elsewhere been given the convenient name "*paradata*," to distinguish it from "*relatively static metadata*" (National Science Digital Library, 2012).*

| |
|---|
| Gradual, localized, accumulation of information |
| Focus on information about the end user |
| Purpose-based use of data, in support of a pragmatic web |

**Table 6:** *Criteria from Gordon McCalla*

These ideas coalesce into a scheme (Table 6) for capturing use data related to learning, by "*attaching*" a model of the learner to each object the learner uses. This model is considered to have "*characteristic*" and "*episodic*" aspects (*metadata* about the user, and *paradata* about the specific use event). The data is to be used by "*purpose-based*" data clustering and data mining algorithms.

**Good contributions and good questions** Q&A sites make reasonably transparent use of paradata (upvotes, downvotes, and accepted answers, as well as editing behavior) to award points and badges of markers of progress. They provide a concrete example of *marker-based stigmergy* in action, and a partial implementation of McCalla's vision. Since pragmatics is at issue, it's worth asking: what makes a good contribution on a Q&A site? The first basic finding is that good online contributions tend to mirror good offline contributions. Tausczik and Pennebaker (2011) showed that strong mathematicians (according to offline metrics) are more likely than other participants to ask highly rated questions on the mathematics research Q&A site MathOverflow. Anderson, Huttenlocher et al. (2012) examined MathOverflow's big sister, Stack Overflow, and found that questions that appear together with multiple different answers brings lasting value to the site. Another striking finding was that the events that take place within the first hour after a question is posted are reliable predictors of its subsequent interest up to a year later.

Reflecting on her experiences with students in an offline setting, Alison King (1999) captures the essential message: "*thought-provoking questions promote high-level discussion, which has been found to result in high-level learning.*" King divides questions into two categories, *comprehension questions* – which seem well-suited to dealing with factual

*The term "paradata" was previously used in an approximately similar manner in the context of survey-based research, see Safir et al. (2001).

material – and *connection* or *integration questions*, which are more suited to analyzing and integrating ideas. Combining these with *strategic* questions, peer learners can tackle ill-structured problems.

**Still more mathematical meaning** One can look for meaning throughout a wide range of actions and interactions, for example, solving a problem, asking a question, and giving or receiving a hint. These activities may have further subjective features, including "*processing level*," "*quality of self-explanation*," and "*interest in exploration*" (Hosein, 2009). Relational or contextual features of behavior are likely to be particularly significant. Chan, Lee, and van Aalst (2001), drawing on Scardamalia (2000), described "*Working at the cutting edge*," "*Collaborative effort*," and "*Identifying high points in the discourse*." There are many ways in which this sort of distributed collaboration can be recognized and persisted to add value to a resource; *cf.* Corneli and Mikroyannidis (2011).

**Sticking things together** "*Local solutions to local problems*" is the watchword for Elinor Ostrom's theory of commons governance (Ostrom, Burger, Field, Norgaard, & Policansky, 1999). Local features are also important in McCalla's proposal to implement educational systems using *localized* accumulation of information. Martin Nowak's work is tremendously useful because it points to a kind of glue – "*cooperation*" — that can be used to stick local regimes together (Nowak, 2006). Nowak's five rules for the evolution of cooperation are quite simple:

1. **Cooperate with agents to whom you are sufficiently related.**
2. **Cooperate with agents who help you.**
3. **Cooperate when it improves your reputation.**
4. **Cooperate with agents in your locale.**
5. **Cooperate with agents in a designated "group" that you belong to.**

**A pragmatic web** King's (1999) discourse patterns for collaborative inquiry provide a simple implementation of "*the gradual, localized, accumulation of information* [...] *in support of a 'pragmatic web'*." Following Geertz we might accept a more typical name for this sort of technology: "*culture*" (Geertz, 1973, p. 5).

The next section examines the practical features of culture formation in peer production communities, where knowledge is no less "*localized, embedded, and invested in practice*" than in the corresponding case of firms (Carlile, 2002) – but where "*integrating devices*" and "*representational capacity*" are perforce more explicitly artificial.

## 4.5 Architectures for collaboration

In Aaron Krowne's overview of the principles that informed the design of PlanetMath in its original incarnation, he states:

> *The basic, universal goals of digital libraries are to provide a logically organized, conveniently accessible, and (if possible) easily **actionable** collection of digitized **knowledge** in some field or fields for an audience of **learners**.*
>
> (Krowne, 2003B), original emphasis

As we've retraced his steps in the Planetary project, we've had the opportunity to re-examine the huge amount of often pioneering technical work that went into Noösphere. This work was nevertheless "of an era" – an era that can be understood more completely with reference to two other CBPP-related projects that started around the same time: Wikipedia/MediaWiki (Barrett, 2008) and Drupal.org/Drupal (Melançon et al., 2011). The key point, as we reflect back on the foundations of PlanetMath, is that there was a certain way of working, and indeed, a certain *theory of collaboration*, implicit the early architecture. The most noticeable tenets of this theory were associated with practices for "*homesteading the noosphere*" (Raymond, 1998) – namely object ownership and the corrections system. These features set PlanetMath apart from Wikipedia, and situate it closer to the typical practices associated with open source software.* PlanetMath, Wikipedia, and Drupal.org are all concrete examples of large postmodern organizations, built by many individuals who are asynchronously updating a shared database. The software systems behind the websites define and delimit many of the connections and boundaries between people, topics, and goals; *cf.* (Siemens, 2005), (Star & Griesemer, 1989).

**Building large knowledge bases by mass collaboration** The framework presented in Table 7 comes from Matt Richardson and Pedro Domingos (2003), who wrote about the particular challenges of building large knowledge bases through a process of mass collaboration. They also propose a specific approach to this problem, which cannot be given the same level of acclamation as their theoretical framework: it is hard to imagine inputting rules and facts as Horn clauses as a particularly popular pastime. The framework can be usefully applied to describe the process of "*building a digital library the commons-based peer production way*" (Krowne, 2003B), and, by extension, used as a lens for examining the social-technical-computational systems that *do* peer production.

Krowne's paper shows that a distributed approach can directly help to satisfy the criteria listed by Richardson and Domingos. For example, when compared with a centralized approach, commons-based peer production can help with *relevance*, insofar as:

* *Cf.* HTTP://DRUPAL.ORG/PROJECT/PROJECTAPPLICATIONS

**Relevance.** *In a distributed setting, ensuring that the knowledge contributed is relevant – and that volunteers' effort is productive – is a* [...] *significant problem.*

**Quality.** *Ensuring the quality of knowledge contributed by many different sources, when little is known about most of them, is likely to be very difficult.*

**Scalability.** *To achieve its full potential, a collective knowledge base must be able to assimilate the work of an arbitrarily large number of contributors, without the need for centralized human screening, coordination, or control becoming a bottleneck.*

**Consistency.** *As the knowledge base grows in size, maintaining consistency between knowledge entered by different contributors, or even by the same contributor at different times, becomes increasingly difficult.*

**Motivation.** *Following the example of open-source software, collective knowledge bases should allow user-developers to enter knowledge that is first of all relevant to solving their own problems.*

**Table 7:** *Criteria from Richardson and Domingos*

> [K]*nowledge is distributed unevenly and/or widely; a centralized effort can be inflexible; experts may be too busy with other work to commit entirely; and digital library builders themselves are not experts.* (Krowne, 2003B)

**The architecture of theories** To better understand the theory of collaboration underlying sites like PlanetMath, it is useful to enlist the help of Charles S. Peirce, who began his 1891 paper on "The architecture of theories" with a critique of those theories built from one idea, as if they were "*houses made of paper*" (Peirce, 1891). This would seem to be an especially important point for theorists of mediated collaboration to consider. A more recent critique focused on overly-constrained paradigms for the learning technology sector was advanced by Bernard Lisewski and Paul Joyce (2003), who argued that "*there are inherent dangers in them becoming too dominant a discourse*" or "*grand narrative*." They repudiate the uncritical use of theoretical systems. Table 8 shows how the framework proposed by Richardson and Domingos applies to the systems in operation at PlanetMath, Wikipedia, and Drupal.org. This table offers reassurance that each project has several important contributing ideas. Even so, a system needs to keep evolving in order *not* to become monolithic; *cf.* (Peirce, 1891).

**Dictionary vs Encyclopedia** Umberto Eco (1986) talks about written works that are organized along the model of the *Porphyry tree*, and works that are organized along the model of the *rhizome* or *labyrinth*, echoing (Deleuze & Guattari, 1980, pp. 5–7). Elsewhere, Deleuze emphasizes the associational aspects of cultural systems.

| | PlanetMath | Wikipedia | Drupal.org |
|---|---|---|---|
| **Relevance** | Relevance depends on peer review, and irrelevant content may be deleted. The mechanisms that would ensure that relevant content *will* be added need improvement. | People contribute articles about what they're interested in; apart from this, rules like WP:WEIGHT come into play. | Anyone can upload projects, although "full projects", require one time approval. Changes to the core require more vetting. |
| **Quality** | Quality control is handled with corrections and the "orphaning" mechanism in case of nonresponsive authors. Some articles are world-writeable, as in the wiki model. | Automated tools for spam and vandalism detection combined with a system of editorial oversight, in which Jimmy Wales has last say. | In addition to bug reports and feature requests handled through the issue tracker, modules can make use of an automated patch testing system. |
| **Scalability** | Peer review is distributed. Links are handled automatically. Caching is deployed where relevant; in particular, interlinking features are kept up to date. | The database and other infrastructure is massively scaled. There are many bots that help with small tasks. | In theory, anyone can join. Earl Miles, NYC-Camp 2012 keynote: "*There are no insiders, except Dries; there are no outsiders, only resumes.*" |
| **Consistency** | Although automatic links and corrections can help with consistency, mainly PM relies on standards for proof and expository quality. | NPOV is the key rule, which works together with templates and other process tools to maintain community standards about style and content. | The project issue queues are the place to go when one module's changes breaks another's. The core of the project has considerable oversight in these matters. |
| **Motivation** | People are solving some of their learning, exposition, and social needs on the site by writing and reviewing articles and posting in the forums. | As of 2006, over 50% of the site had been written by less than 1% of the users; these days, paid editing is somewhat notorious. | Miles continued: "*To build a resume, find someone who needs help, and help them. Find something that needs doing, do it.*" |

**Table 8:** *Criteria from Richardson and Domingos applied to describe three communities where mass collaboration happens (PlanetMath, Wikipedia, and Drupal.org)*

> [Hume] *gave the **association** of ideas its real meaning, making it a practice of cultural and **conventional** formations* [...] *rather than a theory of the human mind. Hence, the association of ideas exists for the sake of law, political economy, aesthetics, and so on. People ask, for example, whether it is enough to shoot an arrow at a site in order to become its owner, or whether one should touch the spot with one's own hand. This is a question about the correct association between a person and a thing, for the person to become the owner of the thing.* (Deleuze, 1953), original emphasis

Is PlanetMath best approached as an associationist dictionary or as a rhizomatic encyclopedia; *cf.* [§2.3, p. 30]? The specific example of *ownership* mentioned above is quite relevant in forming a prognosis. Following Raymond (1998), ownership after the Lockean property model was an important consideration in the design of PlanetMath/Noösphere (Krowne, 2003A, p. 5). As James Leach explains, in the Enlightenment tradition after Locke,

> *Human society is based on the ownership of property, as relationships between individuals **came into being** because of the appropriation of resources and the need to institutionalize that appropriation.* (Leach, 2007, p. 105), original emphasis

This way of thinking is based on the "*separation of spirit from matter, of appropriator from the substance appropriated*" (Leach, 2007, p. 105).* As described above [§4.1, pp. 47-48], this kind of thinking can truncate or distort important features of adaptivity. If "[t]*he knowledge needed to make land productive is part of a complex arrangement in social groups, and between them, whereby land and people become parts of one another*," and if people, more so than property, are "*the projects of other people*" then one "*distinguishes himself and gains authority through connection, not through exclusive control*" (Leach, 2007, pp. 111–112). At the very least there are some ideas here worth considering.

The next section treats other issues to do with relationship, community, and motivation that have relevance to paragogy.

## 4.6 Building online communities

Communities are often places where smaller communities form and dissolve again when their purpose has been served – recall the small naturally asynchronous groups of Goggins et al. (2013). One initial hypothesis that could help to understand communities is that they are most attractive when they work in a recursive fashion: *i.e.* healthy communities help their participants to develop sub-communities. In "Participation in an online mathematics community" Tausczik and Pennebaker (2012) use a framework adapted from Dholakia, Bagozzi, and Pearo (2004) to describe the reasons people participate in the research-level online mathematics community, MathOverflow (Table 9).

*Quoting (Pocock, 1985, p. 43).

| *Getting information* |
|---|
| *Giving information* |
| *Reputation building* |
| *Relationship development* |
| *Recreation* |
| *Self-discovery* |
| *Constructive Feedback* |

**Table 9:** *Criteria from Tausczik and Pennebaker*

Tausczik and Pennebaker examine each of these various motivations, using MathOverflow's built-in stigmergic mechanisms (score, downvoting, accepting answers, and comments) together with a textual analysis of comments conducted using the Linguistic Inquiry and Word Count (LIWC) methodology (Pennebaker, Francis, & Booth, 2001), which connects psychologically meaningful words to activities representative of each motivation.

Their findings report that "[t]*he strongest, most consistent evidence was found for the importance of reputation building in encouraging contributions*" however, there was only "*weak evidence that relationship building acted as an incentive to contribute*" and indeed that it "*may be another form of reputation building*" (Tausczik & Pennebaker, 2012). They emphasize the motivating features of concrete "*social benefits*," similar to Benkler's (2006a, 2002) notion of "*social-psychological rewards*." The model used on sites like MathOverflow seems to be an updated version of Eric Raymond's (1998) perspective: he suspected that "*the reputation-game gift culture is the globally optimal way to cooperate for generating* [...] *high-quality creative work*."

This would tend to go against the recursive hypothesis advanced above. Nevertheless, following Bateson, it is clear that relationships of one form or another are required in order for MathOverflow to support learning. The key point is that these relationships may be what Karin Knorr Cetina calls "*postsocial*" (Knorr Cetina, 1997): relationships not with people, but with mathematical objects and a distributed process of inquiry.

**Personal motivations** Daniel Pink (2010) believes that external rewards are often simply distractions, and proposes a theory of motivation based on "*autonomy*," "*mastery*," and "*purpose*." Thomas Malone (1981) advances a related framework for building motivating learning activities based on elements of "*fantasy*," "*challenge*," and "*curiosity*."

**Building new online communities** Even if all users are not motivated to participate in community-buliding activities, some are – and some may participate even without an explicit motivation. It is worth mentioning some points of alignment between Table 9 (particularly, reputation building and relationship development) and the advice from Resnick, Konstan, Chen, and Kraut (2012), who make these pragmatic suggestions to aspiring community builders:

> *– Drawing attention to external publicity and endorsements can raise expectations about future success.*
> *– In synchronous spaces that are not always active, a schedule of 'expected active times' coordinates visitors and can become a self-fulfilling expectation.*
>
> (Resnick et al., 2012)

**Almost Wikipedia** "Almost Wikipedia" was a thought-provoking talk given by Benjamin Mako Hill (2011A) at Harvard's Berkman Center, in which he says, in order to be as successful as Wikipedia, you should first make something people are familiar with – and second, make it easy for them to contribute.* In these terms, rebuilding PlanetMath as a collection of peer produced mathematical textbooks would make a good amount of sense. Sub-communities would also be readily recognizable (*e.g.*, "the authors of the Calculus textbook", *etc.*). In a recent paper, Hill and Monrey-Hernández (2012) point out a related finding: "*attributes of works associated with increased generativity are associated with decreased originality, and vice versa.*" Something that is easy to understand – and is, in that sense, perhaps not so novel – will be relatively easy to build upon.

## 4.7 Other related frameworks

Gregory Bateson explained the existence of a class of parallels arising in "*communicational and organizational process*" by reference to the "*the mysterious and polymorphic relation between context and content*" (Bateson, 1972, p. 163). There are many frameworks in the broader space of learning, communication, and organization theory that bear some analogy to the paragogy framework. These include Bateson's own criteria of mind (Bateson, 1980, p. 92 *et seq.*), and the functional framework of social movement theorists in the tradition of Luther Gerlach (Palmer, 2007). In short, if the paragogy principles are a rediscovery, they appear to be a very common one. As Harman (2008) remarks, "*anyone who looks seriously for fourfolds will start to find them everywhere*," and the same seems to go for 4 + 1 or 3 + *n*; *cf.* Genosko (2002, pp. 217–226).

The analysis that follows will draw upon some of these approximately-parallel frameworks, including: Pease and Martin's (2012) interpretation of Imre Lakatos's (1976) categories of mathematical activities, which is discussed in Chapter 7; Marshall McLuhan's tetrad of media effects (McLuhan & Powers, 1989), which is used to examine new features of Planetary in Chapter 8; and Stafford Beer's criteria for viable systems (Beer, 1981), which helps to frame discussions around projects in Chapter 9. In introducing the cybernetic notion of culture that was adopted in Chapter 1 [§1.4, p. 9], Geertz explained his motivations – which continue to apply here:

> [W]*e need to replace the "stratigraphic" conception of the relations between the various aspects of human existence with a synthetic one; that is, one in which biological, psychological, sociological, and cultural factors can be treated as variables within unitary systems of analysis.* [...] *It is a matter of integrating different types of theories and concepts in such a way that one can formulate meaningful propositions embodying findings now sequestered in separate fields of study.* (Geertz, 1973, p. 44)

*HTTP://CYBER.LAW.HARVARD.EDU/EVENTS/LUNCHEON/2011/10/MAKOHILL

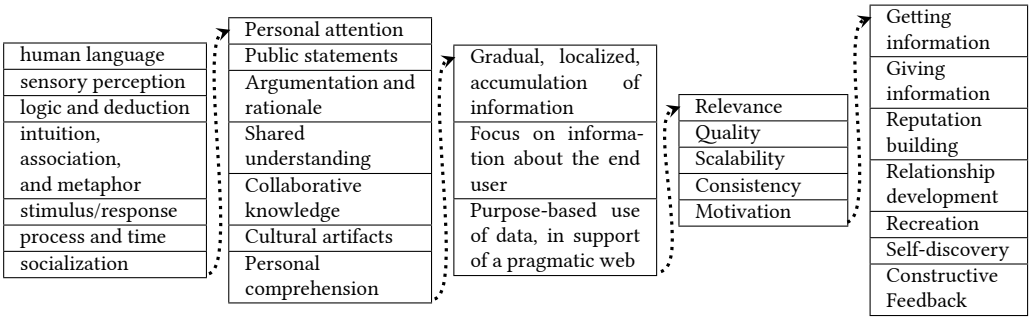



**Figure 6:** *Links between the key dimensions coming from the literature*

## 4.8 Summary

This chapter has contextualized the key pieces of research literature that are relevant to peer learning and peer production in mathematics. The framework provided by paragogy allowed a wide net to be cast, and provides an initial set of interlinked dimensions to consider when building a software model or further developing a paragogical approach in mathematics.

Arguing against any overly simple *narrative*, the aim has been to give paragogy a useful *form*. One possible instantiation is presented in Figure 6, which pulls together the main thematic ideas from the central figures discussed above, and shows that they fit remarkably well in a hierarchy. Socialization is important to Thurston: how does it work? This is explained by Stahl. And Stahl's critical personal comprehension? Perhaps this is built in a ecological way, as with the systems considered by McCalla. These rely on the idea of a pragmatic web, which could be described in terms used by Richardson and Domingos. Finally, the system as a whole is fueled by motivation, which has been examined in detail by Tausczik and Pennebaker. Whereas Maslow (1943) would presumably say that mathematics is pursued by generally healthy individuals as part of their efforts at self-actualization, this figure provides a more refined view. The relevant behaviors are inherently social – and the development of mathematics takes place across individuals. What we typically think of as the life of the mind may simply be "*the cutting edge of the life process itself*" (Ingold, 2000, p. 19).

Notice that the picture does not include the word "mathematics." It is possible to be more specific by thinking about how this scheme works in the mathematics case. After Halliday, understanding mathematics as a semiotic domain suggests that the core processes are not simply concerned with abstract symbol manipulation, but with making meaningful experiences. And after Bateson, it's clear that this requires the voicing of $\mu$ functions in which meanings can subsist. What is at stake in taking this forward is the matter of "*becoming relational*" (Durie, 2006, p. 183).

# Part 3
# Methodology and preliminary studies

# 5 Methodology

**The core research methodology aims to build a "thick description" of mathematical social life in the form of a working system for peer produced peer learning.**

## 5.1 The importance of building a working model

The following chapters examine "*man-made information sources for the directive ordering of human conduct*" (Geertz, 1973, p. 251) under conditions of emergence. This work retraces the layers of mediated sociality that were reviewed in Chapter 4: however the methods that will be applied are generally quite different from those of the authors examined there. Chapter 6 will exhibit a method for modeling the social antecedents to learning using statistical analysis. This is not a conceptual model of learning, but a way of showing, on average, what kinds of mediated acts bring about or catalyze learning outcomes. Chapter 7 uses focus group and interview-based methods to try to understand what potential users of PlanetMath would consider to be pragmatic for their purposes. Chapter 8 describes the design and development of the Planetary system, which is intended to be feature complete relative to Noösphere, with some key differences that that make it more suitable for peer learning applications. In Chapter 9, motivations to use PlanetMath/Planetary are investigated through discourse with users, who are involved in thinking critically about the affordances and limitations of the software. Research project does not focus on the processes of proof invention, heuristic discovery, and good old fashioned textbook problem solving, with which mathematical social life is often occupied, but rather the matrix in which these activities can be embedded. This does not mean that day-to-day mathematical activities are construed to reduce to a determination from the macro level (DeLanda, 2006A, p. 5) any more

than we would say that the theorems of Euclid or Archimedes derive from the floor plan or scribal practices at the Library of Alexandria. A emergent multi-layered approach, after (DeLanda, 2011) engages mathematicians and others in thinking carefully and critically about the "*the basic mental infrastructure of mathematics*" (Thurston, 1994), which as Chapter 4 has shown, has important social and cultural dimensions. Frayling (1993) describes "*Customising a piece of technology to do something no-one had considered before, and communicating the results*" as an example of "*research through art and design.*" This is a crucial aspect of the approach taken here, however, as the current effort aims to produce an actor-oriented model, what is most important is that the development of a prototype opens the way to consultations with users about the customizations they would like to see made. The findings comprise key steps on the way to the "*thickest possible description*" (Ryle, 1971) – in this case a software system that the research subjects would not only recognize as a reasonable account of relevant social experiences in mathematics, but which at least some of them would ultimately find useful. The thematic focus of the project as a whole is on software development in a social context. This approach affords the opportunity to build and jointly critique a working model of mathematical social life.

## 5.2 Review of primary methods

### Microgenetic analysis: studying the process of learning

Open online learning environments provide researchers with access to most if not all of the data that participants use to communicate with one another (Stahl, 2006); *cf.* [§4.2, p. 50]. These data can provide detailed evidence of learning and development. This kind of data – and the informal, *ad hoc* interactions that generate it – matches many of the ideas and techniques of *microgenetic analysis*. The microgenetic approach studies learning as a process, rather than the outcome of a process (Lavelli, Pantoja, Hsu, Messinger, & Fogel, 2005). The approach aims to examine moment-by-moment changes, often observed over a hort period of time, and often across a high number of separate observations, but not necessarily subject to the same staged treatment patterns found in longitudinal studies. Observations tend to be analyzed intensively, both qualitatively and quantitatively. In particular, microgenetic approaches have been used to take into account the social process of development in which individuals learn concurrently in a distributed fashion (Fischer & Granott, 1995). Broadly speaking, microgenetic analysis seems particularly well suited to our informal peer learning context, where a pre-test/post-test method for assessing learning quality would be inappropriate or unfeasible. The study in Chapter 6 can serve as a proof of concept illustrating the microgenetic measurement of learning, as adapted to the PlanetMath setting and dataset. Statistical

analysis of this sort of data presents some unique challenges (Cheshire, Muldoon, Francis, Lewis, & Ball, 2007). One particular challenge for modeling is that some users may not log in for long periods at a time, and may be strongly motivated in short bursts of activity; *cf.* (Barabási, 2005).

Chapter 6 examines additions and modifications to the PlanetMath encyclopedia taking place between Oct 6, 2001 and Dec 16, 2010, along with forum posts and corrections made during this time. The population of interest is comprised of the 445 people who made edits to the encyclopedia in this time frame. For the purpose of the study, changes to the encyclopedia in which a contributer uses a new word (on a per-person basis) is the outcome of interest. The analysis is based on a recently developed statistical method that is well-suited to handle bursty dynamics (Teravainen, 2014).

## Focus group and field work: Gathering user desiderata

Learning about and responding to users' design criteria is well studied in software engineering, where this is usually referred to *requirements analysis*.

> *Project participants or stakeholders may assign to the design object one or more goals – optative statements describing a change in the environment that the design object is desired to produce. A* ***requirement*** *is a feature of a design object that is necessary to achieve a goal.* [...] *More generally, suppose g may be achieved by n design objects* $(D_1, D_2, \ldots D_n)$ *each with features* $(F_1, F_2, \ldots F_n)$*. The set of requirements* $R_g$ *may then be defined as the intersection of the properties* $(R_g = F_1 \cap F_2 \cap \ldots \cap F_n)$. (Ralph, 2013)

In Chapter 7, I focus on what Ralph (2013) terms *desiderata*, rather than requirements. Chapter 8 revises this material into a minimal intersection of features for which the term *requirements* is more appropriate. In initial fieldwork, I aimed to understand users' goals, especially, but not only, the ones that could be achieved using software. At implementation time, a minimal kernel of features was developed, although this had as much to do with the limitations of time as with the empirical structure of user goals.

Chapter 7 is built around a focus group with fellow postgraduate students, and a brief period of field work and interviews with domain experts. This chapter uses the paragogy framework from Chapter 3 to map out the dimensions of change that seemed most relevant to these individuals as representatives of their respective stakeholder groups (students, mathematicians, and researchers with an interest in mathematical practice). The choice of populations to work with approximately follows (Corneli & Mikroyannidis, 2012). Inspired by Geertz, I adopt an interpretive perspective.* I relied on the idea of coherent change as an interpretive principle. The questions I returned to with research

*"[I]*nterpretation in my view is fundamentally about getting some idea of how people conceptualize, understand their world, what they are doing, how they are going about doing it, to get an idea of their world.*" Clifford Geertz, quoted in (Panourgía, 2002); reprinted in (Panourgía & Marcus, 2008).

subjects were "Would you find this sort of tool suitable for use in your work? (And if not, how would it have to change in order to be useful?)" The aim at this stage was to engage stakeholders in brief critical discussions about the potential usefulness of PlanetMath, from which I could extract a range of "*optative properties*" (Zave & Jackson, 1997). Statements about PlanetMath were also, explicitly or implicitly, statements about the stakeholders themselves.

For the initial focus group, 6 participants were recruited via email sent to the postgraduate student mailing list at the Open University (which reaches a population of around 250 local postgraduate students), and with an in-person announcement at the weekly postgraduate forum at the Centre for Research in Computing. The respondents, who as it turned out, were all from the computing department and who were all known to me previously, met in one focus group. I was able to make an audiovisual recording of the one-hour meeting. After a short presentation about PlanetMath, participants were asked to discuss three sets of questions in a variant of the "fishbowl" format.* Participants were divided into two sub-groups of 3 persons, who then were seated facing each other, at either side of a table. Group A was comprised of individuals who self-identified as "more mathematical" and Group B of those "less mathematical" in terms of their research interests and background. First Group A discussed a set of questions, while Group B listened, and then Group B discussed these questions while Group A listened. This design was employed to help ensure that everyone had a chance to talk, and to let differing views develop in the relative secure context of in-group dialog. The three sets of questions were increasingly vague, aiming to elicit exploratory interpretation on the part of participants (Figure 7). Each of the two-part rounds lasted 15 minutes. Participants were also asked to complete a brief questionnaire, which included space for free-form feedback.

My field work with domain experts again began with a presentation about PlanetMath, this time taking part in a seminar series at the School of Informatics at the University of Edinburgh. My primary contacts, a mathematician and a researcher who studies mathematical practice, who had both agreed to do interviews with me, were in the audience. During my four day research visit, discussions often happened in public (for example, after the seminar, or at a restaurant), and in the first private interview, the interviewee preferred not to be recorded. During the four-day visit, I therefor relied for on a daily journaling protocol as a way to create a record of what I was learning. This protocol (Figure 8) was adapted from a set of questions in a follow-up email from one of the ("less mathematical") participants in the initial focus group. Selections from the video transcript from the focus group and sources for quotes appear in Appendix C, and brief sample excerpts from my field notes in Appendix E.

*HTTP://WWW.CO-INTELLIGENCE.ORG/Y2K_FISHBOWL.HTML

**First set of questions ("Easy")**

- Would you find this sort of tool suitable for use in your work or studies?
- If you're an instructor, how might you have to change your teaching or assessment strategies if PlanetMath becomes popular?
- If you're a student, how might PlanetMath change your study strategy?

**Second set of questions ("Medium")**

- How do you currently solve problems?
- What do you do if you get stuck?
- What tools and strategies do you use to learn?

**Third set of questions ("Hard")**

- Describe your learning style.
- How do you deal with people whose viewpoints are very different from yours?
- How important is it to you to "own" what you create?

**Figure 7:** *Questions for discussion, posed to participants in the initial focus group of post-graduate students*

1. What were your key hypotheses at the start of the day?
2. How did you analyze the data you collected?
3. How did the results change what you knew before?
4. Did any of your hypotheses get corroborated as a result of your data analysis? If so, which, why and how strongly?
5. Did any of your hypotheses get challenged as a result of your data analysis? If so, which, why and how strongly?
6. Did you create or encounter any new hypotheses during or after the day's events? If so, please specify what they are and how you are going to test them.
7. How did any of the hypotheses relate to learning?
8. In hindsight, which questions proved to be the most or least salient?
9. Followup studies?

**Figure 8:** *Outline of journaling protocol*

## A basic grammar for interaction

Table 10 presents a summary of PlanetMath's *entity relation diagram* ("*ER diagram*"), showing the kinds of interactions that will be supported in PlanetMath/Planetary.* The research agenda is rooted in the idea of iterative development, whereby the basic interaction design will gain additional detail and expressive thickness.

Hence this central methodological proposition:

> A sufficiently extended version of the "grammar" presented in Table 10 should be able to transform mathematical texts and interactions into computationally accessible objects in a coherent way.

The five-fold division in Table 10 is again inspired by the paragogy principles from Chapter 3. Although the table provides a useful initial plan for implementation, from the point of view of modeling mathematical behavior, it is certainly not complete. For instance, one rather simple extension to the table would add subscripts to the arrows to code for the different agents involved – this would be needed for modeling the use case described in Figure 3 [§2.3, p. 31], for example.

To narrate the table's current entries: the encyclopedia, made up of interlinked articles and concepts provides the central/decentered shared context. The simplest kind of contextual glue is relatedness.† A link $A \overset{\ell_1}{\leftarrow} A'$ between two articles is only useful if the two articles are, indeed, *related*. Here, $A \overset{\ell_1}{\leftarrow} A'$ denotes a connection from $A'$ to $A$ via a standard hyperlink, whereas $A \leftarrow A'$ means that article $A'$ has been added as an attachment to article $A$.

| Context | Feedback | Quality | Structure | Heuristic |
|---|---|---|---|---|
| $A \leftarrow A$<br>$A \overset{\ell}{\leftarrow} A$ | $X \leftarrow T$<br>$S \leftarrow R$ | $X \leftarrow Q$<br>$A \leftarrow C$ | $A \leftarrow P \leftarrow S$<br>$L \leftarrow A, P$<br>$M \leftarrow A$<br>$Q \leftarrow A$ | $S \leftrightarrow H$<br>$G \leftrightarrow U$<br>$Q, T \rightharpoonup C, W, P$ |
| $A$ article<br>$\ell$ link | $X$ object<br>$T$ post<br>$S$ solution<br>$R$ review | $Q$ question<br>$C$ correction | $P$ problem<br>$L$ collection<br>$M$ classification | $H$ heuristic<br>$G$ group<br>$U$ user<br>$W$ request |

**Table 10:** *A paragogical decomposition of PlanetMath's activities: "production rules" in the grammar of mathematical behavior*

* *Cf.* Figure 1 [§1.4, p. 12].
† *Cf.* Nowak's rules of the evolution of collaboration [§4.4, p. 59].

Learning can happen in many ways on PlanetMath, but learning about learning seems most likely to be mediated by informal feedback, and developed in informal discussions. On the other hand, quality control develops through more structured peer review. Problem solving applications represent another layer of semantic structure. Deeper organization is driven by the use of heuristics, embedded in problem-solutions ($S \leftarrow H$), currently in a way that makes them hard to extract. Social heuristics involve purpose-created groups that solve problems together. Finally, these processes are accompanied by a curatorial or administrative process that mutates informal discussion into objects with more explicit meanings in the system ($Q, T \rightharpoonup C, W, P$). Time flows from left to right for all of these expressions. The table suggests various vectors of data on user activities that could be pulled out of the database: *new links between articles added*, *new questions asked*, *new corrections submitted*, *new problems proposed*, *new group memberships activated*, and so on. Rebuilt on this design, PlanetMath will become a large-scale recording device for interactions related to learning mathematics. In addition to mathematical relationships between articles and other objects, the "production rules" in Table 10 serve to code for various features of human collaboration. For example, by attaching a comment to an object, the user collaborates with the object creator by offering feedback. By attaching a correction to an article, the user collaborates not only with the original author, but more broadly with the readership, by helping to maintain quality and consistency throughout the encyclopedia. Attaching a problem to an article or organizing a set of articles and problems in a collection helps to structure the corpus, creating a small subarea with coherent meaning. People can use groups to collaborate as co-authors or in *ad hoc* teams. Collections and groups are also useful for specifying and sharing learning and production goals, and for breaking them down into smaller components. Chapter 8 describes the development of the Planetary system, which progressed along the lines indicated by Table 10, and further informed by the findings from Chapter 7. Chapter 8 uses Marshall McLuhan's (1989) tetrad of media effects to highlight the differences that the implemented features are expected to make.

## Design patterns and emergent design

Peer production is not a panacea. On the one hand, "*many successful peer production projects exhibit strong inequalities of participation and deeply entrenched leadership*" (Shaw & Hill, 2014). On the other, many free/open software projects never attract collaborators at all: "[t]*he benefits of collaboration become something to understand, support, and work towards, rather than something to take for granted*" (Hill, 2011b). This challenge will be approached here in a methodical way through the use of *design patterns* to describe problems and develop solutions within a paragogical context. This application of design patterns extends David Cavallo's (2000b) thinking about the *emergent design* of learning environments, rendering it somewhat more systematic.

**Title**: Encapsulate the idea – possibly include a subtitle.

**Definition**: Explain the idea and the context in which it is meaningful.

**Problem**: Explain why there's some issue to address here.

**Solution**: Talk about an idea about how to address the issue.

**Challenges**: Talk about what can go wrong.

**What's Next**: Talk about specific next steps.

*Examples, Illustrations, Objectives, and References are optional.*

**Table 11:** *Pattern template from the* Peeragogy Handbook

Christopher Alexander first introduced design patterns as part of an holistic approach to architecture, urban planning, and construction (Alexander, Ishikawa, & Silverstein, 1977; Alexander, 1979). The design pattern methodology spread from architecture to software (Vlissides, Helm, Johnson, & Gamma, 1995; Coplien & Schmidt, 1995; Gabriel, 1996), and later, to other fields, including public affairs (Schuler, 2008) and education (Bergin et al., 2012). A relevant recent survey is included in (Kohls, 2013).

Alexander's patterns are presented in tree-like structure called a *pattern language*, ordered in a top-down manner from large-scale to small-scale levels of application, with each pattern presented in terms of a *picture*, a *context* (including links to relevant larger patterns), the *problem* that the pattern addresses, the *solution*, a *diagram*, and *links to smaller patterns* (Alexander et al., 1977, pp. x-xi). A relatively convincing implementation of Alexander's idea of patterns as a "*living language*" (Alexander et al., 1977, p. xvii) was realized with one of the earliest applications of wiki software developed by Ward Cunningham: the Portland Pattern Repository.*

Taking the idea of a pattern language as a point of departure, the *Peeragogy Handbook* (Rheingold et al., 2014) includes a catalog of patterns for peer production and peer learning that the authors discovered while working on the book. The patterns are presented using a template (Table 11) that is closely aligned to the structure coming from Alexander. Each pattern is essentially a short textual abstract, and describes an active dimension of the Peeragogy project. This emphasis on the active aspect is made particularly clear through the "What's Next" facet, which concretely links the patterns in the *Handbook* to the current activities being conducted within the aegis of the Peeragogy project. The *Newcomer* pattern is presented as an example in Table 12.

*HTTP://C2.COM/PPR/

***Newcomer: Welcoming "beginner's mind."***

**Definition**: Unless there is a new person to talk to, a lot of the "education stuff" we do could grow pretty stale. Many of the patterns and use cases for peeragogy assume that there will be an audience or a new generation of learners.

**Problem**: Some of the problems are well summed up with a quote:

> **Régis Barondeau**: I joined this handbook project late, making me a "newcomer." When I started to catch up, I rapidly faced doubts: Where do I start? How can I help? How will I make it, having to read more than 700 posts to catch up? What tools are we using ? How do I use them? Etc. Although this project is amazingly interesting, catching the train while it already reached high speed can be an extreme sport. By taking care of newcomers, we might avoid loosing valuable contributors because they don't know how and where to start, and keep our own project on track.

**Solution**: It is good to try to become aware of what a newcomer needs, and what their motivations are. Another quote can illustrate:

> **Charlotte Pierce**: Joe was working a lot on the book, and I thought "this is interesting hard work, and he shouldn't have to do this alone." As a Peeragogy newcomer, I was kindly welcomed and mentored by Joe, Howard, Fabrizio, and others. I asked naive questions and was met with patient answers, guiding questions, and resource links. Concurrently, I bootstrapped myself into a position to contribute to the workflow by editing the live manuscript for consistency, style, and continuity.

**Challenges**: Newcomers in the Peeragogy project have often complained about feeling confused, suggesting that our project roadmap may not be sufficiently clear, and that more work has to be done the project accessible. Even in the absence of actual newcomers, we need to try and look at things with a "beginner's mind."

**What's Next**: We recently revised the "How to Get Involved" page, listing the top ten sites we use. Another reasonable thing to post would be a top-ten list of activities, so that people can get an easier view on the kinds of things we do in the project.

**Table 12:** *The* Newcomer *pattern from the* Peeragogy Handbook

The main features of the Peeragogy pattern template are analogous to the paragogy principles [§3.1, p. 36]. The *definition* is meant to connect the pattern to a particular context; specifying the *problem* provides the opportunity to explain why there's something here to learn; a *solution* usually involves interactions with peers; *challenges* arise in practice due to the nonlinear nature of real-world problems; and writing down *what's next* indicates that after identifying a pattern, there is usually still work to be done.

This approach seems to be a novel interpretation of the design pattern methodology. For comparison: the Public Sphere project uses a detailed template to elicit input for their pattern catalog. They aim to distill existing, ambient, "*civic intelligence*" in an effective form.[*,†] However, although this pattern language continues to develop as a work in progress, the patterns are not explicitly linked to actions taken in the project. Table 13 presents a succinct overview of the Peeragogy pattern catalog, with references to the Minskian heuristics from Table 4 [§4.2, p. 53]. This helps to show how the Peeragogy patterns are relevant as a set of heuristics for social problem solving. Further details are in (Rheingold et al., 2014).[‡]

The Peeragogy project continues the tradition of reflexive research-through-writing that was used by Tomlinson et al. (2012). Both of these efforts used their own collaborative writing and research activities as primary sources of data. Building on this bootstrapped material, the application here is comparatively straightforward.

In Chapter 9, feedback arising in the user evaluation of PlanetMath/Planetary is analyzed in terms of the Peeragogy pattern catalog, and the catalog is extended with three additional patterns. In the work reported in Chapter 9, PlanetMath users were not involved directly in doing pattern-based analysis or coming up with new patterns: rather, I wrote new patterns based on what I learned in our interviews. In preparing the analysis, I used a "path metaphor" similar to that employed by Christian Kohls (2010, 2011), who describes patterns in the language of constrained optimization problems, considering the initial state, end state, and forces acting.

At the level of process: I successfully recruited 6 people who completed a 10 day study on PlanetMath that took place two months after the launch of the new software system. The participants were 3 PlanetMath users and 3 mathematics postgraduate students from the Open University. Two of the PlanetMath users had prior familiarity with PlanetMath/Noösphere, and two Open University Masters students were familiar with online distance learning. Four participants use mathematics professionally. (A seventh participant was recruited from the *Peeragogy in Action* Google+ community, but he disappeared before completing the planned tasks or an interview.)

[*] HTTP://PUBLICSPHEREPROJECT.ORG/PATTERNS/CHEAT-SHEET.PHP

[†] HTTP://WWW.PUBLICSPHEREPROJECT.ORG/TEMP/YOUCANHELP.PHP

[‡] And in the online version of the book, at HTTP://PEERAGOGY.ORG/PRACTICE/.

We *simplify* things for a **Newcomer**.

We *change focus* by using a **Roadmap** to guide us from one step to another. Every roadmap is associated with **A Specific Project**, and the specificity is important. In addition, the project's **Heartbeat** leads us to let go of our focus at one moment, and resume with another point of view later.

We *change description* first of all by having a **Wrapper** who describes the new state of the project. For the Peeragogy project, that often meant summing up the high points that we saw over a given period of time. It seems possible that with a rich enough **Pattern Language**, the description would itself be made in terms of patterns.

We *divide* work up not only "horizontally" among different **Roles**, but also "temporally" by using the **Roadmap**. Someone who is moving ahead with the Roadmap is likely to be "working at the cutting edge".

When we *find an analogy*, we are basically **Creating a Guide** of some sort. This can be used as a form of "exploration", as we look at how one form of engagement may or may not map onto other forms of engagement.

When we *ask for help*, we may avail ourselves of some **Moderation** service that will decide how to deal with our request. One simple way to ask for help is **Polling for Ideas**. Obviously once we start to get help, we're working in a regime of "collaborative effort".

If you *know the answer*, then you may be able to reuse it (which is the basic idea of **Use or Make?**). Someone who knows the answer and who is good at self-explanation may also have a good idea about how to get from the current state to the goal state; alternatively, this may be broken down into steps in a sub-**Roadmap**, and moving from step to step would then illustrate "progressive problem solving".

It is important to *give it a rest* so as not to over-exhaust oneself, busting one's own **Carrying Capacity**, or, alternatively, overwhelming the group.

It seems that one of the things that *experts do* is **Discerning a Pattern**. This allows them to simplify their processing.

Finally, again, if we *know why it is hard*, then we may be able to **Create a Guide** that will help get around, or at least better cope with, the difficulty.

**Table 13:** *Overview of the central patterns in the* Peeragogy Handbook

**(Step 1)** You'll join an hour-long online focus group, where I'll brief you on the tasks for the next step, and give a more detailed description of the study, as well as gather some initial feedback about the site in an open discussion.

**(Step 2)** You'll try out the new system features that we've developed – this can happen as you have time, over a week or two. We'll be able to talk about issues or concerns as they arise, in the PlanetMath forums.

**(Step 3)** You'll meet me with me for a small group or one-to-one interview to talk about your experiences with the system; this will again take less than an hour.

**Figure 9:** *Steps in the user evaluation (from an email sent to 30 high-scoring and recently-active PlanetMath users)*

The steps in the study are indicated in Figure 9. In an initial meeting held via Skype, I walked study participants through a set of screenshots from PlanetMath/Planetary (presented here in Chapter 8). As a set of tasks to complete over the next 10 days, I asked participants to try each of the features described in the screenshots. During the meeting, I also asked users to introduce themselves, and to describe their mathematical background. I suggested that they work together in case of any difficulties, attempting in this way to foster a peer learning culture within the group. Participants who didn't yet have PlanetMath accounts made them at the start of the study. Over the next 10 days, most participants tried many of the tasks, sometimes posting comments about their experience in the PlanetMath forums, and sometimes emailing me directly to ask for help. After the 10 day period had elapsed, I conducted individual follow-up interviews.

As in the initial study with stakeholders, the main question I focused on in the interviews was: would PlanetMath be relevant to the way you study and do mathematics? I also tried to elicit ideas that might help to improve the site, making use of shared familiarity with the system.

During the interviews, I took graphical notes and made audio recordings (when possible). Interview data was analyzed twice: first on a thematic basis, by connecting user feedback to the main dimensions of the design of PlanetMath/Planetary discussed in Chapter 8, and again by looking for matching patterns in the Peeragogy pattern catalog (Table 13). The idea in the pattern analysis was that either the existing catalog of Peeragogy patterns would describe the topics under discussion, or new patterns would emerge with links to the existing patterns showing how they are similar or different. In either case, a contribution will have been made to enhancing the usefulness and descriptive power of the pattern catalog.

## 5.3 Summary

The patterns that guide behavior in the PlanetMath context are related to historical mathematics practices, but are understood to have features that are emergent in the new setting. This invites a "DeLandan" perspective on culture that looks at the emergent dynamics within and across several levels of analysis. The primary methods are statistical analysis of historical data, qualitative field work, implementation of a software system for collaboration, and emergent design.

Several novel methodological points were introduced, including the simple idea of using vocabulary growth as part of a statistical model of learning in a technical domain, and the more complex idea of using design patterns to represent the heuristic, and emergent, aspects of paragogical projects. Paragogy was useful in developing an initial entity relation diagram for PlanetMath/Planetary, and new schemes for analyzing qualitative data. In this way, paragogy has been baked into the design of both the software system and the research process.

The overall aim is to do research through iterative development within an emergent design framework. A key limitation is that the following chapters are only *one* research cycle. This work necessarily focuses on the dimensions which are currently accessible to study, for example, "vocabulary use" as opposed to "heuristic use" – as well as the most basic technologies and use cases. The primary merit of the approach is that it opens the way to a participatory, computer-mediated, study of mathematical culture.

# 6 An Analysis of Learning Interactions in PlanetMath/Noösphere

**The first study focuses on understanding the effects of peer support on learning in the course of encyclopedia authoring activities and forum discussions on the original PlanetMath.**

## 6.1 The fundamental problem of instrumentation

This chapter looks at what causes a change in propensity to use new vocabulary words among contributors to the PlanetMath website. In addition to new vocabulary, in practice, there are of course many other nonlinguistic, embodied, behaviors that people learn when they learn mathematics. The statistical methodology employed here would be equally applicable to the effect of behavior on word use, or the effect of word use on behavior. The focus is not on the internal or embodied organization of learning, but on the social covariates, antecedents, and possible precipitating causes. The particular strength of the method that was employed here is that, unlike standard time series methods, this approach is designed to filter out effects that are due to "calendar time", like seasonality, leaving only effects that are tied to the "gap time" between events. The technique works with any pair of time series vectors, one representing the *treatments*, and one representing the *outcomes*.

From a qualitative perspective, describing a given antecedent, for example, giving instruction, as a "*cause*" of learning is rather thin: it may be better to speak of statistical effects as "*catalysts*"; *cf.* (DeLanda, 2006B). After all, we could always go back further, looking for earlier and deeper contributing causes. What the method used here shows is that there was a significant social contribution to learning in PlanetMath/Noösphere; and it shows the differential effect of two different catalysts.

Due to it's generality, the method exhibited in action here is a candidate solution to the fundamental problem of instrumentation: *how do we know if it's working?* For example, if we were to use PlanetMath to make recommendations, as in the use case from Figure 3 [§2.3, p. 31], we would want to know that the recommendations were having the desired effect. The method demonstrated here could help.

## 6.2 Design and related work

The study aims to demonstrate that peer interactions in PlanetMath/Noösphere had a tangible and measurable effect on learning outcomes. New types of interactions may produce vastly different, and stronger, effects. It will be important to have a framework for analysis that can be used to gauge the effect of these interactions on desired outcomes.

The study offers a preliminary quantitative analysis of learning on PlanetMath, using legacy data from the site's first 10 years of existence. Even though PlanetMath was not designed to function as a learning environment during this phase, by design it had an explicit "*pedagogic slant*" [§2.1, p. 21] andr it is reasonable to hypothesize that people were, in fact, learning through their interactions on the site.

It is important to mention that during this time period, the growth of the PlanetMath encyclopedia, and, in its metadata layer, an associated mathematics lexicon, was the key measurement of success. Practices of peer review and norms like "one definition per concept" and "proofs should be self-contained" afforded a degree of integration and rigor that would not be directly possible with a fully-distributed information source (like the blogosphere).

Learning new things and sharing existing knowledge were two possible motivations for contributing to the encyclopedia. In some sense the latter motivation would have to be typical for encyclopedia authors – although the two motivations are not mutually exclusive, as expressed in the adage "*The best way to learn is to teach*" (Luk & Wang, 2012).

Distinguishing between the use of new words at the personal and community levels might help to distinguish between these two motivations. Someone who often uses new words on a per-person basis that are not new on a community-wide basis may be a more typical learner. Someone who consistently adds globally new words is more likely to be an expository writer or researcher.

This study focuses on the use of new words on a per-person basis, without distinguishing the words that are globally new from those that are only personally new. The analysis focuses on the immediate catalysts for the use of a new word, comparing a self-

started effect (posting in the forum) and an other-started effect (receiving a correction). For the purpose of the study, any use of a personally new word is thought of as a positive outcome, and considered to indicate learning in some form.* Even if in many cases this is learning in the course of practicing something that one already knows reasonably well, it is interesting to understand the social catalysts for such practice.

In order to facilitate the comparison between the two effects, I processed all of the data describing contributions to the PlanetMath encyclopedia from the first 10 years of the website, and found the timestamped changesets where each user first uses a term that is part of PlanetMath's technical lexicon. These technical terms include the titles of articles, as well as synonyms noted by the author in article metadata. Along with this word usage data, timestamped entries in the database showing when authors had either posted in the forum or received a correction.

This data was then analyzed using the recently developed computational method from Timothy Teravainen's PhD thesis *Semiparametric Estimation of a Gaptime-Associated Hazard Function* (Teravainen, 2014), discussed in overview in (Wu et al., 2011). This work falls within the field of statistics known as survival analysis; *cf.* (Fleming & Harrington, 2011). The method estimates the increase in likelihood associated with the specified outcome (here, using a new word) after the treatment event (here, either the self-initiated treatment of posting in the forum, or the other-initiated treatment of receiving a correction). The change in likelihood is considered in comparison to a otherwise equivalent user who did not receive the treatment, and the results are averaged across all contributing users and all time.

For comparison: Albert-László Barabási's paper on "The origin of bursts and heavy tails in human dynamics" (Barabási, 2005) uses a large collection of email correspondence as a data source. Barabási's models use power law functions, which are generalized by the Ornstein-Uhlenbeck process at the core of the computational model employed here (Behme, 2011, p. 12).

There are many relevant techniques for dealing with social networks and time series data; *cf.* (Rajaraman & Ullman, 2011, Chapter 10), (Quenouille, 1949). Standard cross correlation is the most typical approach to studying time series data, but it cannot distinguish between gap time effects and calendar time effects. For this study, the typical gap time is of interest, rather than seasonal effects. Many modeling efforts consider only the relative *sequence* of events, discarding the timestamps – *e.g.* Abbott (1995) – but this approach is not well-suited to studying activities that are distributed over long periods with bursts of activity.

* *Cf.* Leung [§4.2, p. 51], Halliday [§4.2, p. 52] and Minsky [§4.2, p. 53] for some of the considerations involved with using vocabulary growth as a proxy measure for learning.

Various corpus techniques could be used to extract meaningful patterns from a text (Pustejovsky, Anick, & Bergler, 1993). Any refinement to the input data would be compatible with the statistical model used here, as long as the data can be mapped to a time series. Work with named entities is particularly straightforward and well-studied (Nadeau & Sekine, 2007). The LIWC, mentioned in Chapter 4 is an example of this sort of approach, and focuses on psychologically meaningful words, including pronouns, and words related to emotions and causality; *cf.* (Pennebaker et al., 2001).

## Methods

The last ten years of versioned encyclopedia articles were loaded into Git, which made it relatively easy to extract line-by-line differences between versions, using `git log` and `git show`. These change sets were analyzed using a term spotting approach, with technical terms drawn from the thesaurus-like metadata from PlanetMath's encyclopedia articles. This was done using *ad hoc* code implemented around a regular expression based search.* The functionality is similar to that of PlanetMath's autolinking tool, NNexus (Gardner, Krowne, & Xiong, 2006), and with a few changes to NNexus, the method used here could be readily adapted to make use of transcripted NNexus data. Subject to the general caveats above, that detected words may in many cases be exposition events rather than elementary learning events, the outcome of interest is any encyclopedia edit that includes a new word from the lexicon (per-user, considered relative to the history of their previous contributions to the encyclopedia). During the time period under observation (Oct 6, 2001–Dec 16, 2010) for the population of 445 "committers" under consideration – exactly the population of persons for whom we have edits on record – there were a total of 8051 forum posts made, 14064 corrections received, and 3867 modifications to the encyclopedia in which any non-zero number of new (per user) technical terms were detected.† The method applied to analyze this data was a slight variation on that studied in Teravainen (2014), in which the primary application was to sequences of events with a statistical self-retriggering property that did not present the need to distinguish between "treatment" and "outcome". Here, the computation first identifies gaps between treatment and outcome events, and then aims to estimate the effect that the most recent previous treatment had on a given outcome. It does this by fitting the data to a physical model with two primary dimensions: an initial impulse and a damping effect (an "Ornstein-Uhlenbeck process"). Teravainen considers stochastic generalizations of this model that were not applied in the computations here.

*HTTP://METAMESO.ORG/~JOE/MATH/THESIS-SUPPLEMENT/

†PlanetMath defines quite a few more than 3867 terms, and one might expect an aggregated per person usage statistic to be still bigger than the total number of terms available. The reason for a lower figure may be accounted for by clarifying just what was counted as a "usage event." First, articles can define multiple terms, and if multiple terms are introduced in one editing pass, then this would only count as one instance of "new word usage." Second, many articles have titles like "*proof of . . . theorem*" which never appear in the article's body, and which consequently wouldn't be detected in the changesets.

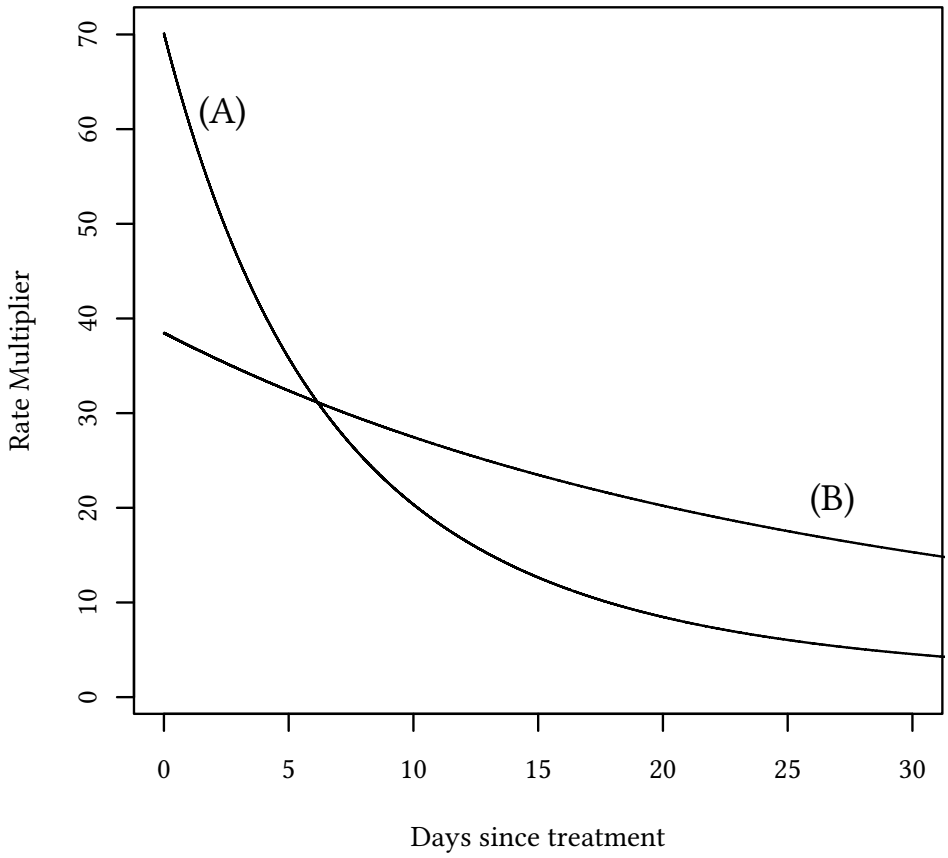



**Figure 10:** *A plot showing the differential effect of (A) an exogenous treatment* (receiving a correction)*, and (B) an endogenous treatment* (posting in the forum) *on* using new words in the encyclopedia

## 6.3 Estimating effects

Intrinsic self-motivation is a well-known catalyst for learning. It is interesting to consider the differential effect on learning outcomes from stimuli that are self-driven from stimuli that are initiated by peers. Two trajectories along these dimensions are examined:

**Trajectory A.** The effect of *corrections received* on *new technical terms used.*

**Trajectory B.** The effect of *posting in the forum* on *new technical terms used.*

Trajectory A is other-motivated; Trajectory B is self-motivated, but occurs within a social context. As shown in Figure 10, Trajectory A has a strong initial effect, but wears off relatively quickly, whereas Trajectory B has a much more moderate effect,

and wears off slowly. This suggests that people who are motivated to post in the forum are also motivated to keep adding new concepts in encyclopedia articles – and it also shows the strong value of peer support even among this self-selected group of highly self-motivated people.

## 6.4 Potential refinements

In the first place, without considering any semantic extensions to the underlying model, it would be useful to have confidence intervals around the curves in Figure 10. Such estimates are forthcoming.*

It would be useful to extend the model to study several different treatments in parallel; this could help to tease out the differential effect of a socially-motivated post in the forum *versus* an edit to the encyclopedia. At the moment, the model compares a given treatment to "no treatment"; it would be useful to be able to run more elaborate simulated control studies.

The current model does not make use of the "networked" aspect of the PlanetMath encyclopedia or the underlying social network. A more ideal model of an individual's learning patterns would consider the geometric features of the system that they are studying. It seems reasonable to hypothesize that person would be more likely to learn terms that are near terms that they already know, and especially likely to (serendipitously or opportunistically) learn things that are adjacent to topics they are currently working on. A simple prior distribution could be employed, saying that, all else equal, terms are most likely to be learned in proportion to their PageRank, or inversely in proportion to their mathematical depth (which could be estimated by other graph-theoretic distance from known elementary concepts).

Apart from modeling issues to do with personal learning history, the modeling problem is complicated by the fact that people are different from one another in ways that have nothing to do with the domain. One possible adaptation of the current scheme would use a Dirichlet prior that takes into account the fact that there are "different kinds of people." For instance, we could categorize users according to the apparent strength of their intrinsic motivation. Some users may require more attention if they are to stay involved. More broadly, we might want to consider individuals with a range of features, per (Broderick, Mackey, Paisley, & Jordan, 2011).

An alternative interpretation of the PlanetMath data would come from transposing the matrix, so that instead of looking at new-terms-per-user, we could look at new-users-per-term. This approach would useful for studying community-level linguistic change.

*Timothy Teravainen, personal communication.

Vocabulary growth has previously been studied with the aim of predicting user migration. According to Danescu-Niculescu-Mizil, West, Jurafsky, Leskovec, and Potts (2013), "[a]*fter an initial period of adaptation to the language of their community* [...] *individuals' language patterns slowly rigidify until the moment they abandon the community.*" It would be interesting to see whether there are explicit strategies for lifelong learning in a deep domain like mathematics that help maintain user interest.

It would also be interesting to look at the developmental course of learning: do individual users undergo intensive periods of learning, similar to the burstiness studied by (Barabási, 2005)? Do users learn new nonlinguistic behaviors along with new vocabulary, per (Iverson & Thelen, 1999)?

## 6.5 Summary

This chapter has employing a new statistical method to estimate the effect of two different treatments on learning outcomes. An analogous study could be performed in any context where learning can be modeled by vocabulary growth. This includes technical domains, and potentially any book with an index.

The significance of the approach is considerably more general, since, as suggested in principle by (Wu et al., 2011) and now illustrated here, the same techniques can be used to model the effect of any time-delineated series of "*blip treatments*" on a time-delineated vector of outcomes.

The techniques can be applied directly to study discrete online interactions that are likely to have effects on learning outcomes or other valued goals. With further refinements, this sort of analysis could inform varied diagnostic and recommendation services. Although these issues are not developed further in the thesis, this chapter does illustrate the potential usefulness of gathering varied data on user interactions, and this point is taken up again in Chapter 8.

# 7 Surveying Stakeholder Perspectives

**The second study employs qualitative methods to build a richer picture of the stakeholders in a peer learning environment for mathematics.**

## 7.1 Overview

The purpose of this study was to understand the perspectives on PlanetMath's rebuild from several likely user and stakeholder groups – students, mathematicians, and "reflective practitioners." The latter group is understood to include philosophers, social scientists, educators, and computer scientists whose work contributes to an "*inquiry into the epistemology of practice*" in the mathematics domain; *cf.* (Schön, 1999, p. viii).

Asking potential PlanetMath users what they would expect from a "peer learning environment for mathematics" offers the opportunity to learn something about what they think about mathematics *per se*, and to understand the role that computer-mediated communication plays in shaping their perspectives.

I have endeavored to convey the understandings that my research subjects offered in terms of the meanings that are operant for them in our conversations. The idea of PlanetMath constitutes a minor perturbation to the life-worlds of these research subjects; their framing of "the question" of PlanetMath is of particular interest.

> *A meaning-reaction is* [...] *a fitting adjustmental response which individuals acquire through the direct influence of the surrounding objects and conditions. It is in this way that a meaning-reaction becomes the means for bringing about an especially fitting adjustment of the person to his surroundings.* (Kantor, 1921)

In some cases, the subject's responses to PlanetMath, and what it represents, were enthusiastic – and in other cases, highly skeptical. Wherever these responses were not entirely explicit (whether they were favorable or unfavorable), I have expanded the discussion to help put them in context. For example, perhaps the most "interpretative" move on my part was to point out the connections between students' explicit mention of *trust* and the unspoken issue of *credit*.

At this point, it is not entirely appropriate to think of the collateral adjustments that are indicated for PlanetMath as "*requirements*" in the typical engineering sense; I have called them "*desiderata*," following (Ralph, 2013). The feedback elicited at this stage points to long-term goals that are not wholly realized by the implementation work described in Chapter 8. At the same time, the work presented in this chapter usefully conveys a sense both that this implementation work is "on track," and a meaningful indication of what it is pointed towards.

This research is based on a focus group I ran with 6 postgraduate students in computing at the Open University, and a 4 day research visit to Edinburgh, which was spent partly at the Department of Informatics at the University of Edinburgh and partly at Heriot-Watt. In Edinburgh, my primary informants – who have agreed to be non-anonymous in this report – were Murdoch James Gabbay and Alison Pease. I scheduled my visit at a time when I was also able to speak with Ursula Martin, Timothy Gowers, Mohan Ganesalingam, and Andrew Aberdein, and this chapter benefits from those discussions. From a practical standpoint, a standard of non-anonymity among the professionals I spoke with is justified because connections to published work are discussed in the chapter; from an ethics standpoint, this choice was seen as justifiable, since the points discussed were not contentious or especially personal. Had I been interested in a detailed ethnographic description of the lives of these informants, I would have opted to disguise the responses with pseudonyms.

Although the fieldwork methods used in the two settings were somewhat different – a "fishbowl" style discussion around my prompts in the student focus group at The Open University, and several days of relatively immersive discussions with professionals in Edinburgh [§5.2, pp. 71-73] – I have presented the findings in a uniform format. Inset quotes are paraphrases of things that respondents said; the relevant issues are then expanded in my own words. References [C1]–[C15] point to supporting data in Appendix C. The conclusions presented in the text were often, but not always, arrived at jointly with the research subjects.

The findings are presented in detail in the sections that follow, where they are mapped to the paragogical framework of dimensions of change from Chapter 3. A brief "composite" overview is presented just below.

### Student

As long as their university courses provide everything that current students are seeking, they are unlikely to use PlanetMath, even as an ancillary tool. And yet it seems clear that PlanetMath could be used as a form of alternative credentialization – through a transparent record of contributions, including problems solved – or as a textbook, or even as a course management tool.

### Mathematician

For a mathematics researcher, contributing to public discussions about mathematics is not typically done directly for the purpose of "getting credit" (as in, something to put on one's CV), but rather because it helps with research. There are many ways this could be better supported, both on PlanetMath and with tools that work across the web. Working in the capacity of instructor, a professional mathematician needs to help students learn relevant social as well as technical skills. One way to encourage this is to make suitably structured peer interaction a required component of courses. Some of these interactions could take place online.

### Philosopher

Philosophers of mathematics working within the empiricist tradition are interested in the ways in which understanding arises. This can be studied using various kinds of models, ranging from heuristic frameworks imposed over data generated through human interaction to computer simulations. Insights gained through this work can help to improve education, tool design, and can also contribute to the field of artificial intelligence. PlanetMath could be both a source of useful information, and a consumer of the philosopher's insights.

## 7.2 A student perspective

**Changing the nature of the space** The issue of trust was raised even before I had shared any details about which factors might already exist that might instill trust. This minor incongruity suggests that the issue of trust might be conflated with another deeper issue. I suspected that this was the issue of credit. In a university course, students can trust their instructors and TAs to adhere to clear contractual obligations. They can trust that if they put forth a reasonable focused effort, they will earn an appropriate mark. So it is clear that in the traditional university setting, trust and credit are closely linked. However, without explicit attention to creating suitable mechanisms, in online peer production systems, neither trust nor credit is there

> *"Can we trust the people who are contributing?"* [C1]

by default. Accordingly, systems for distributed production, including Q&A sites, open source software projects, and online encyclopedias – some to better effect than others – have all created their own mechanisms for building trust and maintaining quality [§4.5, p. 61]. It seems that trust and credit go well together online, as well as offline. If it is easy to see what someone has contributed, and perhaps what others thought about it, then it's also reasonable to expect their engagement will continue roughly in the same vein.

**Changing what I know about myself** Small moves with continuous feedback at each stage could help build the user's confidence. The feedback may reveal that they know the answer to the problem, for example – and if not, taking small steps means that the user is never too far from what they do know. Indeed, mathematics naturally breaks down into small steps, but it may not be obvious how this works for a beginner. Learning *how* to break a task down into small, individually-masterable steps is a useful skill, and can help make mathematics less daunting (Mighton, 2003).* Students also need to learn persistence, but persistence through many small steps is one route to this.† More broadly, teaching that mathematical ability can be developed through hard work tends to have a lasting positive effect (Dweck, 2000, 2007).

*"Does PlanetMath provide support for progressive problem solving?"* [C2]

**Changing my perspective** In what might be called instrumental problem solving, some piece of information is needed as soon as possible; it is of fleeting interest and limited application. In non-instrumental problem solving, the needs are in some sense simulational rather than informational. What is at stake is building a lasting mental model of how things work. The nominal use of an encyclopedia is for lookup – the encyclopedia can play the role of the model. On the other hand, peer interactions open up the world of non-instrumental problem solving. It would be possible to make a further distinction and say that instrumental problems can be solved *transactionally*, whereas non-instrumental problems can only be solved *interactionally*.

*"When I'm programming, I look up error messages online; but in my Calculus course, I don't want to ask a question until I've really thought a lot about the answer."* [C3]

**Changing content or connectivity** Participants had many ideas about how the site should give feedback – ranging from having hints available to building rich "*ontology of problems*." It

*"Are there good mechanisms for getting (and giving) feedback?"* [C4]

* *Cf.* Gowers [§4.2, p. 50] and Leung [§4.2, p. 51].
† Jim Pitman, personal communication

occurred to me that since they had many creative ideas about this, in the long run it would be as important to be able take new feedback on board as to give it out. A rich give-and-take of feedback gets to the crux of (non-provisionist) peer learning. Tools like Get Satisfaction* are the typical way to elicit and record users' ideas about system features. Of course, actually acting on the feedback tends to be the real sticking point, and the only way to keep feedback coming. Transforming a flow of often highly transient attention into usable code calls for deep consideration of "*the match between cognitive processes and the statistical structure of the environments in which they function*" (Dhami, Hertwig, & Hoffrage, 2004) – across rather different populations. A project usually will not look the same to programmers and end users; *cf.* (Aboulafia, 2012, §5).

**Changing objectives** One might hypothesize a class of environmentally-distributed problems that are too complicated to simulate easily, and for which the environment is the best model.† On this basis, an economics of problem solving begins to unfold that is reminiscent of Coase's (1937) theory of the firm. There is a class of problems that is best solved "in-house," by interacting with and developing one's personal mental models – because building the mental model is the real point. There is a class of problems where the answer can be retrieved from one's existing mental model. This is the standard case of *knowing* – or what might in a litigious frame of mind be called "intellectual property." There is a third class of problems that are best solved through a transaction with the environment, by lookup or search – in this case, the only aspect of the mental model that matters is knowing where, or how, to look. This is the connectivist's actionable knowledge.‡ And there is a fourth class of problem which is again best solved interactionally, but this time, by interacting with others after the model of the commons, rather than that of the firm – in a meeting of minds. In practice, it seems there will always be a degree of sociality, in the sense of being "*betwixt and between*" [§1.3, p. 8] – a savant will think for a moment before answering, for example. The relevance for PlanetMath is that some people will want to follow relatively well determined channels which offer reasonably assured results, whereas others will want to experiment with new models, or tackle problems with greater risks. Outside of compulsory education, the student is typically someone who wants to earn a credential of some sort, which puts some constraints on the degree of newness. The "standard" methods should be low hanging fruit for PlanetMath.

> *"Some people are more social in their problem solving than others – and some problems lend themselves better to social problem solving."* [C5]

*HTTPS://GETSATISFACTION.COM/

†*Cf.* Clark (1998), also Feynman (1982).

‡*Cf.* Siemens [§4.1, p. 47] and Krowne [§4.5, p. 60]

## 7.3 A mathematician's perspective

*"It would be useful to have better systems for giving and receiving feedback."* [C6]

**Changing the nature of the space / Develop empirical studies and a critical apparatus** As Jamie explained to me, the relevance of changing communication technology, for a mathematician, can be discussed by analogy to considerations about a very different sort of technology: a commuter bicycle. A bicycle with a hybrid design and an internally-geared hub rides about 20% slower than a bicycle with true road geometry running a derailleur system. A communication technology can have similar beneficial or detrimental effects concerning knowledge transfer and synthesis. As this is the mathematician's bread and butter, it is a subject of some considerable concern. These concerns are not limited to computer mediated communication, however. A mathematician is sensitive to problems in exposition and direction – problems that are associated with an even more basic communication technology, language. In the seminar room, the audience asks questions both out of curiosity and to prompt the speaker. Over coffee, a non-expert may help to improve a research paper with a naive question. These are the kinds of interactions that designers of computer systems for mathematical communication need to consider. They are mostly missing from current systems, such the popular e-print repository ArXiv.org. * Jamie would like to being able to point out a problem in "*paragraph 7*" of a given text – or point to a completely different resource that does a better job.

*"Highly-abstracted and non-expository articles are hard to understand: they suffer from a kind of impenetrable quasiformality."* [C7]

**Changing what I know about myself / Find companions for the journey** We looked at the definitions of the mathematical term *ultrafilter* on PlanetMath and on Wikipedia (see Figure 11). In PlanetMath's definition, an ultrafilter is defined to be *filter* with certain properties. Wikipedia's definition describes the object first in terms of basic features of sets. It then develops several different equivalent conditions, and finally gives a definition in terms of filters. The PlanetMath definition takes up roughly the same amount of space as Wikipedia's "preliminaries", and manages to squeeze in a terse treatment of "types of ultrafilters." However, after the preliminaries, the Wikipedia article goes on to deal with *completeness*, *generalization to partial orders*, *types and existence of ultrafilters*, *applications*, *ordering on ultrafilters*, and *ultrafilters on* $\omega$. It also provides some interesting-looking references, for instance, to a paper on "Arrow's Theorem and Turing Computability". In short, the Wikipedia definition appears to be superior to the PlanetMath definition in all the relevant ways: it is relatively intuitive, complete, and

*HTTP://ARXIV.ORG/

A COLLECTION $\mathcal{U}$ of SUBSETS of $X$ is an *ultrafilter* if $\mathcal{U}$ is a FILTER, and whenever $A \subseteq X$ then either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. Equivalently, an ultrafilter on $X$ is a MAXIMAL filter on $X$. More generally, an ULTRAFILTER OF A LATTICE is a maximal proper filter of the lattice. This is indeed a generalization, as an ultrafilter on $X$ can then be defined as an ultrafilter of the power set $\mathbb{P}(X)$.

**Types of ultrafilter** For any $x \in X$ the set $\{A \subseteq X \mid x \in A\}$ is an ultrafilter on $X$. An ultrafilter formed in this way is called a *fixed ultrafilter*, or a *principal ultrafilter*, or a *trivial ultrafilter*. Any other ultrafilter on $X$ is called a *free ultrafilter*, or a *non-principal ultrafilter*. An ultrafilter on a finite set is necessarily fixed. On any infinite set there are free ultrafilters (IN GREAT ABUNDANCE), but their existence depends on the Axiom of Choice, and so none can be explicitly constructed.

An ultrafilter $\mathcal{U}$ on $X$ is called a *uniform ultrafilter* if every member of $\mathcal{U}$ has the same cardinality. (An ultrafilter on a singleton is uniform, but this is a degenerate case and is often excluded. All other uniform ultrafilters are free.)

HTTP://PLANETMATH.ORG/ULTRAFILTER

Given a set $X$, an ultrafilter on $X$ is a set $U$ consisting of subsets of $X$ such that

1. The empty set is not an element of $U$.
2. If $A$ and $B$ are subsets of $X$, $A$ is a subset of $B$, and $A$ is an element of $U$, then $B$ is also an element of $U$.
3. If $A$ and $B$ are elements of $U$, then so is the INTERSECTION of $A$ and $B$.
4. If $A$ is a subset of $X$, then either $A$ or $X \setminus A$ is an element of $U$. (Note: axioms 1 and 3 imply that $A$ and $X \setminus A$ cannot *both* be elements of $U$.)

A characterization is given by the following theorem. A FILTER $U$ on a set $X$ is an ultrafilter if any of the following conditions are true:

1. There is no filter $F$ finer than $U$, i.e., $U \subseteq F$ implies $U = F$.
2. $A \cup B \in U$ implies $A \in U$ or $B \in U$.
3. $\forall A \subseteq X$: $A \in U$ or $X \setminus A \in U$.

HTTP://EN.WIKIPEDIA.ORG/WIKI/ULTRAFILTER

**Figure 11:** *Definitions of "ultrafilter", from PlanetMath and Wikipedia*

well-referenced. There could still be some specialized look-up purposes for which the PlanetMath is better – but one thing it does not provide is intuition. This is not to say that the PlanetMath article is "wrong." It may well be that after unwinding the recursive definitions, and expanding the attached articles, one could come to a clear and complete understanding of filters and ultrafilters.* A "*quasiformal*" style of writing is much more prevalent on PlanetMath, although it is not absent on Wikipedia. Wikipedia's "anyone can edit" policy tends to produce articles that are "good enough" on average (Krowne, 2005). It seems fairly likely that critical feedback about PlanetMath's ultrafilter article, of the kind described here, hasn't made it to the author(s) yet. However, the problems may more properly lie at the platform level. On PlanetMath, it is possible to create

*However, for technical reasons, attempting this wouldn't be so easy. The word *filter* in the PlanetMath definition was autolinked to the definition of the more specialized concept *lattice filter*, rather than the actual definition of a filter, putting the reader on a bit of a chase to find the relevant definitions.

an account, submit a correction, wait for the author to respond, and then check their response – but this workflow runs counter to the design principle that states "*with each click you require, you lose half your audience.*"* On Wikipedia, you could just make the change directly. Flaws in exposition point to the salutary (but missing) effect of contact and discussion with other people.

**Changing my perspective / Work with real users** An instructor who would like to make use of a peer pedagogy must deal with the resistance that students feel toward speaking with one another. It would be helpful to convince the students that there are benefits to the extra work and exposure to critique. Students could be convinced that they will need to practice social skills in order to improve their workplace-readiness. They could also be convinced that group work or peer review can improve the quality of their understanding and their final marks. It may be easy to demonstrate that interaction with others helps to avoid the more obvious unthinking mistakes. This might be done by assigning homework to be completed in, or discussed by, small groups. This model stands in contrast with the more fully disaggregated model used in online Q&A systems. Incentives in the disaggregated model tend to produce Zipf-like distributions: in an instructional context, it's important that the greater portion of any "long tail" pass above a minimum threshold of quality. Jamie has previously tried using a standard Q&A system for course support. The small-group model asks for a different set of social skills.

> *"Students are not comfortable sharing their work with other students. They are only somewhat comfortable showing it to their professor, whom they regard as one might a medical doctor."* [C8]

**Changing content or connectivity / Study and build nonlinear interfaces** It may take months or years to get a given model "*loaded into your head*," but once you do, you can then see how manipulating it in one way "*over here*" will produce a change "*over there*." First and foremost, a mathematician looks for tools that will help them build effective mental models. This often means creating useful high-level abstractions. This is different from quasiformality, since the aim is to provide an intuitive, sharable, handle on a topic. For example, a picture is an abstraction that can help a reader to instantly grasp the notion of a Cauchy sequence. If some formalism is used, something very light-weight – e.g. the structured proofs of Lamport (1995) – is preferable to something heavy – e.g. Hales's (2006) Flyspeck.† Parallel points apply to user interface design. As already mentioned, being able to edit a Wikipedia page without logging in is seen as an asset; being able

> *"Formalisms are only useful when the model can't be held in your head."* [C9]

*John McLear, personal communication.

†HTTP://CODE.GOOGLE.COM/P/FLYSPECK/

to quickly click "Was this post useful to you?" helps create a digital trace that might be re-used in various ways; and even a dynamic layout on a web page tends to look "*more professional*" than a fixed-width layout. Related points apply to the technological accouterments of peer pedagogies. Students will not be interested in procedures, like using a course wiki or course forum that do not provide a clear and tangible benefit. Jamie, and others at his institution, seem to agree that the essential skill is "*thinking carefully and critically*" [§4.2, p. 49].

*"Wikipedia seems to have better coverage, better exposition, and a nicer-looking interface."* [C10]

**Changing objectives / Limit philosophizing** Of the three implied goals, "quality of exposition" seems like the hardest one to deal with, as it may require rethinking the entire way in which editing and feedback works on PlanetMath. But implementing a *standard* for quality is perhaps not so complex. On Wikipedia, templates are used to indicate that an article "*needs attention from an expert*" or that it "*may not meet Wikipedia's general notability guideline*." We could implement a set of templates for quality in mathematics exposition, so it would be easy to make it clear that a given article "*needs an example*," or "*needs a more intuitive presentation*." Even if without allowing edits from non-logged in users, it would be relatively straightforward set things up so that users could tag articles using a constrained vocabulary.

## 7.4 A reflective perspective

*"Empirical studies of mathematical discourse are a key research focus."* [C11]

**Changing the nature of the space / Develop empirical studies and a critical apparatus** In Alison's research for her PhD (Pease, 2007), she implemented Lakatos's theory of "*informal, quasi-empirical*" mathematics that grows "*through the incessant improvement of guesses by speculation and criticism*" (Lakatos, 1976) in a collection of intelligent computational agents, testing and extending the approach. She also finds significant value in computer mediated communication between humans. Transcripts of contemporary mathematical exchanges provide a more detailed picture of what happens when people do mathematics together than what is found in Lakatos's dialogues, or historical records of mathematical exchanges in old letters. As she put it, this is as close as she can get to having access to mathematician's brains. In "Seventy Four Minutes of Mathematics" (Pease & Martin, 2012), user comments are coded as *concepts*, *examples*, *conjectures*, *statements related to proof*, and "*other*" (an apparatus inherited from Lakatos): Table 14 reproduces the exemplars with which these categories are introduced in this paper. In "informal logic," the focus is usually on proof, but other broad topics of interest for Alison are listed in Table

**CONCEPTS** - "*Since the points are in general position you could define the* ***wheel*** *of* $p$*,* $w(p)$*, to be a radial sequence of all the other points* $q \neq p$ *around* $p$*.*"

**EXAMPLES** - "*If the points form a convex polygon, it is easy.*"

**CONJECTURES** - "*Is it possible to prove that any point and any line will do?*"

**PROOF** - "*Maybe the strategy should be to take out the convex hull of S from consideration; follow it up by induction on removing successive convex hulls.*"

**OTHER** - linking one thread to another, certifying the proof as complete, etc.

**Table 14:** *"Concepts, examples, conjectures, proof, and other" - five categories used by Alison Pease and Ursula Martin in "Seventy four minutes of mathematics: An analysis of the third Mini-Polymath project"*

**EXPLANATION** - "*This seems to be right, but there something I don't understand. Please see if you can help me with it*[.]" [...] "*If I understand well your example: the problem is that you must give an orientation to the line. Then, left and right are define with respect to this orientation*[.]" [...] "*Got it! Kind of like a turn number in topology. Thanks!*" (Pease & Martin, 2012)

**UNDERSTANDING** - But how does understanding relate to explanation? E.g. often explaining something in greater detail will confuse the listener.

**MOTIVATION** - Which can be considered in a sociological sense; for example when and where are people motivated by "addictions"?

**LINGUISTICS** - Including linguistic moves that help to build community, like talking about "*our*" instead of "*my*", etc.

**Table 15:** *"Explanation, understanding, motivation, and linguistics" - topics of interest in Alison's research*

15. This work is focused on studying what takes place in existing systems rather than on system design. However, platforms that more explicitly represent the kinds of things Alison and her collaborators find in the content would make certain aspects of her job easier; *cf.* (Martin & Pease, 2012).* Lange (2007) describes work on a semantic wiki, and Lange, Bojārs, Groza, Breslin, and Handschuh (2008), an ontology for argumentation that could likely be adapted for this purpose.

*Although they are not system building in the software sense, note the phatic, channel-creating, role of the "linguistic" moves mentioned in Table 15; *cf.* (Elyachar, 2010).

**Changing what I know about myself / Find companions for the journey** Data from PlanetMath could be coded at the text level or at the hypertext level, and modeled as a forest of interconnected concepts, examples, conjectures, and proof steps. At the hypertext level, the grammatical moves described in Table 10 [§5.2, p. 74] could be given "senses" associated with the categories listed in Table 14, for instance:

*"How does experience build towards understanding?"* [C12]

$$A \xleftarrow{\text{omits concept}} C, \quad X \xleftarrow{\text{for example}} T, \quad A \xleftarrow{\text{suggests that}} P, \quad P \xleftarrow{\text{partially solved}} S_0$$

If PlanetMath hosts interactions that are of interest to researchers, this would be a good sign that it is doing something that is in a certain sense new – even if this is achieved by simulating the features of older models. Anything researchers are able to learn about the way mathematical discourse works can inform future work on the system, and help to boost the relevance for typical end users.

**Changing my perspective / Work with real users** An example from one of Alison's earlier studies was the term *landmine*, introduced in a combinatorial problem solving context. This term almost begs to be made into a neologism once again, to describe the knock-on effect of a given new usage. We could measure the impact of a given post (or term) by measuring the number of follow-up posts (or uses). Similar features are modeled in the Behavior Ontology described in Angeletou et al. (2011), using the property `oubo:PostImpact`. This metric could be used to promote important posts to a place where newcomers would easily spot them. A focus on terms means that a forum conversation can be straightforwardly modeled as a graph that is covered by the trees which form its threads.* Graph theoretic techniques could be applied to compute and show the *centrality* of a given person, post, term; or example, conjecture, *etc.*, if this has been formally coded.

*"It would be useful to be able to detect the first use of a word within a given episode of discourse, and to measure it's impact."* [C13]

**Changing content or connectivity / Study and build nonlinear interfaces** As discussed above, quasiformality can be difficult for a reader who often relies on informal or non-mathematical language, or pictures, to gain a sense of understanding. One concrete application of network structures within the PlanetMath encyclopedia would be to show the distance of the terms used in a given exposition from a set of basic definitions which are taken as axioms. This would essentially be a topographical map, and gradients and

*"Can we build more detailed models of mathematical discourse?"* [C14]

* *Cf.* Eco [§4.5, p. 61].

flows could be computed. Such a model could help explain the subjective experience of expository quality in terms of linguistic features (e.g. anaphora) and cognitive models (e.g. cognitive load). With this as a basis, we could then take further steps to explore how knowledge grows and spreads. A graphical approach could be used by research mathematicians to help manage posts and metadata. The user could browse through topics as bubbles, "*open them up*" to zoom in on details of interest, "*attach*" a notebook to a given topic at a given point, or "*connect*" topics with additional comments. This would give a graphical interpretation to maneuvers that currently take place in textual and hypertextual form. A promising related development in the cartography of mathematics is described by Dörrie and Kohlhase (2013), following earlier work by Dave Rusin.*

**Changing objectives / Limit philosophizing**

*"Even if computer mathematics did a good job of simulating human mathematics, it won't do things the same way humans do."* [C15]

For comparison, even though there are computers that are very good at chess, they generally do not play like humans. Timothy Gowers and Mohan Ganesalingam (2012) have been working to teaching a computer how to do "*routine mathematical problems*" – that is, problems where "*the obvious thing works*." This, and most other research in computer mathematics, is more or less distinct from research on informal logic. Typical theories of argumentation also focus on proof and truth (Pease & Aberdein, 2011), leaving issues associated with expository quality and collaboration to one side.† Appendix B reworks two examples presented by Gowers and Ganesalingam, aiming to illustrate how social theories focused on learning and collaboration might be applied to solve standard mathematics problems. PlanetMath may be able to offer a hybrid digital/analog tutoring system long before a comparable fully computerized system exists that can perform at the same level. Indeed, in contrast to computer chess, it is possible that the relevant target for computer mathematics is not "human mathematics" at all, but social mathematics. "*We may compare a man in the process of computing a real number to a machine*" (Turing, 1936) – but "*the program is best conceived of as a little trial court, or as an evidence-collecting and evidence-weighing procedure*" (Minsky, 1967).

> [W]*hen we write a large program, with many such courts, each capable if necessary of calling upon others for help, it becomes meaningless to think of the program as a "sequence."* [...] [O]*nce past the beginner level, programmers* [...] *write for the individuals of little societies or processes. For try as we may, we rarely can fully envision, in advance, all the details of their interactions. For that, after all, is why we need computers.* (Minsky, 1967)

*HTTP://WWW.MATH-ATLAS.ORG/

†Walton's agential theory of "critical questions" (Walton & Godden, 2005; Walton, Reed, & Macagno, 2008), while still focused on truth claims, is closer than earlier theories, like Toulmin's (1958).

## 7.5 Summary

Table 16 summarizes the findings from qualitative fieldwork with potential users and stakeholders. It illustrates how the paragogy framework can be used to draw out interesting and relevant design themes.

These findings can be applied directly to guide the implementation work described in Chapter 8. Given the ambitious nature of some of the points that were raised, it is likely that the issues discussed here will inform the development of PlanetMath/Planetary for time to come.

This chapter incorporates views from three important categories of potential users: students, mathematicians, and researchers who study mathematical practice. Chapter 10 will return to these findings again to compare them with the views of users of the realized system.

| Student | Mathematician | Philosopher |
|---|---|---|
| α. | β. | γ. |
| "Trust" and "credit" are two important and potentially related issues for users/contributors. | We need an efficient system for integrating user ideas and comments. | Researchers can study the content or the system in almost interchangeable ways. |
| δ. | ε. | ζ. |
| Progressive problem solving and incremental feedback are important for users, particularly new users who are building their identity as an effective problem solver. | Discussion and critique have a salutary effect; deficiencies in exposition may be a result of their absence. | We should be able to build research-ready data flows as long as the system is actually used. |
| η. | θ. | ι. |
| People have instrumental, transactional, information needs and noninstrumental, interactional, learning needs. | People need to learn and practice social skills through interaction. | Graph theoretic methods can be used wherever we have explicit semantics or even simple models of terminology use. |
| κ. | λ. | μ. |
| Users and potential users have a lot to say about systems: how can we empower them to actually make changes? | Users want to be able to make adaptations that leave traces, and simple customizations that don't. | Visualizations of graphical knowledge structures could be used directly in the UI. |
| ν. | ξ. | ο. |
| Social problem solving is suited to some tasks more than others. | Coverage is good to work on, but quality of exposition is harder. | Computer understanding and formal theories of quality and effective social heuristics are good long-term goals. |

**Table 16:** *Overview of desiderata gathered using the paragogy framework*

# Part 4
# Results and discussion

# 8 Implementation

**PlanetMath's software system is rebuilt along the lines specified by the paragogy model, augmenting the collaborative authoring environment with tools for peer supported problem solving.**

## 8.1 The Planetary system

In 2010, in parallel to the exploratory research described in earlier chapters, the Planetary project launched. One of the project's key milestones was to rebuild and modernize PlanetMath's software. The project also aimed to create a software system that could be deployed and used in settings unconnected with PlanetMath. In order to facilitate this, Planetary uses the popular content management system, Drupal, as its basis, and connects Drupal with other open source components. The primary participants in this project were Michael Kohlhase of Jacobs University, Bremen, several of Kohlhase's students, and the current author.*

With regard to the system's deployment on PlanetMath, the main objectives were to be feature-complete relative to Noösphere, and to add new features relevant for learning mathematics. These new features include the introduction of problem and solution data types, collections for assembling content into interactive structures, along with other user-visible changes like a new syntax-highlighting source code editor and fast in-browser rendering, and new semantic models in the backend.

My role in the project was to lead the development and deployment of the version of the system needed for the PlanetMath use case. This involved integrating contributions

*HTTP://TRAC.MATHWEB.ORG/PLANETARY/WIKI/PEOPLE

| | **Basic** | **Enhanced** |
|---|---|---|
| *Interaction* | Drupal, ACE2/ShareJS editor | +JOBAD |
| *Storage, Access* | Drupal, Virtuoso | +TNTBase, Git |
| *Representation* | XHTML+RDFa | +OMDoc |
| *Processing* | LaTeXML, NNexus, PyRDFa | +JOMDoc, Krextor |
| *Content* | LaTeX (*e.g.* PlanetMath, ArXiv) | +sTeX |

**Table 17:** *Components of the Planetary system*

from other members of the team where relevant. A schematic of the system architecture is presented in Table 17, adapted from the earliest system paper describing Planetary (David, Ginev, Kohlhase, & Corneli, 2010).

Further enhancements to the system are being explored in deployments at Jacobs (David, Kohlhase, et al., 2010; Iancu, Jucovschi, Kohlhase, & Wiesing, 2014), integrating earlier work by members of the KWARC research group on knowledge management tools including Krextor (Lange, 2009), TNTBase (Zholudev & Kohlhase, 2009), OMDoc (Kohlhase, 2001), sTeX (Kohlhase, 2006), and JOBAD (Kohlhase, Giceva, Lange, & Zholudev, 2009). In 2011, an early prototype of the system was selected as a finalist in Elsevier's Executable Paper Challenge (Kohlhase et al., 2011). The system is currently in active use, not only as a web portal on PlanetMath, but also as a course management tool for Kohlhase's introductory computer science courses, and on the new "flexiformal" (Kohlhase, 2012b) mathematics website, MathHub.info.*

PlanetMath/Planetary moves in the directions indicated by the survey presented in Chapter 7, and Drupal's extensibility opens the door to further improvement. For example, the use case sketched in Figure 3 [§2.3, p. 31] has been brought considerably closer at the level of knowledge representation, but the system currently lacks the reasoning support that would be needed to finish the job. I collaborated concurrently on another project that added a recommender service to Drupal (Mulholland, Wolff, Zdrahal, Li, & Corneli, 2013), and integration of the corresponding features into Planetary appears to be simply a matter of time.

What is most essential for current purposes is that Planetary can be discussed, studied, and developed further together with PlanetMath users. The renewed attention to system development restores an empirical aspect to PlanetMath: more than "*kit*" for doing math-on-the-web, PlanetMath/Planetary is a "*research device*" for studying mathematics as a cultural system; *cf.* (Rheinberger, 1997, p. 81). This research programme is developed further in Chapter 9, but the system itself constitutes an important research result, and not just a means to an end. PlanetMath/Planetary is the most concrete synthesis of the ideas that have progressively developed in the course of this work.

*http://mathhub.info

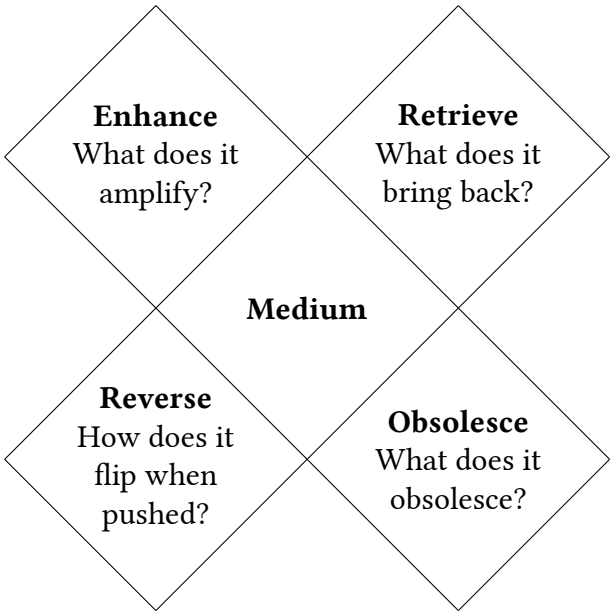



**Figure 12:** *Marshall McLuhan's tetrad of media effects*

It is a functional actor-oriented model of the way people do mathematics – although it is, of course, only a partial model. The paragogy framework is used below as an organizational device [§3.3, p. 41] to describe the system's high-level specification.

1. **PlanetMath/Planetary should be a special-purpose tool that is customized to suit the challenges of learning mathematics.**
2. **Learning mathematics is an active process that is centered on problem solving; to get the added benefits of peer support, participants have to interact with one another.**
3. **Users will contribute problems and solutions; discussions about problems should spur new engagement among encyclopedia editors and teachers.**
4. **To be viable, the system should facilitate mathematical meaning making; the foundations for this include emergent structure, pragmatic knowledge building, ongoing redesign and evolution at the system level, and the development of new content-focused subcommunities.**
5. **The system should support increasingly refined user and domain modeling, in order to become an increasingly high-fidelity actor-oriented model of mathematical behavior.**

The next section expands these points based on consultations with PlanetMath users and new stakeholders, as described in Chapter 7. After describing the implementation strategy, the following sections present Planetary's new features and analyze them using Marshall McLuhan's (1989) tetrad of media effects (Figure 12).

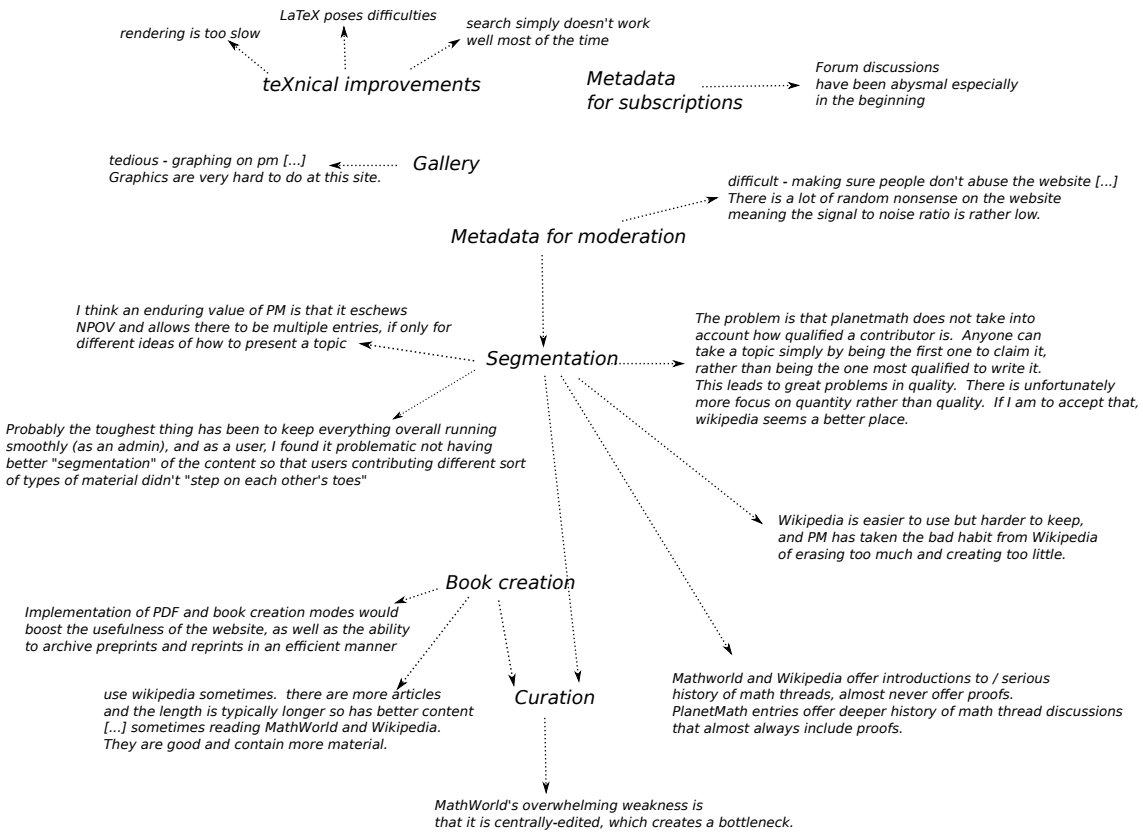



**Figure 13:** *Participatory design: surveying users of the legacy system*

## 8.2 Fleshing out the specification

### Participatory design

On December 16, 2011, a questionnaire was circulated by email to 30 of PlanetMath's historical top contributors, inviting feedback on an early "alpha" version of the new website as a work-in-progress, which had been deployed on alpha.planetmath.org. The questions aimed to discover the mathematical goals and interests of top contributors, as well as their sentiments about PlanetMath and other related websites, like Wikipedia. Feedback was provided by 10 respondents, and this is summarized in Figure 13. Several central themes are summarized below.

**Geometric and visual thinking** The frustration people experienced working with images on PlanetMath/Noösphere underscores the importance of visual, analog, thinking for making meaning in mathematics. Improved support for creating and sharing images could only help – particularly for learners who are using the site to build intuition, not just to check the details of a definition.

→ Better support for mathematical diagrams is one feature of Dimension 1 of the specification: the mathematical symbol system includes both analog and digital features.

**Different levels of qualification** If one were to concede the point that Wikipedia's encyclopedia is a better reference resource – which is convincing enough to be considered – that still leaves the question of a place for everyone who is not yet an expert to engage. In mathematics, these people are often students, and their traditional mode of becoming more mathematically qualified is to solve problems.

→ Adding problem solving support to cater to non-expert users is the essential feature of Dimension 2.

**Breadth, Depth, and Quality** In-depth treatments of individual topics and broad coverage of many different topics are both considered valuable. Breadth and depth are subject to social and technical constraints, including availability and incentives for experts, and suitable tools for dealing with complex presentations. High-quality exposition and structured presentations are important for anyone learning mathematics. Some people doubt whether a "crowd" can build things like. Better support for making and critiquing books, problem sets would certainly help, along with techniques for semantic modeling that can assist in quality control.

→ One of the key ideas behind Dimension 3 is that feedback from students and careful attention to their needs will help editors find places where exposition could be improved.

**Moderation difficulties** Related to the point just mentioned, there is tension between having something online *per se* (which people often like, since it potentially helps them gain exposure, get feedback, and find potential collaborators) – and having something in the encyclopedia, where others end up having to interact with it. For example, my own uploads from APM-Ξ didn't fit well in PlanetMath's encyclopedia, but I would have liked to find some place for them on PlanetMath (Chapter 2).

→ Suitable strategies for segmentation would help with many of the sub-points of Dimension 4.

**PlanetMath's unique model** The ownership model blends "Wikipedia" and "Scholarpedia"* approaches. Raymondite homesteading is part of this model, and the intent is that this will give experts the chance to manage their own content. However, this comes with some of potential quality control issues: what if instead of improving an article, an owner only "squats it"? In fact, users who don't respond to corrections must orphan their articles – but the ownership model still slows down change relative to Wikipedia. An improved co-authoring workflow could take care of some basic points of conflict. In

*HTTP://WWW.SCHOLARPEDIA.ORG

addition to ongoing discussions with users, better data gathering and analysis should help detect places where users are running into difficulty.

→ Understanding and better serving user needs is one of the key applications of Dimension 5.

## Connections to stakeholder desiderata

Table 18 further expands the specification by referring it to the desiderata identified in preliminary field work (as summarized in Table 16 [§7.5, p. 104]). This yields a detailed outline of work for the implementation project.

This expanded view shows how the five focal dimensions of change at the system level are, here, projections of a more complicated change process. Rather than "dimensions" it may be more accurate to speak of them now as "subspaces," or "subprocesses," with the further caveat that these are not wholly distinct, but that they will interact with one another. Each of the five subspaces traces a virtual engagement between the dramatis personae Student, Mathematician, and Philosopher, in the (changing) context of the system-under-development. All of these figures are themselves involved in their own processes of change [§3.3, p. 41]. When compared with Chapter 7, what's new here is that it is now seen how these change processes can be related to each other.

The descriptions presented in Table 18 were not obtained by reading across the rows of Table 16, but by refactoring its elements to tell a more coherent story. For example, along Dimension 3, through the hinge-pin of *motivating users to contribute problems and discuss solutions*, trust and credit ($\alpha$) are now seen as connected with discussion and critique ($\varepsilon$) as well as data-driven research looking at topics like the way understanding develops in dialog ($\zeta$). As the nature of the student's environment changes so that he or she can get credit for participating in activities online, the mathematician and philosopher must increasingly understand their roles to be connected to discourse. The system, correspondingly, should adapt to support participation at varying levels of familiarity and literacy, not only with regard to mathematical concepts and proof techniques, but also the relevant forms of social engagement.

This broad programme still only points in a general way to what needs to be implemented, rather than to a manifest of specific implementation tasks. Geertz notes that the separateness of plans of this type from actual cultural practice "*makes them less useful as models for the interaction of cultural patterns*" (Geertz, 1973, p. 250). On analogy with Vygotsky's view that "*the primitive child is a healthy child*" (Vygotski, 1929), the implementation approach will focus on the minimum kernel of features that is needed to build a concrete working model, rather than an abstract picture. Interacting with a realized system will render the subsequent discussions and critique comparatively concrete.

**Dimension 1 (Desiderata η, λ, μ): "A special-purpose tool customized to suit the challenges of learning mathematics."** The system must deal with both informational and interactional needs of users. It should allow users to make learning-relevant adaptations and customizations. It should help people visualize and interact with mathematical knowledge – diagrammatic thinking is one basic aspect of this.

**Dimension 2 (Desiderata δ, θ, ο): "An active and interactive process centered on problem solving."** This should include features of progressive problem solving and incremental feedback, as well as the opportunity to learn and practice collaboration skills. We should build better models of expository quality and effective problem solving heuristics, and the relationship between them. As students become better at solving problems, they become more mathematically qualified: ideally, the system should be useful wherever the user is in their mathematical literacy lifecycle.

**Dimension 3 (Desiderata α, ε, ζ): "Motivate users to contribute problems and discuss solutions."** In order to accomplish this, we will have to address the issues of *trust* and *credit*. If we want to motivate new levels of engagement from editors and teachers, we will have to highlight the efficacy of discussion and critique. We should identify the meta-mathematical research problems associated with discourse in the space, and help researchers work on these problems. Following Timothy Gowers, let's ask: "Is massively collaborative mathematics instruction possible?"

**Dimension 4 (Desiderata β, γ, κ): "Support mathematical meaning making within the system and through ongoing system development."** Will user comments be heard and acted upon (whether at the level of "system" or "content")? Can users themselves contribute to development? Can we develop a rigorous approach to system change and development? One concrete problem is the tension between making it easy for contributors to add material, and making the things the reader sees high-quality. This invites an innovative approach to segmenting the site.

**Dimension 5 (Desiderata ι, ν, ξ): "Support user and domain modeling."** Can we discriminate between the tasks where social problem solving works better than individual effort, and vice versa? Can we understand the behavioral and semantic features of high-quality exposition? How can we facilitate graph theoretic analyses of questions like these? How can we understand the impact of system features (like the ownership model) on the user experience?

**Table 18:** *Design dimensions*

## 8.3 Implementation strategy

Figure 14 connects the specification described in the previous section to concrete system features. As can be seen by comparing with Figure 13, many long-standing feature requests have been addressed in the process of this conversion as well.

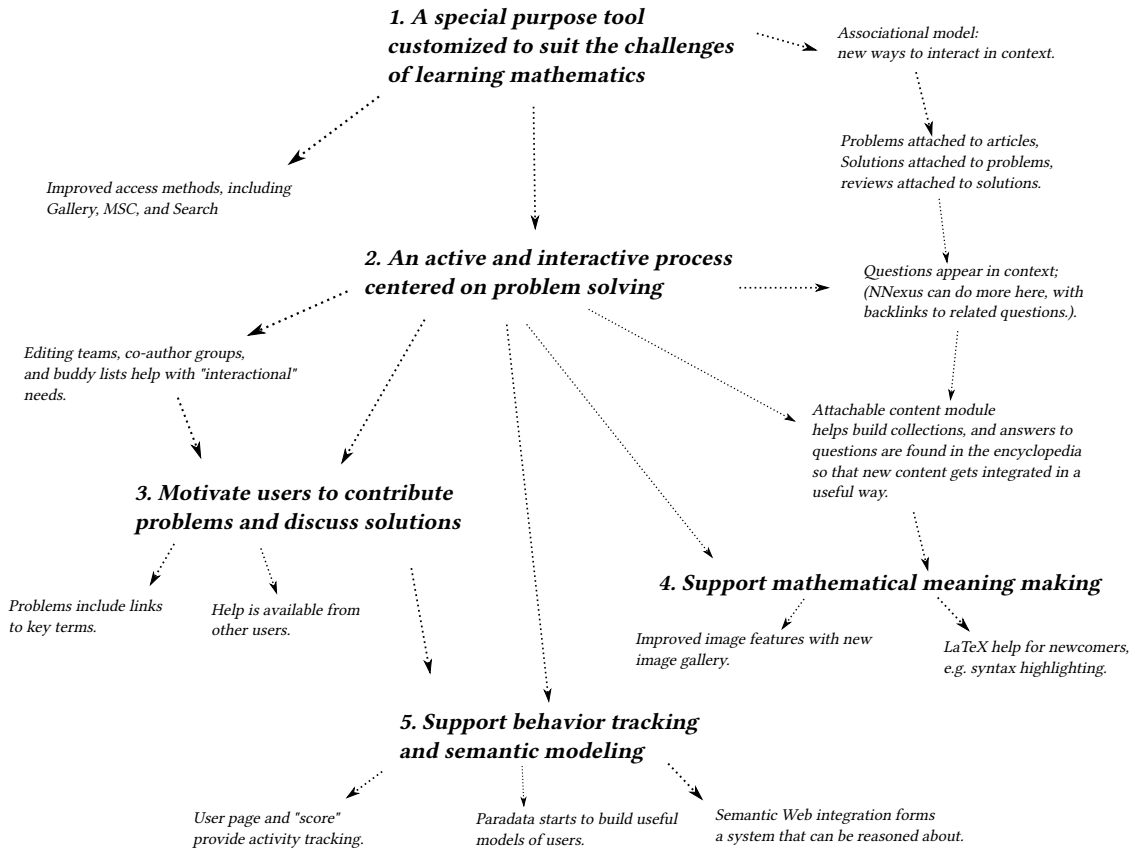



**Figure 14:** *Connecting the design specification to concrete system features.*

In order to realize this design and develop a solid foundation for further work, development began by rebuilding the main features of Noösphere from the ground up – using the extensible content management system (CMS), Drupal 7.* This allowed us to modernize and modularize the system in the process of building our enhancements. It cannot be emphasized enough how much an easy to install system built as customizations to a mainstream CMS helps with extensibility when compared with a difficult to install custom codebase.† This is important in light of the fact that we could not realistically do everything that everyone would have like to see done in the time available. An extensible modern system that combines feature-parity with Noösphere and new problem solving features was sufficient for current purposes. Not only is the system immediately usable

*HTTP://DRUPAL.ORG/DRUPAL-7.0

†Full installation instructions for Planetary are available on Github: HTTPS://GITHUB.COM/KWARC/PLANETARY/WIKI/INSTALLATION-INSTRUCTIONS

– featuring some noticeable improvements over Noösphere – it also allowed collaborating developers to build new working prototypes that open up interesting directions for further work. Development took place on Github to facilitate distributed revision, and developers made use of the integrated issue tracker.

In addition to Drupal, two other core components come together to form the new Planetary system. These are: (1) LaTeXML,* which we used to provide a modern, fast rendering system that supports semantic markup; and (2) the Virtuoso triple store,† which allowed us to begin to make use of semantic markup in a computational manner.

When compared with Noösphere, Planetary is better able to expose semantics at the fine-grained textual and bulky hypertextual levels. This means that in the future, much more information will be available for analysis along the lines indicated in Chapter 6.

As is the nature of open source collaboration, Planetary builds on a considerable body of prior work; *cf.* (Kohlhase et al., 2011). PlanetMath is the first major public demo of the Planetary platform. After a short overview of the system as a whole, this chapter will focus on *problem solving support* (problems, solutions, reviews, collections, questions), *data management* (personalized tracking, SPARQL integration via Virtuoso and PyRDFa), and *support for groups*, which I've been primarily responsible for developing in order to build PlanetMath/Planetary. Details on other features of the Planetary system – including real-time collaborative editing, Javascript-based interaction for readers, mathematics-aware search, and expressive semantic markup via sTeX – can be found in papers by other members of the Planetary team (Jucovschi, 2012; Kohlhase, 2012a; Kohlhase et al., 2009; Kohlhase, Matican, & Prodescu, 2012; Kohlhase, 2006, 2008).

## Overview of Planetary's features

After a first round of prototyping with Vanilla Forums‡ (Kohlhase et al., 2011), we settled on using Drupal 7 as the basis for the system. Drupal is a popular open source content management system, known for its modular approach. Many contributed free/open source modules are available, and the Drupal developer community offers a range of professional services. For processing mathematical text, we used LaTeXML, a parser that takes in LaTeX source code and generates XML, and from that, HTML+MathML.§

> *MathML is a low-level specification for describing mathematics as a basis for machine to machine communication which provides a much needed foundation for the inclusion of mathematical expressions in Web pages.* – http://www.w3.org/Math/

*http://dlmf.nist.gov/LaTeXML/
†http://virtuoso.openlinksw.com/dataspace/dav/wiki/Main/
‡http://vanillaforums.org/
§http://dlmf.nist.gov/LaTeXML/

From the user perspective, a significant advantage of MathML is that it transfers and renders much faster than other alternatives. It is rendered natively by an increasing number of web browsers, and the Javascript-based MathJaX provides a fallback solution in browsers where native MathML rendering is not available. MathML also lays the foundation for interactive in-browser services (Kohlhase et al., 2009).

We were able to use several contributed Drupal modules with zero configuration to reproduce some of the old features from Noösphere. Examples include the contributed privatemsg and watcher modules. Other modules could be used with some minor customization (*e.g.* scorekeeping via userpoints) or integration via an API (*e.g.* groups support using the Organic Groups module, og). Some features naturally had to be custom-built, for example, a new corrections module, and the drutexml module, which communicates with LaTeXML running as a web service.

Constantin Jucovschi used the wysiwyg module to add a custom real-time collaborative editor, based on ShareJS.* He is extending the editor with "bots" that provide a range of in-editor services (Jucovschi, 2012). The only code remaining from the legacy PlanetMath codebase is the NNexus auto-linking module, which Deyan Ginev has refactored into object oriented Perl, and adapted to run as a web service along the lines of the LaTeXML web daemon, which he also maintains; *cf.* (Ginev & Corneli, 2014).†

*Problems* and *solutions* are the key new objects, along with *questions*, *collections*, and *groups*. Working with problems and solutions, and integrating them with expository material from the encyclopedia requires new workflow and metadata management features. Table 10 [§5.2, p. 74] summarized the main activities planned for the new system, and connected these activities with the five dimensions of paragogy. Figure 14 now provides a significantly expanded view of a paragogical design for mathematics learning.

Many of the new features are familiar from other wikis, online communities, or offline problem solving contexts. Collections do require a brief introduction. The available interactions are similar to those associated with YouTube playlists,‡ *i.e.* collections are designed to help users organize articles, problems, and other content. In the mathematics context, collections are useful for sharing reading lists, problem sets, and more complex curricular objects; *cf.* [§1.2, p. 6]. A similar feature was developed by Silvan Reinhold (2006) for the WikiTrails system, which was designed to augment a MediaWiki installation with navigational and learning pathways. Trails could be created in "*automatic, semi-automatic, as well as manual modes.*" Christoph Lange (2007) points out that the knowledge embedded in wiki pages is not used by the WikiTrails system. In the current prototype, the collections mechanism integrates information about user interactions and

*https://github.com/jucovschi/ShareJS
†https://github.com/dginev/nnexus
‡http://www.youtube.com/yt/playbook/playlists.html

the hypertextual context. Deeper semantic integration is left for future work.

In the following section, the system features that were presented in capsule form in Figure 14 are unpacked and explored with screenshots from the live system deployed on PlanetMath.

## 8.4 System walk-through

Figures 15–25 show some of the key features of the developed system. The pages can also be browsed on PlanetMath.* The screenshots and discussion are organized in terms of the five dimensions of the design described in Table 18.

The clusters of features associated with each of these dimensions is reviewed in terms of the McLuhan tetrad (Figure 12), which helps to focus on specific, localized changes, in contrast, for example, to the more global concerns addressed by Richardson and Domingos (2003) [§4.5, p. 61]. Some of the pragmatic features associated with building large knowledge bases are enhanced by the new features and where something can be said about this it will noted – but there are other cultural factors that are relevant to peer produced peer learning that McLuhan's formulation is better at capturing.

### 1. A special purpose tool customized to suit the challenges of learning mathematics

Figure 15 illustrates the first point where a user might encounter one of the key new features of the system. When browsing an article, a block containing links to attached problems appears, if the article has attached problems. If there are no attached problems, the reader is invited to add one. The figure illustrates additional interactions that are available to any logged-in reader, when browsing an encyclopedia article. The reader can add a correction, ask a question, or add the article to a collection. Both the text of the article and the "look and feel" are worth commenting on.

In the past, PlanetMath used a range of different rendering systems. The earliest of these was LaTeX2HTML,† a system that converts mathematical expressions into in-line images. The most recent system used by Noösphere was jsMath, the immediate predecessor to the more widely-adopted MathJaX. Javascript-based client-side rendering with these tools is considerably slower than native rendering of MathML within the browser, which is currently best supported by Firefox's C++-based Gecko engine. Native support for MathML is not present in all modern browsers, so Planetary includes MathJaX as a fallback option. LaTeX source code and rendered PDFs are also made available.

*Up-to-date screenshots, together with links to live pages on the system, are collected at HTTP://PLANETMATH.ORG/PLANETARYRELEASENOTES.

†HTTP://WWW.LATEX2HTML.ORG/

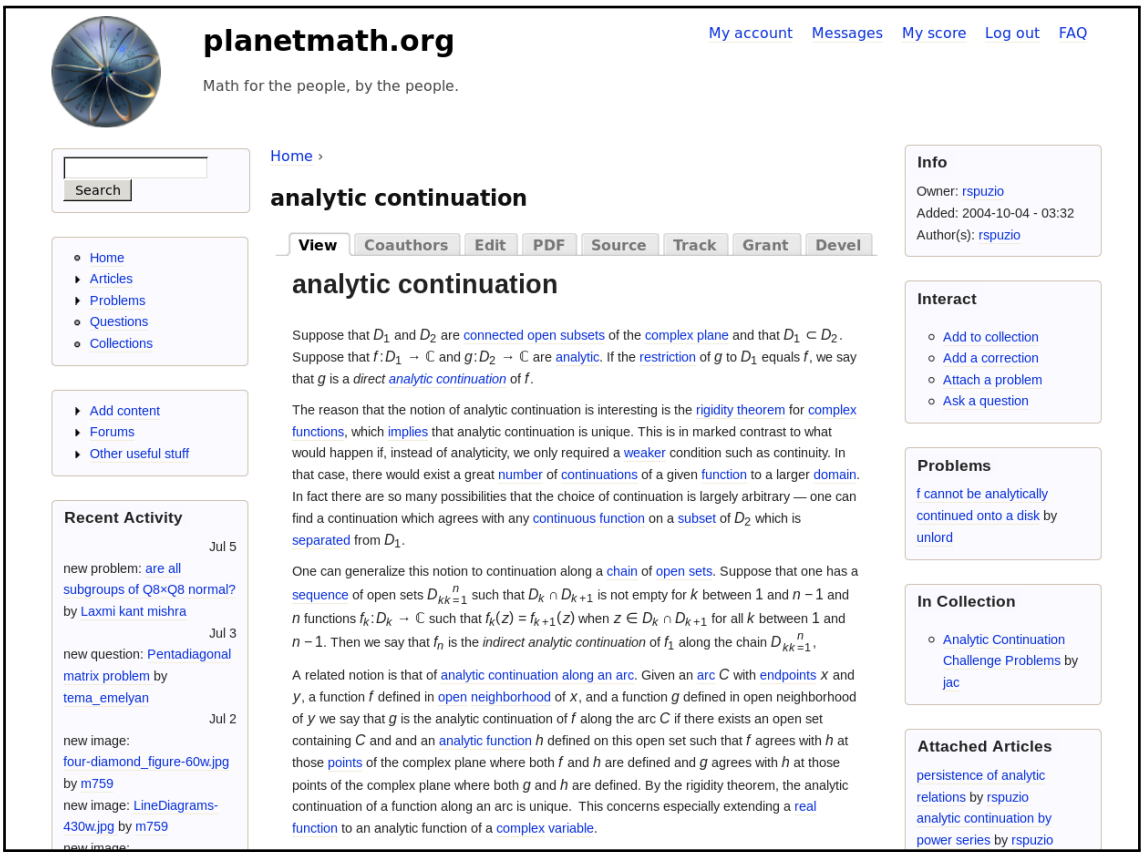



**Figure 15:** *"Article UI" (reader view), including problems attached to articles.*

The three-column theme was implemented by Alex Dumitru using Drupal's popular zen theming system (version 7.x-3.1).* The idea was to provide the basic interactions and site messages in a simple version of a "*heads-up display*".† One designer I spoke with was not greatly impressed, and said that the site has an "*early nineties look.*"‡ As noted in Chapter 7, a layout that used all of the space on the page for text might look "*more professional*" [§7.3, p. 99]. Some additional theming and Javascript work would enable us to make the sidebars stay hidden unless needed, but at the outset of the project there were other more pressing issues to solve.

The HTML produced by LaTeXML expresses mathematical symbols using their Unicode representations. Some of the "obscure" symbols (like Fraktur fonts) occupy registers beyond the Basic Multilingual Plane (0x0000 - 0xFFFF) supported by MySQL's UTF8 encoding. Planetary seems to be the first system to use LaTeXML together with a MySQL database. A resolution was reached by upgrading MySQL and converting to the richer UTF8MB4 encoding,§ and creating a (local) patch for Drupal's core database module.

*HTTPS://DRUPAL.ORG/PROJECT/ZEN
†HTTP://EN.WIKIPEDIA.ORG/WIKI/HEAD-UP_DISPLAY
‡Anna Keune, personal communication
§HTTP://DEV.MYSQL.COM/DOC/REFMAN/5.5/EN/CHARSET-UNICODE-UTF8MB4.HTML

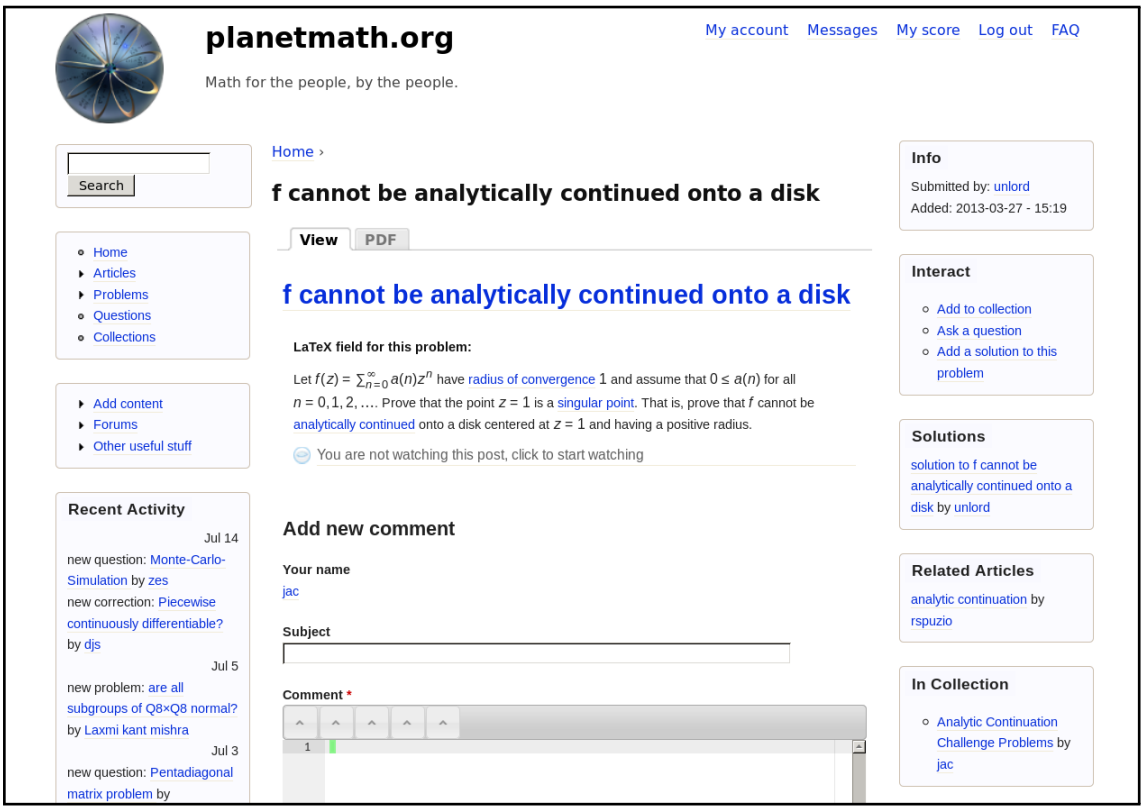



**Figure 16:** *Problems are attached to articles; solutions are attached to problems*

Figure 16 shows a problem, with its own set of interactions, including the ability to add a new solution. Once solutions have been added, they will be listed in another block. Problems also come with a "Related Articles" block that shows the articles to which the problem has been attached (if any). Some users might prefer *not* to know that solutions are available until they specifically ask for a hint; Drupal would make it easy to accommodate this sort of user preference.* In terms of our ability to control system features at this level, we can compare Planetary with WorkingWiki, a MediaWiki extension for interacting with source code of various forms (including LaTeX).† WorkingWiki is an excellent program that pushes the envelope of what can be done with wikis – and, like Planetary, it uses LaTeXML for rendering. The ease with which different kinds of content can be created and connected together in Drupal confirms that using a CMS is a good choice for the Planetary project. Interconnections between different kinds of content, and interactions around these connections are the essential features of the design (Figure 1 [§1.4, p. 12]). Using LaTeXML opens the door to interaction with individual pieces of content, via "deep links", since each element on the page is supplied with a unique identifier (`xml:id`).‡,§

*https://github.com/KWARC/planetary/issues/343

†http://www.mediawiki.org/wiki/Extension:WorkingWiki

‡http://lists.jacobs-university.de/pipermail/project-latexml/2013-April/001291.html

§https://groups.google.com/forum/#!topic/planetary-dev/xP0PiSQVQNE

***Enhance*** The improvements described here target the relational features of the PlanetMath system. Not only are objects related to other objects in new ways, the text itself is enhanced, which facilitates new ways for people to interact with system contents. For instance, it is now possible to link to and discuss individual elements of the page. At both "micro" and "macro" level, the system has better support for user interactions.

***Obsolesce*** The first target of the Planetary project has been to obsolesce Noösphere by building a system with feature parity. A potential longer-term goal is to obsolesce other online or offline mathematics resources, by growing to include feature parity with more of them. At this stage, the key point is to support customizability and extensibility in Planetary system at all levels.

***Retrieve*** Reference material is now connected explicitly to a problem solving context. The sense of mathematics as a subject that is only learned actively is brought into focus in the new system.

***Reverse*** When the relations between different topics are developed further, it may be easier to conceptualize PlanetMath's content not simply as a textual or hypertextual corpus, but as a graphical image or interactive mathematical object. The MathMap project by Jan Wilken Dörrie at KWARC offers an early prototype for this experience, and currently provides an alternative access method to PlanetMath's content.*

## 2. An active and interactive process centered on problem solving

Figure 17 shows the new LaTeX editor in action, with the user in the process of adding a solution to the problem that was shown in Figure 16. The text of the problem statement has been automatically retrieved and copied onto the screen above the LaTeX field, so the user doesn't have to open multiple tabs to keep track of the basic information they need to solve the problem. This is achieved by using a custom *view mode* and a custom template for rendering the solution entry form. The new '`content`' view mode produces only the content of the problem, without the perpheral features that appear when the problem is rendered as a webpage.

By following the links to technical terms used in the problem (here, to PlanetMath's articles on "*radius of convergence*", "*analytic set*", and "*analytical continuation*"), the user can remind him or herself of the definitions. In the future, we could do much better, by generating a "course packet" on the fly to assist the problem solver. The latest version of NNexus embeds RDFa into links, which makes the connections between pages

*HTTP://MAP.MATHWEB.ORG/

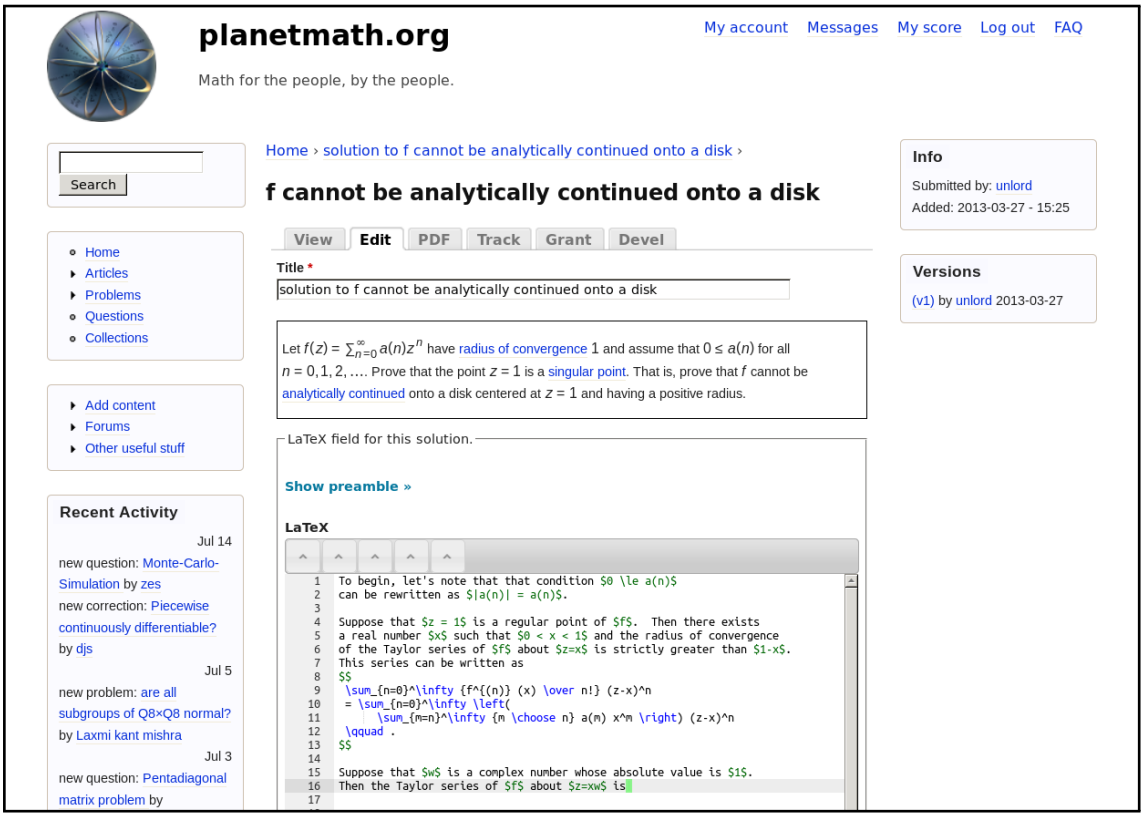



**Figure 17:** *Solving a problem*

explicit; the relevant information for building this sort of on-the-fly course packet is now available via a SPARQL query.* However, to deliver good results, it will be important to develop better strategies for NNexus to use to steer links accurately, since there are often different definitions for the same term: some of these strategies include steering based on the Mathematics Subject Classification, entropy-based approaches, and adding mechanisms for users to give feedback on mis-directed links.† Improving NNexus's accuracy is currently a work in progress (Ginev & Corneli, 2014). In the event that a problem is still too hard to solve, the user can ask a question via one of the interactions offered in Figure 16. This is in line with the example use case for combined peer learning and peer production workflows that was described in Figure 3 [§2.3, p. 31]. Links to easier (or harder) related problems would be useful to add, and with more metadata (for instance, the "mathematical depth" of the terms used in the problem [§6.4, p. 88]), these too would be a SPARQL query away.

Although automatically-created reading lists and problem sets are just beyond the Planetary system's current capabilities, we have stable support for building them manually, via the collections mechanism.

*https://github.com/KWARC/planetary/issues/323
†https://github.com/dginev/nnexus/issues/34

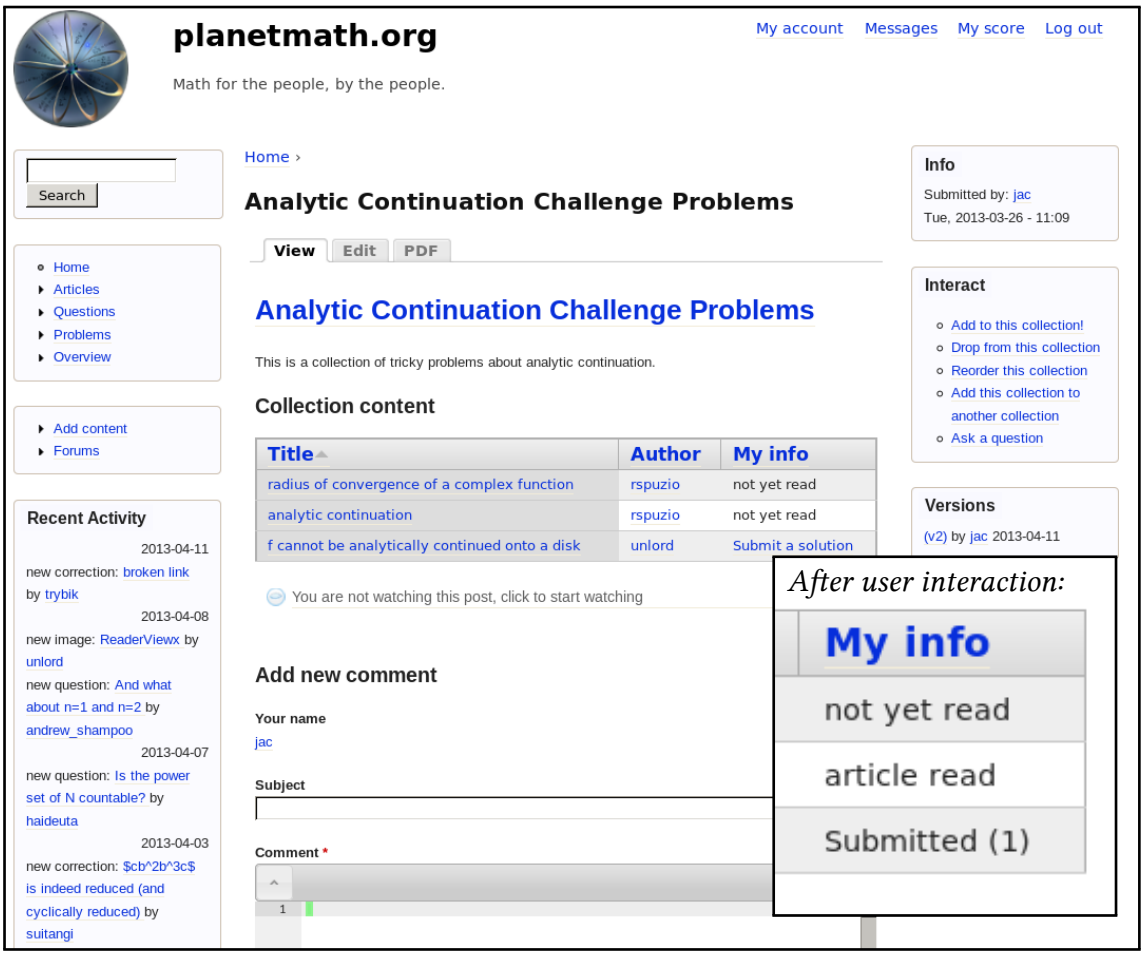



**Figure 18:** *Paradata that shows up as a user works their way through a collection*

Articles, problems, and other content can be added to collections, which function as "playlists" for math content. As illustrated in Figure 18, when the user works their way through the problems in a given collection, personalized metadata (or, more precisely, *paradata* [§4.4, p. 57]) shows up in a field labeled "My info". If another user adds a review for a contributed solution, this information will appear as well.

Although collections are just ordered lists, they can also contain other collections, allowing the user to create and share a tree-like outline, or more general network structure of paths through the content. There are not yet any specific UI features for managing sub-collections in PlanetMath, but our early prototype included "Mathematica-like" folding and unfolding interactions, to show and hide various elements on the page (Kohlhase et al., 2011). This would make the user's interaction with complex structures built using the (sub-)collections facility more intuitive.

An "expand all" feature is the obvious next step, since this would give users an experience closer to reading familiar PDF documents. This could be implemented using the content view mode described in connection with Figure 17.

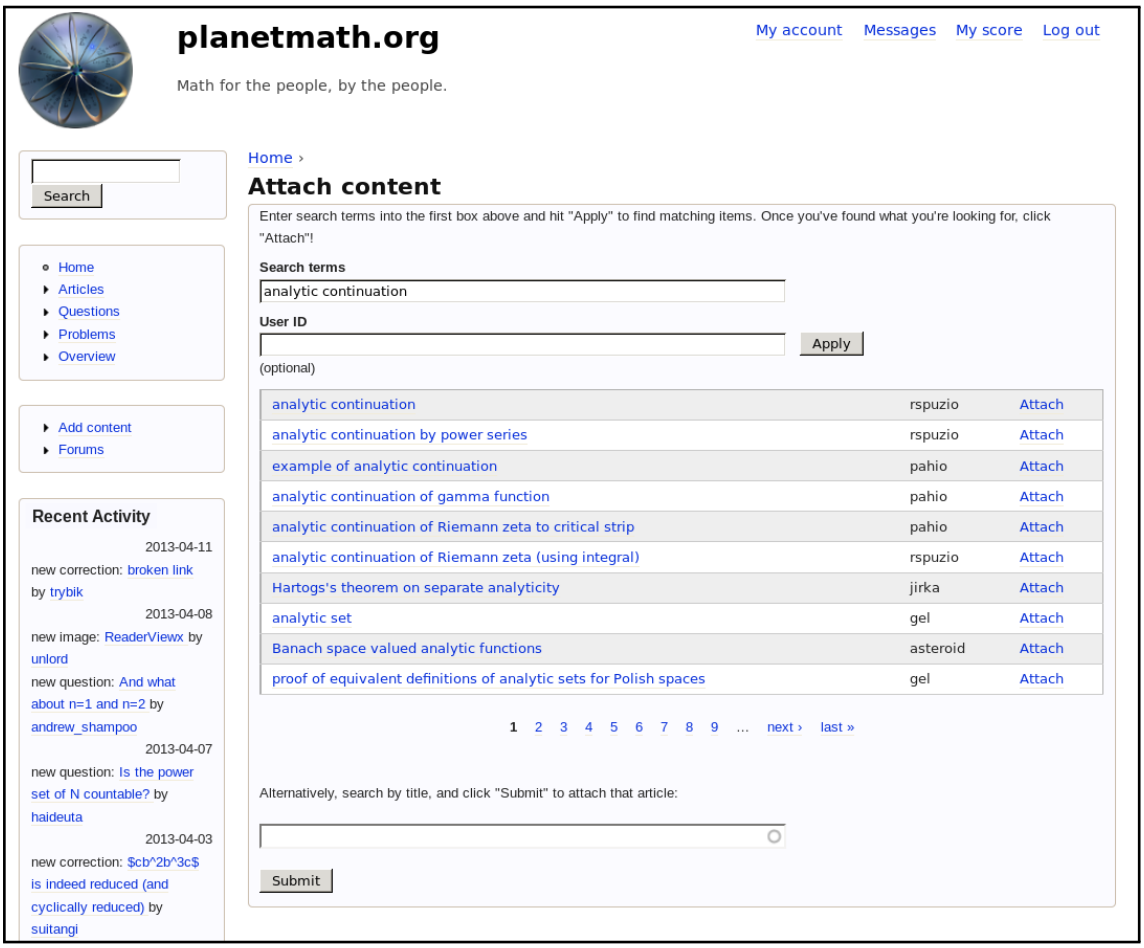



**Figure 19:** *Search being used to add items to a collection*

Figure 19 shows a common UI that is used for building collections, adding content to groups, and also for attaching answers to questions. These interactions were developed in the planetmath_attachable_content module. They make use of dynamic Drupal views that are built using free-text search, which is provided by Drupal's integration with Apache Solr and the associated apachesolr_views module.* Only minor changes were required to index mathematics documents properly,† but integration with a truly mathematics-aware search engine, like MathWebSearch (Kohlhase et al., 2012) remains for future work.

New objects can also be added to a collection by autocompleting on the title, if the title is known – or added *in situ* while browsing by using the link in the "Interact" box (indicated in Figures 15 and 16). New localized interactions are easy to add: so far, there are custom interactions defined for articles, groups, problems, solutions, questions, and collections. The "Interact" box can display different contents for authors/owners and other users; at the moment it does not display at all for non-logged-in users.

*HTTPS://DRUPAL.ORG/PROJECT/APACHESOLR_VIEWS
†HTTPS://DRUPAL.ORG/NODE/1827260

On a whole, the problem solving support is designed with these core features in mind:

- *A low floor*: keywords will be automatically linked to their definitions, which should help users get past a "cold start"; it is also easy to participate just by asking a question; and,
- *A high ceiling*: the collections feature can be used to assemble problem sets or books; users can explore advanced topics, and help others;
- *Support for independent study*: Built-in activity tracking conveys a sense of personal progress, and an increasingly rich collection of metadata and paradata will be useful for recommendations;
- *Support for teachers*: Teachers will be able to use the system to run their own courses – compare earlier classroom experiments run by David Smith (2002) and Robert Milson (2005); Planetary can be used either "as a service" on PlanetMath, or downloaded and further customized in a local installation elsewhere.

***Enhance*** The new problem solving features in PlanetMath/Planetary develop richer dynamic relationships between objects and users by adding the ability to build structure on top of user-contributed materials, and by modeling learning progress.

***Obsolesce*** The collections feature is intended to obsolesce paper-based problem sets. However, with some relatively minor enhancements, it could also make *textbooks* obsolete. PlanetMath content is free and sharable, so it can clearly undercut other offerings in price. Furthermore, unlike textbooks, PlanetMath comes with peer interaction built in.

***Retrieve*** Whereas a hypertext encyclopedia tends to be labyrinthine [§4.5, p. 61], collections can help make the progressive dimension of learning more explicit. They restore the sense of linear order that is familiar from problem sets, textbooks, and state mathematics curricula.

***Reverse*** As PlanetMath builds a more complete catalog of problems, collections, and solutions, we accumulate a machine-readable archive of solution strategies and learning pathways. This archive could potentially be used by artificial agents working on mathematics problems. One application of such a system would be to support the development of a knowledge-rich intelligent mathematics tutoring system. Rather than users teaching the system by adding and structuring content, the system would be actively engaged in teaching users.

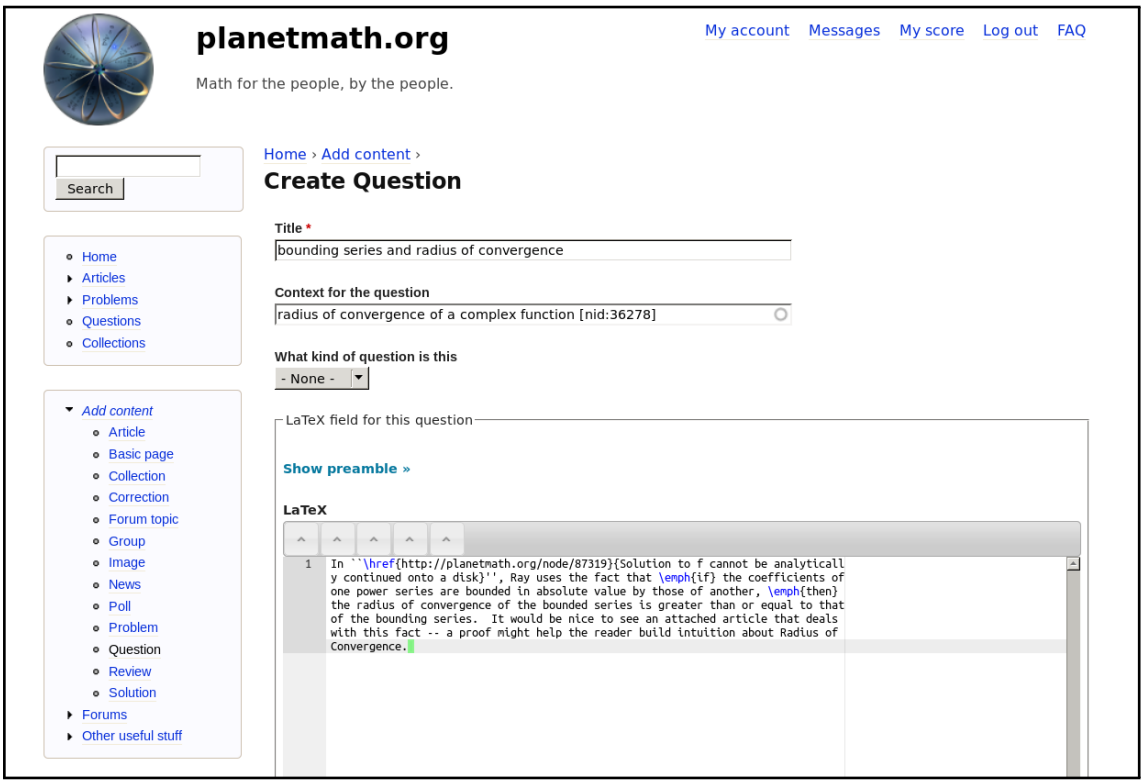



**Figure 20:** *Adding a question*

## 3. Motivate users to contribute problems and discuss solutions

Figure 20 shows a question that is being added "in context" – the user would have gotten here by clicking on the "Ask a question" UI element that appears in the sidebar on articles, problems, collections, and solutions. The user can also ask a question without specifying the context by using the "Add content" menu item from the left sidebar directly; in this case, the "Context" field is hidden from the UI and is left blank when the user presses "*save*."

Questions from across the site show up in a special-purpose feed.* This is one place where we deploy a Drupal *view*. Views offer the possibility for relatively straightforward theming. With more work, we would be able to reproduce many of the same features from StackExchange or its existing open source clone, OSQA.†

Some questions will be useful student-level exercises, like the problems discussed above; but many can be thought of as problems of exposition, and will suggest improvements to the encyclopedia. As with other PlanetMath data types, questions are discussable and can have comments attached – however, *answers* are supposed to be presented in encyclopedia articles, as will be illustrated shortly.

*HTTP://PLANETMATH.ORG/QUESTIONS
†HTTP://WWW.OSQA.NET/

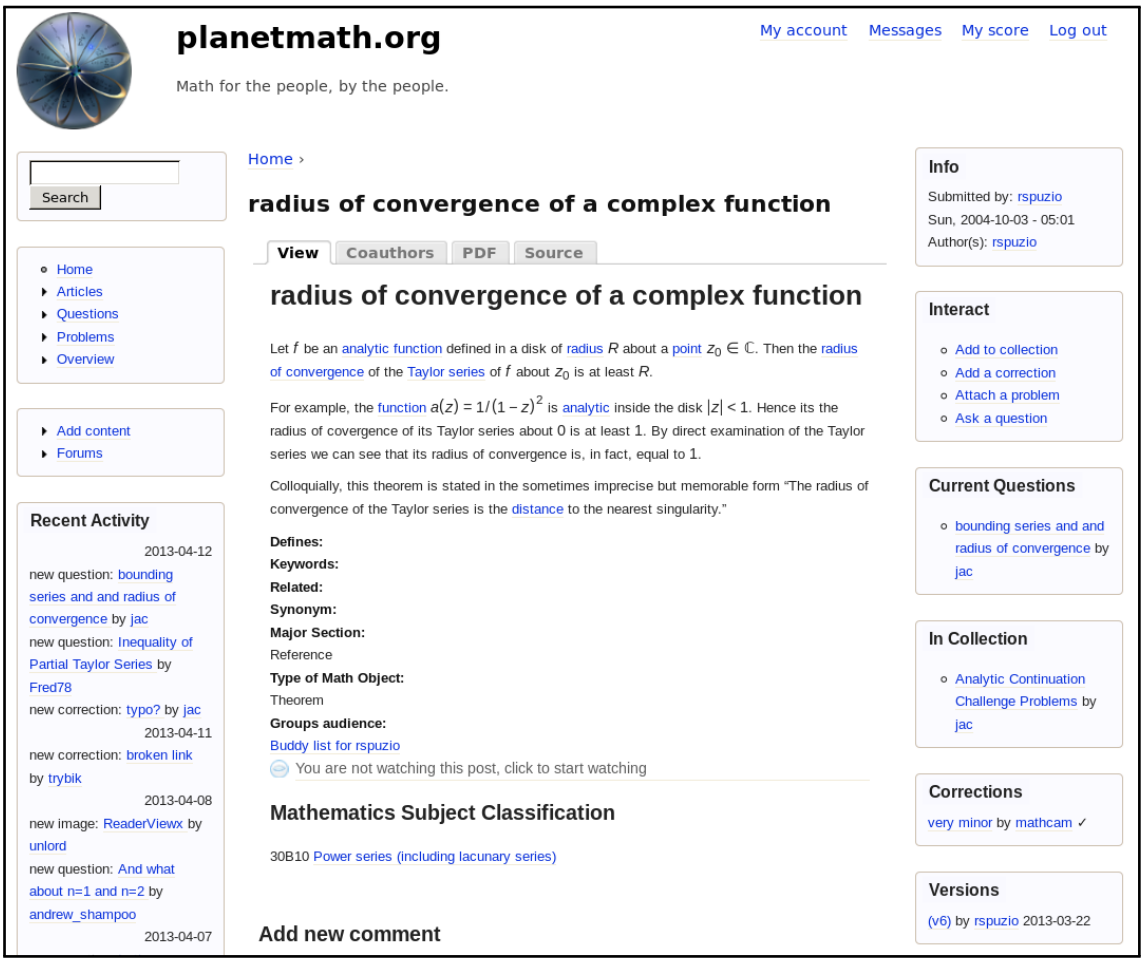



**Figure 21:** *Current questions appear in context*

As depicted in Figure 21, if a question is asked in context, a link to the question shows up in the right-hand sidebar for everyone browsing the relevant object, as a "Current question." New questions also show up in the left-hand sidebar along with other recent activity, and this is persisted across the site, providing a form of light-weight free advertising for users who contribute questions.

Using more refined notions of context, we could adapt this model, so that recent activity would show up sorted by MSC topic. As we shall see in Chapter 9, there are many reasons to reconsider PlanetMath's current orientation towards only the most-global and most-local contexts.

Although contextual questions serve a similar functional role to that of corrections, they are meant to appeal to a more naive reader. A flow of naive questions may a very good way to find and address flaws in exposition. This is consistent with the view that "*the best way to learn is to teach*" (Luk & Wang, 2012). Any constructive reader interaction should help to enhance quality "at the boundary" of author-controlled articles; *cf.* [§1.2, p. 5].




**Figure 22:** *Making improvements to the encyclopedia when answering a question*

As mentioned above, in the question answering model deployed on PlanetMath, all answers are supposed to be supplied in encyclopedia articles. When a contributor indicates that they would like to answer a question, they are presented with an interface like the one in Figure 22. This is the interface from the planetmath_attachable_content module, similar to what was shown in Figure 19. Additional explanatory text and a link are presented when the user indicates that they want to answer a question:

> *The way question answering works around here is that answers are found in encyclopedia articles! If you can't find an existing article that answers the question using the tools below, you can* ***create one*** *and it will be automatically attached as the answer when it's saved.*
>
> planetmath_attachable_content module, emphasis indicates hyperlink

The answerer can either point to an existing article, and add a brief memorandum to the reader – or add a new article. In practice, many users seemed to find this aspect of the interaction confusing. Contributed questions were often answered in *comments* attached to the question, rather than by connecting the question to articles. This does not immediately invalidate the idea of using questions to improve the encyclopedia, but does suggest that the workflow will need to be streamlined. As another approach, one that is particularly interesting in light of the fact that a highly-effective workflow related to question answering already exists on sites like math.stackexchange.com and MathOverflow, various integrations with these sites may be worth pursuing. For example, it should be possible to develop machine learning techniques that provide automated or semi-automated means of connecting questions (and answers) to relevant articles.

***Enhance*** With the introduction of an explicit "question" object, texts contributed to the site start to become more explicitly typed. This moves in the direction of the ontologies for argumentation developed by Christoph Lange (Lange et al., 2008); (Lange, 2011, Chapter 3).

***Obsolesce*** Questions may contribute to obsolescing the old corrections mechanism, at least in part. The question-answering mechanism makes the explicit point that unclear aspects of one article may be explained better in another. The question mechanism also directly obsolesces the old "requests" mechanism from PlanetMath/Noösphere – old requests are mapped into the new system as a question sub-type.

***Retrieve*** Unlike other Q&A sites, wiki discussion pages, or the combination of Q&A with wikis, this system couples questions closely with content. This restores a spirit of camaraderie to the process of addressing user concerns, and points in the direction of collaborative R&D instead of simply Q&A. In short, it retrieves a robust sense of "We". (Compare Table 2 [§3.1, p. 35].)

***Reverse*** Once many questions have been asked and answered, there will be less reason to ask questions anew, since the old answers can just be read. If pushed far enough, the question-answering tool will produce a set of very clear expositions, in the form of repeatable instruction sets. More aspects of the mathematical experience would become algorithmatic – comparable, perhaps, to sewing patterns, cooking recipes, or driving directions.

## 4. Support mathematical meaning making

Two new features of Planetary are particularly geared towards meaning making. A new gallery allows users to share images between articles. Formerly, in Noösphere, each article had its own "file box", to which images could be added. This was useful for adding images to individual articles, but it did not provide a central place for learning about and discussing mathematical images. As noted by legacy users (Figure 13), making these images can be quite a challenging process, often involving custom scripting or programming. The current gallery provides the early beginning of a strategy to address these issues. Groups and collections are intended to address several aspects of meaning making that were identified in the research literature. In short, collections are useful for structuring content, while groups are useful for structuring activities. Planetary includes three kinds of groups: teams, buddy lists, and co-author groups. Groups provide a more intuitive way to control permissions than Noösphere's old UNIX-like Access Control Lists – and they also give a platform on which to continue building features that offer improved support for organizing around shared interests.

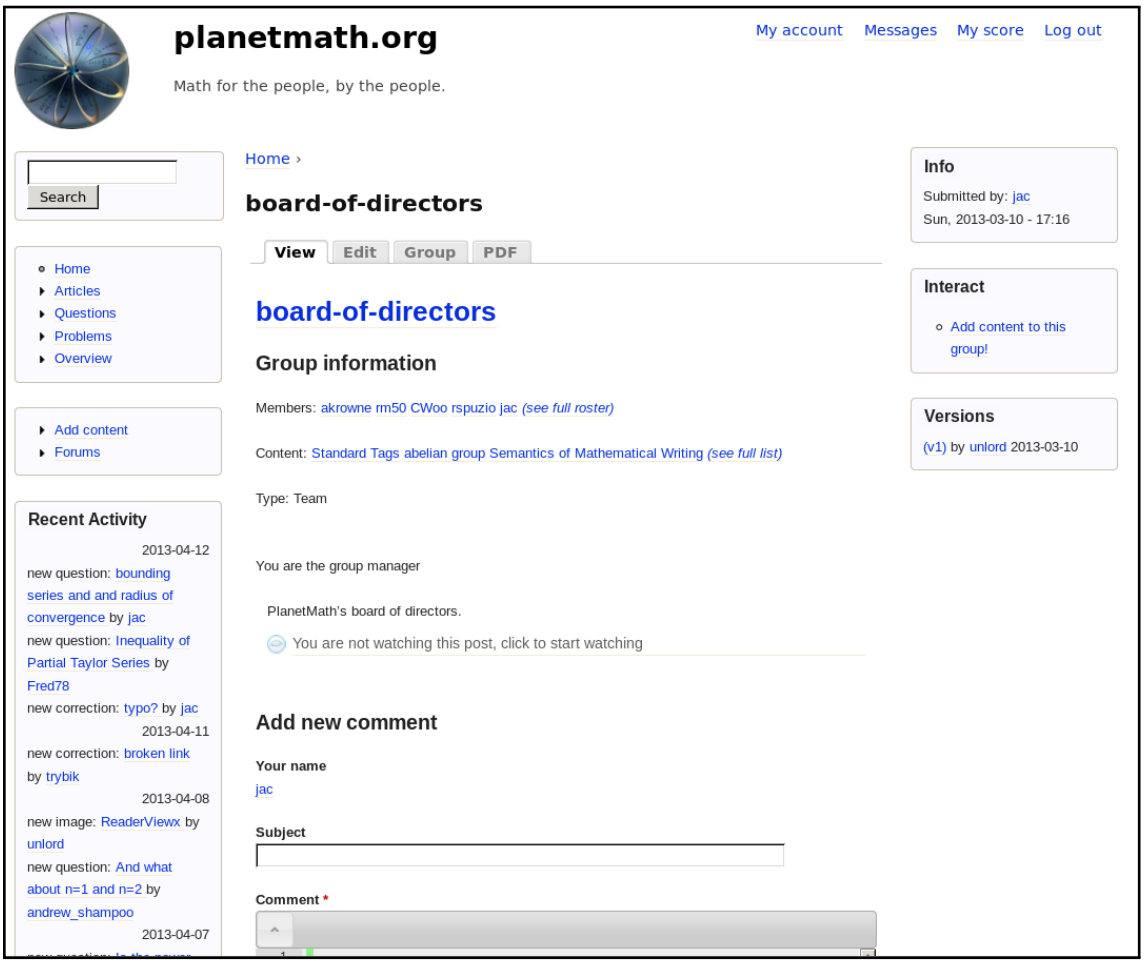



**Figure 23:** *Groups workflow: a team's shared content*

Figure 23 shows a team's content. Any content that is added to a team is made editable by all of that team's members – content can only be added to a group by someone who already has editing permissions. Buddy lists are similar. The members of an individual's buddy list have write access to all of his or her articles. Optionally, a team can be given access, instead of a list of individuals. In particular, if a user wants to give everyone access to their articles, they can designate the World Writeable team as their buddy list.

Co-author groups are specified on a per article basis. Permission to edit an article granted via one of the mechanisms described above *does not* automatically make you a "coauthor," although all edits are recorded in the article's revision history. This allows trusted members of a buddy list or team to make small corrections, without assuming full co-author status. Groups were implemented using Drupal's Organic Groups module.* Various desirable enhancements (like per-group forums and mailing lists) are available as contributed modules, but are not always well supported. Drupal Commons is a suite of modules, also using og, that could provide further enhancements or inspiration.†

*HTTPS://DRUPAL.ORG/PROJECT/OG
†HTTPS://DRUPAL.ORG/PROJECT/COMMONS

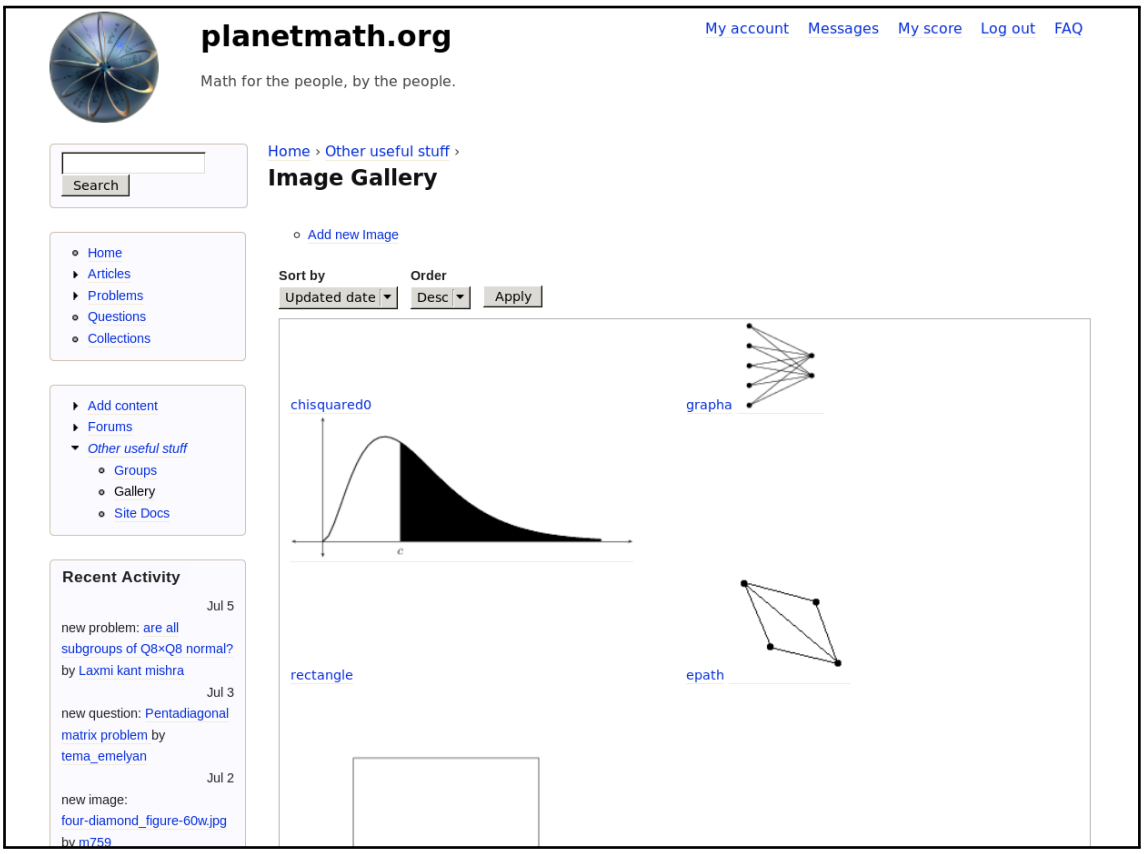



**Figure 24:** *A page from the new image gallery*

Figure 24 shows a page from the new image gallery. The look and feel is currently quite basic: in fact, the gallery is implemented as another Drupal view. Individual images can be included in articles with the LaTeX `\includegraphics` command. When including an image in an article, LaTeXML also deposits RDFa into the page to indicate the inclusion relation to downstream processes. In general, RDFa is useful when we need explicitly work with semantic details of user-contributed materials that would otherwise remain hidden in text. To illustrate one potential application: Since we know which articles use a given image, it would be possible to bring up images in search results by using the texts from the articles that include a given image as a source of index terms in the search engine. A content-focused next step would be to enhance images with instructions on how to generate the image: this is one place where a large set of examples would be very useful. Some recent thinkers on mathematics education have proposed to expand the typical acronym "STEM" (Science, Technology, Engineering, and Mathematics) to "STEAM" (Science, Technology, Engineering, Arts, and Mathematics) (Robelen, 2011). The current gallery is a modest foray into this space. It is also a prototype for other kinds of "shared" objects, for instance a centralized bibliography that would include citations from across the website in one database.

***Enhance*** Groups are designed to help people organize according to their interests. The notion is that it will be easier to integrate user contributions to smaller domains, with the benefit of support from local experts. The gallery, or subsequent extensions, are analogous insofar as images that are shared between articles become a place to organize further work.

***Obsolesce*** Groups partially obsolesce (or at least, make increasingly redundant) PlanetMath's front page, with its stream of posts from across the site. Relative to most specific goals, activity on a per-group or per-topic level is likely to be more coherent and in that sense, more meaningful.

***Retrieve*** Groups offer the sense of a sub-system that is smaller than the whole, but larger than the individual. The aim with features like this is to retrieve a meaningful sense of relationship or even community.

***Reverse*** If groups become strong in their own right, more than bringing people together, they could instead break up into little balkanized islands.

## 5. Support behavior tracking and semantic modeling

The reader will note that the "goal" and "strategy" data types proposed in Figure 1 were not implemented as specific data types. On the other hand, we've built features that give us the foundation for understanding goals and strategies as they are embodied in groups and collections. Thus, for instance, personal goals could be directly modeled as a collection of "things I'm working on." Paradata begins to convey a sense of progress. Collections can also be used to specify a sharable localized learning project, in the form of a reading list that helps build an understanding of a given topic. Teams and projects, along with an associated division of labor focused will be important if site contributors are going to successfully collaborate to build a rich paragogical learning environment. This point will be developed in more detail in Chapter 9.

At the level of domain modeling, one place to begin is with the Mathematics Subject Classification (MSC) is a taxonomic domain model for mathematics. Since the early days of PlanetMath, it has provided a way to browse the site's content "by subject." Reimplementing PlanetMath's MSC browser was an opportunity to develop and demo an integration between Drupal, LaTeXML, and the Virtuoso triple store (Lange et al., 2012), a first step toward exposing more of PlanetMath as Linked Open Data. This infrastructure is not yet integrated with the site's features for behavior tracking, although it would be desirable to do so. This applies to both collection-level paradata described above, and to site-wide activity tracking that uses the userpoints module.

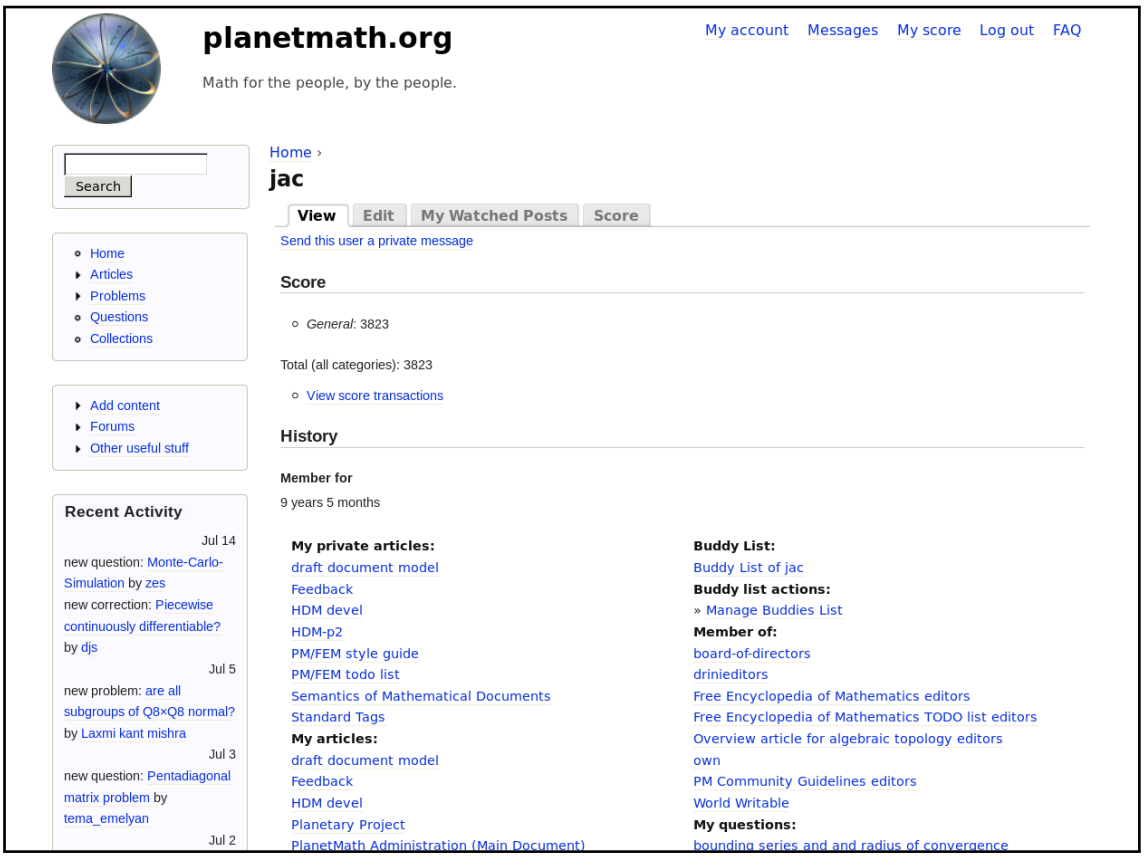



**Figure 25:** *User homepage: showing contributed content and the new buddy list feature*

The user homepage provides a simple listing of the user's contributed content (Figure 25). The user's private articles are only shown if the person viewing the user page is either the user that the page describes or a site admin. Basic biographical information appears at the bottom of the profile page, if the user chooses to supply it.

The user homepage includes links for managing the user's buddy list, as described above. It also includes links for managing watched pages and for sending private messages, which are supplied by the watcher and privatemsg modules, respectively.[*,†] The user's score is managed via the userpoints module; a summary of point-scoring activities is available via the "Score" tab on the user page.[‡] Editing the user page gives the user control over default settings that do not appear in this screenshot, including control over whether or not Planetary will use userpoints to track page reading.

Since some users have contributed hundreds of articles, a more ideal system would include tabs, tags, and other mechanisms for interactively organizing contributed content. This would help users keep track of and manage their contributed material, and help them control how it appears to others.

[*] HTTPS://DRUPAL.ORG/PROJECT/WATCHER
[†] HTTPS://DRUPAL.ORG/PROJECT/PRIVATEMSG
[‡] HTTPS://DRUPAL.ORG/PROJECT/USERPOINTS

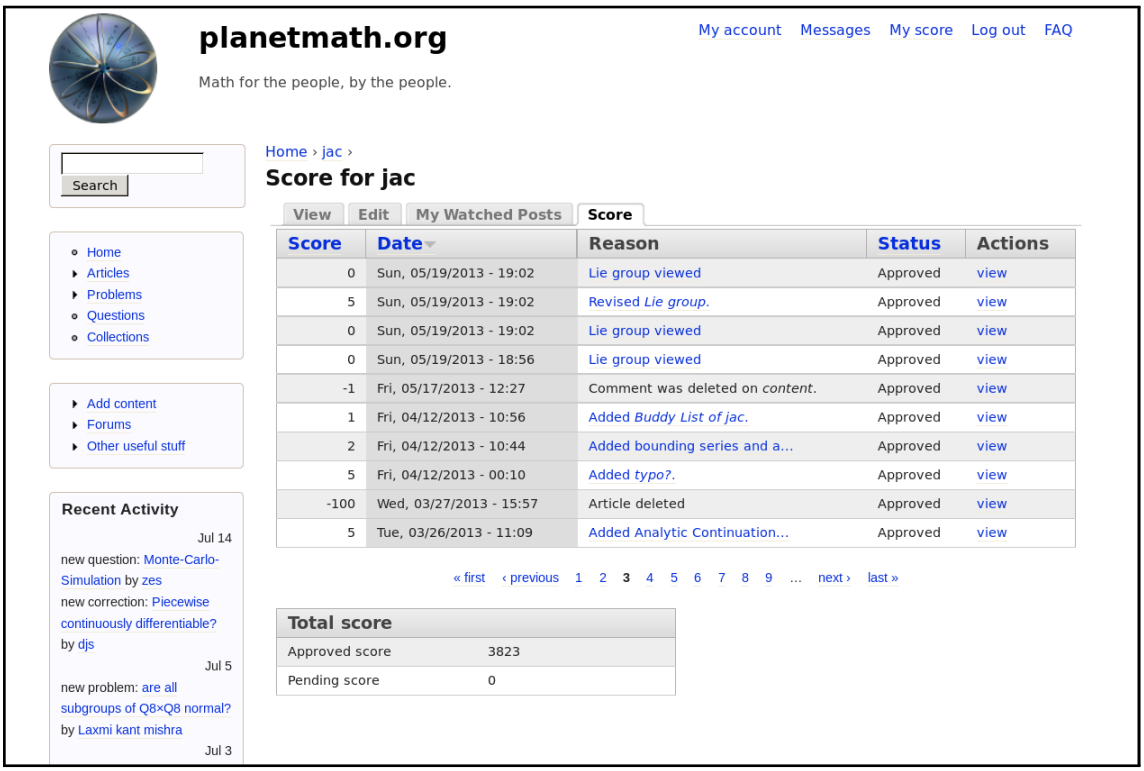



**Figure 26:** *Tracking individual actions with the* userpoints *module*

Figure 26 shows one of the pages from the "Score" tab, which contains a detailed breakdown of the user's contributions to the site. If the user has not opted out of page tracking, page reads will appear in the history with a contributed value of zero. Assigning and adding custom point values that correspond to particular interactions is quite easy with the userpoints module, which provides a function expressly for this purpose (`userpoints_userpointsapi`).

This record includes all of the substantial actions of the users, in some cases going back over a decade (although page reads were not logged in Noösphere). Additional interaction logging provides more data that is amenable to analysis using techniques from Chapter 6. However, to be useful within that framework, the data needs to be separated into vectors specifying the "treatment" and the "effect." This analysis could be made easier if activities were partitioned into meaningful "episodes," rather than simply collected into a large body of potentially unrelated work. Without user intervention, identifying the meaningful episodes and relevant causal factors remains a research topic, although it is one that has been investigated for some time; *cf.* (Faro & Giordano, 1997). One immediately useful idea at the level of user interface would be to display the history of interactions visually, using a module like timeline or views_timelinejs.[*,†]

[*] HTTPS://DRUPAL.ORG/PROJECT/TIMELINE
[†] HTTPS://DRUPAL.ORG/PROJECT/VIEWS_TIMELINEJS

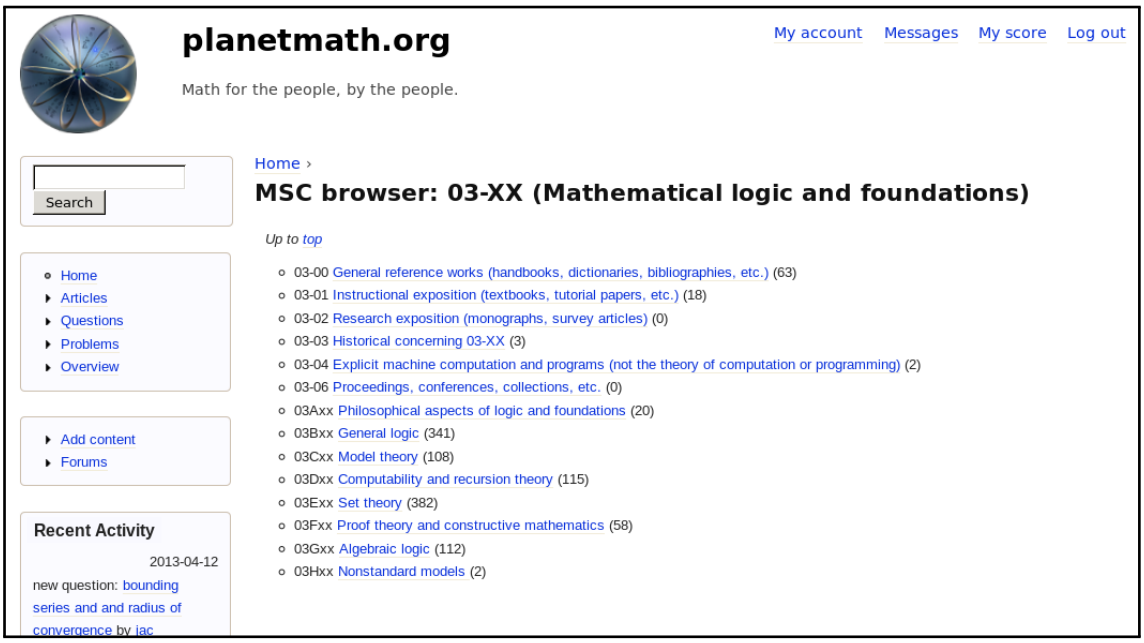



**Figure 27:** *Browsing articles via the Mathematics Subject Classification*

The primary source of top-down structure in PlanetMath/Noösphere was the Mathematics Subject Classification. This system has been replicated in Planetary, but with a new backend that will allow considerable expansion. Figure 27 shows the user browsing the encyclopedia by subject.

For the MSC implementation, we opted not to use Drupal's built-in support for taxonomies,* but instead used the new representation of the MSC as a Simple Knowledge Organization System (SKOS) ontology (Lange et al., 2012), served by an instance of the Virtuoso triple store that we connected to Drupal. Browsing the MSC generates pages and counts articles in various categories using SPARQL queries (Figure 27 shows the category 03-XX and the number of articles in each of its subcategories). The SKOS version of the MSC now includes MathML and multi-lingual category labels. Since the MSC is a core navigation feature, this is an important step towards eventual internationalization of the site.

Drupal serves pages as HTML+RDFa, where the RDFa we're most interested in is generated by LaTeXML from the LaTeX source code, which is further enhanced with links by NNexus. When pages are saved, we run a hook in the new pyrdfa module, which is a wrapper for a local instance of the pyRDFa distiller.† The distilled metadata is then sent to Virtuoso. This allows the system to keep track of which pages link to one another, which articles include which images, which articles are in a given category in the MSC, and to keep track of other provenance-style metadata like author and title.

*https://drupal.org/documentation/modules/taxonomy
†http://www.w3.org/2012/pyRdfa/

Using a triple store opens the way to clever ways to browse and recommend content, for instance, using nearby MSC categories to find related problems. We may also be able to use NNexus to assign MSC classes to otherwise unclassified texts, like forum posts and contributed problems, by clustering the MSC classes of the text's technical keywords. We could then index this content in subject-specific feeds, or attach related forum posts to similar articles or problems as a light-weight recommendation. One of the key issues here is disambiguation of terms with multiple technical definitions; *cf.* (Ratinov, Roth, Downey, & Anderson, 2011).

As mentioned above, currently quite a bit of relational metadata (and paradata) is maintained in Drupal and the underlying MySQL database rather than the triple store. This includes personal history data gathered with the userpoints module, and the hypertextual network of interlinked articles, problems, and solutions. This additional data will eventually be required for inference and recommendation facilities; the approach taken by Mulholland et al. (2013) is similar, and relevant code for moving data to and from the triple store could be reused.[*,†] Once we have a large enough collection of problems with induced MSC categories and depths, we should be able to recommend easier or harder related problems, taking both individual and aggregate paradata into account [§2.3, p. 31]. These models could build on literature related to adaptive testing (van der Linden & Glas, 2000) and the psychology of the "*growth mindset*" (Dweck, 2000).

***Enhance*** More explicit object types together with more data on user interactions adds to our ability to use diagnostic tools, like the one described in Chapter 6. Additional user data around problem solving and peer support could provide the basis for a detailed study of learning-related effects.

***Obsolesce*** Detailed record-keeping on PlanetMath could potentially obsolesce mechanisms like the course transcript or list of publications. As a history of work, the data we can gather resolves to a finer level of detail than either of these.

***Retrieve*** As the system gathers more information about individual contributions, we gain a better picture of who our users are. A user's interaction history should not simply be thought of in terms of pre-determined "badges" or "achievements." It provides something more like the index to a laboratory notebook; *cf.* Bradley (2009).

***Reverse*** If activity on PlanetMath becomes reified as a mechanism for producing value, we may see further effort expended towards mining this value, after the manner of the computations that drive the market for Bitcoins.

[*] https://github.com/ekilfeather/open-storyscope
[†] https://github.com/kmi/decipher

## 8.5 Preliminary system evaluation

A user-focused evaluation of the system is detailed in the next chapter; here the focus is on the technical aspects of the system in their own right.

When compared with Noösphere, Planetary is much easier to install, configure, and extend. Drupal's profile subsystem allows Planetary to be set up "out of the box" as a course management tool, or as the home to a larger PlanetMath-like learning community. Drupal's content management features, like the ones that we've used to build the groups, collections, and problem set workflows, help to set Planetary apart from other mathematics-capable tools. In addition to documentation generated by our team, many members of the worldwide Drupal developer community are happy to help new Drupal developers. The features we've developed cast PlanetMath's original mission – to be "*a central repository for mathematical knowledge on the web, with a pedagogical slant*" – in a new light.* Several legacy feature requests have been addressed in the process – and work on others is in progress.†

One could argue that the *combination* of features from other sites (*e.g.* Wikipedia plus StackExchange) could provide a system with the same functionality as PlanetMath. To some approximation, that is certainly true. This begs the question: what benefits do we get from putting various features under one roof, with a purpose-built system? The primary benefit is that integrative functions can be performed more easily. This is demonstrated well with the new collections feature. In the future, it would be attractive to increasingly integrate the platform with what others are doing elsewhere on the internet. For example, NNexus could be used to index mathematics blogs, and provide a topic-specific blogroll attached to each page in PlanetMath's encyclopedia (Ginev & Corneli, 2014): recall that the usual meaning of "planet" on the blogosphere is planet-as-aggregator.

I would cite two particular points of difficulty associated with using Drupal 7 as the basis of the new site:

1. *Drupal is known to have a steep learning curve.* This slowed our progress at the start, and sometimes things that should in theory be simple still ended up not working at all (thus, for example, dynamically changing menu items would work for a while, but then crash the site, so those features had to be turned off).

2. *We had to patch Drupal's core.* This is justifiably "taboo" amongst Drupal programmers, since it makes updating the site difficult, but we needed a new hook (`hook_post_save`) to supply our RDFa extractor with *rendered* HTML+RDFa; we also

*http://planetmath.org/history.html

†https://github.com/holtzermann17/planetmath-docs/issues/62

needed to modify Drupal's database interface to add support for the characters that are needed to display mathematical symbols.

Some of the technical issues will be resolved in Drupal 8 – but the system doesn't show signs of becoming simpler to learn. Luckily, the Planetary developer team has accrued significant wisdom, some of which can be passed along to future contributors. There are some "add-ons" for Drupal that are quite helpful for development work – for example, Drupal is fully scriptable via Drush,* and it can be interacted with using a Read-Eval-Print Loop (REPL), care of Boris.†

As a matter of building a usable system that can also be the basis for future work, the project has been a technical success. There are many places where new features, ranging from format shifting to research on recommenders for learning would be useful. Planetary is interesting partly because of what it makes possible – including the user study presented in the following chapter, which will help evaluate and better understand the paragogical model embodied in PlanetMath/Planetary. Future work on the system will progress subject to demand. As one small example: the activity data we store is similar to the format proposed in the ActivityStreams project, but use-cases for downstream consumption of data about activities on PlanetMath are not yet entirely clear, so we have not yet put effort into serving bona fide Activity Streams.‡ When the use case is there, the system is ready to adapt accordingly.

## 8.6 Summary

The Planetary system is an extensible open source platform for doing mathematics on the web. PlanetMath, which was previously oriented around authoring and editing a mathematical reference work, has been rebuilt using the Planetary platform. The new system hosts online problem solving interactions in a knowledge-rich, wiki-like environment. New features include custom collections, which can be used to build reading lists and interactive problem sets. Planetary records information about problem solving and other interactions, making it a useful tool for learning more about the way people learn mathematics.

While PlanetMath/Noösphere considered logical correctness, encyclopedic coverage, and expository quality to be matters of importance, mathematical problem solving did not receive explicit attention. The implementation work presented in this chapter opens new horizons for the PlanetMath community, with a focus on relevance and usefulness for a broader population of users. The development work has been informed by con-

*http://drush.ws/docs/shellscripts.html

†https://github.com/tobiassjosten/boris-loader

‡http://activitystrea.ms

tact with system users and stakeholders and by the theory of paragogy developed in earlier chapters. As a computer program, Planetary must follow strict rules; if anything, mathematics "*is much less formally complete and precise*" (Thurston, 1994). However, its expressive thickness derives not from the fact that all of the semicolons are in the right place and most if not all of the critical bugs have been eliminated – although this is certainly important – but rather from the fact that it can now be used, critiqued, and improved.

The discussion of screenshots framed using McLuhan's tetrad helps to add media-theoretic depth to the design dimensions and user desiderata discussed earlier in the chapter. The screenshots are, in a way, figurative depictions of typical mathematical contexts, but with a difference. In offline mathematics we are familiar with scratch paper, journals, and blackboards for building, sharing, and discussing mental models; gradebooks, transcripts, and CVs for keeping track of progress; problem sets, textbooks, and monographs for organizing learning materials; and libraries for browsing existing mathematical works. Planetary does not reproduce these technologies flawlessly, but the parallel online context has the advantage of being connected to a robust meta-level discourse and an ongoing process of technological change. This chapter has described the progress from design elements to working code deployed on a public website. The next chapter will explore the system's underlying dynamics in more detail, by examining the user experience.

# 9 User Evaluation

**Qualitative findings from a user study after the public release of the new system demonstrate the feasibility of the paragogical approach, while suggesting a range of improvements to the software that will be needed for the project to successfully scale up – in particular, the further re-structuring of PlanetMath's activities into "sub-projects."**

## 9.1 Research design

Planetary was deployed to PlanetMath.org on February 13, 2013. In April, I began recruiting participants for a 10 day study on PlanetMath. From emails sent to 15 all-time top-scoring and 30 other recently active PlanetMath users, I successfully recruited three persons. I also recruited three postgraduate students in mathematics at The Open University, contacted through their department.* The main question I focused on in the interviews was: would PlanetMath be relevant to the way you study and do mathematics? I also tried to elicit any ideas that might help to improve the site. Participants in the study obliged, with quite a bit of constructive critical feedback about the system. For example, one noted his resistance to creating collections, another talked about difficulties working with LaTeX, a third pointed out that PlanetMath would be more useful if it provided a feature for blog hosting, a fourth suggested new features for navigation. Figures 28 and 29 sum up the research design that helped to give order to this type of feedback, using a visual metaphor. The study aims to describe the relationship of system features to emergent patterns of use, here visualized as "paths in the grass."

*One person also joined in response to an announcement in the *Peeragogy in Action* Google+ community, but he dropped out of the study before completing an interview.

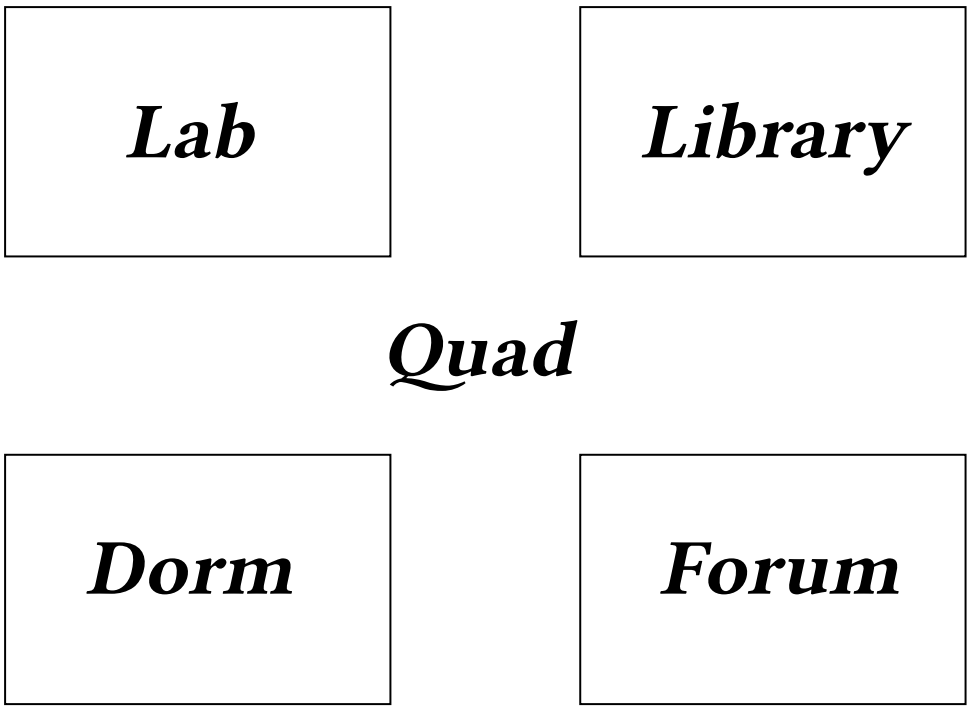



**Figure 28:** *Map of a virtual campus*

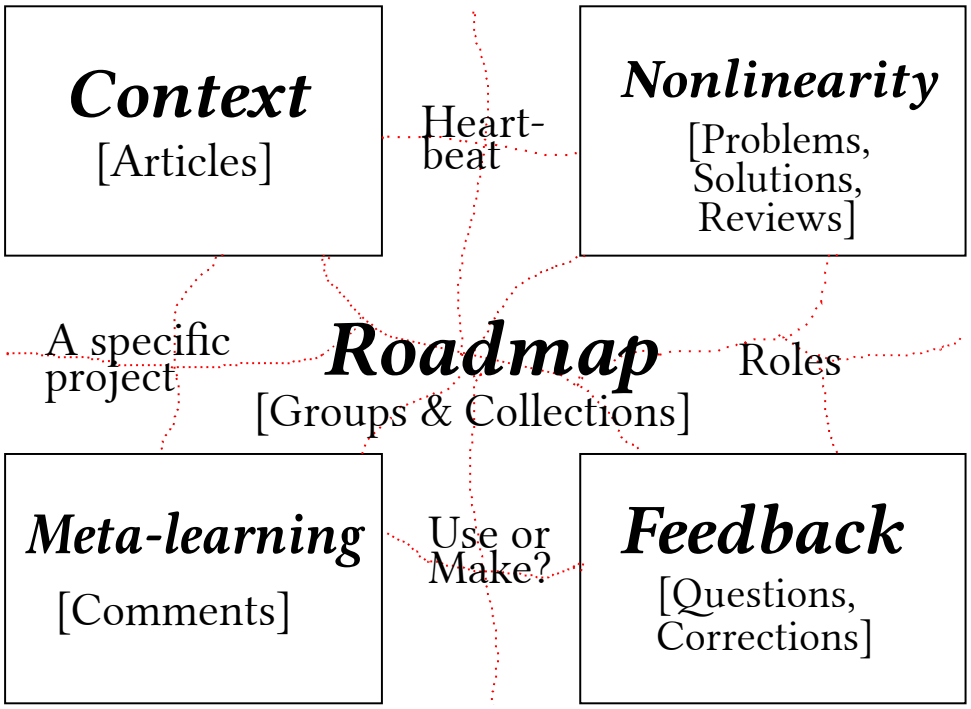



**Figure 29:** *Peeragogy patterns as loci for "paths in the grass"*

The Peeragogy project's catalog of patterns for peer learning (summarized in Table 13 [§5.2, p. 79]), which is used to seed the current study. If actual behavior were as depicted in Figure 29, we would expect to see people moving between "Articles" and "Problems" by way of "*A specific project*", or from "Question" to "Collection" by way of a division of labor among several "*Roles*." At a narrative level, this makes a certain amount of sense. In practice, the emergent structure of use may be more elaborate. More than a collection of hypotheses, Figure 29 represents a "meta-model" that can be used to identify the dynamic patterns of use that emerge within in a socio-technical system.

David Cavallo explained the idea of *emergent design* as a practice of "*probing for skills and knowledge resident in a community and using these as bridges to new content*" (Cavallo, 2000A). The aim here is to find not just new content, but new form – which necessarily involves transformation.

The study's phases are described in Table 19. In the first phase, participants (Table 20) attended a focus group where the study was described in detail, and where participants were asked to describe their mathematical backgrounds and to say what interested them about the study. I also walked them through a sequence of screenshots similar to those in Chapter 8. In the second phase, participants tried the new features of Planetary, on the public PlanetMath.org website. In the third phase, I conducted individual interviews. The participants were recruited in two cadres, so these three phases all ran twice (on May 2, 2013 and May 15, 2013). Participants were awarded a £32 Amazon.co.uk voucher upon completion of the study.

The analysis focuses on material gathered from the interviews; selected quotes appear in Appendix D. The initial analysis connects this feedback to the specific features of Planetary by way of the dimensions of design discussed in Chapter 8. Patterns offer another heuristic method to understand and organize new ideas coming from users. At the outset, several patterns (*Heartbeat*, *Roles*, *Use or make*, *A Specific Project*, and *Roadmap*) that came up particularly frequently in discussions in the Peeragogy project seemed likely to describe the user experience here. During interviews, I used the layout from Figure 29 as a template for visual notetaking.

A pattern-based analysis is rigorous in the sense that either the existing patterns match the topics under discussion, or new patterns should emerge. Design patterns are far from being a "universal" symbolic language in the Leibniz sense: nevertheless they can be used in the spirit of *mathesis universalis* as understood by Deleuze; cf. (Deleuze, 1968, pp. 229–230, 241). Design patterns can help build culture where before there were only constraints, symbols where before there was only a landscape; *cf.* [§1.4, p. 10]. In the course of analysis, the user evaluation of the site was encapsulated in three new design patterns that contribute to a new, emergent, design for the site, including concrete indications of "What's Next".

| | | |
|---|---|---|
| **1** | In an initial focus group, I will present the background of this study. | I will aim to discern what the *Roadmap* for individual participants looks like. |
| **2** | I explain the tasks that I would like people to try over a 10 day period, and explain that I will be available for discussion in the forum or privately by email. | I will aim to discern whether people use the four basic patterns (*Heartbeat*, *Roles*, *Use or make*, *A Specific Project*) as they orient themselves to the new software. |
| **3** | I will meet with each of the participants for an individual debriefing after the 10 days has passed. | I will aim to discern whether people use any other more refined patterns as they used the tools. |

**Table 19:** *Phases of the structured component of the user evaluation*

| | |
|---|---|
| **AA** | UK resident, male, chemistry background, currently working in media; hoping to enroll in the Open University's mathematics masters degree (contacted through PlanetMath) |
| **AB** | UK resident, female, current postgraduate student in mathematics (contacted through OU mathememathics department) |
| **AC** | German resident, male, mathematics PhD, staff scientist and programmer, long-time PlanetMath user (contacted through PlanetMath) |
| **RSP** | US resident, male, physics PhD, long-time PlanetMath user and active volunteer (Raymond S. Puzio, requested non-anonymity) |
| **BA** | UK resident, male, engineering background, enrolled in the Open University's mathematics masters program (contacted through OU mathematics department) |
| **BB** | UK resident, female, engineering and software background, enrolled in the Open University's mathematics masters program (contacted through OU mathematics department) |
| **BC** | UK resident, male, teaching and media background (contacted through the "Peeragogy in Action" Google+ community; BC dropped out of the study partway through) |

**Table 20:** *Study participants*

## 9.2 Thematic analysis

Individual interviews offered the most information and are examined in detail in this section. Detailed quotes from the interviews are presented in Appendix D and are referenced here in square brackets.* The analysis also draws on observation of participant interaction with the tool, as well as forum posts and emails exchanged with participants during the study. The most useful way to begin to organize the analysis is under the headings coming from the initial dimensions of the design (Chapter 8), and to think in terms of the concrete system features that instantiated this design (Figure 14 [§8.3, p. 114]). The key ideas are collected into emergent themes.

### 1. A special-purpose tool customized to suit the challenges of learning mathematics

**"Necessary but not sufficient"** In our interview, **AA** – who came to PlanetMath with mostly undergraduate-level questions – said that the site is only as good as the *people* who use it. **AB** – who was thinking about research-level problems – suggested that it is only as useful as the *content* that it provides. Working software is clearly another "necessary but not sufficient" criterion. A model devised by **RSP**, connected with earlier discussions among core volunteers in the PlanetMath project, came up again in our interview [RSP1]. A schematic rendition of this model is presented in Figure 30. *Content* and *Community* features are familiar from other social websites. A *Catalog* is similar to what one finds in libraries: a resource that provides pointers to other available resources. **RSP** is particularly interested in using existing library catalogs to find additional content that can be used on PlanetMath [RSP2]. Inasmuch as the catalog helps to find content, it represents an essential feature. **BB** repeatedly emphasized the importance of functioning search – otherwise we will have "*masses of very good math, masses of very good theorems and results and definitions and so on*" – that are effectively useless because you can't find the one you need [BB1]. Thus, a tool that supports learning mathematics should at least support these three core features. Two additional components, *Code* and *Organization*, provide the basis for the existence and evolution of such a tool, but most users will not interact with these components directly [RSP1].

**Challenges with writing mathematics** At the moment, the most prominently visible mathematics-specific aspects of the system have to do with *content* – and particularly with entering and displaying mathematical *symbols* in textual form – although mathematical *figures* were also discussed. As it stands, even mathematical symbols presented

*Recordings from **AA** and **AB** are missing due to a problem with the recording software; accordingly, data from these interviews is embedded in diagrammatic form in my handwritten notes. **BC** dropped out of the study before completing an interview.

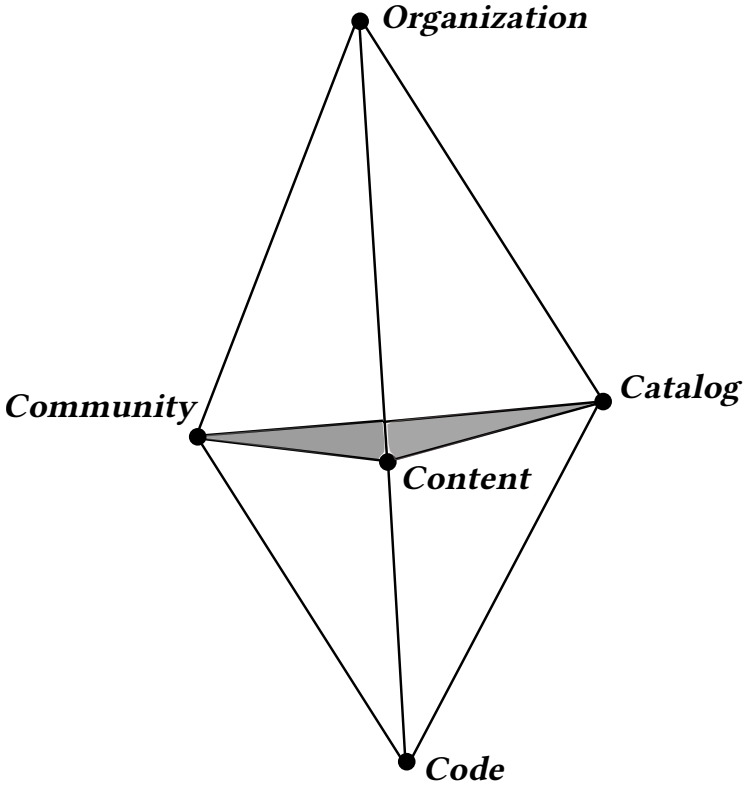



**Figure 30:** *Raymond Puzio's "3C" model, with elaborations from our conversations*

various difficulties for some of the users. **AA**, who was new to LaTeX, emailed me several times during the study asking for typesetting help. **BB** and **BA** were familiar with the "What You See Is What You Get" (WYSIWYG) LaTeX frontend LyX,* having used it to complete their course assignments. They managed with PlanetMath's current web editor, but **BA** indicated that he missed the familiar WYSIWYG workflow. With the current web editor, he felt concerned that the browser might time out, and that he would lose any text that hadn't been saved [BA2].

Currently, LaTeXML "*fatal errors*"† (where no resulting HTML can be generated) aren't reported to the user at all and result in a blank screen, causing a strong sense of uncertainty and doubt when things go wrong – even for an experienced LaTeX user like **AC** [AC1]. On the other hand, **AC** emphasized that having LaTeX itself is quite useful [AC2]. Autosave and auto-preview would not be terribly hard to add to the current PlanetMath editor, particularly since both the editor and the LaTeXML daemon have WebSocket support. Simple point-and-click entry of mathematical formulas would also be an advantage for new users. Novice users might also prefer to use handwritten mathematics (Lo, Edwards, Bokhove, & Davis, 2013). **BA** indicated that scans of handwritten papers can be submitted in Open University courses, which lowers the barrier to participation.

*http://www.lyx.org/

†http://dlmf.nist.gov/LaTeXML/manual/errorcodes/

**"Nice to have"** In the interview with **AB**, we came up with a way in which an improved gallery could be useful for search. She was looking for a paper in graph theory – but wasn't sure what search terms to use, although she knew what the graph she had in mind looked like. Our idea was that if PlanetMath's gallery had more semantic structure, allowing her to find the figure, it could also be used as a way to find the relevant definitions. Currently, however, the gallery needs many basic improvements, and image search does not work at all. **BA** found the workflow of uploading to the gallery difficult and suggested that images wouldn't often be reused and that the global gallery was of limited importance. He suggested to switch to "article-local" interaction with images (as in Noösphere). It should be possible to support intuitive local uploads that integrate with the global gallery. As time goes by, we may see user interest in additional features for remote collaboration, including various mathematics-specific features, like a interactive computer algebra system [RSP3].

## 2. An active and interactive process centered on problem solving

**Progressive problem solving** Several subjects brought up interesting points and questions about the meaning of "problems" and "solutions." This helped to show that the entity relation diagram in Table 10 [§5.2, p. 74] could be made more complete by adding conjectures and partial solutions [AC6]. It seems clear that interactions centered on research problems and textbook problems would be rather different. **BB** outlined an ethical stance on questions about homework problems [BB2] that is very different from **AC**'s orientation towards sharing both questions and partial answers to research problems.

**BB**'s understanding of the right thing to do in the case of *questions about homework problems* was consonant with the design decision to have "answers" be links to encyclopedia articles [BB2], [BB4]. Linking to encyclopedia articles is effectively a hint, indicating relevant and ostenibly common knowledge, which should put the questioner on the right track, but which may still leave them something to work out. However, when it comes to *her own solutions to problems*, like **AC**, **BB** was comfortable with uploading and discussing conjectures and partial solutions as work-in-progress [BB3].

It was also clear that unless users are comfortable sharing incomplete answers, partial, or failed solutions, then this sort of knowledge will be underrepresented – which may represent a missed opportunity for learning (Tsovaltzi et al., 2010). Nevertheless, there is a barrier to overcome. **AB** said she found it nice to be able to help – but she was concerned that if she admitted that she couldn't figure out the answer to a given question, or if she posted an incomplete answer, people would get upset. She reported spending an hour attempting to work out a tricky integration by parts problem that another user had uploaded, before giving up – no trace of this work showed up on the site. Regard-

ing another question having to do with differential equations and interest rates, she said she would have liked to help, but "*I haven't looked at that in years.*" Certain comments from **AB** and **BB** were aligned with an idea that I had previously discussed with the Planetary development team, namely the creation of an Etherpad-like website for mathematics [BB3]. This would be a stand-alone system built using an improved version of the PlanetMath editor for creating ephemeral (potentially anonymous or throw-away) mathematical artifacts, the electronic equivalent to a chalkboard. However, if this collection was (optionally) taggable and searchable, the material could be re-used, and could, furthermore, provide a real-time guide to topics of interest around the site; *cf.* (Corneli, 2010), (Mary, 2013).

**Personal history, constructivism** Planetary currently presents an overview of the user's "History" on the user homepage. It tracks more detailed interactions using the userpoints module, but **AC** found the presentation of this information confusing [AC11]. **BB** pointed out that history is not only important for the user home page, but could be usefully integrated into the browsing experience, using appropriate breadcrumb trails rooted at important topics [BB7]. Meaning making is associated with "constructive" features, and collections represent a light-weight tool for constructive meaning making. Subjects generally agreed that collections provide a useful way to organize material, and also that there were various ways they could be improved, for instance: by adding the ability to "expand all"; by being able to display sub-collections in a nested format; by being able to include links to outside resources; or by adding narrative "glue" between components. For **BB**, who was thinking primarily of reference-related user interactions, PlanetMath's existing narrative *topic articles* that summarize important mathematical topics present a compelling alternative to collections [BB8]. **BA** felt that collections should be managed by someone with a designated "editor" role [BA3].

## 3. Motivate users to contribute problems and discuss solutions

**Regulating learning in a social/mediated context** Although she described herself as "*not very self-motivated*," **BB** seems to be highly motivated in the right context – and she is very interested in understanding how she learns best [BB5]. After our interview, she sent me a description of her own learning patterns [BB6], which seemed to closely match the "*forethought*", "*performance*", and "*self-reflection*" cycle from (Zimmerman, 2002; Zimmerman & Campillo, 2003). As Zimmerman and Campillo (2003) indicate, there are differences between problem solving in "*formal*" and "*informal*" contexts. Although mathematics is a relatively formal domain, PlanetMath can also help people solve a range of "*ill-structured*" informal problems. For example, **AA** was particularly enthusiastic about PlanetMath's potential as a place to network with people with similar interests, but pointed out that there needs to be a "*sense of urgency*" about answering

questions that are uploaded to the site. I mentioned that the StackExchange system seems to have done a good job in that respect. **RSP** also expressed interest in networking [RSP4], but pointed out that simply displaying a log of user activities wouldn't really be sufficient for finding collaborators, since even very prolific contributors "*might not be in the mood to talk about it*" [RSP5]. He suggested that a map of user activity would be more useful if it included more detailed social networking information, beyond just mathematical interests or a state of "co-presence" online [RSP5], [RSP7].

**Comparison with roles in other contexts** People are motivated to use systems that help them achieve their aims. People are relatively *unlikely* to report software bugs if they don't have a sense that the bug reports will be responded to. I encouraged **AC** to point out any software problems he encountered with PlanetMath/Planetary, even though I had to admit that the Planetary development team is not "*available 24/7*," [AC7]. On the software level, if we make it easy to report bugs, if we respond to bug reports, and if we make it easy for others to get involved with the software, then there's a good chance that the software will eventually become satisfying for site users. "Bug reports" and "feature requests" present a useful model, not just for software, but for mathematical content – indeed, this is precisely what *corrections* and *requests* were supposed to accomplish in Noösphere. However it may be time to introduce a new abstraction layer, that helps people discuss and engender more broad-sweeping change (*i.e.*, above the level of individual articles, which is what corrections target). An "open" production model comes with the ability to redesign the platform and content as a work-in-progress. This is quite different from the model used in OU courses, where, for instance, the forum moderator tries to "*dispel any talk*" about the Tutor Marked Assignments [BA6].

## 4. Support mathematical meaning-making

**Readily available feedback** Study participants pointed out in different ways that if there are to be interactions with other people, then people have to show up. **AC**, who had first joined the site around the time of PlanetMath's peak in activity in the mid-2000s (Figure 2 [§2.1, p. 25]) noticed that things were fairly quiet around the site these days [AC3]. He pointed out some questions that the PlanetMath ownership model poses regarding interaction with content once article owners have left the site [AC4]. Relating back to **AA**'s point about a "*sense of urgency*," **AC** noted that it is very important to be able to "watch" a problem, so as to get notifications when its status changes [AC5]. In OU courses, the tutor is available for live help on an individual basis: as **BA** indicated, "*The student can contact them by email, phone – and, um, whatever.*" We can contrast this with the experience in PlanetMath, where, for instance, an active volunteer like **RSP** expresses a somewhat distant hope for time to help newcomers [RSP10].

**Concreteness as a criterion of quality** Concreteness applies at both the system and content levels: people need to know what they can do, and what it means. Some of the study participants found the proliferation of options and types of data on the home page confusing. It seems possible that something like a "start menu" would help people see what they *can do* on PlanetMath – and if someone wants to do something that isn't possible yet, they should be instructed to use the action, *send a message to the developers requesting a new feature.* **BA** described the OU courses he had taken, and told me that he had quit one of them part way through [BA7]. Even though "*some people*" can find their way around functional analysis, his course was presented in a way that made the topic seem "*so abstract*" that he was not able to follow it. **BA**'s experience points to the importance of being able to visualize the topic that is being studied, and understand how it can be applied. Adding concreteness is one "service" that a knowledgeable teacher or mentor can provide – if one is available.

## 5. Support behavior tracking and semantic modeling

**Personalization and localization** **AC** was particularly enthusiastic about the ways in which PlanetMath could, at least in theory, be personalized to the user's interests. In his view, PlanetMath shows more potential in this regard than sites on the StackExchange network [AC8]. This perception is striking particularly because there is still considerable room for improvement in implementing personalization features. Some of these features would be within easy reach, for instance, after the model of dashboard module used on Drupal.org.* **AC**'s perception of personalizability may have to do with PlanetMath's *openness* [AC9] and its subject-specific focus. He also had some specific technical suggestions. One of these was to incorporate blogging tools in PlanetMath user pages [AC10]. **RSP** suggested a similar point, namely, to incorporate familiar social networking tools into user homepages, using a system like Friendica† [RSP6].

One of the ways to localize around specific interests is to "segment" the site, per requests from legacy users, see Figure 13, [§8.2, p. 110]. **BA** suggested that people working at different levels may not need to interact much [BA5]. **BB** made a similar remark that indicated that segmentation would be most useful when combined with a localized, faceted search functionality [BB9]. Connecting this to the theme of personalization, **RSP** pointed out that all of PlanetMath's important features always "localize" to any given user at any given point in time [RSP8], although they may fail to do this gracefully. He noted that individual projects had not been particularly well modeled or well supported in PlanetMath/Noösphere [RSP9].

*HTTPS://WWW.DRUPAL.ORG/PROJECT/DASHBOARD

†HTTP://FRIENDICA.COM/

## 9.3 Pattern analysis

As a secondary level of analysis, the interview data is further analysed in relationship to the Peeragogy pattern catalog (Table 13 [§5.2, p. 79]). The notion of pattern-finding as a process related to, but distinct from abstraction, is described by Richard Gabriel, who emphasizes that the "*patterns and the social process for applying them are designed to produce organic order through piecemeal growth*" (Gabriel, 1996, p. 31). This criterion is here applied to the growth of the pattern language itself. This section draws on the thematic discussion from the previous section, and aligns user feedback with the existing pattern language, in the process of describing three new patterns. A discursive presentation is presented first, followed by a summary using template from Table 11 [§5.2, p. 76] in Tables 21–23, which appear at the end of this section.

### Frontend and Backend

Although mathematics is a relatively formal domain, many of the motivations for using PlanetMath map onto Zimmerman and Campillo's (2003) dimensions of "*informal*" problem solving. Informal problems are "*are personally defined*" and possess "*open-ended boundary conditions*": they are situated within an open world. "Formal" motivations can typically be addressed by a generalized look-up approach: for instance, by reading the manual, or running the algorithm. An acquaintance with the formal features of mathematical problem solving is usually seen as a prerequisite for engaging constructively with the more informal activities of mathematics research, which involves scratch work, serendipity, and various human factors. This dichotomy suggest a new and important pattern, linked to the 3C+2 model developed in conversations with **RSP** (Figure 30). The pattern could be called *Frontend and Backend*. A similar distinction was proposed by Reuben Hersh (1991), after Goffman (1959). Hersh compares the "*front*" of mathematics to the seating area of a restaurant, and the "*back*" with the kitchen. In this metaphor, the suggestion is that the front is misleadingly neat and the back realistically messy. This corresponds with Thurston's perspective on written mathematics [§4.2, p. 49].

However, rather than focusing on degrees of formality, the *Frontend and Backend* pattern is perhaps best understood in relation to the *Newcomer* pattern. Typically one will not expect the user of a system to know how to, or be motivated to, work with backend features until they have mastered at least some of the frontend features. For example, you have to eat before you can cook, and it would be rare to find an auto mechanic who did not know how to drive. David Cavallo (2000B) wrote about an "*engine culture*" in rural Thailand in which structurally open systems brought aspects of what residents of more modernized regions think of as "backend" features into the "frontend". Thai farmers routinely move one engine between different pieces of farm machinery, for

example. In mathematics, textbook problems are stated in terms of a certain vocabulary, but for all but the simplest problems, knowing the definitions of the terms is not enough to solve the problem. One needs to understand how the ideas fit together. The *Frontend and Backend* pattern clearly lends itself to standard service provision, as, for example, in Open University courses. It can also be part of paragogical activity, as in **RSP**'s proposal to focus energy on supporting individual users, by helping them develop a high-quality sub-site on their topic of interest [RSP8], [RSP10]. This would simultaneously inform the development of various replicable backend features, and help raise the profile of the site as a whole. The pattern is in this way associated with *A Specific Project* and with the *Roles* pattern. *Frontend and Backend* also connects with the *instrumental* and *interactional* modes of engagement discussed in Chapter 7 [§7.2, pp. 94-95]. If you only want to know the answer, or if you wish to avail yourself of a service that can be rendered in a transactional manner, then you can usually stay at the level of the frontend. If you want to understand what's really going on – or if the service can only be rendered interactionally – then you may have to take a look at what's happening in the backend.

## Spanning Set

This pattern connects with **BB**'s thoughts about using topic articles as a map to the content. You may be able to get what you need without digging – but if you do need to dig, it would be very good to get some indication about which direction to dig in. **BB**'s proposal is similar to an idea described in a semantic web context by Katerina Tzompanaki and Martin Doerr (Tzompanaki & Doerr, 2012A, 2012B).

> *In order to overcome the problem of effective searching* [...] *we propose a querying system for semantic networks based on a few fundamental categories (FCs) and (binary) relationships (FRs). These categories are "base classes" covering the domain, and the relationships are deductions from complex path expressions of all sorts of deep relationships and documentation alternatives in a much richer and more specialized semantic network*[.] (Tzompanaki & Doerr, 2012A)

**BB**'s proposal and the interpretation above point to another pattern, related to but distinct from *Frontend and Backend* – the *Spanning Set*. This may be made up of media objects, or of people. In a standard course model, there is one central node, the teacher, who is largely responsible for all course communication. In OU mathematics courses, this model is scaled up; as **BA** indicated,

> [E]*veryone's allocated a course tutor, who might take on just a half-dozen students – so, they're not the overall person in charge of the course, by any means.* [BA6]

Given sufficient demand, the model could be extended to courses or learning platforms

that are arbitrarily large. For instance, this can be done via a master/apprentice system, in which every apprentice is supervised by a certified master. In the typical online Q&A context, these roles are made distributed, and are better modeled by power laws than by formal gradations.

There are intermediate modes. Stephen Downes is correct that a *star graph* $S_k$ (with $k$ students and 1 teacher) does not work well for learning purposes as $k$ grows.* This is a problem of *Carrying Capacity*. However, a $(j, k)$*-biregular graph*, in which every one of $j$ teachers has $k$ students does not have the same problems if $j$ grows along with $k$. A given discussion group of 100 persons that is divided according to the so-called 90/9/1 rule,† would have 90 lurkers, 9 contributors, and 1 content creator. This is what one might observe, for example, in a classroom with a lecture format. The system would presumably shift if it was broken up into smaller groups. For example, suppose each of the 9 contributors leads a small discussion group of 10 persons. The worst case scenario for at this scale is a "9/0/1 rule" but it seems reasonable to expect "8/1/1" or better.

David Cavallo wrote that in his research with Logo, the goal "*is not only* [working] *for the benefit of others*" but also "*to help the developers themselves formalize and make robust the knowledge required to accomplish the project*" (Cavallo, 2000B, p. 127) This principle can be applied at every level in paragogical projects. It may be embodied by the members (or elements) of a Spanning Set. For instance, it would be worthwhile to make the PlanetMath encyclopedia into an effective Spanning Set for a wide range of mathematics questions. In some cases, a Spanning Set already exists in a concealed form. For instance, the PlanetMath project has an existing *Heartbeat*, in the form of quarterly board meetings and weekly informal discussions. This functions as a Spanning Set for staying up to date in the project's activities – for those who attend. It should be possible to scale up the informal discussions somewhat simply by advertising them more widely; as **RSP** remarked:

> [T]*hings like restarting the community discussions, for example our having our Thursday things once a month, with more people showing up. None of these really require us to build any new software or do anything too time consuming.* [RSP12]

Similarly, "Previews" of features-in-development would help illustrate existing priorities to people who may want to know what's coming next, and how they can get involved.‡ These examples show how in many cases the idea of a Spanning Set can refine the pattern for *Creating a Guide*. Making it clear what people *can* do will help keep them from having to go too far out of their way when it comes time to decide what *to* do.

*http://www.slideshare.net/Downes/the-connectivist-learning-environment (Slide 6)
†http://www.wikipatterns.com/display/wikipatterns/90-9-1+Theory
‡https://github.com/holtzermann17/planetmath-docs/issues?labels=PREVIEW

## Minimum Viable Project

The Minimum Viable Product approach to software development is about putting something out there to see if the customer bites (Ries, 2011). Another approach, related to the pattern just discussed, is to make it clear what people can do with what's there and see if they engage. We might call this the *Minimum Viable Project*, an adjunct to the *Roadmap* pattern, and a new interpretation of the earlier pattern *A Specific Project*.

What makes a project "viable"? Beer (1981) emphasizes the feature of recursivity: viable systems are typically made up of smaller viable systems. This again suggests that one way to strengthen the PlanetMath project as a whole would be to focus on support for individual projects. The front page of the website could be redesigned so that the top-level view of the site is *project focused*. Instead of collecting all of the posts from across the site – or even all of the threads from across the site – the front page could collect succinct summary information on recently active projects, and list the number of active posts in each, after the model of StackExchange questions or Slashdot stories. For instance, each Mathematics Subject Classification could be designated as a "sub-project" and there could be many other cross-cutting or smaller-scale projects. This points to an important syntactic modification to Table 10 [§5.2, p. 74] – introducing the *project* as a container for other productions. Rather than having all of the chains show up in one container, each project would present a distinct context for constructions. Using the grammatical metaphor, relationships between projects would be similar to anaphora as a relationship between sentences.

A viable system is always "*organized with respect to what it is not*" (Deacon, 2014). In other words, each project will have a *Frontend and Backend*. This suggests that projects model their outcomes to some degree of fidelity, and that they are made viable by features that connect to the motivations and ambitions of potential participants. In addition to the basic feature of recursivity, Beer's (1981) dimensions of viable systems (which can be applied to study any project – or pattern) are:

1. **The elements that produce the system.**
2. **Information channels for communication between the elements.**
3. **Internal control/regulation systems.**
4. **Sensors and regulators along the outer boundary.**
5. **Policy decisions governing the system as a whole.**

***Frontend and Backend: A separation of concerns***

**Definition**: The part of the system users see is often connected to a "backend" that they don't interact with as much. Working with the frontend is more formal and rulebound, while working with the backend is relatively informal.

**Problem**: The idea of Frontend and Backend is related to the *Newcomer* pattern: typically one will not expect the user of a system to know how to, or to be motivated to, work with any of the backend features of a system until they have mastered at many of the frontend features. "Users" tend to expect a level of service provision – and *new* users often require some hand-holding.

**Solution**: The pattern of Frontend and Backend lends itself to standard service provision and transactional models of exchange. However, it can also be part of commons based peer production, if people are willing to put in the work. Sophisticated oldtimers can focus energy on supporting individual newcomers by helping them develop a high-quality suite of content and tools focused on their topic of interest. Reflection on this process can inform the development of the system backend. In addition, the new content can help to raise the profile of the site as a whole. This pattern is associated with focusing on *A Specific Project* (following the interests of the newcomers) and with the *Roles* pattern, since it requires a committed and knowledgeable mentor who learn from working with newcomers.

**Challenges**: Simultaneously mentoring newcomers and working on system features constitutes a major commitment. If this work can be spread out among several volunteers – or possibly paid staff – this could have some advantages. Depending on the nature of the process, providing a single point of contact for the user may still be the most straightforward.

**Example**: David Cavallo wrote about an *engine culture* in rural Thailand, in which structurally open systems made tinkering with the "backend" features of internal combustion engines a part of daily life. Cavallo observed that people who were familiar with tinkering with engines tended to be able to learn how to tinker with software, suggesting that there are some common underlying informal reasoning skills.

**What's Next**: At PlanetMath, we have an *open engine*, but not yet an *open engine culture*. Along with continued work to serve and understand specific users by focusing on specific and concrete use cases, we will want to build pathways for meaningful user involvement in work with the software system.

**Table 21:** *The* Frontend and Backend *pattern*

***Spanning Set: Follow the paths in the grass***

**Definition**: With a well-constructed information access system, you may be able to get what you need without digging. If you do need to dig, it is very good to get some indication about which direction to dig in. At the level of content, this may be achieved by using high-level "topic articles" as a narrative map to the content. In general, the Spanning Set may include people as well as less dynamic media objects. A spanning set is comprised of a set of fundamental actions (e.g. asking a question) and fundamental relationships between resources.

**Problem**: People need to know what can be done with a given resource, and this isn't always obvious. Relying on a single knowledgeable guru figure isn't always possible.

**Solution**: A system's features, categories, and relations can be comprised of many different kinds of components, and it needs to be organized in appropriate ways. For example, a "start menu" or pop-up window showing keyboard shortcuts can show what can be done with a given tool; a schedule of office hours can show people when and where they can find help; and topic-level narrative guides to content can show people where to read more.

**Examples**: *Creating a Guide* and *Heartbeat* both produce spanning sets. In a standard course model, there is one central node, the teacher, who is responsible for all teaching and course communication and answering student questions. In large courses, this model is sometimes scaled up through tutors and TAs. In a large MOOC, a "spanning set" of peer tutors could help give everyone personal attention. In Q&A sites, these roles are disaggregated and distributed.

**Challenges**: From the perspective of *Frontend and Backend*, principles and features are visible as part of a system's "frontend" – but the spanning set of relevant behaviors tends to be emergent. If any individual tries to span too much, they will get spread too thin; see *Carrying Capacity*.

**What's Next**: As a project with an encyclopedic component, PlanetMath can be used to span and organize a significantly larger body of existing material. The high-level idea is that of a "cross-index" to the mathematics literature. Initial prototypes should be built for smaller domains (for example, Calculus or Mathematical Biology).

**Table 22:** *The* Spanning Set *pattern*

***Minimum Viable Project: Hard work for no pay? Where do I sign up?***

**Definition**: The Minimum Viable Product approach to software development is about putting something out there to see if the customer bites. Another approach, building on the notion of a *Spanning Set*, is to make it clear to *Newcomers* what they can do with what's there, and see if they engage. Whereas a *Roadmap* lays out a plan for *A Specific Project* that could go nowhere, a Minimum Viable Project is one that people will actually engage with.

**Problem**: Minimum Viable Projects may have a some features in common, and these patterns could be studied using the features of viable systems in general. However, the proof is in the pudding, so we especially need a methodology for trying things out.

**Solution**: The proposal is to take a project-oriented view on everything. Understand actions and artifacts as being embedded within projects, modeling projects in terms of user experience and system features (per *Frontend and Backend*). Where possible, project updates can be modeled with a language of fundamental actions (per *Spanning Set*). Projects themselves model their outcomes to some degree of fidelity, and they are most likely to be viable if they have features that connect to the motivations and ambitions of potential participants. The practical side of this proposed solution will be to build systems that can express all of these aspects of projects, and study what works in practice.

**Challenges**: There are many different kinds of projects, operating at different scales, and with different kinds of actors involved. Each project will have its own *Frontend and Backend*, which will define interfaces to other projects, and this could get complicated.

**What's Next**: The front page of the website could be redesigned so that the top-level view of the site is project focused. That is, instead of collecting all of the posts from across the site – or even all of the threads from across the site – the front page would collect succinct summary information on recently active projects.

**Table 23:** *The* Minimum Viable Project *pattern*

## 9.4 Translating to a pattern-based design

The design dimensions that guided implementation in Chapter 8 can be reassessed in light of this study. Further work on the project can be translated into to a pattern-based design, centered on the three patterns summarized in Tables 21, 22, and 23, where connections to other patterns from Rheingold et al. (2014) are indicated.

Rather than describing the implementation project in terms of (1) *a special-purpose tool customized to suit the challenges of learning mathematics* that would support (2) *an active and interactive process centered on problem solving*, these ideas could instead be described using the idea of a *Spanning Set*. The tool is special purpose just insofar as it makes the actions that are relevant to learning mathematics sufficiently clear. For instance, the notion of a *Spanning Set* could be deployed to progressively clarify the complex notion of problem solving by focusing on the fundamental categories and relationships that make the search for solutions effective. At the heuristic level, a pattern catalog could be used for this purpose, and applied to reassess the implementation.

Rather than focusing on the issue of (3) *motivating users to contribute problems and discuss solutions*, we could instead focus on how users become involved in *A Specific Project*, thinking of these as typically *Minimum Viable Projects*. Minimal, first, in sense that they are often based on relatively small-scale, one-off experiments that will not need to be repeated once the project has run to completion. And second, in the basically microeconomic sense that users will only be attracted to projects which are seen to convey benefits in marginal utility. The issues of "trust" and "credit" that were highlighted in the initial focus group are two small but important facets of what might make a project viable. Users also need examples to follow that show how they can get involved. Question-answering – which comprised a key set of features that were intended to motivate users to contribute problems – addresses primarily informational needs. This sort of feature does little to address interactional needs, which robust support projects would accomplish. Groups on their own do not seem to provide sufficient structure. All of the dimensions of viable systems (Beer, 1981) should be given due consideration in further design and implementation work.

Finally, individual projects – including the software implementation project – can (4) *support mathematical meaning making* – but the details of just how this is supported would be made more clear using the *Frontend and Backend* pattern. In particular, *Frontend and Backend* is directly associated with (5) *support for behavior tracking and semantic modeling*. If PlanetMath users were more easily able to cross between the mathematical and computational ways of thinking, including work with data and analytics (as in Chapter 6), we would be well on our way to building an "open engine culture."

## 9.5 Summary

The layout of Figure 29 is reminiscent of the model of group informatics (Figure 5 [§4.3, p. 55]) from Goggins et al. (2013), but it adds more flexibility to describe the emergent category these earlier authors called "*contextualized interactions*." The approach provides a "meta-model" that looks for emergent order relative to given boundary conditions. As new structure forms, it becomes part of the boundary conditions for future iterations.

Figure 29 explicitly suggests an "open world" model in the sense that projects can be hooked together. The design pattern methodology offers a practical form for a theory of peer learning and peer production in distributed projects. Caroline Wagner (2008) highlights "*emergence*" as one of the key features behind the rise of a "*new invisible college*" among globally networked scientists. The approach seems to be a general and replicable method for doing ongoing emergent design in a peer produced peer learning context.

In the current setting, three design patterns, *Frontend and Backend*, *Spanning Set*, and *Minimum Viable Project* summarize the key take-away from user engagement with PlanetMath/Planetary. In particular, the "What's Next" sections of these patterns suggest practical steps that directly address the main research question [§1.2, p. 5]:

- At PlanetMath, we have an *open engine*, but not yet an *open engine culture*. Along with continued work to serve and understand specific users by focusing on specific and concrete use cases, we will want to build pathways for meaningful user involvement in work with the software system.
- As a project with an encyclopedic component, PlanetMath can be used to span and organize a significantly larger body of existing material. The high-level idea is that of a "cross-index" to the mathematics literature. Initial prototypes should be built for smaller domains (for example, Calculus or Mathematical Biology).
- The front page of the website could be redesigned so that the top-level view of the site is project focused. That is, instead of collecting all of the posts from across the site – or even all of the threads from across the site – the front page would collect succinct summary information on recently active projects.

# 10 Discussion

## 10.1 Recapitulation

I have argued that mathematics is a socially shared, symbolically – and technologically – mediated cultural activity. In light of this, I asked: Could learning mathematics become more like the learning that happens in the course of developing free/open source software? While theories of pedagogy can help us understand conditioning practices, they fail to model the productive, explicating, way in which learning unfolds in online peer production communities, which is described here with a theory of *paragogy*. The paragogy framework I introduced suggested a way to organize and survey prior work in adjacent fields, and formed an organizational tool useful across several scales throughout the work. The core research methodology aimed to build a "thick description" of mathematical social life in the form of a working system for peer produced peer learning. The first study focused on understanding the effects of peer support on learning in the course of encyclopedia authoring activities and forum discussions on the original PlanetMath, documenting the catalysis of learning outcomes. The second study employed qualitative methods to build a richer picture of the stakeholders in a peer learning environment for mathematics, using the paragogy framework to derive directions for change. Informed by this work, PlanetMath's software system was rebuilt along the lines specified by the paragogy model, augmenting the collaborative authoring environment with tools for peer supported problem solving. Qualitative findings from a user study after the public release of the new system demonstrate the feasibility of the paragogical approach, while suggesting a range of improvements to the software that will be needed for the project to successfully scale up – in particular, the further re-structuring of PlanetMath's activities into "sub-projects."

| Context | Feedback | Quality | Structure | Heuristic |
|---|---|---|---|---|
| $A \leftarrow A$<br>$A \overset{\ell}{\leftarrow} A$<br>$X \hookrightarrow \mathbb{X}$ | $X \leftarrow T$<br>$S \leftarrow R$<br>$\mathbb{X} \rightarrow \mathbb{X}^\sharp$ | $X \leftarrow Q$<br>$A \leftarrow C$<br>$X \rightarrow X'$<br>$\mathbb{X} \vDash \mathbb{X}^\star$ | $A \leftarrow P \leftarrow \mathbb{J} \leftarrow S$<br>$L \leftarrow A, P$<br>$M \leftarrow A$<br>$Q \leftarrow A$ | $G \hookleftarrow U$<br>$S \hookleftarrow H$<br>$Q, T \rightharpoonup C, W, P$<br>$G \hookrightarrow \mathbb{E}$ |
| $A$ article<br>$\ell$ link<br>$X$ object | $T$ post<br>$S$ solution<br>$R$ review | $Q$ question<br>$C$ correction | $P$ problem<br>$L$ collection<br>$M$ classification | $G$ group<br>$U$ user<br>$W$ request<br>$H$ heuristic |
| $\mathbb{X}$ project | $\sharp$ update | $'$ fork<br>$\star$ outcome | $\mathbb{J}$ conjecture | $\mathbb{E}$ ephemera |

**Table 24:** *Entity relation diagram for PlanetMath 3.0*

## 10.2 PlanetMath 3.0

Table 24 presents a revised version of Table 10 [§5.2, p. 74], which now includes the insights gained in the user evaluation of PlanetMath/Planetary discussed in Chapter 9. Any object, or activity, that takes place within the system should be thought of as being part of some project. This carries over to the "micro" level, so that individual proof steps or computations are to be thought of as small projects. In particular, one important class of "project" is a conjectural solution to a problem, which can generally be thought of as a (fallible) process for arriving at a solution. In the process of working on problems, groups would like to be able to form on an *ad hoc* basis and create ephemeral content, the online equivalent of a chalkboard that can be wiped clean. Articles – and presumably other forms of project-embedded content – should be forkable. Projects should be updatable, with revised goals or with concrete indications of progress.

Projects can be modeled both in terms of user experience and computational features. Project updates can be modeled with a language of fundamental actions. Projects themselves model their outcomes to some degree of fidelity, and such models provide an interface for other projects and for potential participants. Projects are made "viable" by features that connect to the motivations and ambitions of participants.

Chapter 6 showed results from instrumenting the corrections and forum posts and analysis aggregated data from PlanetMath's contributing population over a decade. It would be useful to instrument the entire system in order to be able to offer real-time feedback on learning relevance of interactions across the system. For example, a sufficiently instrumented system might be able to generate charts of progress on solving specific problems on the fly; *cf.* (Schoenfeld, 1987).

## 10.3 Channels for communication between the strata

As per Hill (2011a), it may be most convincing to market the rebooted PlanetMath project as an effort to build a familiar kind of resource in a new way: for example, to create a free/open clone of a familiar set of textbooks, like the Schuams outline series or the Springer Graduate Texts in Mathematics series, by assembling them out of smaller pieces. Another familiar mode of engagement is the mathematics course. Traditional courses could be run in 1-to-1 (or many-to-1) correspondence with the textbook series mentioned above. One can compare the earlier experiment conducted by Robert Milson, in which he used a local instance of Noösphere to run a graduate-level mathematics course (Milson & Krowne, 2005). However, it would be even more propitious if a course adopted PlanetMath itself as an interdisciplinary "learning project." How might this work? For comparison, consider William Thurston's reflections on "Mathematical education" (Thurston, 1990) and Frank Quinn's perspectives on "A science-of-learning approach to mathematics education" (Quinn, 2011).

For Thurston, an ideal teaching environment was a course on "*Geometry and the Imagination*" that was "*taught as a team, shunning lectures and emphasizing group discussions among students*," culminating in a "*geometry fair*." Quinn's ideal instructional context is "*over a thousand hours of one-on-one diagnostic work with students in the Math Emporium at Virginia Tech*", covering material "*designed so that pre-existing learning errors will cause serious difficulty*," with the intention "*to expose these errors so they can be diagnosed and repaired.*"

Appreciating one of these perspectives need not lead us to abandon the other. After all, "*understanding comes on many different levels*" (Jaffe & Quinn, 1994). PlanetMath may find new purpose in fulfilling the request voiced by Thurston (1990) for "*the creation of channels for communication between the strata*." Surely methods that work – and both Quinn and Thurston have strong points here – should be precisely what is shared and developed through such channels.

One of the participants in the initial focus group I ran at The Open University felt that PlanetMath would be more successful if it was more limited in scope.

> *I think that the planetmath project needs to be sharpened in terms of what it is trying to achieve. That may be done by narrowing down the range of users that it is directed to. Considering the fact that lots of financial resources are spent from formal institutions (like the OU) to develop online platforms for delivering their services, it might be quite hard for planetmath to compete with such platforms.*
>
> Anonymous study participant, excerpt from questionnaire response

It would be useful to sharpen the focus of the PlanetMath project, but not by limiting the range of users it serves, topics it covers, or features it offers.

Rather, PlanetMath needs to maintain a sharp focus on the goal of global mathematical literacy if it is to be successful in establishing effective communication between state school and Ivy League; between first and third world; between researchers, teachers, and students at all levels; and between many individual peer learning projects with a specific local scope. The project as a whole may meet with ambivalence from some parties with deeply embedded institutional commitments; *cf.* (Nietzsche, 1878, pp. 51–52). Even so: "[T]*he emergence of novelty requires that objects be at once both in the old system and in that which arises with the new*" (Mead, 1932).

## 10.4 Teaching mathematics the commons-based peer production way

> *To teach is to honor our commitments by repaying what we owe the world for our formation. In short, teaching (and not ethnographic writing) is the other side of participant observation: there cannot be one without the other, and both are indispensable to the practice of anthropology as an art of inquiry.* ***To teach anthropology is to practise anthropology; to practise anthropology is to teach it****.*
>
> (Ingold, 2013, p. 13), original emphasis

One of the outcomes of the Peeragogy project was a general-purpose framework for building peer learning projects (Corneli, Keune, Lyons, & Danoff, 2013). The Peeragogy project is currently using this syllabus as the basis of a new mutual-aid effort for free/open/peer produced projects called the Peeragogy Accelerator.* This framework can also be instantiated as a syllabus for participation in PlanetMath, as a design for "*active learning*"; *cf.* (Freeman et al., 2014). The idea would be to study mathematics and mathematical anthropology together, and in the process, build and share resources with others. Rather than a classroom culture [§1.3, p. 9], teachers and students who look at mathematics this way would find themselves part of a global culture – but it is one "*without any central committee that would have to, or even could, tell the active what their next operations should be*" (Sloterdijk, 2013, p. 402). This underscores a key refinement to the notion of paragogy introduced in Chapter 3 that has been emphasized in the design coming from Chapter 9, namely the need to focus on individual projects. The model in paragogy is not simply one of unbounded productive emergence, but always an emergence with limits. In regard to productivity, its paradigm is the "*thermodynamic work cycle*" (Kauffman, 2001). This is not made sufficiently clear by the five paragogy principles, which rely on the word "context" and the idea of realized motivation to indicate a sense of scope. The quote from Ingold points to a "*cyclic revolution of continuity*" (Baudrillard, 1976, p. 155). Not only do we now have "*a word for teaching and learning at the same time*" (Kelly, 1994, Chapter 5), but a corresponding set of practices.

*http://commonsabundance.net/groups/peeragogy/

**1. Setting the initial challenge and building a framework for accountability among participants.** Identify a challenge that mathematics or computing could help you solve. How have people approached this sort of problem in the past? What role do technology and media play in previous approaches to this sort of problem? What's at stake now? Share your intention and refine it in discussion with others. Be prepared to go through several rounds of revision as you convert the initial idea into a manageable project. Again, you'll want to consider the way other related projects have been organized.

**2. Other people can support you in achieving your goal and make the work more fun too.** Create a project on PlanetMath and find join a few adjacent or related projects. One place to find of related projects is the list of previews of new features in PlanetMath's issue tracker.* Individual MSC classes can also be considered to be content-related projects. Look around on the internet for other relevant projects. What other problems are people working on that are related to the problem you have identified? Can you collaborate with them? What do you need to learn in order to be helpful in these projects? What help are you likely to need?

**3. Solidifying your work plan and learning strategy together with concrete measures for 'success' can move the project forward significantly.** Find or develop collections or sub-projects that specify the reading material, exercises, and prototypes that will help you move toward your goal. Arrange these into a project roadmap, and revise it as you go. Is anyone interested in your intermediate steps? If you run into difficulties with content or software communicate them to the relevant people. Figure out how you can get take part in resolving the difficulties. If you get stuck, write down and share an antipattern describing the problem, and look for a suitable work-around.

**4. Wrap up the project with a critical assessment of progress and directions for future work.** What did you learn? What roadblocks or bottlenecks did you have to overcome? How will you change your approach when you tackle the next problem? Be sure to upload a project summary with pointers to any new code or content. Assessment is easier if it's ongoing: each of the subprojects you've created should be assessed, and some of them should be wrapped up before the main project is complete. Once you've finished a sufficiently large project, give yourself some time off before you start again on something new.

## 10.5 Limitations

This thesis has been restricted to developing the following three claims.

- A theory of peer produced peer learning was developed, which was given the name paragogy.
- A working model of a paragogical approach to mathematics learning has been developed and deployed as PlanetMath/Planetary.
- A user evaluation of the deployed system has been conducted using principles of emergent design, which has resulted in refinements to both model and theory.

Apart from this clearly circumscribed focus, there are other limitations.

The clearest limitation of this work is that it does not include a classroom study or any other longitudinal analysis of learning under the paragogical model. Accordingly, we cannot say: "prior to the introduction of PlanetMath/Planetary and the paragogical approach, student achievement on such-and-such standardized test was at this level, and now it has shifted on average by Y%." Uptake of the new tools on PlanetMath (Figure 1 [§1.5, p. 12]) has not been sufficient to warrant a study of learning using the methods of Chapter 6. From the point of view of studies of this nature, the work presented here is a necessary preliminary.

Planetary is already used as a course support tool – but evaluating the paragogical model in a mathematics learning context rather than a design context would mean running a course like the one outlined in Section 10.4, and for that, more quite a bit more groundwork is needed. Some form of longitudinal evaluation would be important for developing any policy considerations that might follow from the idea of paragogy.

Although conducting a classroom based evaluation seemed, at times, to be almost within reach, simulation work artificial agents has remained entirely out of scope. Many of the ideas discussed in the thesis may be germane to such studies, but this claim has been in no way tested. This means that even though the thesis points toward a computational model of mathematical collaboration, the usefulness of this model remains conjectural; and its empirical validity stands, for now, only on its own apparent self-consistency.

Inviting PlanetMath users to become familiar with the pattern catalog and write new patterns would have been an interesting opportunity to more fully involve participants in co-design, and to learn more about whether design patterns are indeed suitable for use in the mathematics domain. However, when the studies ran, the pattern catalog was still in a relatively formative state, and would not have been as easy to share or

explain as it is now. My methods evolved along with the research project; *cf.* (Tom, 1996). The Peeragogy community provided a "*structurally parallel situation*" (Cavallo, 2000B, p. 141) in which to develop many of the design ideas. However, the engagement between Peeragogy and PlanetMath and has so far remained mostly "virtual" – and as yet there has been no formal study *evaluating* the pattern methods as employed in either setting.

Spending time working on the Peeragogy project and, earlier, running courses at P2PU, gives a range of experiences to compare with, and, accordingly, helps to make the work theoretically robust. However, exploration comes with costs, and in particular, empirical work trades off against hours spent programming. There were certainly many hours spent on the programming effort, along with many trips to meet with collaborators at Jacobs University. Nevertheless, my initial hope to build actionable semantic domain and user models, as represented by the use case in Figure 3 [§2.3, p. 31] was not realized. Much of the relevant *basic* infrastructure now exists, but recommendations and automatic tutoring have to be left for future work.

Regarding recommendations and diagnostics, the model employed in Chapter 6 has yet to be extensively tested in limiting cases, for example, to ascertain how well it works on small data sets. The model does not yet include confidence intervals. The method would have to be modified further to work in real time as part of a recommendation system. Work with simulated learners would be one route to testing and improving the model, *e.g.* informed by (McCalla & Champaign, 2013).

To ground paragogy as a scientific theory and not just an "approach," it will be necessary to make it falsifiable. This may be most easily achieved by working with a collection of paragogical "micro-theories" presumably based on collections of design patterns (Corneli, 2014), but the precise method for doing the testing is left for future work. Without further development at this level, and without further empirical studies of problem solving, the idea that the work discussed here could add insights into social problem solving beyond those developed, for example, in (Pease & Martin, 2012), remains conjectural – although it does offer a new approach.

To summarize: the primary methods employed in this thesis were statistical analysis of historical data, qualitative field work, implementation of a software system for collaboration, and emergent design. As applied, each of these has certain limitations which might be addressed through further work. However, the greatest limitations of the work might be addressed with methods that were not employed in the thesis, in particular methods coming from artificial intelligence. These were explicitly set aside in order to pursue what I see as groundwork. I genuinely hope that the ideas will be taken forward by others. I outline my own planned contributions below.

## 10.6 Future work

**In computer science:** In order to build a computational study following from this work, one would need: a reasonably large computer-accessible body of mathematical content (perhaps along the lines of my earlier work in mathematical knowledge representation [§2.1, p. 22]), along with computational agents that are able to navigate the relevant mathematical structures (as outlined Table 24), able to apply "standard" (Table 4 [§4.2, p. 53]) and "social" problem solving heuristics (Table 13 [§5.2, p. 79]), sufficiently metacognitively aware [§1.2, p. 9] as to be able to set and solve problems by generating and applying new heuristics in the form of design patterns (Table 11 [§5.2, p. 76]) – and ideally able to annotate, reflect on, diagnose, and extend the overall process (see Appendix F). Appendix B contains two examples that have been worked out in a preliminary manner – but more work is needed in order to fully describe a mathematical proof or exposition as a social-computational processes. To further develop examples of this nature, it would be useful to thoroughly reassess the five dimensions of change described in Chapter 3 – for example, are they really the right ones for dealing with mathematics at the micro level? – and to develop a catalog of design patterns expressly devoted to proofs, problem solving (Posamentier & Schulz, 1996), and problem posing (Brown & Walter, 2004). In particular, the notion of problem posing gets at the essence of pattern-making: we will need patterns about patterns in order to obtain the kind of closure alluded to in the introductory comments on the pattern analysis in Chapter 9 [§9.3, p. 149]. The plan would then be to implement a system that can work with these patterns using a society-of-mind style architecture (Minsky, 1988). Related work using artificial agents in a narrative reasoning context was discussed in (Singh & Minsky, 2005), with a detailed development in (Singh, 2005), drawing on the earlier notion a computational critic (Sussman, 1973). This work emphasized reflexive aspects of mentation and multiple ways to think about problems (Minsky, 2007).

**In humanistic computing:** PlanetMath should at least provide a more satisfactory learning platform for beginning mathematics students than Wikipedia. The idea that peer supported online learning might also be "better" than standard classroom participation, in the sense of Bloom's 2-sigma problem,* does not require a great stretch of the imagination. What it does require is the opportunity to carry out the necessary research.† David Smith (2002) writes that "*learning mathematics is, first of all, learning,*

*Students tutored one-to-one were observed to perform *two standard deviations better* than students who received conventional instruction; Bloom (1984) asks if there are methods of group instruction that can be as effective as individual tutoring.

†As motivation, one can compare results from studies of the JUMP math project from Mary Jane Moreau, and Tracy Solomon et al. (forthcoming); see (Mighton, 2014) and the overview in "A Better Way to Teach Math", David Bornstein, April 18, 2011, The New York Times, HTTP://OPINIONATOR.BLOGS.NYTIMES.COM/2011/04/18/A-BETTER-WAY-TO-TEACH-MATH/

*and only secondarily about mathematics*." It is not necessary to wait for the results of further studies of learning on PlanetMath to pursue related work in other sectors (Corneli et al., 2014).

**Focusing on content:** PlanetMath now hosts, on a demonstration basis, a copy of a recently published free/open textbook on Homotopy Type Theory (Univalent Foundations Program, 2013), and a retrodigitized 20th century calculus book (Davis & Brenke, 1912) both of which have been assembled using interconnected collections.* The sections of these book are discussable, the problems from the book are available to be solved, and some solutions have been added. The contents are presented in MathML, which brings with it the possibility of deep linking and expression-level annotations. The reader's interface could certainly be improved, and integration with Git should be added – after the manner of the deployment of Planetary on MathHub.info (Iancu et al., 2014) and as preferred by the original authors of the HoTT textbook.† And, of course, more books should added as well. Digital libraries are an active area of interest for the mathematics community (National Research Council, 2014). Planetary may prove to be quite useful in this distributed effort, since the system can be straightforwardly adapted to support disparate use cases. Naturally, the recent National Research Council report focuses on research-level applications, but as outlined in Section 10.3, PlanetMath can work on making this effort relevant to stakeholders and constituencies they see as secondary, including "*students below the advanced graduate student level*" and "*researchers outside of mathematics*" (National Research Council, 2014). The National Research Council report expresses disinterest in the large-scale retrodigitization of public domain materials – but, even so, considerable progress could can be made with off-the-shelf OCR tools and small amounts of funding to support a crowdsourcing approach. David Cavallo's focus on finding "*bridges to new content*" (Cavallo, 2000a) is particularly interesting at large scales. Rather than viewing Wikipedia, math.stackexchange.com, and MathOverflow as competitors, they could seen as potential collaborators and contributors to the same commons resource (they all use the same CC-By-SA license as PlanetMath). Perhaps PlanetMath's best application at the moment is not to generate new content, but to focus on organizing and integrating content from these sources and from the public domain.‡

**Hacking the good hack:** Having laid considerable groundwork for the use case outlined in Figure 3 [§2.3, p. 31] both in this thesis and in related work (Mulholland et al., 2013), it is worthwhile to finish the job and add a suitable recommender system to PlanetMath, particularly in connection with the content-focused work outlined above.

*http://planetmath.org/users/pmbookproject

†https://github.com/HoTT

‡http://meta.wikimedia.org/wiki/Grants:IEG/PlanetMath_Books_Project

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# A DIY Math: Course description and post mortem

**COURSE TITLE:**

DIY Math

**COURSE TWEET (200 CHARACTERS OR LESS):**

This course is designed to build independent study and peer-support skills for mathematics learners at all levels. It will require both self-directedness and active participation.

**FACILITATOR:**

Joe Corneli (Board Member at PlanetMath.org; PhD student at the Knowledge Media Institute, The Open University, UK; `HTTP://METAMESO.ORG/~JOE`)

**COURSE DESCRIPTION (NO MORE THAN 500 WORDS):**

This is a mathematics course open to all topics and levels (unless participants come up with a compelling reason to focus during the course). The main "pedagogical" reasons for such radical openness are:

(1) To give people one extra reason to take initiative in their own learning. One might say "I've always wanted to learn about *P* vs *NP*", another might say "I've always wanted to learn precalculus" – both are overcoming a mental and emotional hurdle. I'm not suggesting that it's the *same* hurdle, but there are probably some similarities. In the same way that a writer's workshop can serve writers with different interests, this "DIY Math" workshop can serve mathematics learners with different interests by providing a place to talk about their challenges and successes.

(2) To help people develop skills at co- or peer-to-peer teaching. Ultimately people have to do their own homework exercises and so on, but one can also learn a lot in mathemat-

ics by helping others. My "teaching philosophy" is that there are no stupid questions, even if there are a lot of dumb answers. Let's see if we can get better at answering questions together.

There is no official text book, but participants may want to take a look at "How to solve it" (or other books) by George Polya. Participants will get access to a new "beta" grade libre software platform being developed to support mathematical communication; we'll also have a mailing list available as a fall-back mechanism (and if you're interested in the course, please sign up for the list now to talk about your ideas or any questions you have about the course).

**PREREQUISITES:**

The willingness to learn. Also, you'll need to come up with some realistic goals for a 6-week mathematics course!

**AUDIENCE:**

I hope the participants will want to both *ask* and *answer* questions. Collectively I hope that they will have varied backgrounds and aspirations.

**EXPERIENCE:**

I have a bachelor's degree in mathematics from New College of Florida (2002), and I did a couple years of math grad school before switching to a more "DIY" approach. My current PhD project has to do with "Crowdsourcing a Personal Learning Environment for Mathematics" and this course can be considered to be an informal dry run.

**POST MORTEM:**

Dear DIY Math participants:

I feel like this course is effectively over (and maybe was even "over before it began," though I think none of us saw that). Rather than trying to squeeze more out of the course than it has to offer, I'd like to share my reflections on "what went wrong."

This mailing list, of course, continues to be a place to discuss "DIY Math" in general, and perhaps the course will rise again sometime. I'd like that, and I am not discouraged by what I've learned so far. Further, I wish to salute all of you for your efforts to be self-directed learners. But I think we need a different approach.

The following is my own post-mortem analysis of the course. If you would like to add to it please do. If you feel like the course is not actually over, now would be a good time to speak up about that, too.

My assessment of the course was that participants knew *more or less* what they wanted to focus on, but they didn't exactly know "how" to start, and they didn't particularly want to discuss questions about "how to do things."

Unfortunately, "how questions" were supposed to be the central focus of this *discussion-based* "DIY" course.

At this point, I can see several profound shortcomings in the way I designed the course, not the least being that the implied tension between "doing it yourself" and "discussing it with others" was not given enough attention up front. I had hoped this would be a productive tension or "dialectic," but it doesn't seem to have turned out that way. There is a certain degree of commitment required to work with(in) such a dialectic, and no one who signed up for the course seemed particularly interested in that. Rather, applicants for the most part "just wanted to learn math." I did not do a very good job explaining why I think working with the diy/p2p dialectic is *how* to learn. Moreover, my personal philosophies aside, I simply didn't do enough to get people talking in the course, partly because I rather blindly assumed that the desire to discuss would be shared by all participants. Or, more precisely, I assumed that those who wanted to discuss the most would carry the conversation forward.

Aside from these issues, which I take responsibility for, some of the shortcomings of the course itself point to design considerations that would best apply at a higher level; something like "P2PU best practices." One could say that course-level and P2PU-level issues have some degree of "trade-off" between them.

As one very simple example: If P2PU was taking an ongoing survey of the "wished for" course topics (to be tagged and voted up or down by site visitors), I would presumably have been able to create a mathematics course tailored to the interests of pre-self-selected participants, instead of fashioning a course that was supposed to serve all mathematical interests simultaneously.

As it turned out, I learned at the beginning of the DIY Math course that the interests of the applicants were approximately equally distributed between (1) General interest, e.g. learning how to learn mathematics; (2) Technical interest, e.g mathematics for computer programming; (3) Mainstream mathematics, e.g. precalculus and calculus; and (4) Advanced mathematics, e.g. probability and statistics.

One obvious thing to do "next time" would be to simply run a "standard" mathematics course for each of these four segments. This would be a lot of work to set up, but by no means impossible: perhaps I could find some other people who would want to co-facilitate these courses. My guess is that with significant work and a design that draws on P2PU's collective experience of how to teach (and how *not* to teach) "academic subjects" in a peer-to-peer setting, these courses would all be reasonably successful.

I'll also note that it would be possible to focus on "how questions" in any of these courses (perhaps especially obvious in the case of the first group, since one could copy an existing course on mathematical foundations). Nevertheless, I think the course would most likely need *a lot* more structure in one form or another. There are undoubtedly several different ways to go about structuring something like this, but my guess is that *either* we'd need a more step-by-step curriculum crafted to ease people into thinking about "how they learn," *or else* we'd need to know in advance that participants were really committed to asking and answering "how" questions.

Regarding this issue of "commitment" – which in fact seems quite important independent of the course's "peertagogic" style – in the latest P2PU community call, Alison Jean Cole again reminded me about the idea of setting up a *social contract* for the course.

I said to her that this seemed like another issue that could potentially benefit from support coming from a level above individual courses, e.g. by involving an "course mentor" who would help facilitate the creation of a social contract for the course. This could be someone who had previously run a "successful" course.

Such an idea might go along with re-thinking and nurturing the "social contracts" implicit to the P2PU structure itself (*i.e.* we would ask ourselves what we share as a broader learning community). Some other grassroots organizing styles are very strong on this aspect, both those that are "radicalized" and others that are more politically moderate.

To sum up: I think a course built around "how to learn mathematics" is a great idea. Trying to involve mathematics learners at all levels in discussing this question is also probably a good idea, but a course may not be the best way to have such a discussion. Certainly people need to know what they are getting into in any case.

To conclude: Hopefully we can find the "right" group to make progress on these issues in the future. This course has brought up some of the core problems that apply to peertagogy. We should cut ourselves some slack, because this field is so new that I had to come up with a neologism here and now to describe it! If that's to be a lasting positive outcome of the course, I suppose we should offer a more formal version, "paragogy" (not to be too heavily confused with the etymologically related "paragoge"). Whatever we decide to call it, the key point of DIY Math as a genre is to do away with the artificial distinction between pedagogues and learners. This is a power struggle par excellence, easy enough to mock perhaps, but one I think we should not ignore.

Thank you for your attention,
Joe

# B A paragogy-inspired coding for two mathematical proofs

This appendix applies the paragogy framework to a few short mathematical texts, to help show how the ideas about peer learning developed in the early sections of this thesis relate to the mathematics domain.

As example texts, I've chosen two mathematical proofs where "the obvious thing works", as presented by Timothy Gowers in a recent talk on *Modelling the mathematical discovery process* (Gowers & Ganesalingam, 2012). The detailed small steps presented in this talk give concrete examples of Gowers's notion of the "quantum of progress" as described in Chapter 4. The individual steps are coded as (1), (2), (3), (4), or (5), by matching into this scheme:

| | *Nowak* | *Paragogy* | *Lakatos* | *Peirce* |
|---|---|---|---|---|
| (1) | kin selection | context | concepts | – |
| (2) | direct reciprocity | feedback | examples | induction |
| (3) | indirect reciprocity | quality | conjectures | abduction |
| (4) | spatial selection | structure | proof | deduction |
| (5) | group selection | heuristic | other | – |
| | "social theories" | | "logical theories" | |

This discussion can be connected to the Minskian heuristics for problem solving and the design patterns that were introduced in Chapter 4 and Chapter 5. This leads to the short outline of heuristics for social problem solving, which appears at the end of the appendix. This can be taken as a very preliminary sketch of potential "implementation details" for an agent-based system that would use design patterns to do proofs or tutoring in mathematics.

**To prove: A compact subset of a Hausdorff topological space is closed.**

*First: maybe it isn't so important to know the meaning of the terms.*

[(5) - This is an off-handed remark that is almost certainly not true, but it does suggest an important heuristic: restricting the unknown to whatever it is that we *do* know.]

*Second: This is not true in non-Hausdorff topological spaces.*

[(1) - This looks like an aside, or potentially an extra lemma to prove; however, the purpose it serves in this context is to ensure that we have the *concepts* right.]

*The first step is slightly non-obvious. We'll try to prove that $A^C$ is open.*

[(1), (4) - This is an equivalent definition of the problem statement, however, it's "slightly non-obvious" why this is being used as a proof step. Later in the proof, we'll see why – for an expert who has a feel for the structure of proofs like this, this choice of phrasing is relatively obvious.]

*So, given a point $y \notin A$, we need an $\epsilon$-ball around $y$ that doesn't intersect $A$.*

[(3) - The idea will be to map this hypothetical $y$ to some conjectural but concrete feature, the open ball that doesn't intersect $A$.]

*Take a cue. You have a library of things that can be used in the proof. Use an equivalent statement.*

[(5) - Again, we're reminded to restrict the unknown to what we know.]

*Let $y \notin A$.*

[(2) - This introduces the example we need to operationalize our main conjecture so far.]

*Now expand something else. Because of* **compactness**, *every open cover of A has a finite subcover.*

[(1) - Mechanical expansion of the definition.]

*Finding an* **open cover** *of A would mean finding a collection of sets $\cup$ such that $\forall x \in A \Rightarrow \exists U(x) \in \cup : x \in U(x)$, and U open.*

[(1) - Mechanical expansion of the definition.]

*Because of the* **Hausdorff property**, *$\forall u \neq v$ are such that there are disjoint open sets U and V such that $u \in U$, $v \in V$.*

[(1) - Mechanical expansion of the definition.]

*We now know that "$U(x, y)$" containing $x$ and not $y$ exists because of the Hausdorff property (and because $x \neq y$). Symmetrically for $V(x, y)$. Let $\cup$ be the smallest set of*

$\{U(x,y) \mid x \in A\}$. *Similarly, let* $\mathbb{V} = \{V(x,y) \mid x \in A\}$.

[(3), (2) - We're mapping the hypothetical $y$ to some additional constructs, $\mathbb{U}$ and $\mathbb{V}$, building a more complete model of the space. As part of this complex gesture, the placeholder $x \in A$ is introduced as a example.]

*Let* $\mathbb{F} = \{U(x_1,y), \ldots, U(x_n,y)\}$ *be a finite subcover of* $\mathbb{U}$.

[(4) - We now apply the definition we expanded earlier.]

$w \in A$ *implies* $w \in U(x_i,y)$ *and* $w \notin V(x_i,y)$.

[(4), (2) - Applying previous definitions to move around the unknowns.]

$y \in V \subset \bigcap_{i=1}^{N} V(x_i,y)$. *This would be pulled out of the library.*

[(1), (4) - The fact that finite intersections of open sets are open is a key "concept" that can be applied. We have now positioned the unknown $y$ in an open set that does not intersect $A$. There is also a bit of deduction implied.]

*(All the way through, everything we did was extremely routine.)*

[(5) - This certifies the proof.]

**A 'Magic Leap' problem: compute the 500th digit of** $(\sqrt{2}+\sqrt{3})^{2012}$**.**

*What is the* $500^{th}$ *digit of* $(\sqrt{2}+\sqrt{3})^{2012}$*?*

[(1) - This is just the statement of the problem.]

*Even this, eventually, a computer will be able to solve.*

[(5) - This is an aside, but it points out that there is something here that is non-obvious – *i.e.* an "unknown".]

*For now, notice that total stuckness can make you do desperate things. Furthermore, knowing the origin of the problem suggests good things to try. The fact that it is set as a problem is a huge clue.*

[(3), (1) - We initially guess that we're "totally stuck", but then think more about the context, trying to move beyond the page to see what might be implied. Note that the *social* context is referred to, rather than the *semantic* context.]

*Can we do this for* $(x+y)$*? For e? Rationals with small denominator?*

[(5) - We have now made a critical step, moving from a "known" blob of text to a statement with a useful unknown in it. We can then fill in things that we have some hope of doing calculations about in place of this unknown – for instance, $(1/2)^{2012}$ or $(1/10)^{2012}$.]

*And how about small perturbations of these? Maybe it is close to a rational?*

[(3) - We now begin to form a conjecture.]

*$m^{th}$ digit of $(\sqrt{2}+\sqrt{3})^n$?*

[(2) - Introduce variables as a generalization of the problem statement. (We can't deal with the conjecture directly, so we'll explore.)]

$(\sqrt{2}+\sqrt{3})^2$*?*

[(2) - Try a particular example. We're now exploring the generalization of the problem.]

$(2+2\sqrt{2}\sqrt{3}+3)$

[(4) - Just computation.]

$(\sqrt{2}+\sqrt{3})^2+(\sqrt{3}-\sqrt{2})^2=10$

[(2), (3) - A somewhat clever idea: we've built an example that we can understand. However, this step is also abductive – we may have found a class of expressions that we can understand in a similar way.]

$(\sqrt{2}+\sqrt{3})^{2012}+(\sqrt{3}-\sqrt{2})^{2012}$ *is an integer!*

[(4) - All the terms with odd powers in the second product also have odd exponents on the factor −1, so they cancel with corresponding terms from the first product.]

*And* $(\sqrt{3}-\sqrt{2})^{2012}$ *is a very small number. Maybe the final answer is "9"?*

[(3) - We refine our earlier conjecture.]

*We need to check whether it's small enough.* $(\sqrt{3}-\sqrt{2})^{2012}<(\frac{1}{2})^{2012}=((\frac{1}{2})^4)^{503}=(\frac{1}{16})^{503}<.1^{503}$*, so we're in luck...*

[(4) - This is important to check! (In fact, we could show that $(\sqrt{3}-\sqrt{2})^{2012}$ is still smaller, but it is not infinitely small.)]

*The answer is indeed 9.*

[(5) - We've confirmed that $(\sqrt{2}+\sqrt{3})^{2012}$ is very close to, but just less than, an integer (by the right about)].

The following page collects the key ideas developed above and connects them with Table 4 [§4.2, p. 53] and Table 13 [§5.2, p. 79]. One can also compare the system design overview from Chapter 8 [§8.2, p. 113], which offers a very different but complementary interpretation of what are at their root the same five dimensions.

(1) One of the things that "*experts do*" is *expand the definition.* Experts also have the ability to appropriately "scope" the problem, to explain why it is feasible or important, and to convey some sense of purpose to others. An expert can find the relevant related concepts, establishing the context of the problem. In some cases, technical definitions can simply be "*pulled out of the library*" – but an expert knows which portion of the library to search in.

(2) "*Polling for ideas*" can help generate a set of examples. If you don't know quite what to do, you can try an example to generate ideas by considering a special case or a particular instance of the problem. In this way, in exchange for modest effort, examples offer feedback on the problem solving process. Pacing oneself and cultivating appropriate diversity among instances comprise an important class of heuristics related to example selection. Within a proof, variables can be introduced as examples, which can have either "global" or "local" significance. In some cases, finding even one relatively comprehensible example can suggest a theory that leads to the solution.

(3) Conjecture has to do with outlining a plan, sighting a path, or "*creating a guide.*" Concerning textbook problems, as "*The fact that it is set as a problem is a huge clue.*" Constructions provide analogies, and tentative explanations provide models. These models can subsequently be evaluated on their fitness-for-purpose. Such evaluations allow a problem solver to progressively build and refine an argument. Helpful conjectures are "rewarded" with further attention – non-helpful conjectures (e.g. "*total stuckness*") are abandoned.

(4) "*Changing the description*" of a construct – by symbol swapping and by using suitable definitions – can help shed to light on a problem. A new definition provides a concise "*wrapper*" for an idea or method. Deductions or proof steps are typically meant to be straightforward applications of the definitions. Although the structure of a proof reads "linearly", the way a problem solver finds the right proof steps is often nonlinear.

(5) A "*roadmap*" is a shared strategic plan that is allowed to change. It often comes with rules about the group of agents who are, in fact, allowed to change it. When formulating a proof, we often begin by restricting ourselves to the parts that we know – leaving little archipelagos of the unknown on our map, which can then be divided up and examined. The relevant operations in proofs are not simply about systematically breaking down the unknown, but also include the skillful introduction of unknowns.

# C Selected quotes from initial fieldwork with stakeholders

Sections C1–C5 contain portions of dialog from the initial focus group with computing post-graduates that is described in Chapter 7. Discussants were **M1**, **M2**, and **M3** (self-identified as "more mathematical") and **L1**, **L2**, and **L3** ("less mathematical"). Sections C6–C15 briefly summarize the key features (setting, speaker, etc.) of comments by professionals that inform the later sections of Chapter 7. Primary discussants were Murdoch James Gabbay (**MJG**) and Alison Pease (**AP**). These discussions were not recorded; brief samples from my notes are presented in Appendix E.

## C1 Dialog: Would you find this sort of tool suitable for use in your work or studies?

**M2.** *OK, so, yeah, the first thing that comes to mind is why would somebody would look for something like that? Especially if you assume they're not aware of this PlanetMath website, right. So this should be an ideal first place for someone to go, online, and look for some help. So, in my case, I'm studying, part time, a degree in maths here, at the Open University, and this – it's an online distance lecture thing – and therefor they provide the platform, you can have some kind of interaction with other students and stuff. So, like, the first point of reference, I would, like, go, in case I encounter problems – so, I'm working through the book, I'm trying to understand a method in Calculus, I have a problem, and I don't understand this – so I go to the forum to see if someone else had the very same problem and see if there's an explanation about that problem, and that would be the first thing I do. And if I don't get an answer there, I'll try to post something on that forum, which is already there. So, all I'm trying to say is that the way I open up is also related to the things I know. So, I wouldn't, you know, go online to look for a forum that I don't know, it might take some*

*time also to familiarize myself also with the whole structure and how it works. and, you know, you need, like, when you concentrate, you're trying to understand something, you need to be, like, on a roll.*

**M1.** *But there's one extra point, because if you're having, say, a problem with Calculus, would you trust the forum? So, do the users have a rating in terms of experience?*

**M2.** *Well, you have the – how do you call, it not regulators.*

**L3.** *Moderators.*

**M2.** *Moderators, yeah.*

**M1.** *But, we didn't mention anything about moderators or central control. Because, say, in Wikipedia or in other wikis you have some sort of moderators and central control just to ensure some sort of quality. Or can they just go and define something and then wait for others to say, well, this is wrong?*

**L1.** *Like, I think the discussion about, like, a course forum versus a more general PlanetMath type forum is an interesting thing to think about, um, because what I would think about, again, not from a math background, is kind of, instructor buy in, I think, like, the benefit of you going to your course forum is that your instructor is there, and your classmates are there, and you're using the same texts. And, again, I'm not knowledgeable about math, but, say if I were to plug in something about long division, there are different ways of teaching and learning long division, so I guess – the "pro" of going to something within the course is that you'll be taught, and you'll know, you'll have similar parameters or similar methodology. And whereas with PlanetMath, I think if I were going there as someone who is a total math newbie, it would almost overwhelming to parse the type of information.*

## C2 Dialog: Describe your learning style

**L3.** *If I approach this thing from, I want to increase my mathematics skills, I don't exactly know where I am, because I stopped doing school courses 10 years ago, so I've gained some and I've lost some. I'm interested in this, and that, and generally very interested, maybe not in this, but this – but I don't search specifically for a topic, but just something that will get me further. And getting me further sort of implies that it's at my skill level, so that I can understand the text. So that requires a little bit of searching around, probably looking at different things. What really would would help me is trust, again, trust in the fact that if I don't understand something, or I'm stuck, it's my own fault, it's not due to an error in the problem or article. So that's where I would start, and then, presumably it would be nice to have something like a map or a histogram or a list or a chart of the area – the charted area, or the uncharted.* [...]

**L3.** *What is it like, you get a question, the square root of 3 for example, and then you answer it – if the question's more complex, what does the answering look like, do you sort of go away, and do your scribblings on some paper, and you find the answer and you post it in, or is it more like, is the problem solving, progressive, in steps?* [...]

**L3.** *So I think that for people like* **L2** *and me who like to do prototyping, and do iterative solutions, and approximate a solution* [...]

## C3 Dialog: How do you currently solve problems?

**M3.** *It's not practical for logic, but when you do some programming, you have to parse some text, you would look if someone else had solved it and use their solution, just to save some time.*

**M2.** *I think what's important is the immediacy of trying to solve it. You're there, you're concentrating. You don't want to lose this concentration, get familiar with a website, that would totally destroy my concentration. The first thing I would do is stick with it.* [...] *And it could be, like, an exercise that you are trying to solve for the assignment that you have to submit, and you wake up with the solution in your mind. So that's one way of solving the problem. Then, you might need to go to the forum and see if someone else had the same problem and – sometimes there are students who have the same problem, and there are a lot of students and they reply and they get it right. Or maybe the moderator comes in the end and says this is how it works and maybe you want to look at that.*

**M1.** *It depends on what you're using the forum for – I use both strategies depending on the problem. Sometimes you just get a generic error message and it's not really clear and you just have the message on that. So you go to Google and you find dozens of posts on forums that will give you the solution almost instantly. But sometimes if you're trying to develop something that's a bit complex, that's not the problem, you're just developing it and it takes time. And you need to just work at it.*

**L1.** *To me it depends a little bit on where I am on in my own lifecycle of research. Like, when I'm starting out, and I'm doing more general thinking like – again, this is not a math example, but say like which theorists I need and are kind of essential for foundational knowledge – I think what I would do to solve that problem is different from what would do when I'm deep into my topic, and I then I do more, kind of, introspective problem solving, or I'll email a former trusted adviser, who I know kind of shares my orientation – I think I refer more to personal sources when I'm more deep in it, versus when I'm doing more general, then I'll turn to forums or do Google search. I think it depends a little on stage of problem solving. Or, maybe, also the kind of sophistication of problem solving.*

**L2.** *Yeah, I also think that it depends on the topic.* [...] *Motivation is important. Looking*

*at examples – then again, it depends on the person, it's quite general.*

**L1.** *It always depends on the type of problem that you run into – is it a boring problem that I just want to get out of the way, is it a personally interesting one, and you want to play with it.* [...] *Do I expect to get something out of it, or do I just want to get it out of the way?* [...] *There are also formulated problems that come to you in a formulated way, like an error message for example, or problems that you have to first formulate. And also, old problems that people have had before, or new problems that nobody has had before.*

## C4 Dialog: Describe your learning style (continued)

**L2.** *To me, with the learning style, I like the learning by doing, and I have to say, with maths, usually there is this problem, that usually they teach without getting hands on, and for people who learn by doing, sometimes that's kind of an issue.*

**L3.** *How important would you say immediate feedback is?*

**L2.** *Yeah, visualizations, or...*

**M3.** *Can I just say one thing about learning styles? I think it's not just important for you to know everything, but it's very important to know how to find things when you need – I think that's more important, rather than trying to understand everything. I mean, in general. I think the skill would be nice – you cannot kind of capture, understand everything in all domains, it's too ambitious.*

**L2.** *The platform, also, in this case can play a role, right, of making it easy for you to find problems.* [...]

**L3.** [When compared with workbook style problems] *that's a way more interesting problem, of how do you scaffold complex problem solving. I mean, one easy approach would be "making aware of possible approaches" – can you break the problem down into subproblems, can you do a sort of root-cause analysis (I don't know if that's used in maths, but other areas certainly). Just probing the... prompting the user with different ways to think about the problem, if it's not trivial.* [...] *Can you use substitution? Made me think about, can you build an ontology of problems, make an estimate of the size of problems in terms of the number of steps usually required.*

## C5 Dialog: How do you deal with people whose viewpoints are very different from yours?

**M2.** *I was thinking about the learning styles – yeah – what kind of learning? You have opportunities to learn different things in different situations, right. And normally you learn something after you have defined certain aims. So in my case, I want to understand the*

*mathematical theory behind the computational model I'm using, and therefor I decided I have to build the mathematical background. And how am I going to do that? I'm going to build it from from scratch. So, you know, I defined my aim, how am I going to learn something. And that's one way to do it, I'm going to follow it, I'm just doing it there. You have other aims, like, sometimes, you realize, oh my god, you're having a conversation, and you realize... like, this guy learns this way, and I'm not really doing this. Like, for example with* **L3** *yesterday, he was – we were having a discussion, and he was telling me, you interact with other people and this is how you learn. And if I think of myself, and me doing the PhD, over the past how many, 3 years, I haven't really done this to, um, a similar extent that* **L3** *does it, I mean,* **L3** *does this as a practice – I mean, I'm interacting with my supervisors, and I have this kind of thing, but you have much more opportunities to get out there to see how people learn, and bla bla bla. So, that also gets at the other thing, which is, how do you deal with people whose viewpoints are very different from yours? So, I think the different viewpoints has to do with the aims that people have in mind. They, basically, they use a certain kind of learning that they have identified as a proper one, in terms of getting somewhere, that, there is an aim behind it. And therefor, you see, OK, these people have different aims and different learning styles for achieving these aims...*

**M1**. *Yeah, sorry, it might also be a misconception so, hence, sometimes it's a different viewpoint because it's just a different point...*

**M2**. *OK, yeah, yeah, how, with regard to a particular...*

**M1**. *Problem let's say, and they would like to solve the problem this way, but they've got to conform to the standards.*

## C6 Comments on online mathematics

"*It would be useful to have better systems for giving and receiving feedback.*"

**MJG**, comments after my talk presented to the Mathematical Reasoning Group at the University of Edinburgh, and reemphasized in our interview at Heriot-Watt. **MJG** told a story about how he had tried to learn about ultrafilters from various sources around the web, some that were very incomplete, and some that included outright mistakes. Generally, these sources do not come with easy ways to comment.

## C7 Comments on encyclopedia articles

"*Highly-abstracted and non-expository articles are hard to understand: they suffer of impenetrable quasi-formality.*"

**MJG**, discussing ultrafilters (Figure 11 [§7.3, p. 97]), *etc.*, in our interview.

## C8 Comments on student attitudes toward peer learning

"*Students are not comfortable sharing their work with other students. They are only somewhat comfortable showing it to their professor, whom they regard as one might a medical doctor.*"

**MJG**, commenting on his experiences with "peer pedagogies" in our interview.

## C9 Comments on computational tools for mathematicians

"*Formalisms are only useful when the model can't be held in your head.*"

**MJG**, in the discussion following my talk at University of Edinburgh.

## C10 Comparison of PlanetMath and Wikipedia as reference resources

"*Wikipedia seems to have better coverage, better exposition, and a nicer-looking interface.*"

**MJG**, discussion in our interview, based on a side-by-side comparison of the two websites.

## C11 Alison's description of her approach to research

"*Empirical studies of mathematical discourse are a key research focus.*"

**AP**, first interview, describing her approach to research. Further details appear in Table 15 [§7.4, p. 100].

## C12 Alison's description of her approach to research (continued)

"*How does experience build towards understanding?*"

**AP**, first interview, discussing themes in her research.

## C13 Tools that could be useful for studying mathematical discourse

"*It would be useful to be able to detect the first use of a word within a given episode of discourse, and to measure it's impact.*"

**AP**, second interview, looking at how PlanetMath could be directly helpful in her research.

## C14 Comments on computational tools for mathematicians

"*Can we build more detailed models of mathematical discourse?*"

Informal discussion with Mohan Ganesalingam and Sir Timothy Gowers, of Cambridge University, visitors to the Mathematical Reasoning Group at University of Edinburgh, who spoke with me about linguistic modeling of the mathematics domain, and interactive software.

## C15 Comments on artificial intelligence for mathematics

"*Even if computer mathematics did a good job of simulating human mathematics, it won't do things the same way humans do.*"

**AP**, first interview, reflecting on the joint work of Gowers and Ganesalingam (Gowers & Ganesalingam, 2012).

# D Selected quotes from interviews with participants in the user evaluation

This appendix presents a selection of illustrative quotes from interviews with participants in the field trial described in Chapter 9. These quotes are referred in brackets in the main text: for instance, [AC3] references the third quote from subject **AC**. Table 20 [§9.1, p. 142] presented a brief description of the interview subjects; the key data is repeated below. Recordings from **AA** and **AB** are missing due to a problem with the recording software; accordingly, data from these interviews is embedded in diagrammatic form in my handwritten notes, and not included below. **BC** dropped out of the study before completing an interview.

## AC Selected quotes

German resident, male, mathematics PhD, staff scientist and programmer, long-time PlanetMath user (contacted through PlanetMath).

1. *But these are escape things, is there any documentation. . . because they're not necessarily LaTeX stuffs.*

2. *Yeah, that's great, when you can use everything that's available in LaTeX with PlanetMath, I think that's really positive, because you can't do that with other places, with Wikipedia, you can't build such things.* [. . . ] *Because, I'm thinking, if I'm blogging Wordpress, and if I'm doing it, I have to convert everything I've written in LaTeX in Wordpress, and that's a lot of work.*

3. *And, recently you know, I was looking at the activity in PlanetMath, in the past 5 days, since I posted something new, there wasn't much activity in PlanetMath itself.*

4. *But, the funny thing is, I mean, assuming that somebody wrote an article like 10 years ago, what would happen then? He's, like, not anymore active in PlanetMath.*

5. *Another question is – regarding the problems – if I post a problem, can I watch on it? Because I haven't seen this possibility. Let me try again, I cannot remember, but that was one of the issues that was lingering in my mind. Alright, yeah I'm not able to watch* [...] *Oh, no no, I am - "you're not watching this" - so, I'm able to watch, so maybe that's my mistake.*

6. *Well, I was really seriously thinking of putting some really you know... but these are articles, I do not know the validity of them or something... so I can put them in articles... I can put them... I'm not sure if I should put them in problems or so, so I was just a bit confused – I'm not sure where they should land.* [...] *Well, I mean, they are, like you say, open problems... you can put it in problems, right – but suppose you are working on an open problem, and you have certain theories, and you're not really sure if these theories are at all correct. And you just want to put stuffs.*

7. *Every time I have some such issues, I won't hesitate, I'll just bombard you with screen-shots or questions. I know you're quite busy with these things, but it might help. Do you have a bug tracker?* (I tell him that the project is developed on Github, and we use the integrated bug tracker there.) *Oh really! Is it, I mean, do people look at it, I mean, some bug trackers they're just...*

8. *I think what StackOverflow, or, I mean I think it's called MathOverflow, there's also this Math-StackOverflow something, it's fine, but it's not too personalized. With PlanetMath, you can log in, and then you can, maybe you can, it's easier to find, or to categorize or to organize your problems there than in StackOverflow.*

9. *Yeah, I think it's more user friendly when it comes to users having problems with their homework or asking question. For instance, there's this MathOverflow, they filter out, they only entertain people who are a bit of, above graduate level, but in PlanetMath you don't have this problem, I mean this you don't have this kind of...*

10. *Well, you know, for instance I'm doing some private research, mathematical research, purely mathematical, that has no application at all, you know, it just interests me. I was thinking, you know, I could write a blog on it, because I'm not really interested in having publications, because, the problem I see with publications, is that you have to wait for... almost a decade until all the reviewers suddenly say, OK, this is fine and then you can publish it. I mean, I've had experience with it in my Ph. D. years, that you know you'll have to wait a long time because the reviewers don't really have time for it; and then after that, you will have to do the changes that the reviewer asks you, and then after that, it's published. But I would prefer to have it already online,*

*I'm really not interested in having my work officially published, because in the end, the people who are getting benefit from are actually the press, and it's not really the scientists who are involved, because in the end, people have to buy my articles, which is not interesting for me. So I was interested in writing blogs about this thing. But now that PlanetMath comes to... you know, I was thinking of, maybe it would be possible to do such, write my researches in PlanetMath.*

11. *I think the problem is PlanetMath lacks some of the most trivial things, like, for instance, if I posted something, like about Lie algebra or something, then after one month, I do not see it in my history. Unless it's an article. If I post it in, like as a question, or in a problem set or something... I never see it in my history and if I search for it, it's a catastrophe. You cannot – you're not able to search it.* (I show him the "Score" tab.) *... but this includes also the things that I've viewed.*[*]

## BA Selected quotes

UK resident, male, engineering background, enrolled in the Open University's mathematics masters program (contacted through OU mathematics department).

1. *Um, right, I started out with an article* [...] *I did find, however, that when I was typing into it, I was pressing "preview" to see what it looked like, but it would have been really convenient to see what it looked like in some ways, as I went along. But, it wasn't clear – I ended up doing a lot of "save" operations and then carrying on, because I wasn't clear as to whether the thing was going to time out on me, and then all the data would be wiped – because, it's quite a slow process, obviously – if I had left it a few hours, and then went back, and wanted to make some amendments to it, because I couldn't actually see the article, you know, you had to sort of go in, make your amendment, do a preview...*

2. *Um, yeah, because there are so many categories, I found it a bit confusing. Yeah, so, something like "collection", I would think that ought to be something where you would have an editor, who would move articles around, into collections, say, rather than a writer choosing which collection to be in.*

3. *Yes, it's a – it's a different approach really. So, I mean, something – probably something like a collection – it would probably be good if an editor could at least move something, collect articles together, perhaps place some blank articles there – you know, to try to get someone to actually write one.*

[*]More detail was added to user pages after this interview.

4. *Yeah, I mean, that's another thing – you might have a collection, where you have something at, say, at school level, and another one at, I don't know, postgraduate level. And there really isn't going to be any overlap at all.*

5. [E]*veryone's allocated a course tutor, who might take on just a half-dozen students – so, they're not the overall person in charge of the course, by any means.* [...] *It is called Elluminate, spelt with an E, I think it's some Canadian organization; it's got lots of Java in it, but um, it's essentially a whiteboard, with very limited ways in which people can talk, apart from the tutor.* [...] *In OU terms, the forums, essentially, there's a moderator, and they try to dispel any talk about these Tutor Assignments, at all, on these forums.* [...] *Well, you – either* [solve the problems] *yourself – or if you're in difficulty, you should contact your tutor. Well, they're quite helpful, in that they'll give you a hint to set you off in the right direction. And, obviously, if all else fails, these questions are normally in parts, so if you can't do one part, you can just move on to the next. So that is not, if you like, exclude you because you've sort of... failed to get enough marks on your assignment.*

6. *This one, I dropped before the end, and that was Functional Analysis, it was just too, if you like, abstract for me.* [...] *I think that Functional Analysis is just, if you like, it's really something that is so abstract – that, it would be very d... – I think there are some people who can find their way around that, but, for some people, you reach a certain level of abstractness, and they're just lost.* [...] *Well, as it turns out, my current course - does use a few of the ideas of functional analysis, in the sense that you get finite dimensional linear spaces, rather than infinite dimensional spaces... which you can't really visualize, you can't draw pictures, really... so, I mean, I find that quite straightforward; I mean, it's really just ordinary vector spaces, things like that, which is sort of undergraduate stuff.*

## BB Selected quotes

UK resident, female, engineering and software background, enrolled in the Open University's mathematics masters program (contacted through OU mathematics department).

1. *Because, I mean, what we've got on PlanetMath is masses of very good math, masses of very good theorems and results and definitions and so on; I just found it a bit difficult to find the one I wanted.*

2. *But is a problem – are you thinking of a problem as something that a teacher has provided for a student to look at, and he has the answer that the student can then go and look at, when he's tackled the problem?* [...] *Because if I was in there thinking*

*about answering someone else's question, I want to know, whether that's just a person who wants to know how to do something that I can easily tell them, or whether the tutor has put up something for them to do, and we're not supposed to be answering it, they're supposed to be finding it out for themselves.*

3. *I think it would also be nice to have a place to talk with mathematicians, without thinking that you might be making a permanent copy on the website, that ought to be correct and validated and so on, so that you can say, how about this for an answer... where you might be guessing just as much as the other person... but the forums don't seem to be very much used.* [...] *I don't have any problem with them being public, it's that I want it to be quite clear that I'm just speculating, and not giving a definitive proof or theorem or whatever, because most of the site is definitive material.*

4. *So, you would expect if somebody asked a question, that the person answering it wouldn't answer the question, they would just point them in the direction of how to think about it.*

5. *Well, I certainly go into all the OU forums. I certainly ask questions and answer questions. And actually, answering questions is just as much learning as asking them. Because you have to spot how the person who's asking it is looking at something from a funny angle, or upside down, and then explain the way you should be looking at it.* [...] *But don't you just find it interesting, the way other people approach problems? It gives you a whole new view on the world, to see some other, other, um, attitude to the same question that you've just taken for granted – suddenly you get a new perspective.*

6. *I read a section of my maths text, sometimes trying to understand the proofs of the theorems, but more often just reading and trusting that they really work. Then I try to work through some examples or exercises. This often brings up questions about the original material. For example, I ask myself, "Why does the closed contour theorem talk about a function having a primitive while Cauchy's theorem requires an analytic function? What is the significance of this difference?" I think that it is only when I begin to join up bits of the theory in this way that I am really learning.*

7. *I think it's more, the problem is, that when you go out to look at a particular theorem or whatever, you can't very easy get back, you might do your back button, but you might have to do 10, 12 presses to get back to where you started from – whereas if you could have a collection or a topic article or something in your breadcrumbs at the top, then when you'd been wandering around following leads, you could then jump back to where you were.* [...] *I don't think it would need the whole history, I think it would just need the thing that was the topic, whether that's a collection or a topic*

*article, a sort of high level thing, that when you went into one, it would remember that until you went into a different one at the high level. It would be a bit difficult to have everything listed where you'd been, all the way, like a history, it would be too complex.*

8. *I think they're just a bit difficult to find, which is why I was just suggesting that maybe topics should be brought up to a higher level, and listed separately from articles. So that you can easily find a whole topic, and then like you say that topic gives you the introduction into all the bits that belong to it – probably work better than collections, because being basically an article, you can have all the links, and you can have the words in-between.*

9. *Yeah, also, maybe we could have these sections in smaller steps. You might be able say, I'm looking for a theorem, or I'm looking for a definition, or I'm looking for a method.*

## RSP Selected quotes

US resident, male, physics PhD, long-time PlanetMath user and active volunteer (Raymond S. Puzio, requested non-anonymity).

1. *I should mention, with the 3C's and your elaborations – I figured last night that I should explain to you what I meant, because there's been some confusion. So, you are absolutely right about the importance of the code and the organization, however, I guess the way I'm thinking of it is that those two are more, sort of, in the background, they're what's supporting it. Whereas, what I'm thinking is that the content, the catalog, and the community is what the typical user will be interacting with.* [...] *So, that's what I meant, so the idea is, if they're interacting with the 3, hopefully some of them will get interested in looking in the background.*

2. *Also, as I found when I was rummaging through the Library of Congress catalog last week, there's oodles and oodles of elementary math books of problems out there in the public domain – I'm sure we have... we might even have a million problems if we managed to get all of those in.*

3. *Some people might find the shared document and email enough, others might want a voice channel, others might want something else, maybe they want, let's say some algebra program that's running, and a common place to save files.*

4. [A]*s for other uses I'd like, interacting with people on PlanetMath – a lot of that is going to depend on building the user community, for example if I have an interest*

*in a research-level topic, there's someone with a similar interest to talk about it with there.*

5. *Personally, ok, that information would be good to display, but you also have to involve people, because let's say they posted a lot, but they might be too busy, or might not be in the mood to talk about it – so I would say that improving the interaction along with that would be important.*

6. *So I guess one way of thinking of it is as a replacement of our bio – I mean our user home pages. My idea would be to replace those home pages with some such more robust thing.*

7. *Not just make it easier to respond at your own pace, but also to put that information in a useful form, like here's people's interests, here's the ones who already know you and are contacting you, and ones you might want to contact, etc.*

8. *One thing about content could making it specific. Let's say, we look at* [elided]. *OK, then the community might be, currently the community might be* [**AB**] *and myself and one or two other people. Content, OK, we've got her catalog – her collection – of* [elided]. *Catalog end, I could just extract from the Library of Congress all the books pertinent to that subject. Even if we don't have time to build everything up. So, even if we don't have time to build everything up, at least we could build up that one sub-project really nicely.*

9. *Well, let me tell you* [...] *two specific instances. One thing I did was, we had this project with Matte and Cameron, where we had this project, where the there of us were looking at the counter-examples in topology, and just making sure they were all on PlanetMath. Another one was when – that you probably know – was when Matte tried to do his thing with the real number project, which was basically such a role, the problem was that at that time, it seems he didn't have resources or the support he needed, and he got frustrated.*

10. *So, what I'm thinking is* [...] *I'm not so worried about automating. In the past, I've served that purpose of bridging users together. And I could see myself doing it again, it's just that, for doing that, I should free enough time to make sure I actually have time to interact with people.* (Pause) *So you know, for example, in the past, I've started things like this project we had with topology, and other things, and you know, brought people in. So, one hope I've had is that I'd have an hour or two a day to working on the community side and exactly helping people with such things.*

11. *OK, let's say, Internet Archive, sooner or later, they're going to have them all uploaded as graphics images, if we have the OCR running, then, yeah they could all become*

*TEX files, but it might be that only a handful of those ever gets human edited into proper editions, because only they get enough attention.*

12. [T]*hings like restarting the community discussions, for example our having our Thursday things once a month, with more people showing up. None of these really require us to build any new software or do anything too time consuming.*

# E Illustrative excerpts from field notes

**(Day 2 of field work) Did any of your hypotheses get corroborated as a result of your data analysis? If so, which, why and how strongly?** I got some good insight into "peer supported problem solving" from the point of view of a mathematics researcher and instructor. With his "instructor hat" on – what he wants is for his 100 students to *talk* with each other. But he wants this because he wants them to turn in better papers. He doesn't care if they work together or whatever! He just wants them to learn the stuff better. He seems, in this regard, to be a really amazing instructor. So, we came up with a model that might work. ("Paper mock up" - I'll have to think more about whether I could implement the thing he needs in software.) With his "research hat" on, we saw that expository text is very different from what might be called "quasi-formal" mathematics – and PlanetMath has a lot of the latter. With things like this, you don't see what the point is. One definition points you to another. I used to write a lot like that. It's interesting to think about how this could be improved (either on PlanetMath or Wikipedia, which, on average, is better in this regard already).

**(Day 3 of field work) Did any of your hypotheses get corroborated as a result of your data analysis? If so, which, why and how strongly?** "How could PlanetMath help with that?" was an ongoing question. In some cases the answer was very "hypothetical" but in general there tended to be an answer. For instance, looking at some of the rather "bad" (in Jamie's view) articles through the linguistics of mathematics view of Mohan, I can see how PlanetMath might integrate a tool that could *show* how "formal" a given article is, and help lead authors to present their work with suitable expository details. Some of this I could likely do even more cheaply, without waiting for Mohan's work to be completed.

# F A critical apparatus, applied to developer discussions

> [I]*t is by headlong flight that things progress and signs proliferate.*
>
> (Deleuze & Guattari, 1980, p. 73)

Table 25 collects the themes from Chapter 9 [§9.2, pp. 143-148], and expands them using key points from the literature review conducted in Chapter 4 (Figure 6 [§4.8, p. 66]). The issues identified in Chapter 7 (summarized as α–ο in Table 16 [§7.5, p. 104]) appear as further annotations. This shows that many of ideas from the initial work with potential stakeholders were still points for discussion in the user evaluation, underscoring their importance, while showing that the underlying issues are not fully resolved.

The mapping here confirms the relevance, for PlanetMath users, of some of the notions coming from the literature. For example, in Chapter 7, it was observed that (λ) "*Users want to be able to make adaptations that leave traces, and simple customizations that don't.*" This is connected to McCalla's idea of "*gradual localized accumulation of information.*" Observing that (ο) "*Computer understanding and formal theories of quality and effective social heuristics are good long-term goals*" connects with McCalla's idea about "*supporting purpose-based use*" illustrates that there is quite a bit of work needed to understand the pragmatic purposes that are involved in mathematical activities.

Chapter 7 gathered ideas about user feedback (β) and involving users in technical work (κ) that had not been encountered in the literature review. There is of course a healthy literature on end-user programming and contributions to free/open source software. This material is relevant to a paragogical approach, but it is easy to see why it did not come up in Chapter 4, given the aims stated there [§4.1, p. 48]. This does suggest the need for caution in applications of the paragogy framework.

To be successful, system developers must find ways to mesh what happens on the site with participants' phenomenal experiences of mathematics.

- human language
- sensory perception
- logic and deduction
- intuition, association, and metaphor
- stimulus/response
- process and time
- socialization [ν]
- content
- community [η]
- catalog [η]
- organization
- software [ι]

There will be a boost in participation and learning-relevant measures if users can express their ideas easily in written language.

Improved graphics and interactive features will also boost participation and learning-relevant measures. [μ]

"Projects" should be modeled as progressive problem solving. [δ]

- personal attention
- public statements
- argumentation and rationale
- shared understanding
- collaborative knowledge
- cultural artifacts
- personal comprehension

The process of progressive, informal, problem solving maps onto the history of interactions and curatorial activities. [γ, ζ]

Roles can regulate behavior in the project, but most roles are ephemeral. Collections based on use tracking can help users create more concrete roles.

- gradual localized accumulation of information [λ]
- focused on the end user
- supporting purpose-based use [ο]

When compared with "closed" production, "open" production comes with a host of new problems to solve.

relevance quality [α] scalability consistency motivation [α]

Users need feedback and follow through: consistency is important in social interactions, not just for expository content. [β, κ]

Concrete applications are an indicator of quality. [ε, ξ]

People will be willing to put in effort if it helps them build "literacies" that they can use to reliably achieve their informal goals.

- getting information
- giving information
- reputation building
- relationship development [θ]
- recreation
- self-discovery
- constructive feedback

**Table 25:** *A critical apparatus, with mnemonics*

The idea is not that the heuristic concepts in Table 25 should be "*corrected and improved until at last they are worthy of a place in the list of fundamentals*" (Bateson, 1972, p. 5). Rather, much in the same way in which the "grammar" in Table 24 [§10.2, p. 160] could be used to parse mathematical activities, Table 25 can be used as part of a working language that describes the "para-mathematical" activities that are involved in building and improving PlanetMath. Something like this seems necessary if mathematics is to once again be seen as "*a living science*."* In principle, this apparatus can be applied to connect the problems that still exist in the working prototype with the research literature, and the research and development programme. Like the Peeragogy pattern language, this apparatus should be used as a distributed *Roadmap*, or rather, a set of roadsigns. As with the *Roadmap* pattern, it should be regularly revised.

 A similar lexicon should be developed to code the expository features of mathematical text. The "dangerous bend" sign from Bourbaki is a prototype.

**Application: coding developer discussions** This apparatus is applied here to code eleven outstanding but high-priority issues selected from the Planetary Github tracker. This coding seems particularly suitable for application at the per-ticket level, although it could potentially be used at fine-grained levels to directly annotate text. This helps to connect the theoretical work developed in the thesis with the everyday concerns of developers, and helps to check the completeness of the apparatus.

The dynamics of developer interactions were not studied in detail in the thesis, but the data is online for curious readers, for example: Issue #87 is a prime example of social problem solving. Issue #106 is managed by one person, with updates posted in a blog-like manner. After an initial information-gathering phase, Issue #351 is still open, although preliminary work, on Issue #68, is completed in a branch.† These tickets also help to illustrate my role within the Planetary project, as both "designer" and "software integrator" (I *opened* 332 out of 383, or 87%, of the total issues) as well as programmer (I closed 162, or 78%, of the *closed* issues). The full set of tickets is online,‡ along with additional summary information on developer participation in the Planetary project.§

*Early volumes in Springer's Graduate Texts in Mathematics series include this note in the backmatter: "*A student approaching mathematical research is often discouraged by the sheer volume of the literature and the long history of the subject, even when the actual problems are readily understandable. Graduate Texts in Mathematics, is intended to bridge the gap between passive study and creative understanding; it offers introductions on a suitably advanced level to areas of current research. These introductions are neither complete surveys, nor brief accounts of the latest results only. They are textbooks carefully designed as teaching aids; the purpose of the authors is, in every case, to highlight the characteristic features of the theory. Graduate Texts in Mathematics can serve as the basis for advanced courses. They can either be the main or subsidiary sources for seminars, and they can be used for private study. Their guiding principle is to convince the student that mathematics is a living science.*"

†https://github.com/m-iancu/planetary

‡https://github.com/KWARC/planetary/issues

§https://github.com/KWARC/planetary/contributors

#407 editor needs to be supplemented with a “cheat sheet”

The editor could be made much easier to use, with a cheat sheet and more buttons.

#381 SQLite instead of MySQL

It would be good if contributing developers could get started with an instant deployment, and SQLite would be one way to help with this.

#354 Planetary should do better error reporting for LaTeXML fatal errors (was: silent failure of DruTeXML/LaTeXML for posts containing an unescaped “# symbol”)

It’s very confusing for usrs if they don’t get feedback on errors in source code they submit. Other services, like ShareLaTeX, do a relatively nice job with this, with line-level markup to indicate errors.

#351 killer feature: use git to facilitate instant revisions

With suitable integration, Git could be used within Planetary in order to get the best of both Noösphere-style ownership and wiki-style editing. The work done on the MathHub frontend does not support this sort of feature yet.

#328 Meta-ticket for module cleaning process

To help future developers, the source code needs a lot of refactoring. Drupal has a “submodule” feature that can be used to organize things.

#288 make the user experience faster for contributing authors

Currently, users have to wait for processing which would be better to do in the background.

#88 developer docs

Documentation that will help people who are interested in development work have a good experience may help to bring in more volunteers.

#73 deal with subpath for image conversions (was: send in basepath as LaTeXML flag)

Links/Backlinks between gallery pages and the documents that use the images are not currently available.

#68 SVN/git integration

Even without fancy UI features and awareness of the ownership model, Git could be used directly for storing revisions. This is done in the MathHub branch of Planetary.

#67 build system

🗐, 💾 If LaTeXML changes or if there is some other reason we need to rebuild all of the content, we currently do this with a script, but this does not give feedback on which articles or macros are breaking. There is a separate LaTeXML build system, but this is not integrated with Planetary.

#46 integrate JOBAD into planetary

🔎, 🐞 JOBAD is a Javascript package that has been used on a demonstration basis to build intuitive point-and-click in-document interactions (like per-line corrections and comments).

# Abbreviations

AAR After Action Review

APM-Ξ Austin Problems in Mathematics – Cross-Index

CC-By-NC-SA Creative Commons Attribution-NonCommercial-ShareAlike license

CC-By-SA Creative Commons Attribution-ShareAlike license

CMS Content Management System

FLOSS Free/Libre/Open Source Software

GNU GNU's Not Unix

GNU FDL GNU Free Documentation License

HDM Hyperreal Dictionary of Mathematics

KWARC Knowledge Adaptation and Reasoning for Content

LIWC Linguistic Inquiry and Word Count

MOOCs Massive Open Online Courses

MSC Mathematics Subject Classification

P2PU Peer 2 Peer University

RDF Resource Description Framework

RDFa Resource Description Framework-in-attributes

REPL Read-Eval-Print Loop

WYSIWYG What You See Is What You Get

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# About this document

The typeface is the Linux Libertine font by Philipp H. Poll and the Libertine Open Fonts Project. Libertine is a proportional serif typeface inspired by 19th century book type. It is intended as a free/open source replacement for the Times font family. The document was compiled with XeLaTeX.

# Towards a Creativity Commons

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**Abstract**

A preliminary treatment of a social creativity is presented that aims to be "mechanism independent" in the sense that it could equally well describe the behavior of simulated societies of computational agents or the embodied behavior of humans engaging in commons-based peer production. The idea of project-embedded problem solving is used as a connection between these two domains. Mathematical thinking provides an inspiring example, and the methodology that is proposed is an interpretation of the highly heterodox interdisciplinary theory of design patterns.

# 1 Introduction

From the perspective of creativity, licenses are a blunt instrument. Copyleft licenses (CC-By-SA, GNU GPL, etc.), non-copyleft licenses (CC-By, MIT, etc.), copyright waivers (CC-Zero), and the public domain support a vibrant culture of creative reappropriation. However, if the taxonomy of licenses is understood as our native theory of "free culture," then it seems we are collectively turning a blind eye to the all-important *creativity* that makes it all work. One thing that's not made particularly clear in the legal approach is that creative work has a lifecycle: at minimum, a beginning, middle, and an end. We need to pay more attention to understanding the intricacies of the "middle bit." In short, despite their important role as protectionary measures, licenses are not of great help in explaining or directly facilitating the genesis of cultural works.

Similarly, (general) economic behaviors, like gift exchange, potlach, or division of labor, can contribute incentives, norms, or boosts to efficiency, but they are typically second-order phenomena, since they assume that there is something there to exchange or destroy, or that there is a process to subdivide and enact. Of course, creativity very often develops at the social, symbolic, or economic level, but this is all the more reason not to take these phenomena as fundamental *explicata* for creative culture. In addition, as Shaw and Hill [20] make clear, the particular economic form known as commons-based peer production is in no way a political or social panacea. On the one hand, "*many successful peer production projects exhibit strong inequalities of participation and deeply entrenched leadership*" – on the other, many free/open software projects never attract collaborators at all. As researcher and free software activist B. Mako Hill argued earlier, "*The benefits of collaboration become something to understand, support, and work towards, rather than something to take for granted*" [11].

This paper approaches that challenge by trying to understand creativity in a mechanism independent way. The hypothesis that computers can simulate humans is not directly useful for building a theoretical model of creativity. What is more interesting is a social-computational perspective that says that certain combinations of humans and computers can simulate relatively advanced computer systems. For example, from the point of view of any given user, a peer produced website might be said to roughly approximate an artificial intelligence with an agent-based architecture. Similarly, the history of mathematics might be thought of "as if" it were an analogous long-running computational process, carried out mostly on paper. The fact that mathematics actually developed in the minds of humans is, clearly, not to be neglected. Nevertheless, the complementary fact that mathematics can be communicated between humans means that it can exist, to some extent, *in charta* or *in silica*. Mathematical creativity can accordingly be thought about in a way which sets aside the question of whether it is carried out by humans, societies, or computers, and which focuses, instead, on the fundamental features of mathematical form. These typically include definitions, theorems, and proofs along with explanations, remarks, questions, and so forth. Mathematics is by no means the only domain that can be studied in this way. Mathematics is, rather, an inspiring and well-studied example, and one can point to practical advances in fields like automated theorem proving that are associated with a computational approach. This example helps to motivate the mechanism independent study of the social-computational dynamics that are involved in the creation of software and other cultural works.

## 2 Building a mechanism independent model of creativity

**Embodiment and sociality are the antidote to "as if" thinking for humans and computers.** Tim Ingold [12] makes a distinction between "as if" thinking conducted, for example, by expert systems, and in-the-world thinking carried out by humans. In addition to their crafty minds, one thing that individual humans definitely have, which computers generally do not, is a robust embodiment. Ingold classifies writing in the same "as if" category as expert systems, since written expression tends to be equally disembodied. Manual DeLanda [7] (from whom I've borrowed the idea of "mechanism independence") talks about a potential antidote to "as if" thinking in computer systems: if we can give the computer even a limited embodiment, putting it in touch with a virtual world – then it doesn't have to have everything worked out in advance. In the case of mathematical thinking, individual humans are in contact with a rich phenomenal world, in which they find themselves capable of drawing lines in sand or on slate, discourse with other humans, and visualization or embodied feeling-ones-way through complex mental projections. A minimum mechanism independent model of creativity should include *agents* and *phenomena*. In particular, agents exist simultaneously as part of one another's phenomenal world, and this is referred to as *sociality*.

**Agents relate around works and acts.** Works, skills, acts, and agents themselves can all be created. Property and ownership are some of the rules that govern the relationship of agents to works, and these rules can become arbitrarily complex. Thus, for example, CC-By-SA-NC says "You can do (almost) whatever you would like with this work – as long as you give me credit, share any derivative works under the same license – and don't sell them." However, one ought not assume that creativity itself takes place exclusively in the context of one big commons, for instance, a given license "pool" like that defined by CC-By-SA-NC. Rather, creativity takes place within working groups, families, homesteads, etc., and it may be understood to evolve within nested contexts at several scales. For instance, creative work done within a given kin group might be comprehensible by other members of their linguistic community, and may be shared at the level of expression within a cultural commons, even if the work itself remains the property of its creators [13]. Even works that are shared more broadly tend to be created, enjoyed, and reproduced in smaller local contexts. In his public talks, Creative Commons cofounder Lawrence Lessig often quotes the composer John Phillip Sousa: "*When I was a boy... in front of every house in the summer evenings you would find young people together singing the songs of the day or the old songs.*"

**Mathematical thinking: decompose processes into theories and problem solving projects.** Quite a bit of background knowledge typically goes into stating, formulating, understanding or proving any given theorem. To speak metaphorically, the relevant definitions and lemmas are something like a neural substrate, in which the new theorem is expressed as a thought. Individual proofs can be broken down into subtrees and formal proof steps, but mathematical insight is better studied from the informal perspective of problem solving heuristics. One classic heuristic is to look for simpler versions of a hard problem and to try to solve those. Relevant social heuristics include asking for help. At any given point in a mathematical thought process, there may be many problem solving projects taking place on different levels: for example, one may be simultaneously carrying a digit, factoring a polynomial, finding a root, and designing a mechanical structure. A given mathematical theory or culture will suggest relevant approaches. In general there can be multiple agents, goals, and approaches at work.

**The more general case: emergent design patterns.** If mathematics is the formal theory of patterns, it can be quite useful to compare it to a relatively *informal* heterodox theory of patterns coming from the field of design. First introduced by the architect Christopher Alexander [1, 3], the design pattern methodology spread from architecture to software [9], and later, to other fields, including public affairs [19] and education [4]. Alexander's patterns are presented in a tree-like structure called a *pattern language*, ordered in a top-down manner from large-scale to small-scale levels of application, with each pattern presented in terms of a *picture*, a *context* (including links to relevant larger patterns), the *problem* that the pattern addresses, the *solution*, a *diagram*, and *links to smaller patterns* [3, pp. x-xi]. A relatively convincing implementation of Alexander's idea of patterns as a "*living language*" [3, p. xvii] was realized with one of the earliest applications of wiki software developed by Ward Cunningham: the Portland Pattern Repository.[1] Taking the idea of a pattern language as a point of departure, the *Peeragogy Handbook* [18] includes a catalog of patterns for peer production and peer learning that the authors discovered while working on the book. The patterns are presented using a template that is closely aligned to the

[1] `http://c2.com/ppr/`

structure coming from Alexander (Table 1). Each pattern is essentially a short textual abstract, describing an active dimension of the Peeragogy project. This emphasis on the active aspect is made particularly clear through the "What's Next" facet, which concretely links the patterns in the *Handbook* to the current activities being conducted within the aegis of the Peeragogy project. Connecting the patterns to the project in this way helps to convey their "living" nature, and the pattern catalog evolves together with the project. The notion of pattern-finding as a process related to, but distinct from abstraction, is described by Richard Gabriel, who emphasizes that the "*patterns and the social process for applying them are designed to produce organic order through piecemeal growth*" [9, p. 31]. In its original form, this statement describes the generative use of patterns to create artifacts (buildings, object oriented programs, etc.). However, this criterion is applied here to the growth and development the pattern language itself.

**Title**: Encapsulate the idea – possibly include a subtitle.

**Definition**: Explain the idea and the context in which it is meaningful.

**Problem**: Explain why there's some issue to address here.

**Solution**: Talk about an idea about how to address the issue.

**Challenges**: Talk about what can go wrong.

**What's Next**: Talk about specific next steps.

*Illustrations, and the following, are optional:*

[**Objectives**: Explain the purpose(s) of the proposed solution's functioning, if not fully specified by the description of the solution itself.]

[**Examples**: Present example(s) that have been encountered, if this aids comprehension.]

[**References**: Citations, if relevant.]

Table 1: Pattern template from the *Peeragogy Handbook*

**Example: emergent design for peer learning.** Patterns offer a heuristic but nonetheless relatively rigorous way to do ongoing "emergent" design. Some of the central patterns described in the *Peeragogy Handbook* indicate that peer learning projects often organize around a temporal *Heartbeat* of meetings, divide the work among several *Roles*, and have to decide what, and whether, to *Use or make* (often remixing earlier works). We've observed that even with a high-level *Roadmap* in mind, participants tend to drift until they pick *A Specific Project* and get down to work. With these and other patterns in mind, the analyst has a simple language for talking about what works and what doesn't work in the online peer learning context. The process of analysis is rigorous in the sense that either an existing pattern matches the topics under discussion, or new patterns should emerge. For example, in [6], the Peeragogy pattern language was applied to study user feedback about a new system that was developed for the mathematics website PlanetMath.org, resulting in three new patterns (*Minimum Viable Project*, *Frontend and Backend*, and *Spanning Set*) that expanded on or clarified earlier patterns. Each of these has its own "What's Next" within the context of a specific project.[2] The *Peeragogy Handbook* collects a dozen patterns and a half-dozen antipatterns (using the same template but emphasizing "Challenges") composed after the fashion described here. As an initial set of patterns describing peer production and peer learning, this catalog forms a candidate description of a creativity commons (Table 2, with details at `http://peeragogy.org/practice`). There are hundreds of other patterns that have been written over the years, but generally without the project-embedded focus adopted here – which puts this work in comparatively direct contact with phenomenal experience.

[2]For illustrative purposes, the *Spanning Set* pattern is reproduced in Table 3 in the Annex to this article.

*Patterns*

**Wrapper** – Front end appearance to participants. Consolidate and summarize.

**Discerning a pattern** – Found a pattern? Give it a title and record an example. (Woah, meta!)

**Polling for ideas** – Invite brainstorming, collecting ideas, questions, and solutions.

**Creating a guide** – Overviews expose the lay of the land. Collecting content and stories.

**Newcomer** – Welcoming "beginner's mind."

**Roadmap** – Plans for future work, direction towards a goal, dynamic.

**Roles** – Specialize and mix it up. Play to participants strengths and skills.

**A Specific Project** – Lightbulb moment: Most specific projects involve learning!

**Carrying capacity** – Know your limits, find ways to get other people involved.

**Heartbeat** – The "heartbeat" of the group keeps energy flowing.

**Moderation** – When leaders step back, dynamics can improve; moderator serves as champion and editor.

**Use or make?** – Repurposing, tinkering, or creating from scratch?

*Anti-patterns*

**Isolation** - A tale of silos, holes, and not-invented-here.

**Magical thinking** – "One meeting will (not) change everything!"

**Messy with Lurkers** – What happens when joining is low-cost and completion is low-benefit.

**Misunderstanding Power** – The workload is almost never evenly distributed.

**Navel Gazing** – "I have this really great idea…"

**Stasis** – What drives collaborative projects? (Whatever it is, it doesn't always work.)

**Stuck at the level of weak ties** – Can we deepen the connection?

Table 2: Catalog of patterns from the *Peeragogy Handbook*

## 3 Outlook and next steps

**Heuristic methods.** As Alexander [2] indicates, it will be a challenge for software developers, architects, and others – educators and researchers for example – who are inspired by the design pattern methodology to work together to transform the lived environment. An especial challenge is to find a way to work across different fields and different levels of analysis, "*integrating different types of theories and concepts in such a way that one can formulate meaningful propositions embodying findings now sequestered in separate fields of study*" [10, p. 44]. Although it makes certain epistemological and ontological assumptions that may not be appropriate in every conceivable case (regarding, in particular, the possibility of a "problematic" understanding of reality) – the adapted design pattern methodology described here should nevertheless be widely applicable as an informal dialectical method; cf. [8, p. 203]. The Peeragogy project will continue to develop its catalog of design patterns in practice based on active participation, with a new quarterly peer produced development course beginning in June, 2014, featuring rolling enrollment.[3] Participants will be invited to develop new candidate patterns starting from a simple reporting protocol (adapted from [23]):

> *(1) Review what was supposed to happen. (2) Establish what is happening. (3) Determine what's right and wrong with what we are doing. (4) What did we learn or change? (5) What else should we change going forward?*

**Computational methods.** Keeping in mind the core aim of mechanism independence, a parallel research agenda involves implementing a computational system that can work with design patterns using a society-of-mind style architecture [14]. Although design patterns have been used extensively to guide programming, there are still many challenges associated with moving from an informal heuristic modeling approach to a formal computational model. Related work using artificial agents in a narrative reasoning context was discussed in [22], with a detailed development in [21]. This work emphasized reflexive aspects of mentation and multiple ways to think about problems [15]. The orientation described here suggests a more cultural and less psychological tack. For instance, microtheories may be useful for describing local cultural contexts, and this resonates with the Ostrom-inspired view on the commons [17]. Another key question to examine is the way different local contexts link together. Nowak takes an evolutionary game theory approach to this question [16]. Here, one would rather consider hybrid cultures and blended theories: and the prospect for referring this work to ongoing research on *computational blending* looks good. Mathematics continues to be a motivating example, and the first order of business will be to develop an extension of the pattern catalog oriented to *mathematical* creativity.

## 4 Conclusion

As a collection of design patterns focused on peer learning and peer production projects, the Peeragogy pattern catalog can begin to be used right now as a heuristic theory of social creativity. It is simultaneously rigorous and anti-fragile: individual patterns need to be negotiated and reasoned through relative to the existing catalog, while the catalog as a whole will improve with discussion and debate. As a system connected to practice within the Peeragogy project, the pattern catalog is very much subject to change and further development. The individual "What's Next" steps serve as a distributed *Roadmap* for the Peeragogy project – and as the work done in the context of PlanetMath shows, they may be useful as a "living language" spread across several peer production sites. Although the computational system described above remains projected and conjectural, the design sketch usefully expands on an earlier discussion of multi-agent architectures for computational creativity, hinted at in [5]. This earlier article focused on critical assessments of creative progress. Here, creative progress is viewed as connected to formulating and working on problems, or, rather more profoundly, asking how a society "*is able to pose the problems set within it and to it by the differential relations it incarnates*" [8, p. 235]. The problem posed here, in a necessarily preliminary manner, asks how we can understand peer production in a computational way.

## 5 Acknowledgements

This work has been supported through EC funding for the project COINVENT 611553 by FP7. I would like to thank Helene Finidori for helpful conversations.

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# Annex

***Spanning Set: Follow the paths in the grass***

**Definition**: With a well-constructed information access system, you may be able to get what you need without digging. If you do need to dig, it is very good to get some indication about which direction to dig in. At the level of content, this may be achieved by using high-level "topic articles" as a narrative map to the content. In general, the Spanning Set may include people as well as less dynamic media objects. In general, a spanning set is comprised of a set of fundamental actions (e.g. asking a question) and fundamental relationships between resources. In the case of media objects, it may reduce to the pattern *Creating a Guide*, but the guide may need to be updated regularly to remain relevant.

**Problem**: People need to know what can be done with a given resource, and this isn't always obvious. Relying on a single knowledgeable guru figure isn't always possible.

**Solution**: A spanning set of a system's features, categories, and relations can be comprised of many different kinds of components: for example, a "start menu" or pop-up window showing keyboard shortcuts that shows what can be done with a given tool; a schedule of office hours so that people know how to find help; and topic-level narrative guides to content.

**Examples**: One social version of a Spanning Set is the classical master/apprentice system, in which every apprentice is supervised by a certified master. In a standard course model, there is one central node, the teacher, who is responsible for all teaching and course communication and answering student questions. In large courses, this model is sometimes scaled up through tutors and TAs. In a large MOOC, a "spanning set" of peer tutors could help give everyone personal attention. In Q&A sites, these roles are fully disaggregated and distributed.

**Challenges**: From the perspective of *Frontend and Backend*, principles and features are visible as part of a system's "frontend" – but the actual spanning set of relevant behaviors tends to be emergent. If any individual tries to span too much, they will get spread too thin; see *Carrying Capacity*.

**What's Next**: As a project with an encyclopedic component, PlanetMath can be used to span and organize a significantly larger body of existing material. We have come up with a high-level design for a "cross-index" to the mathematics literature. We're working on a prototype for Calculus.

Table 3: The *Spanning Set* pattern

# PEERAGOGY IN ACTION

by *Joseph Corneli, Charles Danoff, Anna Keune*
and *Amanda Lyons*

*Peer-produced Learning*

We have been writing the missing manual for peer-produced peer learning: the "Peeragogy Handbook" (peeragogy.org). Throughout this work we have asked and aimed to address questions like these: What would a motivated group of self-learners need to know to agree on a subject or skill to learn, find and qualify the best learning resources about that topic, then select and use appropriate communication media to learn it together? What would these people need to know about learning to put together a successful learning programme?

*It is clear to us that the techniques of 'peer production' that have built and continue to improve Wikipedia and GNU/Linux have yet to fully demonstrate their power in education.* We believe that the Peeragogy Handbook can help change that by building a distributed community of peer learners/educators, and a strongly vetted collection of best practices. Our project complements others' work on sites like Wikiversity and P2PU, and builds upon understandings that have developed informally in distributed communities of hobbyists and professionals, as well as in (and beyond) the classrooms of generations of passionate educators.

Here, we present Peeragogy in Action, a project guide in 4 parts. Each part relates to one or more sections of our handbook, and suggests activities to try while you explore peer learning. These activities are designed for flexible use by distributed groups, collaborating via a light-weight infrastructure. Participants may be educators, community organisers, designers, hackers, students, seasoned peeragogues, or first timers. The guide should be useful for groups who want to build a strong collaboration, as well as to facilitators or theorists who want to hone their approach. Together, we will use our various talents to build effective methods and models for peer produced peer learning. Let's get started!

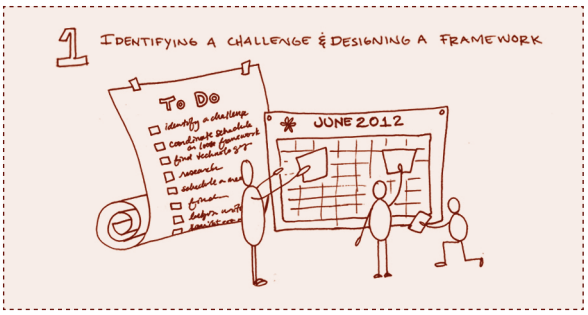
**Part 1: Identifying a Challenge and Designing a Framework**

Setting the initial challenge and building a framework for accountability among participants is an important starting point.

- Activity—Come up with a plan for your work and a 'contract' for your group. You can use the suggestions in this guide as a starting point, but your first task is to revise the plan to suit your needs. Helpful questions can be: what are you interested in learning? What will your main outcome be? What problem do you hope to solve? What steps do you need to take to accomplish this? How collaborative does your project need to be? What sort of support do you anticipate needing personally? What problems won't you solve?
- Technology—Familiarise yourself with the collaboration tools you intend to use (e.g. Wordpress, Git and LaTeX, YouTube, GIMP, a public wiki, a private forum, or something else) and create a first post, edit, or video introducing yourself and your project(s) to others in the worldwide peeragogy community.
- Suggested resources—The Peeragogy Handbook, parts I ('Introduction') and II ('Peer Learning'). You may also want to work through a short lesson called 'Implementing Paragogy', from the early days before the Peeragogy project was convened (https://en.wikiversity.org/wiki/User:Arided/ImplementingParagogy). For a succinct theoretical treatment, please refer to our literature review, which we have adapted into a Wikipedia page (http://en.wikipedia.org/wiki/Peer_learning).

-----➤ Further reading—Boud, D. and Lee, A. (2005). 'Peer learning' as pedagogic discourse for research education. Studies in Higher Education, 30(5):501-516.

-----➤ Observations from the Peeragogy project—We had a fairly weak structure at the outset, which yielded mixed results. One participant said: "I definitely think I do better when presented with a framework or scaffold to use for participation or content development." Yet the same person wrote with enthusiasm about models of entrepreneurship: "freed of the requirement or need for an entrepreneurial visionary." In short, there are trade-offs to be made—hopefully in an informed fashion.




**Part 2: Inviting Others**

Other people can support you in achieving your goal and make the work more fun too.

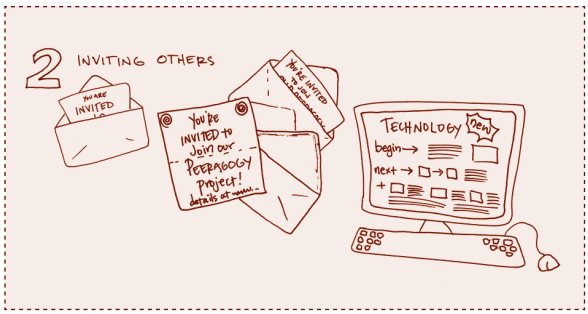
-----➤ Activity—Write an invitation to someone who can help with your project. Clarify what you hope to learn from them and what your project has to offer. Helpful questions to consider: What resources are available or missing? What do you already have that you can build on? How will you find the necessary resources? Who else is interested in these kinds of challenges?

-----➤ Technology—Pick a tool that's new to you and could potentially be useful during the project. Start learning how to use it. Locate some people around the world who share similar interests.

-----➤ Suggested resources—The Peeragogy Handbook, parts III

('Convening a Group') and IV ('Organizing a Learning Context').

- Recommended reading—Schmidt, J. Philipp. (2009). Commons-Based Peer Production and education. Free Culture Research Workshop Harvard University, 23 October 2009.
- Observations from the Peeragogy project—We used a strategy of 'open enrolment': new people were welcome to join the project at any time. We also encouraged people to either stay involved or leave—several times over the past year, we required people to explicitly reaffirm interest in order to stay registered in the forum and mailing list. This choice cut down on 'dead weight'. Nevertheless, the project continued to accumulate content, which gave newcomers the discouraging feeling that there was a lot to catch up on. We've aimed to sum up the high points in the handbook!



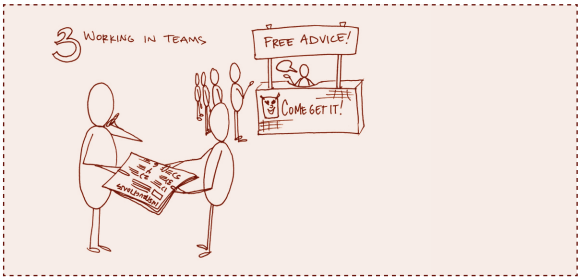

**Part 3: Working in Teams**

Solidifying your work plan and learning strategy together with concrete measures for 'success' can move the project forward significantly. Working in teams and sharing information with others will help you to develop your project.

- Activity—Concretise your ideas by, for example, writing an essay, making visual sketches, or creating a short video to communicate the unique plans for organisation and evaluation that your group will use. Then, edit the pages of the Peeragogy Handbook boldly: by this time you should have identified at

least one section that needs to be improved. Make the necessary revisions.

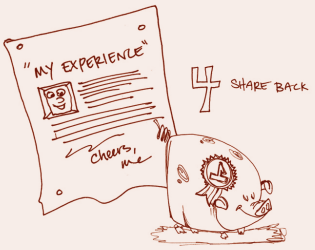
- Technology—Take time to mentor others or be mentored by someone, meeting up in person or online. Pair up with someone else and share knowledge together about one or more tools. You can discuss some of the difficulties that you've encountered, or teach a beginner some tricks.
- Suggested resources—The Peeragogy Handbook, parts V ('Co-Facilitation and Co-Working'), VI ('Assessment'), and part VII ('Patterns, Use cases, and Examples').
- Recommended reading—Argyris, Chris. "Teaching smart people how to learn." Harvard Business Review 69.3 (1991); and, Gersick, Connie J.G. "Time and transition in work teams: Toward a new model of group development." Academy of Management Journal 31.1 (1988): 9-41.
- Observations from the Peeragogy project—Perhaps one of the most important roles in the Peeragogy project was the role of the 'Wrapper', who prepared and circulated weekly summaries of forum activity. This helped people stay informed about what was happening in the project even if they didn't have time to read the forums. We've also found that small groups of people who arrange their own meetings are often the most productive.




**Part 4: Share Back**

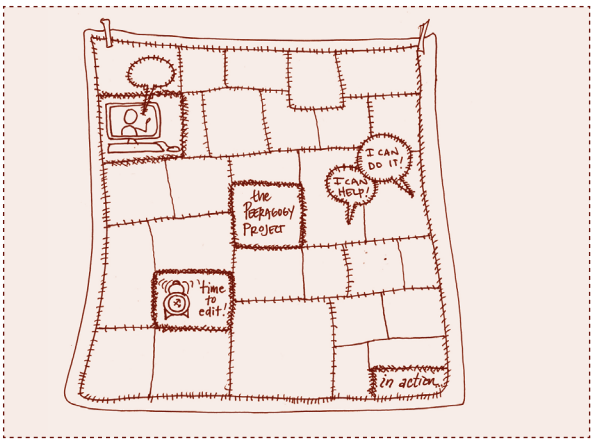
Wrap up the project with a critical assessment of progress and directions for future work. Share any changes to this syllabus that you think would be useful for future peeragogues!

- Activity—Identify the main obstacles you encountered. What are some goals you were not able to accomplish yet? Did you foresee these challenges at the outset? How did this project resemble or differ from others you've worked on? How would you do things differently in future projects? What would you like to tackle next?
- Writing—Communicate your reflection case. Prepare a short written (or video, or photo, ...) essay, dealing with your experiences in this course. Share the results by posting it where others in the broader Peeragogy project can find it.
- 'Extra credit'—Contribute back to one of the other organisations or projects that helped you on this peeragogical journey. Think about what you have to offer. Is it a bug fix, a constructive critique, pictures, translation help, PR, wiki-gnoming or making a cake? Make it something special, and people will remember you and thank you for it.
- Suggested resources—The Peeragogy Handbook, parts VIII ('Technologies, Services, and Platforms') and IX ('Resources').
- Recommended reading—Stallman, Richard. "Why software should be free." http://www.gnu.org/philosophy/shouldbefree.html (1992).
- Observations from the Peeragogy project—When we were deciding how to license our work, various Creative Commons licences were proposed (CC Zero, CC By-SA and CC By-SA-NC). After a brief discussion, no one was in favour of restricting downstream users, so we decided to use CC0. In connection with this discussion, we agreed that we would work on ways to explicitly build 'reusability' into the handbook content.



**Micro-Case Study: The Peeragogy Project, Year 1**

Since its conception in early 2012, the Peeragogy Project has collected over 3700 comments in our discussion forum, and over 200 pages of expository text in the handbook. It has given contributors a new way of thinking about things together. However, the project has not had the levels of engagement that should be possible, given the technology available and the global interest in improving education. We hope that the handbook and this accompanying syllabus will provide a seed for a new phase of learning, with many new contributors and new ideas drawn from real-life applications.




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**Knowledge Media Institute, The Open University, UK**
**School of Art and Design, Aalto University, FI**
**Mr. Danoff's Teaching Laboratory, Chicago, USA**
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# Massively Distributed Authorship of Academic Papers

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## Abstract
Wiki-like or crowdsourcing models of collaboration can provide a number of benefits to academic work. These techniques may engage expertise from different disciplines, and potentially increase productivity. This paper presents a model of massively distributed collaborative authorship of academic papers. This model, developed by a collective of thirty authors, identifies key tools and techniques that would be necessary or useful to the writing process. The process of collaboratively writing this paper was used to discover, negotiate, and document issues in massively authored scholarship. Our work provides the first extensive discussion of the experiential aspects of large-scale collaborative research.

## Author Keywords
Collaboration, writing, crowdsourcing, scholarship.

## ACM Classification Keywords
H5.m. Information interfaces and presentation (e.g., HCI): Miscellaneous.

## General Terms
Human Factors, Management.

## Introduction
The way people collaborate continues to evolve rapidly with technology [21]. Many of the challenges that researchers in fields such as HCI face require

interdisciplinary input and collaboration [12]. In this paper, we argue that wiki-like and crowdsourced [7] models of collaboration can provide currently unrealized value to academic writing and research. We propose that such a model of engaging many people from different disciplines through short-time-scale cycles of collaboration can enable research and provide insights that go beyond what is feasible with the small-group model that is prevalent today.

Many massively multi-authored academic papers already exist, both as targeted experiments, such as the Polymath Project [9], and as the result of large-scale collaborations in particular disciplines, such as Venter et al.'s "The Sequence of the Human Genome" [23] (with contributions from approximately 2900 co-authors). However, these papers do not discuss the experiential aspects of such collaborations.

The current paper was written in a distributed fashion by thirty authors, with approximately ten additional people contributing in some way but ultimately opting not to be listed as authors. At the onset, the only aspects of the paper that were specified were the topic of the paper, the target venue (with format and deadline constraints), and a few principles about collaboration. Writing the paper provided the research context, and the final paper emerged as a research artifact. We have sought to engage in a conscious reflection on the authorial process, and to identify possibilities and challenges that arise at the juncture of academic research, distributed authorship, and digital technology.

The goals of this paper are twofold. The first goal is to discuss the methodology and tools that enable an *ad hoc* group of academic researchers and other interested individuals to gather, work together and produce a scientific paper collaboratively. Working together with a large number of authors, we must take into account the preferences and ideas of a broad cross-section of researchers, a process known to be challenging [10, 19]. The second goal is to experiment with this type of collaboration, including empirically identifying and validating best practices, and present reflections on the process.

Our work provides the first extensive discussion of the *experiential* aspects of large-scale collaborative research. Our findings were established via a research-through-writing approach in line with earlier discussions of research-through-design [26]. This work is of value to CHI practitioners seeking to design tools and environments for large-scale collaboration on scientific research and writing, by revealing the opportunities and complexities of this process.

## Related work

Collaborative writing has been around since well before digital computation, but contemporary projects such as Wikipedia and Crowdforge [13], have demonstrated that crowdsourcing could be used to create written content of broad scope and high quality.

Although work on the social aspects of *coordination in collaborative writing* in the Wikipedia context [24] particularly informed the analysis we give in this paper, we see many differences between writing for a general audience without deadlines, and writing a peer-reviewed submission with deadlines. These differences suggest different interaction designs. The closest empirical paper of which we are aware examined the

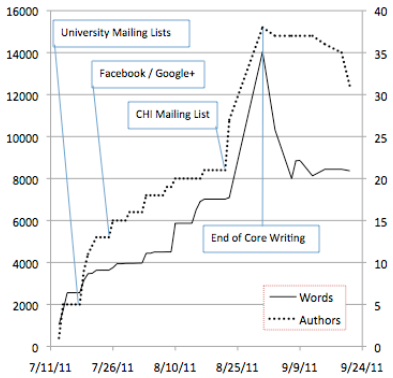



**Figure 1:** The change in word count and number of authors over time. Significant events in the recruiting of authors are noted in callout boxes.

use of a collaborative online tool [3], but for an artificial writing task instead of a research publication. By contrast, we found it most convenient to gather data on collaborative writing in a research context by devising a real research situation in which we could make observations.

*Coordination Mechanisms*

In current academic practice, significant grants are often awarded to multi-institution and multi-discipline groups. In these projects, which can span years, there are often group meetings or phone calls to set an initial direction, with each institution taking a specific aspect of the work and developing it largely independently. Small teams can work more closely, but this takes significant effort (in terms of availability, direction, and roles). These examples suggest different models of coordination, as Van de Ven and others [22] have distinguished: e.g., pooled, sequential, reciprocal, and team modes.

With complex work such as authoring an academic paper, coordination is key to producing coherent and high-quality output. Different circumstances may have different coordination mechanisms that are suitable, for example: markets (first come first served), hierarchical management (project leaders), standardization (pre-defined templates), communication (discussion pages and talk between authors), and shared mental models (initial structure provided by type of community or template) [6, 14, 15]. This project used a mix of coordination mechanisms, as described below.

## Process

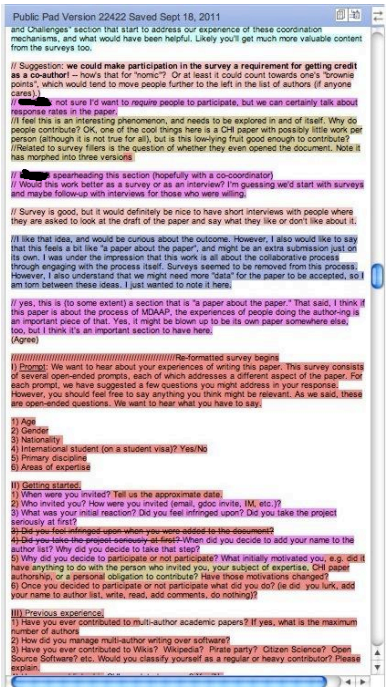
To provide a scaffold for the writing of this paper, a draft of an abstract and a few basic collaboration policies were established by the initial corresponding author (ICA) of the project, and specified in a non-editable web page: http://www.ics.uci.edu/~wmt/CHI2012CollaborativePaper.html.

The initial goal was to write a CHI 2012 full paper. After asking several colleagues for feedback on the core principles, the ICA began recruiting other authors through posts to mailing lists and existing social networks (recruiting in a "snowball" fashion) (see Figure 1).

During the first several weeks of the writing, we the authors collaborated using a Google Doc (viewable at http://goo.gl/RIUwC). However, with an influx of new authors responding to a post to the CHI mailing list, the Google Doc encountered an apparently unfixable error, perhaps having to do with the large number of authors (>25) attached to the project. The production was therefore moved to Etherpad, specifically, to the public installation hosted on PiratePad.net. EtherPad allowed us to finish a complete draft of the paper. Unfortunately, PiratePad.net produced an error that prevented the team from viewing the history of the document.

In the final weeks, an author survey was conducted and integrated into the paper. The paper was formatted and shortened to conform to the page limit, with abundant additional written content, notes, references, etc. remaining available in the online versions of the paper (viewable at http://piratepad.net/Massively-Distributed-Authorship-of-Academic-Papers).

The final edits were completed via an online, shared, and synchronized folder (Dropbox), in which


Public Pad Version 22422 Saved Sept 18, 2011
and Challenges' section that start to address our experience of these coordination mechanisms, and what would have been helpful. Likely you'll get much more valuable content from the surveys too.
// Suggestion: we could make participation in the survey a requirement for getting credit as a co-author! -- how's that for "nomic"? Or at least it could count towards one's "brownie points", which would tend to move people further to the left in the list of authors (if anyone cares!) ;)
// [redacted] not sure I'd want to require people to participate, but we can certainly talk about response rates in the paper.
//I feel this is an interesting phenomenon, and needs to be explored in and of itself. Why do people contribute? OK, one of the cool things here is a CHI paper with possibly little work per person (although it is not true for all), but is this low-lying fruit good enough to contribute?
//Related to survey fillers is the question of whether they even opened the document. Note it has morphed into three versions
// [redacted] spearheading this section (hopefully with a co-coordinator)
// Would this work better as a survey or as an interview? I'm guessing we'd start with surveys and maybe follow-up with interviews for those who were willing.
// Survey is good, but it would definitely be nice to have short interviews with people where they are asked to look at the draft of the paper and say what they like or don't like about it.
//I like that idea, and would be curious about the outcome. However, I also would like to say that this feels a bit like "a paper about the paper", and might be an extra submission just on its own. I was under the impression that this work is all about the collaborative process through engaging with the process itself. Surveys seemed to be removed from this process. However, I also understand that we might need more "data" for the paper to be accepted, so I am torn between these ideas. I just wanted to note it here.
// yes, this is (to some extent) a section that is "a paper about the paper." That said, I think if this paper is about the process of MDAAP, the experiences of people doing the author-ing is an important piece of that. Yes, it might be blown up to be its own paper somewhere else, too, but I think it's an important section to have here.
(Agree)
//////////////

whose names appear on the final paper unanimously responded that they would like to participate in distributed, collaborative online writing in the future.

Some problems arose due to the "recursive" nature of this paper, which confused some co-authors (and reviewers). The writing process resembled a game of Nomic [20], where the game is about changing the rules of the game itself. For example, arguments in one section of the paper might use quotes from another section as evidence. One author suggested that the outcome did a "poor job [of] collectively replicating Malcolm Ashmore's Reflective Thesis" [2]. The self-reflective nature of the papers led one of the CHI reviewers to describe the paper as "Seinfeldish".

### Distributed Authorship, Distributed Authority

Along with the relatively low levels of parallel communication about the paper (most discussion happened in the paper itself, with limited backchannel discussions), authors also felt a lack of any centralized control directing the paper. Contrasting it with other multi-author efforts in which he was involved, one author said that "with this paper, no one's in charge." Not having a centralized voice of authority became problematic in that the central thesis of the paper was not always clear.

Some appreciated this: one author enthusiastically compared the writing process to participation in a seminar. Others pointed to the concerns about editorial authority associated with removing content and finding consensus in multiple voices. In the words of one author, "I think it is quite chaotic to find a common structure and it is sometimes unclear: 3 authors comment and who decides finally? The main author?" These responses suggest an implicit assertion that it is necessary for the author(s) working on each section to understand how that section fits into the larger structure of the paper. The intermittent sense of confusion about the paper's aims reflects a certain assumption that many of the authors on this paper have about the academic writing process: that in order to be effective, all authors must have a clear understanding of the larger argument to which they are contributing.

### Medium of Collaboration

In writing this paper, the authors discussed a number of potential tools —existing and hypothetical— that could support the distributed authorship of academic papers. The most common type of tool considered were systems that support (simultaneous) multi-editing (cf. [17]). Some authors also suggested using a system like Scribtex (scribtex.com), though this would require working knowledge of LaTeX and so could have added a barrier to entry for participating in the collaboration. Other systems such as the CoWord plug-in for Microsoft Word could also have enabled simultaneous writing and editing.

However, it is possible that for the number of authors on the scale of this particular paper, viewing live simultaneous edits may be less important than simply having on-demand access and being able to track contributed text (who wrote what). For this, version control systems, such as Assembla (assembla.com) or Git might have been more effective.

Other tools can be integrated into the writing process to support particular sub-tasks. For example, the authors used Zotero (zotero.org) to collaboratively

build a reference library: https://www.zotero.org/groups/mdaap.

Although online tools appear to be a natural solution to the problem of working across time zones, they are, in their current state, lacking in support for multiple authors and voices. Authors supplemented the messiness of multi-authorship through other tools, such as EtherPad's built-in chat and e-mail. One author told us, "The chat was really nice—it was better than commenting in the paper directly since it didn't add additional 'noise'."

The technology, together with our governance choices, led to an emergent text that only weakly permits tracing of individual contributions. It was not obvious if a particular piece of text came from a graduate student, a researcher in industry, or a professor (which could be seen as a good thing). Figure 2 above illustrates how individual contributions were marked in EtherPad.

## Challenges of Integrating with Existing Practices and Publishing Infrastructures

This experiment helped emphasize how existing practices and infrastructures for the scientific publication process in our discipline are designed with relatively low numbers of co-authors in mind, and generally do not effectively support this kind of writing. For example:

### Online Submission Systems

Manually entering author data for a paper with thousands, hundreds, or even tens of authors can be a tedious process (if managed by one author), as supported by the current submission systems.

### Peer-review Practices

Many journals and conferences require anonymized submissions; however this type of article is difficult to make anonymous because a large number of researchers have been aware of the research effort (e.g. the author recruitment process involves broadcasting to a large community). Reviewer assignment is also challenging; many qualified reviewers may have been involved in the process or may have conflicts of interest with at least one of the co-authors. This issue can become problematic for program committee meetings as well, where members of the committee who have a conflict of interest in evaluating a paper must not participate in the discussion of the paper.

### Impact Measurement Practices

Some current practices for assessing the productivity and impact of researchers do not consider their relative contributions to papers. Two standard indices for measuring the impact of researchers, H-Index and G-Index, do not take into account the differences in the levels of contributions of co-authors. Thus, the growth of this form of scholarship may make these indices less useful as an estimate for academic impact.

The author list is not blank, nor replaced with a collective pseudonym (like *Bourbaki or D.H.J Polymath*), because the authors wished to acknowledge and take credit for their contributions, and agreed to be scientifically accountable for the content as a whole. We agreed that the author list should be ordered by placement "bids" from authors, with the ICA's judgment used to resolve ties.

DIGITAL LIBRARIES AND PUBLICATION FORMAT

Reference formats are also designed for small numbers of co-authors. For example, the APA and the Vancouver convention [11] limit the number of authors. This can demotivate authors who will not be among the top listed authors of a paper. Large author lists take up valuable space in the original publication, but will almost certainly not appear in future citations due to the annotation effort and/or space required. Moreover, even when browsing through papers in digital libraries or indexing systems that provide full metadata of papers, it may be of little benefit to browse through a long list of authors, not knowing how each of them has contributed to the paper.

## Discussion

We will now take a look at various strategies that can be used to mitigate the risks associated with our approach of wiki-like massively distributed authoring of papers. We begin by summarizing the main challenges.

*1. More authors means more content, but also more words thrown away.* Many of the words written by authors were deleted during the ongoing editing process. The sheer mass of deleted words might raise the question of whether authoring a paper in such a massively distributed fashion is efficient.

*2. Technology provides inadequate support for distributed authoring.* During the time we have been working on this project, we have tried, discovered, or created various tools for collaboration, but none of them appeared to be a silver bullet for all our needs.

*3. Task and domain differences.* There is a distinction between "collaboration on research" and "collaboration on writing". Although one normally follows the other, certain models of widespread collaboration might focus on, say, outsourcing experimentation (e.g., ScienceExchange.com), or ask for expert hands/eyes on certain parts of a paper (e.g., literature review, data analysis).

*Strategy 1: Know Thy Collaborator (and Granular Contributions)*

This project had many contributors, most of whom did not know one another. Additionally, several were from outside of HCI, and at least one from outside academia. The nature of the project meant that most participants added miscellaneous comments or perhaps took ownership of a section. At the end of the process, a small set of authors did significant restructuring and rewriting. Traditional collaborators often know and trust one another, with clearly designated leaders or first authors. Future collaborations should further explore the role of management (and associated functions such as task decomposition and assignment), relative to measures such as quality, coverage, coherence, and creativity.

*Strategy 2: Improved Tool Design*

From our experience it is clear that a system for writing massively distributed academic papers needs some way of maintaining and explicating the current state of the paper. At a high level, incoming authors need to understand what "phase" the paper is in (e.g., planning, writing, editing). Future tool design might look to decision support system literature [18] for mechanisms to aid the collaborative process. Sections that are "complete" may also need to be safeguarded, and potential editors pointed towards areas that still need further work. Having a strong sense of structure

could better enable such collaboration—although this paper began with a rough (section-level) outline, the structure became harder to follow as more sections were added and comments were written in-line. We attempted to maintain an up-to-date high-level outline of the paper as time went by, but a more sophisticated tool could have automated this process. More broadly, there is currently no simple way to curate or challenge a portion of the text, elicit responses from the authors, and apply changes that agree with a majority opinion or the opinion of more senior contributors that the authoring community might wish to empower.

*Strategy 3: Develop Suitable Coordination Mechanisms*
In writing this paper, it seems that communication between co-authors happened mostly in the context of the document itself. However, we also found that when authors come and go in the online environment, previous discussions are often brought up that had already been resolved, making the process repetitive at times. It has been a common practice in collaborative authoring tools to provide various forms of annotations to support the coordination of the authoring process [4]. Here, the primary method of coordinating activities was free-form annotations in the document. No annotation schema or rules were defined in advance, and the participants used various methods for distinguishing notes from content, such as using font styles (italic, bold, all-capitals), programming conventions (e.g., starting with //) or changing font color. For higher-level coordination needs, towards the end of the writing process, the ICA began using email to send out alerts and keep writers-up-to-date on process. This switch from the initial "hands-off" approach provided the structure we needed in order to meet deadlines. Further thought about how to best shape and signpost the project (both from a "writing" and a "research" standpoint) would help in future efforts.

## Future work

We have sought to identify challenges and opportunities in massively distributed authorship of academic papers, as well as reflections on the process and considerations for future researchers. Further experiments and repetition are required both to validate the analysis suggested here and to explore the relationship between this form of collaboration and different forms of research. For example, what research disciplines are best supported by this method of collaboration? What forms of research (e.g., user studies, design visions)? Future efforts along these lines might try some alternate methodologies (e.g., discourse analysis [25]).

Future work is also needed to develop the tools that can enable massively distributed authorship. For example, a website to act as a clearing-house for in-progress papers, ideas, or initial findings, would give authors a central location in which to contribute, as well as find new collaborators or stumble across relevant ideas, potentially aiding serendipity, insight, and discovery (as in [1]). Building an authorship community interested in shared process and/or content may help address the indifferent feelings many authors experienced about participation in this experimental effort.

Finally, this form of collaboration could also be extended to other parts of the academic publication process. For example, additional features could be integrated with WikiCFP (wikicfp.com) so that authors

can collectively determine what papers are being written for a given conference. Conferences and workshops could also produce post-event position papers written by the participants collectively, or include it in their proceedings a collaboratively written summary of the event.

## Conclusion

Massively collaborative crowdsourced projects, such as Wikipedia and Linux, affect aspects of many people's everyday lives. Open Data is a related emerging practice in the hard and computational sciences. The corresponding practices for "human sciences" are at present somewhat less clear.

In this paper, we consider one aspect of massively collaborative research that has potential to alter the way we work and discover: authoring. We identify challenges in: participation models for crowdsourced authorship; tool design and coordination mechanisms for supporting stages of a research and authoring process; and piecemeal contribution versus ownership and engagement. There are philosophical and practical questions with regard to contribution and authorship that we have only begun to discuss.

Academia itself is a massively collaborative undertaking, but massively multi-author writing of the kind described in this paper is relatively new. Nevertheless, with the spread of communication tools that facilitate various aspects of the process, and the exploration of new techniques for working together on large scales, this form of scholarship may begin to play an increasingly salient role in the pursuit of knowledge. When deployed well, collaboration can help our community become even more effective in its efforts to "contribute to society and human well-being" [5]. Research funds should be targeted towards developing more effective and rapid means to solve a range of research problems. Without sufficient investment in this domain, we will be bypassing some of the great advantages of living in a networked world.

## Acknowledgments

Many more people contributed to this paper than those currently on the author list. At times, the paper had as many as 38 people on the author list, and many others read drafts and propagated information about the project. The CHI reviewers provided helpful insights as well, and some of their ideas are in this version of the paper (although since they were anonymous, we were not able to invite them to be on the author list.) The authors thank the numerous institutions and other entities that have provided support for their respective participation in this research.

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# PlanetMath/Planetary

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**Abstract.** This paper presents our work in progress on the Planetary system, along with a critical evaluation of the project relative to its stated goals and the goals of one of its main "clients", PlanetMath.org.

**Keywords:** mathematics, learning environments, encyclopedias, Drupal

## 1 Introduction

The metaphor or analogy implied by the title of this paper is that PlanetMath[3] is like an operating system for mathematics, and Planetary[4] is like a kernel for this system (think GNU/Linux). This analogy is both apt and sloppy.

In particular, PlanetMath is not a complete mathematics "userland". Indeed, it is best known for its mathematics *encyclopedia* which contains over 9000 entries, and defines around 16000 concepts (see [1]). Another key feature of PlanetMath is that every entry is *discussable* via its own attached, threaded, forum. PlanetMath has a variety of other features (like mathematics rendering, a term autolinker, and a workflow and authority model suitable to distributed encyclopedia authoring), most of which were developed in a custom system based on Perl and XSLT (called "Noösphere"), which was written up in Aaron Krowne's 2003 Master's thesis [2]. While this feature set has provided a (mostly) stable and functional basis for a popular community mathematics website for over a decade, the custom nature of the software made extensions and adaptations relatively scarce.

In 2010, the present first author was beginning a Ph. D. project on "Semantic Adaptivity and Social Networking in Personal Learning Environments" that aimed to extend PlanetMath so that encyclopedia entries were "connected to exercises and applications, preliminary materials, and resources for further learning", and to develop software that would track individual performance and provide personalized advice based on aggregated data.

Due to the extensive (and intensive) nature of software modifications that would be required to do this well, he welcomed the possibility to collaborate with Michael Kohlhase and his team at KWARC[5] on a complete re-build of Noösphere, using contemporary web frameworks, and integrating the "KWARC

[3] PlanetMath.org (2001-ongoing), http://planetmath.org
[4] The Planetary System (2010-ongoing), http://trac.mathweb.org/planetary
[5] http://kwarc.info

stack" of semantic technologies into PlanetMath. This led to an early prototype of the Planetary system being named a finalist in Elsevier's Executable Papers Challenge [3], but it was not until this year that an end to the PlanetMath rebuild appeared to be within sight.

The rest of this paper will descibe our technical achievements to date, discuss the immediate road ahead, and reflect on Planetary's potential, particularly from the point of view of its use on PlanetMath. (The reader is referred to [4] for a contemporary high-level overview of Planetary as a whole.)

## 2 A new "kernel" for math on the web

As indicated above, one of our main goals behind rebuilding PlanetMath's software was to be able to more easily bring new developers into the project. Another was to integrate new technologies.

After our first round of prototyping, Drupal 7 emerged as a good candidate solution for both of these issues. It is a popular system, with a wide variety of contributed modules – and it also supports a healthy marketplace for professional services. So far, the Planetary team has 14 contributors (most of them computer science students at Jacobs University, Bremen), with the current second author focusing on developing Drupal support for features and workflow similar to those found on PlanetMath.

We have found that there are some modules that can be installed and used directly, with minimal configuration (e.g. *privatemsg*, for the exchange of private messages between users) – others needed to be custom-built (e.g. support for *corrections*, essentially a custom form of bug report used to maintain accuracy and quality in PlanetMath's encyclopedia). Some others, like the *userpoints* module can be installed and used with minor tweaks. All in all, we depend on around 25 existing contributed modules, and have written a comparable number of custom modules. For a few legacy features, we took things in a new direction:

1. For mathematics rendering, we are using LaTeXML[6]. Full support for MathML lays the foundation for many other future services. (For example, our Executable Paper demo integrated JOBAD, a Javascript tool for interacting with mathematical documents while reading.)
2. In place of, or alongside, the legacy autolinking service, we have a new interactive ("semi-automated") autolinker, which should provide greater precision for links – and, again, open the door to a range of new interactive services during the document-authoring/editing process [5]. In addition, this feature is made possible by building on top of a real-time collaborative editor Etherpad, so we will get real-time collaboration on mathematics documents "for free".
3. For access to the encyclopedia by Mathematics Subject Classification (MSC), we used a new Linked Open Data (SKOS) implementation of the classification system [6]. This was motivating partly because it allowed us to develop

[6] http://dlmf.nist.gov/LaTeXML/

and demo an integration between Drupal, LaTeXML, and the Virtuoso triple store, which, again, will be useful in a range of future applications (e.g. we will be able to generate RDFa that can then be used to maintain backlinks, for example from an image to all of the articles that include that image).

There are some other features that have been developed for Planetary that are in use in other installation environments (namely, Michael Kohlhase's Computer Science courses), that are not yet integrated into our work on PlanetMath: specifically, an integration with SVN via TNTBase[7], which provides the ability to edit math on the web without ever opening a web browser, and a new books module, which provides support for lengthy documents. These features may eventually make their way into PlanetMath, but they can already serve to illustrate part of our goal in Planetary: to make a system that is useful in many different math-on-the-web contexts, with various features available in the various environments that need them.

One of our core aims in this regard is to make Planetary install "out of the box"; we are currently in the process of using Drupal's *profile* project to package up our work and support this. In future, this should be very helpful both for users and developers.

## 3 The road ahead

As we can seen in Figure 1, the latest version of Planetary captures many of the same interactions as the legacy version of the site, although the two sites are certainly not identical.

In the short months leading up to CICM in July, we aim to finalize the handful of features that remain unimplemented or partly implemented, and be ready to enter a "beta" with the new platform (i.e., ready to make it the software you see when you browse to PlanetMath.org). From a project management point of view, we have passed our last "milestone" and can now focus on feature-driven development. Our path to deployment on planetmath.org looks like this:

- May 4th: final module tweaking and building for features like requests, private messages, scoring, notices, and the object orphanage.
- May 11th: any final "alpha" features (e.g. finish integrating Etherpad), and a first round of optimizations.
- May 18th: A public "alpha" launch.
- June: Add any "beta" features that we need, in consultation with the PlanetMath user community.
- July: Final improvements readying the site for a switch over to the "beta".

[7] http://tntbase.org/

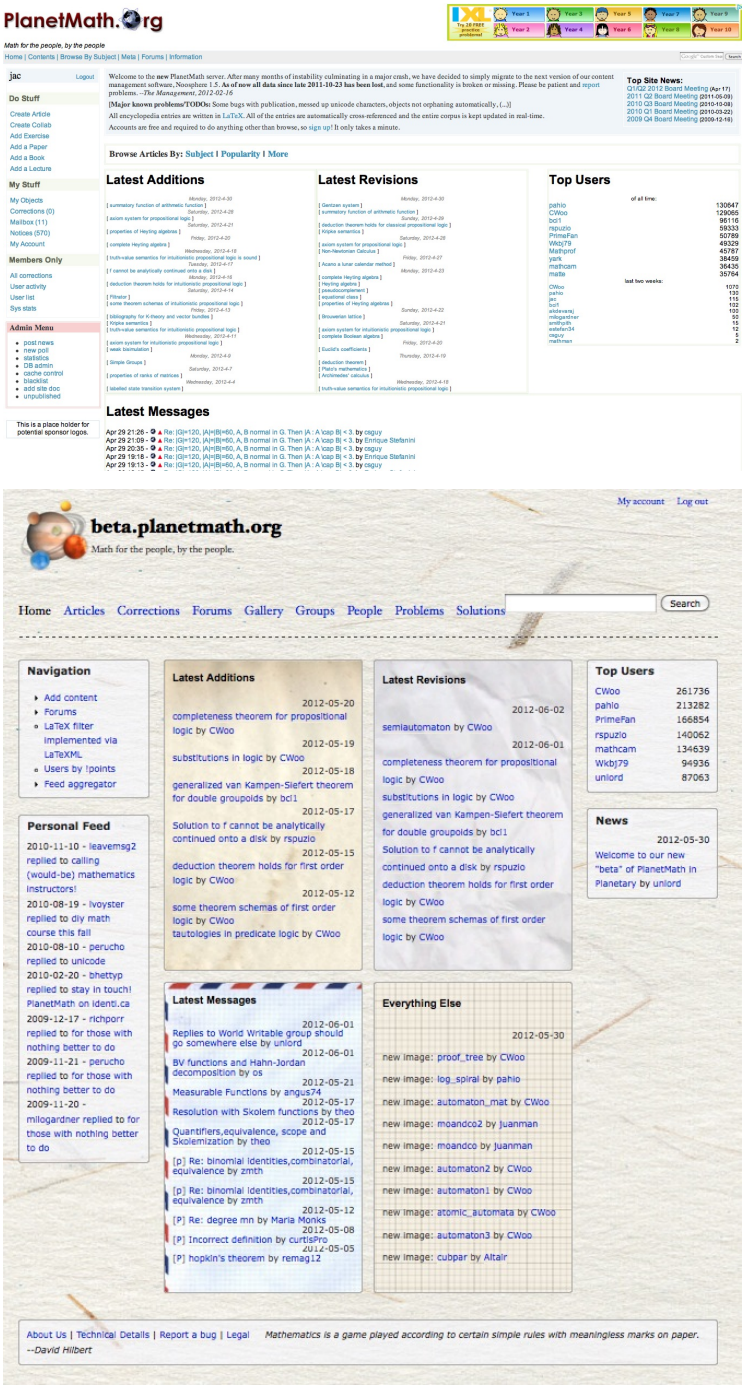



**Fig. 1.** The current PlanetMath webpage under Noösphere 1.5, and the new "beta" version.

## 4 A critical evaluation

The question we take up in this section is: is the Planetary system meeting the aims of its developers, and the needs of its user community? What could it do to improve?

Clearly, releasing the system on PlanetMath, and providing links the code[8] and installation instructions[9] should improve things dramatically. The new PlanetMath will render LaTeX blazingly fast, have better links, and a range of new features that address long-standing user concerns and also some nice surprises.

But our aspirations have in fact been much bigger – and switching to Planetary *should* play a big part in bringing them closer. Specifically:

- It is relatively easy to make new content types in Drupal, and we are introducing "problem" and "solution" node types, and allowing people to attach them to encyclopedia articles, and discuss problems with attached "questions" and solutions with attached "reviews".
- Our aim will be to develop some *semantically aware* activity tracking and "heads up" information for people using this system (see Figure 2). It is within reach to provide "related problems" using the MSC classification, but further analysis of theory dependencies (or approximations to the same) should allow us to give links to "simpler related problems", automating a key Pólya heuristic. Even without sophisticated tools, our hope is that a new generation of *students* will feel more encouraged to participate when problems and solutions become "first class" objects in the system.
- Our hypothesis is that the introduction of problems and solutions will provide a vital quality check, and enhancement. In short: encyclopedia articles that do not have attached exercises (or applications) should not necessarily be presumed to be useful. At the same time, exercises that do not have attached solutions may be too hard, i.e., the relevant subjects in the encyclopedia may not be sufficiently developed.

While we are *not there yet*, this is an example of the sort of thing we expect this technology together with the "encyclopedic approach", puts tantalizingly close to within reach.

[8] https://github.com/cdavid/drupal_planetary
[9] https://trac.mathweb.org/planetary/wiki/DrupalPorting

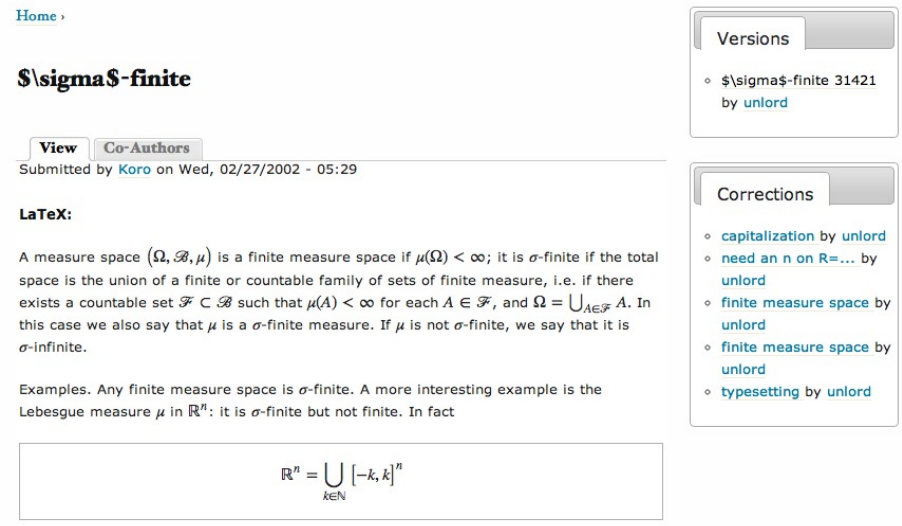



**Fig. 2.** Important information about articles (like outstanding corrections) in a "heads up" style display, shown here for a recent alpha version.

Note that PlanetMath, unlike, say, Wikipedia, is not *constrained* to be either "just a wiki" or "just an encyclopedia" – so, interactive problem sets and/or peer tutoring are welcome in PlanetMath, though they might not fit so cleanly within the existing Wikimedia family. Rather, the encyclopedic approach envisioned here connects interactions to a carefully curated and *systematic* knowledge base – in contrast with, for example, the StackExchange sites, at least in their current implementation.

In any event, the perspective developed above should brings up some big questions: in brief, what happens to mathematics teaching when students have access to a universal solutions manual for their mathematics course work? We may be able to *measure* whether lecture/homework/test is as effective for learning, as, say, participating in applied research projects.

Still, the hazard here would be to imagine that this can all happen overnight. It has taken PlanetMath 10 years to *define* 16000 terms, how much time will it take to provide a good *exposition* of those terms (always assuming that we do find users who want to participate in this process)?

Furthermore, as we have learned in the last few years, programming Drupal is not equally easy for everyone, documentation is not always clear (or available), and development work is generally a slow process (even with skilled programmers onboard). If the potentially revolutionizing changes (sketched above for mathematics education, but relevant also to research) of math on the web are

to be realized, most aspects of this project will have to scale up a lot – and hopefully coding will be less of a bottleneck.

## 5 Conclusion

We have described the Planetary system, and discussed its relevance to PlanetMath's continued project of building "a central repository for mathematical knowledge on the web, with a pedagogical slant." We expect the phase of work we will complete this summer to fully renovate and modernize PlanetMath. But once we have readied and deployed an extensible – and re-deployable – core, in a sense, our main work will just be beginning.

For example, we recall the meaning of "planet" from the blogosphere, i.e. planet-as-aggregator. Thinking in this way, PlanetMath might best fulfill its promise not *just* with a great new platform, but by successfully integrating content from other math on the web projects. This sort of aggregation service has yet to be realized, but forms a highly interesting direction for future work.

Indeed, if we are going to do anything about the "$500 million pricetag" for building a math-capable AI[10], we either have to bring about greater efficiencies, or spread the cost out over a relatively large number of people. Without making promises about just when this will be accomplished, we assert that we have, with PlanetMath and now Planetary, taken some vital steps in this direction.

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[10] http://theconversation.edu.au/if-i-had-a-blank-cheque-id-turn-ibms-watson-into-a-maths-genius-1213

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# Reimplementing the Mathematical Subject Classification (MSC) as a Linked Open Dataset*

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**Abstract.** The Mathematics Subject Classification (MSC) is a widely used scheme for classifying documents in mathematics by subject. Its traditional, idiosyncratic conceptualization and representation makes the scheme hard to maintain and requires custom implementations of search, query and annotation support. This limits uptake e.g. in semantic web technologies in general and the creation and exploration of connections between mathematics and related domains (e.g. science) in particular. This paper presents the new official implementation of the MSC2010 as a Linked Open Dataset, building on SKOS (Simple Knowledge Organization System). We provide a brief overview of the dataset's structure, its available implementations, and first applications.

## Introduction

The Mathematics Subject Classification (MSC [7]) maintained by Mathematical Reviews (MR) and Zentralblatt Math (Zbl) is a widely used system for classifying mathematical documents. All major mathematical journals and digital libraries use the MSC[8], mainly as a means of structuring literature in libraries and for the purposes of retrieving information by topic. As the original format of the MSC2010 hindered its automated use and maintenance, we have reimplemented it as a machine-readable linked open dataset, which will soon be announced as the single official implementation.

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[8] For details about this and about most other aspects of this project presentation, we refer to a full paper published recently [6].

*MSC Usage and Maintenance So Far* Previously, the right MSC class for a publication (e.g. 53A45 for "Vector and tensor analysis") was typically chosen by consulting a human-readable document. Web forms for uploading or searching publications typically required manual input of the desired MSC classes, rather than offering assistance. The source of the MSC had been maintained in one plain TEX file for almost 30 years. From this file, scripts produced several derived forms, including a KWIC index, a printable PDF, and HTML. Quality control of the source file with its custom macros could only be done by few experts, the derived forms did not particularly target machine processing, and the scripts that created them were once more custom, MSC-specific solutions.

*Requirements for a Reimplementation* After the *content* of the MSC2010 had been settled, MR and Zbl decided to technically reimplement the source, according to the following requirements: (1) **facilitate reuse**, i.e. facilitate access, search, queries, and auto-classification; (2) **facilitate maintenance**, i.e. preserve all of the original information, use a standard format supported by existing tools, enable integration of further maintenance information into the master source (such as the changes from MSC2000 to MSC2010); (3) **enable knowledge workers and service developers to adapt and extend** the MSC for their purposes, such as connecting mathematical subjects to related fields, adding descriptive labels in further languages, without affecting the core scheme; and (4) **allow end users to explore such connections** in order to **discover new knowledge**. We chose to reimplement the MSC2010 as an RDF Linked Open Dataset, using the W3C-standardized SKOS vocabulary (Simple Knowledge Organization System [8]), which had been in use in digital libraries for several years.

## Structure of the MSC/SKOS Concept Scheme

SKOS's built-in vocabulary covers a major part of the MSC concept scheme in a straightforward way; however, for full MSC coverage, we had to extend SKOS in an MSC-specific way. We summarize the design below but refer to [6] for full technical detail.

*Basics* The basic concept hierarchy was implemented as a SKOS concept scheme with 63 top-level concepts, which narrow down into 528 intermediate-level concepts, having 5606 final leaves. SKOS also supports collections of concepts besides the main hierarchy, such as "all historical topics". The 5-character MSC class numbers (e.g. 53A45) are used as URIs of the concepts for identifying them and making their descriptions technically retrievable (see below). SKOS allows for attaching multilingual labels to concepts; so far we have English, Chinese, Italian, and Russian, each from authoritative sources, and the RDF data model allows for maintaining further, non-authoritative translations separately. Finally, SKOS supports links across concept schemes, which we have so far used for making explicit the changes from MSC1991 and MSC2000 to MSC2010.

*Advanced Features* We go beyond the SKOS core, but follow established best practices, by linking MSC classes to concept schemes that have not yet fully been implemented in SKOS (the Dewey Decimal Classification). We extended the SKOS vocabulary with partitive relations, i.e. when a link to a related concept is restricted to a certain scope such as "numeric approximations" or "applications in physics". 0.4 percent of the concepts have labels containing mathematical markup, which cannot be expressed in Unicode but requires MathML. RDF supports XML in labels, but it conflicts with multilinguality – a problem unsolved so far. Finally, we attach some information, for which we have not yet designed a dedicated representation, such as co-classification policies, as generic notes to the respective concepts.

## Available Implementations

*The Dataset* We generated the new SKOS master source of the MSC2010 with a Perl script that translated the old TEX source to RDF/XML. We publish the data in four complementary ways, aiming to address a large audience of users and developers. The project page at `http://msc2010.org/mscwork/` contains links to all alternatives. For **linked data access** (i.e. directly retrieving an RDF description of each MSC class by dereferencing its URI; cf. [3]), we split the SKOS master file into one file per MSC class and serve the latter with the `application/rdf+xml` MIME type. For *querying* the dataset, we expose it through a SPARQL endpoint, i.e. a standardized interface for querying RDF. For *browsing*, we offer a wiki frontend. Finally, for application developers, we offer the whole dataset for download as self-contained files in several formats.

Easy and reliable maintenance requires a master source without redundancy, e.g. the concept hierarchy should not be represented both top-down and bottom-up. Linked data browsing, as well as other uses in large-scale applications that use plain RDF without inferencing support for performance, require a richer dataset with most practically relevant entailments expanded. We automatically create this expanded version by applying a set of first-order rules implemented in N3, covering a subset of the SKOS semantics and our MSC-specific extensions.

*In Use* The new implementation of the MSC2010 is currently in use on two websites linked from the project page. Both have been developed independently using the RDF-aware Drupal 7 content management system. On the homepage of the School of Mathematics at Aristotle University Thessaloniki (AUTH), we annotated the scientific fields covered by the courses and the faculty's research interests in terms of MSC/SKOS. The new version of the PlanetMath encyclopedia (shortly entering beta testing) is powered by the Planetary system [4]. In the old version, MSC-based navigation accounted for estimated 5–6 percent of all accesses. We have now reimplemented this functionality in a more generic way: all article metadata (including MSC classifications) are maintained within LATEX preambles, the articles are transformed from LATEX to XHTML+RDFa, from which RDF is harvested into an RDF triple store, which also holds a copy

of the MSC/SKOS dataset. This approach is more flexible than the older static MSC access: our plan is to use SPARQL queries to expose specific slices of the encyclopedia content (e.g. "all articles by my co-authors in algebraic topology"). The following listing, slightly adapted from the actual situation in PlanetMath, shows a SPARQL query that returns the subclasses of the given MSC class (which would actually be a parameterizable variable), and the number of articles classified with them[9]:

```
PREFIX msc:  <http://msc2010.org/resources/MSC/2010/>
PREFIX skos: <http://www.w3.org/2004/02/skos/core#>
PREFIX dct:  <http://purl.org/dc/terms/>
SELECT DISTINCT ?subclass ?notation ?label COUNT(?article) WHERE {
  msc:53Axx skos:narrower ?subclass . # get subclasses; then, for each subclass:
  ?subclass skos:notation ?notation ;                # get the MSC class number
            skos:prefLabel ?label .                   # get the preferred label
  OPTIONAL { ?article dct:subject ?subclass } # get classified articles (if any)
  FILTER langMatches(lang(?label), "en")                   # only English labels
}
GROUP BY ?subclass ?notation ?label   # grouping just needed for COUNT() to work
```

## Conclusion and Future Work

We deliver the first implementation of the MSC that is easily comprehensible to machines. It does not only facilitate maintenance and development of novel services, but the rigorous conceptual modeling approach [6] also helped to uncover issues in the MSC conceptualization, which we have now fixed. We are now planning to do three things in parallel: (1) refining the dataset implementation by making the internal structures of the MSC even more explicit[10] and by adopting further best practices for modeling classification schemes in SKOS [9]; (2) supporting the MKM community in building applications that make use of the MSC, using our new implementation; and (3) interlinking the MSC dataset with further mathematical and non-mathematical datasets, particularly including the OpenMath Content Dictionaries [5] and DBpedia [2].

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[9] The number is determined using the *COUNT()* aggregation function provided by the ARQ extension to SPARQL [1]

[10] So far, our implementation makes two structural aspects more explicit than the old TEX source: the concept hierarchy, which was previously given implicitly by the numbering scheme, and the cross-references, parts of which were previously given in natural language. Our current implementation does not yet explicitly represent co-classification policies.

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# *Detecting mathematics learning online*

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## Abstract

The purpose of the paper is to design a rubric for assessment of informal learning in undergraduate level mathematics. Our proposed assessment strategy would support and parallel the mathematics learning environment we are developing at PlanetMath. PlanetMath is currently an open Web 2.0 system that consists of peer-produced encyclopaedia articles and forum discussions: we will add facilities for contributing textbook-style problems and solutions, and detailed activity tracking to model and support learning.

## Keywords

Mathematics, problem solving, assessment, learning, learning environment, knowledge analytics

## Introduction

We witness an increasing number of open online groups populated by self-organized and self-managed individuals who actively generate content in the attempt to foster learning communities open to all who wish to participate. These groups attempt to deploy the educational affordances (Kirschner, 2002) of Web 2.0 tools and technologies for information sharing, communication, content creation and collaboration. These affordances can enable participatory processes that can support multiple modes of learning in new open spaces, and blur the boundaries between production and consumption (Brown & Adler, 2008; Alevizou, Conole, Culver & Galley, 2009). Calls for learning approaches (especially those linked to socio-cultural theory) that are able to better exploit the educational affordances of Web 2.0 have been proposed (e.g., McLoughlin & Lee, 2007; Brown & Adler, 2008; Alevizou et al., 2010). However, claims about the educational and affective dimensions of online learning, e.g. that "these affordances stimulate the development of a participatory culture in which there is genuine engagement and communication, and in which members feel socially connected with one another" (McLoughlin & Lee, 2007, p. 667) invite further grounding in empirical research. Similarly, there seems to be a dearth of empirical studies addressing the learning-specific impact of Web 2.0.

In this paper, we will begin to tackle these broad issues in one concrete case. We describe an analytical framework for detecting when and how learning is taking place among participants in the open, peer-produced, mathematics website, PlanetMath.org. For this to work, we require learning-specific metrics: for example, "participation" is not in and of itself a particularly salient measure of learning. We follow the popular view that "mathematics is a language". In this view, learning mathematics can be modelled as a process of learning a specialized vocabulary, along with the technical grammar of proof.

We will flesh out this simple understanding in several ways. We agree with David Smith (2002, p. 5) when he writes, "learning mathematics is, first of all, learning, and only secondarily about mathematics." We will draw on microgenetic analysis, a collection of methods and techniques that have been used to study learning "as it happens" in a wide range of fields (Lavelli et al., 2005). Although we have not yet been able to validate this framework in practice, it has informed the design of the platform we are building, as we will describe below.

The paper is organized as follows: we describe PlanetMath and the ongoing project to turn it into a learning environment. We then discuss some theoretical perspectives that are relevant to our project and integrate these into our analytical framework. We conclude with some discussion of the implications of this endeavour for mathematical education.



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## Overview of PlanetMath

PlanetMath.org (http://planetmath.org/) was initially developed by Aaron Krowne during the course of his Master's studies at Virgina Tech. PlanetMath is a virtual community and peer-produced repository of mathematical knowledge (Krowne, 2003). Its central feature is a peer-produced mathematics encyclopaedia, which has been written by a group of around 300 volunteers, and now containing nearly 9000 entries defining over 15000 terms. Similarly to Wikipedia, its contents are available under the terms of the Creative Commons By/Share-Alike License, but as a subject-specific encyclopaedia, it tends to have more specialised content (e.g. detailed proofs) (Corneli, 2011).

## The transition from reference resource to learning environment

PlanetMath's aim is to make mathematical knowledge more accessible, therefore it is a natural step to adapt PlanetMath's strategy of peer production to improve the site's support for learners. The approach underway takes the existing knowledge base as its core resource and extends it with a range of new materials and interactions focusing on problem solving – "the heart of mathematics" (Halmos, 1980). In short, our aim is to build a knowledge-rich online learning environment for mathematics.

A *learning environment* is made of human practices and material objects (Warger & Dobbin, 2009). It is a part of the human ecology wherein learners work alone or with others, using resources to pursue learning goals. It hosts the interactions of different types of learners, who interact, not only with each other, but also with artefacts, technologies and content (Nardi and O'Day, 1999). This "hosted interaction" feature is what distinguishes a learning environment from a learning object repository or digital library. Historically, PlanetMath has indeed hosted some interactions, namely those having to do with authoring and discussing mathematical content: however, its largest user population consists of non-logged-in visitors, for whom it serves as a reference resource. In the future, we aim to provide more active support for hosted interactions that help people learn mathematics.

### The learning design: PlanetMath's objects and environment

We follow Conole (2008) in thinking of "learning design" as a way of creating and representing practice.

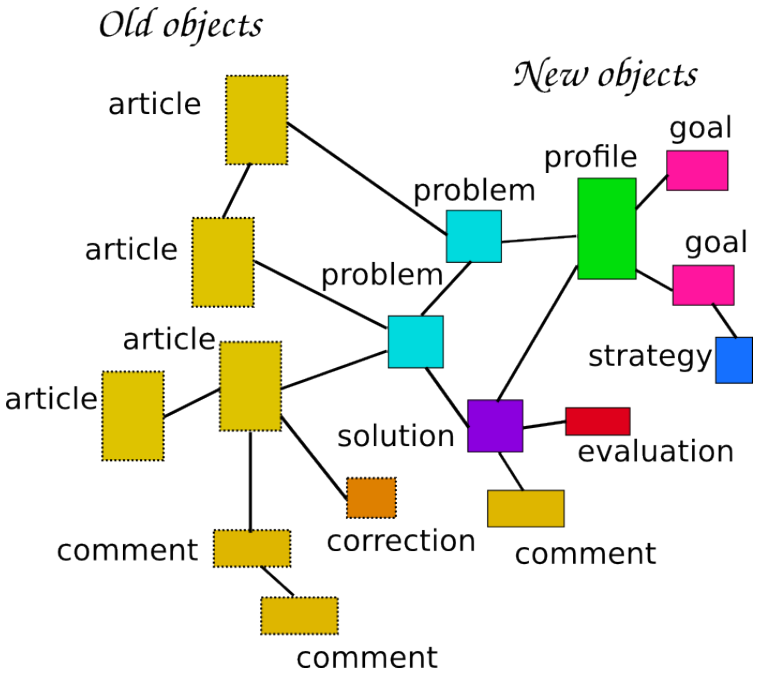



**Figure 1: The learning design in PlanetMath**


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The design we are using to transform PlanetMath from being primarily a reference resource into a new learning environment (Figure 1) will include all of the traditional activities associated with building and maintaining PlanetMath's encyclopaedia (e.g., writing and editing articles, forum discussions, corrections and requests), and a new "problem solving layer", which contains problems, solutions, links from the problems into the encyclopaedia (and vice versa), as well as discussions about problems (including evaluation or marking), and course packets that combine problems with expository material.

Learning in this environment primarily entails solving problems, though numerous other activities will have learning relevance, e.g. asking a question, giving or receiving advice, or improving an expository text. From the user's perspective, the environment will offer some significant advantages over a standard textbook or problem archive. For instance, nothing is more daunting than being faced with a problem to solve and not knowing what the terms in that problem mean. On PlanetMath, we will be able to provide automatically-generated links to the definitions of technical terms in problem statements. Frequently, such definitions are not enough, and the user will be able to make annotations asking for hints about how to use the definitions. These requests will help improve the quality and relevance of encyclopaedia articles. Over time, we expect a large archive of hints and worked examples to accumulate, overcoming the cold-start problem for learners. In particular, as generations of learners interact with the site, sharing and reflecting on strategies for meeting their learning goals, a collection of efficient learning pathways should emerge. Our hope is this archive and guide will help learners connect to mathematics in a meaningful way: rather than having textbook problems serve as a daunting obstacle to application or research, they should, in this context, serve as stepping stones to relevant and meaningful engagement in mathematical practice.

### Theoretical perspective: PlanetMath as epistemic object and space for knowledge building

In this section, we describe two theoretical concepts we are exploring to underpin our analytic framework. Firstly, we suggest that PlanetMath can be conceptualized as an *epistemic object* (Miettinen & Virkkunen, 2005). Miettinen and Virkkunen explain that epistemic objects or objects of inquiry are constructed and are not fixed, but are projected towards something that does not exist yet. We suggest that PlanetMath can become an epistemic object in this sense – an "instruction manual", whose process of development and refinement can contribute to transforming mathematics education. This is a process that develops by critique, questioning, and even dissent at the content level, as well as an ongoing and parallel process of tinkering on the infrastructural level. Conceptualizing PlanetMath as an epistemic object connects us to the socio-cultural historical approach, which examines the foundations of learning not in "abstract rules", but in material artefacts (Miettinen & Virkkunen, 2005). In this view, PlanetMath can be seen as a shared artefact objectifying forms of expression of cognitive processes and patterns that exist outside of the minds of individual participants. The process of learning mathematics can be embodied in this artefact's constituent tools and signs. Drawing on Vygotsky's (1979) concept of mediated action, a participant can internalize these, by interacting with other users, with content, and with tools. PlanetMath becomes a new means of mediating the activity of learning mathematics. One can imagine it providing many of the same features as textbooks (problems and exposition), as well as many features that textbooks cannot provide (like peer-to-peer conversations and mentoring).

Secondly, we suggest that the concept of *knowledge building* is relevant to examining the foundations of mathematics learning in such a concrete setting. Knowledge building refers to the production of objects (Bereiter & Scardamalia, 1996), which, in PlanetMath, include, for example, production of articles and solutions to problems. Similar to work in research and scientific communities, learners build knowledge by asking cutting-edge questions that help the community advance its collective understanding, by engaging in problem solving and by seeking to understand problems at deeper levels (Scardamalia & Bereiter, 2003). Again, this co-created knowledge, which is relevant to the goal of learning mathematics, is embedded in shared artefacts (e.g., structures, algorithms and proofs, explanations and justifications) (Bereiter & Scardamalia, 1996).

## Developing an analytical framework to detect learning

Viewing PlanetMath as a learning environment encompassing interactions among humans, technology and artefacts such as terms, articles, problems and solutions, our proposed analytical framework aims to examine the process of learning in a social context and to support the cycle of knowledge building in which learner-produced materials have an important role to play.

Our suggested framework consists of three layers. The first layer will keep track of what happens in the learning environment at the "micro-developmental" level, in the following way:


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- Examine user activities, such as solving a problem, asking a question, and giving a hint.
- Examine the vocabulary (specialized terms) that people use in texts they submit to the site.
- Assume that when this vocabulary is used in a solution that has been marked correct, then the vocabulary has also been used correctly.

The second layer is inspired by the analytical toolkit (ATK) used by Chan, Lee and van Aalst (2001) to examine knowledge-building inquiry and discourse among high school students using the Knowledge Forum. In addition to providing measures of activities in the Knowledge Forum, including notes written, notes read and keywords used, the toolkit assesses aspects of the development of collective knowledge, using an adaptation of principles designed by Scardamalia (2000). The toolkit aims to detect:

- *Working at the cutting edge*: participants work to advance individual and collective knowledge, and not to reiterate what is already known.
- *Progressive problem solving*: participants pursue problems at progressively deeper level, which means that when participant in the community understands a problem, s/he uses this learning to deepen the understanding of the problem and solve it.
- *Collaborative effort*: participants need to share understanding and to advance collective knowledge.
- *Identifying high points in the discourse*: participants note contributions that help them understand a problem better and advance collective knowledge.

Additional situational or relational features that can describe mathematical problem solving behaviour (Hosein, 2009) which fit at this layer include:

- *Processing level*: deep or surface learning, e.g. as evidenced by attempting to work through problems versus merely browsing solutions.
- *Quality of self-explanation*: the learner is able to explain why they are doing what they are doing.
- *Exploration*: the learner makes use of available tools and content.

We will aim to build software that can identify in an automatic way these aspects of learning and knowledge building. For example, site-wide analysis of vocabulary usage would enable us to identify someone who uses a completely new term to be "working at the cutting edge". We can also use the network of interlinked documents to understand how a given activity relates to another ("progressive problem solving").

We will extend our model of simplified "language acquisition" (Layer 1) and different measures of engagement (Layer 2), with a third layer, which deals with the substance of problem solving itself. Before describing this third layer, we make some remarks on how we plan to analyse the data we will be gathering.

### Microgenetic analysis

Open online learning environments provide researchers with access to all of the same data that participants use to communicate with one another (Stahl, 2006). Researchers have access to a natural data set similar to that provided by "talk-aloud protocols", which can provide detailed evidence of learning and development. This kind of data – and the informal, ad hoc interactions that generate it – matches well with some of the ideas and techniques of microgenetic analysis. The microgenetic approach studies learning as a process, rather than the outcome of a process (Lavelli, Pantoja, Hsu, Messinger & Fogel, 2002). The approach examines moment-by-moment changes observed in a short period of time, often for a high number of separate observations, but not necessarily subject to the same staged treatment patterns found in longitudinal studies. Observations tend to be analyzed intensively, both qualitatively and quantitatively. Microgenetic approaches have been used to take into account the social process of development, in which individuals learn concurrently in a distributed fashion (Fischer & Granott, 1995). This method seems particularly well suited to our informal peer-based learning context, where a pre-test/post-test method for assessing learning quality would generally be inappropriate or infeasible. Statistical analysis of this sort of data presents some unique challenges (Cheshire et al., 2007). A higher-level challenge appears when we try to take what we've learned and apply it to shape practice. We give an indication of how we plan to address this challenge in the following section.

## Applied heuristics: a Minskian approach


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The purpose of this section is to establish a third layer of analysis, which we would position above the use of specialised mathematical vocabulary and below social dynamics like "working at the cutting edge". Namely, here we look for engagement with the methods of mathematical problem. This layer includes the traditional "grammar of proof" (e.g. proof by induction or *reductio ad absurdum*). However, importantly, it also includes heuristics, and this is what we focus on in the current section. The treatment here is not exhaustive, indeed, a key feature of this layer is that it can expand to include any new heuristics we discover.

In one of his memos for the One Laptop per Child (OLPC) project, Marvin Minsky (2008-09) wrote:

> Children [...] learn words for various objects and processes – such as addition, multiplication, fraction, quotient, divisor, rectangle, parallelogram, and cylinder, equation, variable, function, and graph. But they learn only a few such terms per year – which means that in the realm of mathematics, our children are mentally starved, by having to live in a "linguistic desert." It is hard to think about something until one learns enough terms to express the important ideas in that area.

His concern, however, is, not merely with increasing the rate of vocabulary acquisition, but with learning ways to think and problem solve. He quotes Allen Newell (1955):

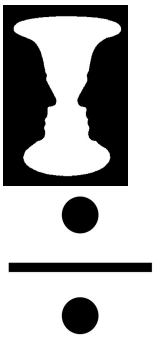
> The essential point of efficient learning is that, after you have solved a problem, it is not enough just to remember the answer: you need to remember the strategies that you used to discover that answer.

Minsky then proposes several example heuristics (ways of thinking about problems) that might be taught, or detected when they are employed (Table 1). He also suggests some meta-level heuristics:

- *Select appropriate representations*: Building an understanding of the problem or goal at levels ranging from deciding its domain, finding suitable parameters or making an intuitive sketch.
- *Find appropriate analogies*: Some of the most useful results in mathematics combine ideas from disparate sub-fields (like geometry and algebra); further, some say that mathematics is the science of patterns, so knowing how to look for useful patterns is a vital skill.
- *Deploy negative expertise*: Knowing what has failed to work in the past can be useful.
- *Construct more realistic self-models*: As one gains experience, one can understand better how one thinks.

These heuristics are of course not specific to mathematical problem solving per se, but the mathematical domain is indeed rather keenly focused on problem solving. In some sense, we could say that all "informal" mathematical speech represents use of heuristic reasoning. As with the four criteria from Chan, Lee and van Aalst (2001) mentioned earlier, we would like to be able to automatically detect patterns like "reasoning by analogy" or "deploying negative expertise". Many of these could be identified from textual features drawn from the informal parts of mathematical speech ("by analogy" or even "it easily follows"), or else via explicit discourse markers, following the example of the "Dangerous Bend" sign employed by Bourbaki (Kranz, 2011) – one use for the icons supplied in Table 1. In analysing a given text, we can also draw on some of the literature about technical language, such as, for example, Trimble's (1985) discussion of technical rhetoric.

If a problem seems familiar, try reasoning by analogy. If you solved a similar one in the past, and can adapt to the differences, you may be able to re-use that solution.



If the problem still seems too hard, divide it into several parts. Every difference you recognize may suggest a separate subproblem to solve.



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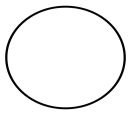
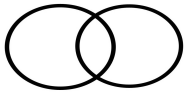
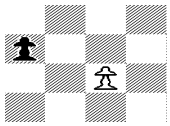
| | |
|---|---|
| If it seems unfamiliar, change how you're describing it. Find a different description that highlights more relevant information. |  |
| If you get too many ideas, then focus on a more specific example – but if you don't get enough ideas, make the description more general. |  |
| If a problem is too complex, make a simpler version of it. Solving a simpler instance may suggest how to solve the original problem. |  |
| Asking what makes a problem seem hard, may suggest another approach – or a better way to spend your time. |  |
| When your ideas seem inadequate, remember someone more expert at this, and imagine what that person would do. |  |
| Whenever you find yourself totally stuck, stop whatever you're doing now and let the rest of your mind find alternatives. |  |
| The best way to solve a problem is to already know how to solve it – if you can manage to retrieve that knowledge. |  |
| If none of these methods work, you can ask another person for help. |  |

**Table 1: Problem-solving heuristics suggested by Minsky, together with mnemonic diagrams**

## Discussion and Conclusion

In this paper we have described our strategy for converting PlanetMath into a learning environment suitable for self-directed, peer-supported study of undergraduate-level mathematics. In this endeavour, we aim to make sense of series (and, indeed, networks) of informal activities, which can include significant high-level thought.

We can contrast the approach that has been developed at Khan Academy (khanacademy.org/), where they have dealt with somewhat similar issues but in a very different learning ecosystem, based on expert-produced videos and workbook-style problems for grade school students. We can also contrast StackExchange and similar Q/A sites, where one-off technical questions are discussed (typically with links to "the answer").

In the PlanetMath case, we are aiming for something very different: co-construction of an increasingly complete mathematical artefact, dealing not just with "the answers" but with the processes and strategies for thinking about how to get to answers. Although we are not in a position to ensure the availability of peer support, we suspect that, on average, learners will be able to use PlanetMath to avail themselves of "observation, coaching, and successive approximation" (Collins et al., 1989). If these efforts are successful, significant changes in the way mathematics education works may follow. In particular, as we mentioned above, we hope that PlanetMath's "super-textbook" will help learners connect to mathematics in a way that is relevant and meaningful to them. Indeed, the idea that we could collect all textbook answers in one place does bring to mind a very interesting question, about why mathematics education in schools and universities has been focused on solving "textbook problems" that have already been solved many thousands of times before. Can improved use


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of technology give us something better? While we believe the answer here should be a fervent "yes", details will be determined in practice, not a priori. Still, we may at this time consider similar questions about other domains of knowledge. Can we develop similar "application layers", similar to our "problem solving layer", that re-use encyclopaedia content from Wikipedia or other knowledge resources in peer produced learning environments or interactive learning tools?

Turning this question around, what might general-purpose "social layers" for Open Educational Resources (such as p2pu.org or OpenStudy.com) glean from a subject-specific project like PlanetMath? It appears that the biggest strength of our proposed approach to detecting learning is its three layer framework: we can detect specialised vocabulary use, socio-technical features like "working at the cutting edge", and manage a catch-all category of heuristics which can be used to detect and talk about different ways of thinking. While there is a somewhat domain-specific mathematical flavour to this approach, we feel it poses an interesting model for educators and educational scientists working in other fields to consider adapting to their purposes. Further work will need to be done to establish whether this analytical framework can effectively detect patterns of learning.

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# Paragogy

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**Abstract.** Paragogy is a theory of peer learning: we endeavor to say how it works, and how it works best. This paper outlines paragogy's contemporary relevance and expounds its principles, showing their connections to other theories. We present an extended example of paragogy in practice, where we use it to evaluate our experiences working at the Peer 2 Peer University (P2PU).

**Keywords:** paragogy, peer learning, P2PU

## 1 Introduction

We use the term *paragogy* to characterize the critical study and practice of peer learning (literally, "para-" alongside, "-gogy" leading, here adapting the classical concept of *pedagogy* and the recent notion of *andragogy* [10] to a peer learning context). The fact that *παραγωγή* is a word in Greek meaning "production" shall not dissuade us from this new usage in English. Indeed, along with J. Philipp Schmidt, executive director at the Peer–2-Peer University (P2PU)[3], we believe that learning is frequently found at the heart of peer production processes [19]. In the case study that forms the heart of *this* work (Section 6), we will use paragogy to evaluate our experiences as course facilitators at P2PU.

Although peer learning has been the subject of various studies, it is typically given a secondary role, within a pedagogical framework. This rather staid definition, from a book that approaches peer learning from the perspective of cognitive psychology, illustrates our point [16]:

> *Peer learning is an educational practice in which students interact with other students to attain educational goals.*

Although this definition is not in itself unreasonable, we are fascinated by the growth and evolution of opportunities for learning outside of formal institutions. A recent article from Fast Company, an influential business magazine, gives an expanded view of peer learning:[4]

[3] http://p2pu.org

[4] http://www.fastcompany.com/1746901/how-mint-execs-new-company-is-going-to-make-teachers-out-of-us-all

> *Just as more and more employees are expected to have basic multi-media skills – the ability to blog, for example, or to shoot images or videos on their smartphones – so will they be expected to have the basic ability to share knowledge with their peers.*

Thus, peer learning can of course take place between non-students, and it can concern productive, as well as educational, goals.

In addition to an increased emphasis on informal learning in the workplace, recent years have seen the rise of open, online spaces that serve the needs of learners via a commons-based approach. Here we cite Cormac Lawler's recent work on Wikiversity ([11], [12]). Lawler uses and advocates an *action research* approach, with thematic questions "What does it take to change a given system? [...] and how does the process of changing a system develop our knowledge about that system?" We have brought these questions to P2PU, and by extension to the education system that P2PU critiques ("The current model of higher education is broken..."[5]).

Our aim is to develop a set of "good practices" around peer learning, suitable for use by everyone involved (individual learners, organizers, administrators). A model is provided by two related works, one from Crowston *et al.* concerning "open source software success" [5] and the other from Resnick *et al.* on "starting new online communities" [17]. We will, however, have to wait for a future work to bring these contributions into one coherent frame with paragogy.

## 2 The challenge

A. T. Ariyaratne's essay on *Rural Self Help* [2], one of the foundational writings of the Sarvodaya Shramadana movement in Sri Lanka[6], begins:

> *Nobody needs to teach rural communities about "group effort" and "self-help". [...] The real question, therefore, is to examine what are the constraints that exist inhibiting the expression of their group effort and self-help qualities designed to improve food and nutrition levels, clothing, shelter, health, sanitation, education and cultural life?*

We approach peer learning in a similar spirit: it is something we all know how to do, but can't always do well. Intuitively, there are bound to be difficulties for a group of peers studying a subject together, outside a traditional classroom or without a teacher. Indeed, peer learning is different from other forms of group effort, the proverbial "barnraising" for example, in which the persons involved can be presumed to know how to build barns – or at least to know someone who knows, and stand ready to take orders. Typically, peers are not experts in learning, didactics, or in the subject they are studying, and are faced with multiple difficulties associated with putting together knowledge about the subject, assembling a suitable learning strategy, and communicating with one another.

[5] `http://okcon.org/2011/programme/hacking-education-p2pu-a-case-study`
[6] `http://www.sarvodaya.org/about/philosophy/collected-works-vol-1`

## 3 What paragogy has to offer

We have five principles, with which we endeavor to both describe the phenomenon of effective peer learning, and to prescribe key aspects of its best practice.

1. **Changing context as a decentered center.**
2. **Meta-learning as a font of knowledge.**
3. **Peers provide feedback that wouldn't be there otherwise.**
4. **Learning is distributed and nonlinear.**
5. **Realize the dream if you can, then wake up!**

Generally the ideas embodied in these principles are not unique to paragogy, indeed, we will try to ground each of them in previously existing literature, while showing their relevance to peer learning.

*1. Changing context as a decentered center.* In paragogy, we recognize that we are not merely teachers or learners, but are actually co-creating the learning context as a whole. The central role of environment is not unfamiliar in constructivist thinking about education ([23], p. 4):

> *Thinking of instruction as an environment gives emphasis to the 'place' or 'space' where learning occurs. At a minimum, a learning environment contains: (1) the learner; (2) a 'setting' or a 'space' wherein the learner acts, using tools and devices, collecting and interpreting information, interacting perhaps with others, etc.*

Again, in the paragogical view, the environment should not be taken as "given" but should instead be viewed as co-created by peers.

*2. Meta-learning as a font of knowledge.* Here we are concerned both with efforts to "learn how to learn", and efforts to learn how to support others in their learning efforts [20]. Further, while it is a good idea for any organization to learn its business well [13], learning about learning is especially vital for those in the learning business. In peer learning, that is all of us.

*3. Peers provide feedback that wouldn't be there otherwise.* Learners must not simply seek confirmation of what they already know, they must confront and make sense of difference as part of the learning experience. Clearly, differences pose challenges but these are worth grappling with. Firstly, for psychological reasons: in many domains feedback is only available from peers (but of course peer learning can be relevant in domains like rock climbing and computer programming, where automatic feedback does exist). Secondly, there are philosophical or political reasons to affirm difference. In a space like P2PU, which aims to provide "learning for everyone, by everyone, about almost anything", we can hardly avoid developing an "understanding of social relations without domination in which persons live together in relations of mediation among strangers" [24].

*4. Learning is distributed and nonlinear.* Learning does not go in a straight line [8]. In particular, involvement in co-creating the learning context becomes an important "strand" in the paragogical understanding of peer learning.

*5. Realize the dream if you can, then wake up!* Without clear goals, there will be be nothing to realize. Without critical thinking about goals (leading us to change them), learning is a mostly passive game. Paragogy calls for a strategy of "deliberate practice" [7].

## 4 Literature review

The paragogical principles were conceived by turning Knowles's principles of andragogy [10] on their edge. In succinct form, these principles are:

1. **That adult learners are self-directed.**
2. **That they bring a wealth of experience to the educational setting.**
3. **That they enter educational settings ready to learn.**
4. **That they are problem-centered in their learning.**
5. **That they are best motivated by internal factors.**

Blondy [4] points out both uses and challenges for each of the Knowles principles, focusing on how they work in online learning environments. For instance, with reference to the first principle, "*Cheren stated that while learners may express a desire to be self-directed in their learning, most lack the required understanding of learning necessary to be self-directed and thus need guidance and encouragement in the learning process.*"

While our principles can be read as a critique of andragogy, it is largely a matter of point of view: thus, unlike andragogy (which takes the view of the adult educator) or pedagogy (which again studies teachers teaching learners), and unlike heutagogy [9] (which focuses on self-directed learners), as we have seen above, paragogy focuses on cases in which learners are actively engaged in co-creating their learning environments. In formulating our first principle, we drew on Nishida's notion of *basho* ("shared context in motion"), which looks at the way a context constrains or supports different types of (inter-)actions, and simultaneously at the ways in which we can (re-)shape the contexts we find ourselves in [1]. Thus, instead of asking whether or not learners are self-directed, we would follow Bingham [3], and assert that self-directedness is only meaningful within a relational context (e.g. within a social field). So much for the first principle, others are subject to a similar re-thinking.

Paragogy is not the only framework that has been used to study peer learning. We'll mention Scardamalia's 12-point framework for Knowledge Building [18] and Mwanza's 8-step process coming from Activity Theory ([15], cf. [14]). Scardamalia's 12 "socio-cognitive and technological determinants of knowledge building" are framed by the idea of collective cognitive responsibility in the workplace. (Collective responsibility for creating a suitable learning context would be

another way to describe our first principle.) Scardamalia's more extensive framework will in general support a more detailed analysis, but may be less intuitive to work with. Mwanza's eight steps map a given situation to Engeström's activity triangle [6], and are used to generate design requirements. This method is less normative than either Scardamalia or the present work, but also less specific. As with the work on software and community-building best practices mentioned in the introduction, we must defer the task of fully comparing and contrasting these approaches with our own.

## 5 Implementing paragogy

How to implement the principles? In this paper we will incorporate a strategy used in the US Army's training programmes: the *After Action Review* (AAR) [21]. As the name indicates, the AAR is used to review training exercises. It is important to note that while one person typically plays the role of evaluator in such a review (and despite the fact that military personnel are differently ranked), the review itself happens among peers, and examines the operations of the unit as a whole. The four steps in an AAR are:

1. Review what was supposed to happen (training plans).
2. Establish what happened.
3. Determine what was right or wrong with what happened.
4. Determine how the task should be done differently the next time.

The stated purpose of the AAR is to "identify strengths and shortcomings in unit planning, preparation, and execution, and guide leaders to accept responsibility for shortcomings and produce a fix." We note here the similarity of the AAR to the action research cycle [11].

## 6 A case study in paragogical evaluation

The paragogy principles provide guidelines on best practices for building successful peer learning experiences. In this section we will apply these principles to evaluate the lessons learned from our work at P2PU as facilitators in 2010–2011. For each of the principles we run through the steps of an After Action Review to look at how well the principle was implemented.

### 6.1 Changing context as a decentered center: mapping system dynamics and semantics

*Review what was supposed to happen.* We both organized multiple courses where participants were supposed to interact and learn about the subject matter: Collaborative Lesson Planning Fall 2010 and Winter 2011 (co-organized with Dr. Majorie King); DIY Math; Math for Game Designers; Open Governance and Learning (co-organized with Marisa Ponti); and, in Spring 2011, Shaping P2PU[7], which was an "intervention" based on a preliminary version of this section.

[7] `http://new.p2pu.org/en/groups/p2pu-the-course/`

*Establish what happened.* Due to critically low participation, the mathematics courses did not run to completion. Participation in Collaborative Lesson Planning and in Open Governance and Learning was minimal, but sufficient for a conversation to be sustained for the entire 6 week session. The theory of paragogy was born in an effort to understand how to produce successful courses. Finally, as of the time of this writing, 32 people have signed up for Shaping P2PU, but so far participation has been very low.

*Determine what was right or wrong with what happened.* In the more active courses, there were nice examples of learning by course participants.[8] Low participation was common across P2PU, as illustrated by Dan Diebolt's graphical analysis, which showed that participation within courses was uneven and falling.[9]

*Determine how the task should be done differently the next time.* Our best experiences as course organizers happened when we were committed to working through the material ourselves. Combining this with gently prompting peers to follow through on their commitments could go a long way towards keeping engagement at a reasonable level – but this only works when commitments are somewhat clear in the first place. The case of Shaping P2PU shows that organizer commitment is not enough. In this case, we feel that further clarification about the aims and intentions of those who are already highly involved in shaping the organization would improve things.

Looking at this another way, the P2PU ecology contains an implicit rubric for learning and engagement: from the time a member signs up for a course, to its completion, peers go through a cycle.[10] As we understand this cycle better, it should be possible to evaluate it for quality. Then P2PU could implement more formal check points throughout the cycle, requiring participants to specify, reaffirm, or adapt their commitments in relationship to judgments about quality.

### 6.2 Metalearning is a font of knowledge: transparency, accountability, and tone

*Review what was supposed to happen.* Support for community members was offered as a P2PU course (Course Design Orientation), in mailing lists, via weekly phone calls, in a Q&A issue tracker, and via other informal channels. Participants in courses were presumed to be ready and willing to contribute in a useful fashion.

*Establish what happened.* Core members do hold themselves accountable, but this behavior is not necessarily transferred or communicated to new members, for whom accountability is low. Course participants frequently disappeared.

[8] E.g. `http://open-governance-and-learning.posterous.com/`
[9] `http://bit.ly/lqPChA`, `http://bit.ly/kH89OP`
[10] See `https://wiki.mozilla.org/Drumbeat/SoW-engagement-ladder`

*Determine what was right or wrong with what happened.* Core members are doing a lot of work, and the project is moving forward, with grant funding, incorporation, and several new staff positions. Apart from contractual agreements within the nonprofit, community members have little or no accountability to one another. Governance follows a "rough consensus" model (after David Clark's "We reject: kings, presidents and voting. We believe in: rough consensus and running code."[11]). As implemented at P2PU, the rough consensus model has its strengths, in particular, it helps avoid tyrannies of the minority in the mailing lists. However, there are a number of ways in which rough consensus seems incomplete.

*Determine how the task should be done differently the next time.* It is typical for online communities to have strictly enforced community norms. It would be helpful to have a concise discussion of these available, together with up to date information on "best practices" for organizers and participants. The current Course Design Handbook provides one starting point, but it falls short of being a complete guide to P2PU.[12] This sort of resource would be particularly useful for newcomers and people who cannot attend the community telephone calls.

### 6.3 Peers provide feedback that wouldn't be there otherwise: dealing with problems in a respectful way

*Review what was supposed to happen.* Discussions about P2PU happen in the community mailing list and other places mentioned above. Bug reports are supposed to go into the Lighthouse tracker.[13]

*Establish what happened.* Discussions about P2PU happen in many places (e.g. in courses). Even within the mailing list, it can be difficult to keep track of the full range of ideas circulating at any given time. There has been some talk about using the Lighthouse tracker for organizational matters, but this hasn't taken off. Earlier experiments, like using a shared spreadsheet to keep track of organization-level tasks, appear to have been undersubscribed.

*Determine what was right or wrong with what happened.* Apart from development work, it can be hard to tell what's happening around P2PU. Presumably participants who have identified critical and unsolvable problems simply leave. The Q&A tracker and mailing list both provide ways to build factual knowledge, but seem less effective for building *strategic* knowledge.

*Determine how the task should be done differently the next time.* In a traditional university, there are typically a lot of ways to resolve problems without dropping out. P2PU's new "Help Desk" could, indeed, help with this issue – if people use

[11] Cf. `http://www.wired.com/wired/archive/3.10/ietf_pr.html`,
[12] `http://wiki.p2pu.org/w/page/27905271/Course-Design-Handbook`
[13] `http://p2pu.lighthouseapp.com/dashboard`

it.[14] The Help Desk and Q&A tracker will also function as a light-weight way to build certain kinds of organizational knowledge. However, there could be more clarity about how to contribute to the process of "shaping P2PU".

One fairly straightforward thought would be to add support for site-wide content tags. Site-wide tags would allow people who are not interested in "meta-discussions" to easily ignore them, whereas a space like Shaping P2PU could aggregate and build upon the already-ongoing platform-level discussions that have arisen in other groups. Tags would provide other learning-specific benefits, including the ability to give and receive light-weight feedback about contributions, and to build a portfolio showing the impact of one's work. This would, in turn, foster a culture of accountability.

### 6.4 Learning is distributed and nonlinear: design considerations

*Review what was supposed to happen.* People are supposed to choose and assemble suitable learning resources (blogs, OER, etc.) for their courses, in which everyone is supposed to learn something.

*Establish what happened.* This is essentially what happened, but it is hard to measure when and whether knowledge was gained.

*Determine what was right or wrong with what happened.* The organization is striving to handle the complexity of life online, for example, by integrating RSS feeds into the site to allow learners to transparently draw in work that they are doing elsewhere. This system is explicitly in an experimental "beta" stage, and quality control has a somewhat precarious meaning in a beta or "eternal beta"; on the other hand, this makes life interesting.

*Determine how the task should be done differently the next time.* In terms of measuring learning, P2PU would have to work hard to use anything but "participation" as a proxy value. In terms of broader issues of quality control, one serious thought is for P2PU core members (including staff) to use the platform to organize their activities – entirely in the open.

### 6.5 Realize the dream if you can, then wake up: high level roadmap

*Review what was supposed to happen.* At one time, the high-level vision was arguably a Declaration of Independence from Formal Education.[15] But arguably each participant has their own vision.[16]

*Establish what happened.* P2PU recently had its first board meeting, but, so far, documentation about the organization's vision and roadmap have not been presented to or affirmed by the user community (nor has the user community presented any stipulations to the organization).

[14] `http://new.p2pu.org/en/groups/p2pu-help-desk/`
[15] `http://www.youtube.com/watch?v=t8wxUbU1W_0#t=12m11s`
[16] `http://www.youtube.com/watch?v=t8wxUbU1W_0#t=13m12s`

*Determine what was right or wrong with what happened.* P2PU has made considerable progress (e.g. in the form of successful grant applications), but without more transparency about these efforts, the ability of non-core members to learn from organizational successes is limited. This, of course, limits the ability of volunteers to contribute to further successes of this sort, and may, to some extent, limit the ability of volunteers to "strike off on their own" to pursue alternative development goals.

*Determine how the task should be done differently the next time.* It is our firm belief that P2PU should work on a public roadmap that leads from now up to the point where the vision is achieved. Both vision and roadmap should be revised as appropriate.

## 7 Conclusion

Reflecting on education-relevant potential of new media, Martin Weller writes: "*It is [...] no easy task to adopt a decentralised model, since it will require massive procedural, economic and professional change in higher education*" [22]. We would argue that what's new here is not simply a disruptive force in the traditional educational landscape: there is also a compelling chance to understand learning better. We hope that further developments in paragogy can contribute to this process in a practical way.

We close with a quote from Young [24] that sums up our current sentiments, and points to a possible wider role for paragogy:

> *If institutional change is possible at all, it must begin from intervening in the contradictions and tensions of existing society. No telos of the final society exists, moreover; society understood as a moving and contradictory process implies that change for the better is always possible and always necessary.*

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# PlanetMath Redux: Web 2.0 infrastructure for mathematical problem solving

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**Abstract.** This demo shows work in progress on a Web 2.0 infrastructure for mathematical problem solving. Our aim is to make undergraduate-level mathematics easier to learn: our strategy is to link problems and solved examples to prerequisite material drawn from an existing free/open mathematical knowledge repository, the encyclopedia at PlanetMath.org.

**Keywords:** mathematics, learning environments, problem solving

## 1 Overview

We are building a new web infrastructure for mathematical problem solving, with the aim of making undergraduate-level mathematics easier to learn. Our strategy is to build a layer of contributed problems and solutions that are linked into the peer-produced mathematics encyclopedia developed and hosted on PlanetMath.org[3]. PlanetMath has been an active content-producing community since 2001, with around 300 authors contributing to an encyclopedia defining more than 15000 mathematical concepts. Support for problem solving would alter the space significantly, making it easier for students to contribute, and opening up new channels for peer tutoring.

The knowledge rich, systematically constructed, peer reviewed encyclopedia will provide relevant material that undergraduate students would not find in online mathematics learning environments like OpenStudy[4], Khan Academy[5], MathOverflow[6], and Virtual Math Teams[7], though we can draw ideas and inspiration from all of these. Some of the core strengths of our system will be:

- A low floor (easy to participate just by asking a question: keywords will be automatically linked to their definitions) and a high ceiling (the possibility to explore advanced topics and help others);
- Simple models of learning (vocabulary acquisition, enactment of various mathematical roles) will help us keep track of students progress, making the system suitable for peer-supported self-study [1];

[3] `http://planetmath.org`
[4] `http://openstudy.com`
[5] `http://www.khanacademy.org`
[6] `http://mathoverflow.net`
[7] `http://vmt.mathforum.org/vmt/`

– Teachers will be able to use the site to run their own courses (cf. earlier classroom experiments by David Smith [4] and Robert Milson [5]);
– Solutions will be available to logged in users only; activity tracking can be used to discourage cheating, and data mined to generate recommendations.

After some initial prototyping [3], we decided to realize the system as a collection of plugins and modules for the popular open source content management system, Drupal. The current demo shows some of the core "Web 2.0" features, and provides a basis for subsequent semantic extensions (such as disambiguation and linking within formulas).

Concretely, in addition to a port of legacy PlanetMath content into the Drupal framework, our demo will show: (1) Autolinking of technical terms from problems and solutions into the PlanetMath encyclopedia (adapting [2]); and (2) learner profiles and basic activity logging (see Figure 1).




**Fig. 1.** A user's contributions as they appear within a learner profile

The realized system is anticipated to help cut time and other costs for both learners and teachers, by being a source of problems and solved examples, cross-referenced with prerequisite readings, all of which can be remixed in purpose-made study guides. We expect that our approach to knowledge reuse and peer-to-peer learning will be applicable in related technical fields.

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# Personalised Peer-Supported Learning: The Peer-to-Peer Learning Environment (P2PLE)

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## Abstract

The Peer-to-Peer Learning Environment (P2PLE) is a proposed approach to helping learners co-construct their learning environment using recommendations about people, content, and tools. The work draws on current research on PLEs, and participant observation at the Peer-to-Peer University (P2PU). We are particularly interested in ways of eliciting explicit, coded, user feedback, and in monitoring the transitions from state to state within the PLE. We discuss the ways in which these ideas can inform the design of a platform for peer-supported study of university-level mathematics.

## Keywords

Peer Learning, Personal Learning Environments, Feedback

## I. Introduction

The Personal Learning Environment (PLE) lies at the forefront of personalised and self-regulated learning online (Chatti, M. et .al. 2007) The use of online forums, blogs, social networks and other Web 2.0 tools for learning purposes is also increasingly typical, even among co-located groups.

In this paper, we will investigate design issues around personalised and peer-supported learning, drawing on participant observation in a new learning-oriented online community, the Peer-to-Peer University (P2PU)[1]. We will make use of the existing PLE literature, and walk through an extended thought experiment where we look at how our ideas could be applied in a mathematics learning context.

The main idea we investigate in this paper is that of the Peer-to-Peer Learning Environment (P2PLE). Our thought is that since learners typically work through tasks together, or with a degree of peer support, they would benefit from access to a sophisticated, personalised, socially-aware task tracking system.

We are agnostic as to whether this system is closely integrated with a given online community, or makes use of the Web-as-Platform model. Part of the view of PLEs is that learners will make use of a wide range of tools and interactions in any case (Willson et.al., 2008). Features of the P2PLE could be implemented in either scenario, and, indeed, we have already seen proofs-of-concept for both (we will discuss this in the following section).

Both self-directed learning and peer collaboration are already quite strongly supported in Web 2.0 environments. What remains to be seen is how to improve the learning experience by making use of this data as we move towards platform with stronger support for semantics (Web 3.0). The P2PLE proposal indicates the scope of data that we can usefully gather, and makes some initial recommendations for how to use this in the concrete instance of mathematics learning.

## II.Background

Personal Learning Environments (PLEs) have shown evidence of facilitating learning and addressing the current limitations of Learning Management Systems (LMS). A PLE can be broadly defined as a facility for an individual to access, aggregate, configure and manipulate digital artefacts of their ongoing learning experiences. Compared to a typical LMS, where the learner is restricted by the lack of adaptability and responsiveness of the learning environment, the PLE follows a learner-centric approach. It allows the use of lightweight services and tools that belong to and are controlled by individual learners. Rather than integrating different services into a centralised system, the PLE provides the learner with a variety of services and hands over control to her to select and use these services the way she deems fit (Fiedler and Väljataga, 2010; Zimmerman, 1989).

For example, n the context of the European project ROLE (Responsive Open Learning Environments - www.role-project.eu), we are supporting learners for lifelong and personalised learning within a responsive open learning environment. Figure 1 shows a PLE developed by the ROLE project for the Open University. The purpose of this PLE is to support students in studying climate change and

[1] http://p2pu.org

sustainable energy by providing them with a set of learning tools in the form of widgets. Within this environment, learners use a search widget in order to find learning materials from a variety of repositories, including OpenLearn (www.open.ac.uk/openlearn), iTunes U (www.apple.com/education/itunes-u), Wikipedia (www.wikipedia.org), SlideShare (www.slideshare.net), and YouTube (www.youtube.com). Learners also collaborate with their peers via widgets offering videoconferencing and real-time document editing functionalities. Learners can personalise this environment by adding tools from third-party widget providers, such as the Google gadget directory (www.google.com/ig/directory).

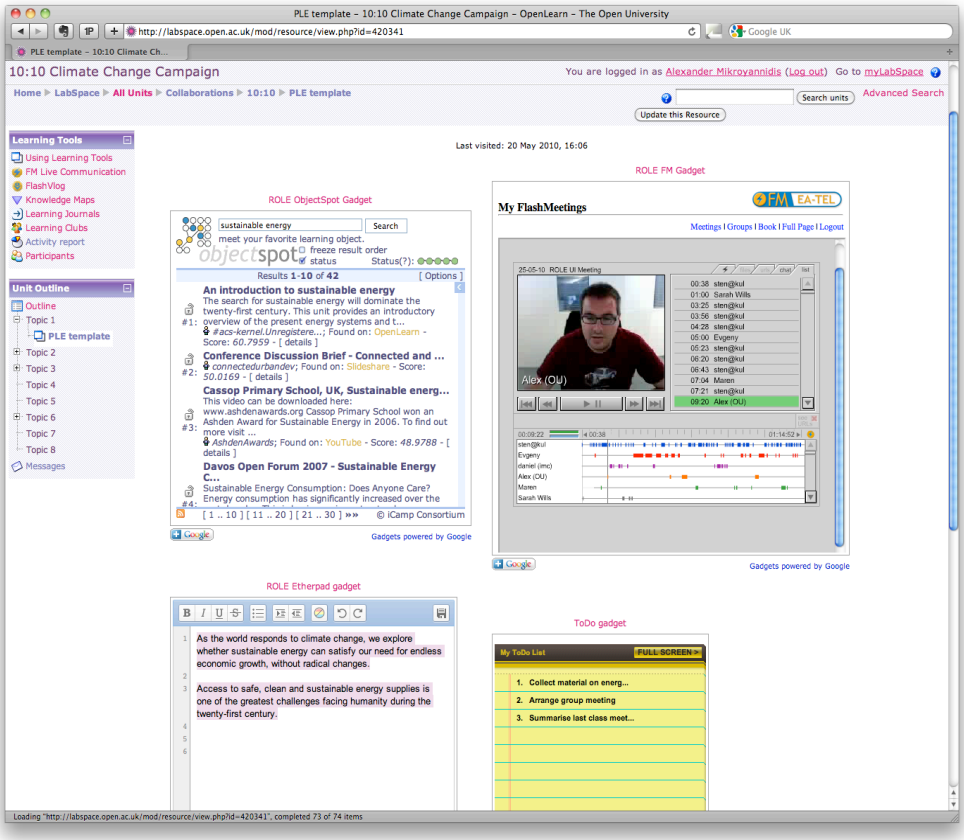



Figure 1. A PLE for finding learning resources and collaborating with peers

How do we think about quality in a PLE? Studies can have been made to determine whether people like their experiences working with a PLE, and the tools can be adapted accordingly; this approach is the state of the art in PLE research.

However, a PLE is all about live, interactive adaptation to the needs of the user. Accordingly, there ought to be a way to measure various aspects of quality within the PLE itself. Here we suggest a lightweight approach to user feedback that uses distributed evaluations of all of the components and activities that comprise the environment.

How might this work? We know that a PLE user assembles various components and performs various activities with these tools. Many of these activities are social – collaborating on a project, giving feedback to one another, and so forth.

Importantly, objectives are selected from a larger set of possible objectives, tools are selected from a larger set of possible tools, peers are selected from a larger set of possible peers, and so forth. Each of these choices represents an implicit judgment about quality – such choices have the potential to be particularly revealing when they change over time.

However, there is a degree of implicitness here that confounds things. Unless we have a way to make the judgments explicit, we won't know whether someone is moving on from a given activity because they are finally satisfied with the results, or because they have grown hopelessly frustrated.

Within the current conception of PLEs in the ROLE project, an explicit evaluation or reflection phase is introduced, which can make more of these judgments explicit – see (Fruhmann, 2010) which builds on the theory of self-regulated learning developed in [8]. Here, we look for ways to wrap evaluation and reflection even more tightly into use and participation.

This requires lightweight ways to communicate sentiments about activities. For example, in response to a given blog or forum post, a reader might say "This blog post was exceptionally well-written, I learned a lot from it", or they might say, "You know, this post was interesting, but it happened to lead me off on a direction that wasn't really relevant to my learning project" -- or they might say, "Reading this post was a waste of my time." In the case of blogs, feedback of all sorts is currently handled through the comment system.

However, feedback could easily be coded. We can start with the popular idea of bookmarking or "starring" content, but we suggest that stars should be used to give feedback, and not simply as bookmarks (Figure 2). In other words, starred content should also show up in the profiles of the people who contributed the content. The set of codes should be expanded, providing readers with a set of ratings that convey significantly more than just "thumbs up" or "thumbs down". The "superstars" lab feature recently mainstreamed in Gmail gives an idea of what the UI might look like (but Gmail doesn't currently propagate information on which posts have been starred back to the content creators).[2]

In a learning context, detailed feedback would be particularly relevant to (peer) producers of content, and also to educators or technologists who are building tools. A peer learner might, for example, request evaluation along a certain set of dimensions: "I'm really trying to work on my writing quality, please let me know if you find my argument cogent or not." A technologist might be happy to learn about novel uses for the tool they've developed.

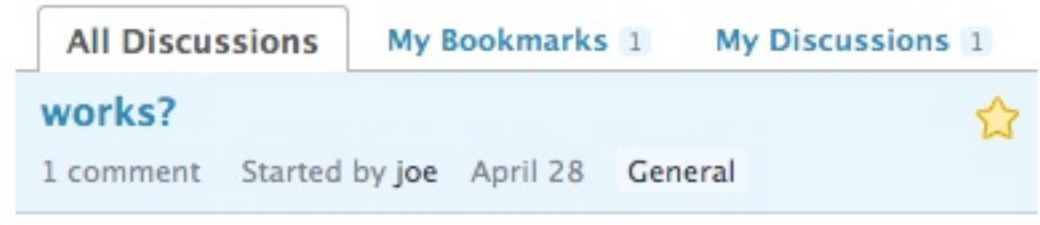



Figure 2: We start with the popular idea of starring content, but suggest that stars should show up not just as "my bookmarks", but also in the profiles of people who contributed the content.

[2] http://gmailblog.blogspot.com/2011/05/3-labs-graduations-1-retirement.html

Thus, a targeted set of feedback options could be requested in addition to open feedback. (Targeted requests for feedback are known to work well in writers' workshops, for instance!) As proof of concept, we note that multi-dimensional judgments have been used to good effect for years in the Slashdot community, while the Facebook "Like" button is a more recent example showing that light-weight quality judgements can be widely distributed.

Of course, such judgments can be applied to all aspects of the learning context: tools, objectives, activities, peers. There is wide precedent for various sorts of feedback presented by technologies ranging from UserVoice to LinkedIn. The real question is how feedback is going to be used. This is the main issue we take up in the rest of the paper.

## III. A simple scenario

Anne, Betty, and Charlie are working together on a paper they plan to present at a conference. Betty and Charlie have a good collaboration session drafting some initial content -- but it turns out that Anne can't make it to the scheduled meeting. She shows up later and does some editing in place, finalising the text. Betty later adds some diagrams. Anne then decides to put together a short film to accompany the presentation. Finally, it happens that the conference is near Charlie's hometown, so he's the one who makes the presentation.

This simple scenario shows three axes: People, Tools, and Activities. The range depicted in Figure 3 is meant to suggest a small slice of data about a much larger environment: in general, there would be more items along each of the pictured dimensions, and more dimensions as well. In particular, here we've aggregated activity over a certain short period of time.

Looking at other slices along the time dimension might reveal that our protagonists collaborate like this frequently -- or perhaps we would learn that they end up collaborating in a similar way, but with other people.

What we don't know just from looking at this picture is how the presentation went, or how any of the protagonists felt about any of the steps in the process. We would need another dimension to learn that Betty and Charlie do indeed find Etherpad to be an effective tool, or that Anne actually prefers it when she gets more help with her video work. But these sorts of judgements could be added as extra dimensions.

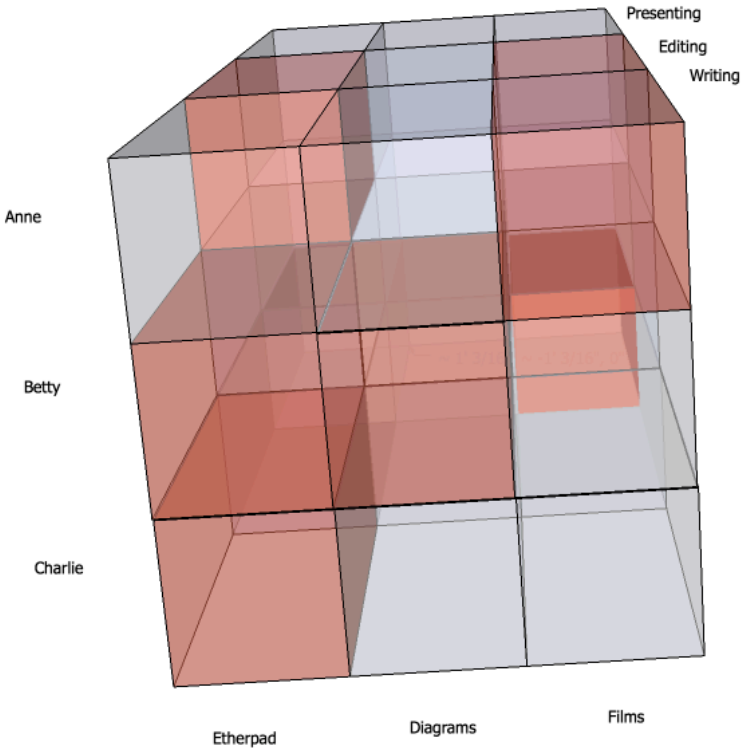



Figure 3: A 3-dimensional grid showing people, technologies, and tasks

## IV. Requirements for Peer Learning

We begin with a couple of basic axioms about feedback.

1. Feedback doesn't do much good if the agent(s) receiving the feedback can't use it.
2. Giving feedback tends to be an "extra step", so we should make it useful for people to give (or they won't do it).

An example of (2) that we discussed above: "starring" items is useful in approximately the same way that bookmarks are useful, but if handled properly, the starring action can provide feedback to content creators. The rest of this section looks at different transitions that can happen in learning. This section is based on recent observational study at P2PU (Corneli,J. and Danoff, J., 2011), in which we developed a framework of five principles that describe successful peer learning experiences. Here, we reframe these principles in active terms, looking at what changes ("Deltas") take place in peer learning.

### 1. Changing the nature of the space: "changing context as a decentred centre").

This could happen at a technological level (e.g. an administrator or programmer adding a new feature to the platform), or, importantly, on a qualitative social level (e.g. by a user making a comment that changes the tone of the conversation).

2. **Changing what I know about myself : "meta-learning as a font of knowledge").**

Users can be kept accountable for their actions and celebrate their contributions by means of a record kept on their user profiles. A user profile that keeps track of things I rated highly, and ratings of my contributions (or recommendations of me) from others would be very useful. In response to changes in my profile, I may update the criteria I'm asking for feedback about.

3. **Changing my perspective: "peers provide feedback that wouldn't be there otherwise").**

Users can have constructively critical views on technology, strategy, or content. Sometimes actions speak louder than words.

4. **Changing content or connectivity: "learning is distributed and nonlinear").**

In a peer-managed environment, "design" can happen on an ongoing, ad hoc, basis. Without changing the "nature" of the space itself, it is possible to change the content, for example by adding or removing a given feed or tool.

5. **Changing objectives: "realize the dream if you can, then wake up!").**

Individual and collective goals and objectives are important as a way to discuss and critically examine progress. Since goals and objectives change (particularly once they have been achieved, but for other reasons as well), there should be ways to users to adjust their goals. Changes to shared goals and objectives should typically be negotiated.

## V. Support For Interpersonal Communication

Is a social context, the five different kinds of changing features discussed in the previous section are workflow and communication issues, *par excellence*.

The various transitions could happen quickly (e.g. by adding a particular tag to some piece of content), or build up more slowly (e.g. by continual feedback from peers about a given topic).

It is important to note that a given piece of content -- like a person -- can be in several categories at once. This leads to the idea of remixing content by sharing it between several courses or groups at once. This idea is akin to the concept of "internationalization", insofar as we consider content that be immediately used a range of "locales".

In this view, a given discourse context, such as a development project, is akin to a simple "language". Indeed we might find many development projects that are mostly distinguishable by their language choices. Until the learning environment supports content that is adaptable in this manner, the environment may suffer from an unnecessary degree of balkanisation.

Since the P2PLE adapts content bi-directionally between several sources, constructive activities can take place in different locales or peer groups, and share across boundaries. Importing changes can be selective based on local evaluations of quality (it's OK if other people use the content in a different way).

The next section will look at a concrete case.

## VI. Peer Interactions in a Mathematics Learning Environment

There are any number of websites that offer help with or learning resources for mathematics. One has been well studied is the Math Forum, where "virtual math teams" have received particular attention (Stahl, 2009). Another more recently developed site is the Kahn Academy, which provides videos and sequenced review problems. At present, Kahn Academy materials do not incorporate peer interactions explicitly, but peer support has been used to good effect in classrooms where the Kahn materials have been deployed.

However, these, and most, mathematics learning platforms focus on pre-university material. We will look back at the ideas generated in earlier sections and think about how they would apply in a P2PLE for university level mathematics.

We will look at the project underway to add facilities for problems and solutions to PlanetMath.org, a community-produced mathematics encyclopedia. This will be viewed through the lens provided by the five points discussed above.

### 1. Changing the nature of the space

As technological facilities develop, new features will be rolled out regularly, moving from basic support for adding and discussing problems and solutions, to automatically linking the problems to related articles in the encyclopedia, all the way through to a recommender system that will mine previous user interactions to suggest useful problems or readings to try next. Although problems will be separate from the encyclopedia, their introduction will change the nature of the space: people will now be able to ask "Which encyclopedia articles are missing problems?" or "Are the encyclopedia articles that are connected to the current problem well-written?", for example. Looking at expository material as part of an ecosystem that contains problems and solutions adds a very useful check on quality. Small independent changes to encyclopedia articles that adjust them to serve the needs of learners are expected.

### 2. Changing what I know about myself

As individual learners accumulate a track record of uploading and solving problems, asking for and offering help, giving feedback on and modifying encyclopedia articles to suit, etc., they should get a better sense of how they learn best. They should be able to ask for specific kinds of feedback and see how their progress improves (e.g. in formulating proofs or demonstrating an understanding of the concept of a limit). They should be able to keep track of particularly helpful and particularly non-helpful suggestions offered by peers or by the recommender system.

### 3. Changing my perspective

Hopefully, peer mentors -- and system developers -- will be able to learn from learner feedback about what's helpful and what's confusing. Feedback should be particularly valuable to learners ("Wow, I didn't even know there was such a thing as spherical trigonometry!"). Ideally, giving and receiving feedback will be comfortable for all involved.

### 4. Changing content or connectivity

In addition to peer-producing mathematical content, our hope is that learners and other contributors will be able to develop their own semantic queries. Such queries could be used to identify holes in the corpus, or interesting relationships between activity patterns. Not everyone needs to be able to build these queries to use them, e.g. to generate a feed showing all the latest additions of problems having to do with tori or klein bottles.

### 5. Changing objectives

A shortcoming that was noted in the previous decade of PlanetMath's existence was that support for individual "projects" was not particularly strong. For example, a project to improve the entries about real numbers chose to base its operations on the organisational wiki rather than in PlanetMath itself.[3] Content quality in PlanetMath has so far been maintained using a "correction system" that points out places where individual articles are mistaken or could be improved. In order to support the production of educational content, it would be good to generalise the correction system to include ranges of content (sub-collections of the encyclopedia or sub-areas of mathematics).

## VII. Conclusion

We have discussed a model of learner interactions that takes into account people, tools, activities, and other dimensions. In particular, we have focused on feedback about these items as an ancillary set of dimensions. Time is one of the most important dimensions, and, in some cases, we may be able to infer judgments from time-delineated data -- however, we've focused on ways people can get and give explicit feedback, keeping in mind two axioms, that the feedback should be useful to both receiver and sender.

We've examined general-purpose criteria for peer learning environments, which have arisen in the context of P2PU, and applied them to generate some initial design requirements for an ongoing to convert a peer produced mathematics encyclopedia into mathematics learning environment.

The thought we wish to conclude with is that in today's global context, we are often in touch with peers from all over the world. There is an increasing need courses that cross natural language boundaries, and for specialized technical literacies. But our ability to understand one another well depends partly on the means that we have to express our thoughts, ideas, and concerns.

Technologies that are able to provide learners with feedback on their expressions – whether coming from a peer, or from an algorithm – can support the learning process. Following the outline of the previous two sections, our suggestion would be to focus on developing technologies (ranging from light-weight mechanisms like rubrics or rating systems, to sophisticated text mining tools) that learners can use to detect, highlight, and share information about the changing nature of the space, its content and connectivity, and their own self-knowledge, perspective, and objectives. We look forward to future work on these issues, building on the ideas in this paper.

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# Collaborative Learning 2.0:

## Open Educational Resources

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# Chapter 15
# Crowdsourcing Education on the Web:
## A Role-based Analysis of Online Learning Communities

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## ABSTRACT

*Learning online has significantly evolved over the past decade due to the emergence of Web 2.0 and 3.0 technologies that facilitate social learning in adaptive online environments. The open content movement and the associated techniques of crowdsourcing (i.e. assimilating several small contributions into resources of high quality) have further influenced education on the Web. This chapter investigates the concept of crowdsourcing in education through an analysis of case studies dealing with two open online learning communities, Peer 2 Peer University, and PlanetMath.org. The case studies proceed via an analysis of the various roles played by the individuals involved in each organization. The outcomes of this analysis are used to extract general recommendations for building online communities and applying crowdsourcing techniques in educational contexts.*

## INTRODUCTION

Web 2.0 and 3.0 technologies are transforming the landscape of learning. These technologies enable learners and pedagogues to co-produce learning environments that adapt to the competencies and motivations of each participant. Building these systems online, in the open, system developers and curriculum authors can make use of both high-intensity, high-cost contributions, and a long tail of smaller and less intensive contributions. This process of assimilating many small contributions into resources of high quality – colloquially known as crowdsourcing – is becoming a key aspect of

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the development of open online learning platforms. The challenge inherent to such efforts is to capture the surplus value of distributed processes of social engagement in a way that permits reuse and further development.

This chapter rethinks Nonaka and Takeuchi's well-known SECI model of knowledge creation (Nonaka & Takeuchi, 1995), and applies it to two case studies in crowdsourced education. The revisions to the SECI model are two-fold. First, what initially appears to be a simple and intuitive shorthand, obtained by mapping Nonaka and Takeuchi's Socialization/Externalization/Combination/Internalization onto Ken Wilber's I/We/Its/It (Wilber, 1997), upon further reflection leads us to a very different way of thinking about things. And, second, Nishida's philosophy of *basho* (summarized in English by Masao Abe (1988)), which was a noted inspiration for the SECI model, plays an even more central role in our version of the theory.

To put it somewhat colorfully, the "Golden Age" SECI is here updated to make it suitable to the analytical challenges present in our "Modern Age". These challenges include organizations that make significant use of commons-based peer production (CBPP) (Benkler, 2005), organizations without a traditional management structure, and collaborations that cut across organizational boundaries. The focus in our analysis is on the various social roles taken on by the persons involved in such settings.

A point of departure for our new understanding of SECI is the critique found in Engeström's "Innovative learning in work teams: Analyzing cycles of knowledge creation in practice" (Engeström, 1999). Engeström makes a convincing case that "SECI" really doesn't adequately represent a cycle, despite the claims of its initial creators. The I/We/Its/It framework doesn't represent a cycle, either. Rather, we use Wilber's terms to describe a given *social role* in terms of its *constituent actions*. So for example, the role of "being a student" might be described as follows:

*"I go to class, we do a class project, the objects of concern ("Its") are things I can add to my portfolio or work-record; and fundamentally it is all about gaining a skill."*

This simple background story gives us a notion of role, persona, or identity: a role that is defined by its constituent actions, relative a given social context. And here, context is conceived of, after Nishida, as a "shared context in motion" (this is the meaning of the term "*basho*").

Our little story describing the role of a student doesn't have much to do with "knowledge creation" or "epistemic action". Still, now that we have a convenient way to talk about roles, we can move on to talk about how roles can change over time, how new roles come into existence, how different roles can conflict, and so on. It is in this respect that we recover the organizational learning dimensions of the SECI model – not as a byproduct of individual learning cycles, but as a complex of ongoing adjustments to the shared context and the social roles that are enacted therein (Engeström, 2007).

Thus, our concern is with the way a given context creates and is in turn created by its constituent social roles. Using this approach, we will develop a theory of organizational learning that is applicable to contemporary educational communities, possessing all of the complexities of our "Modern Age".

The next section will set the stage by looking broadly at contemporary education and its many stakeholders. We then present a case study centering on an informal course that the first author ran at Peer 2 Peer University in Autumn of 2010[1], followed by a second case study, which applies our model to look at possibilities for organizational change in the community-created online mathematics community, PlanetMath.org.[2] We conclude the chapter with some general recommendations gleaned from reflections on these two case studies.

*Table 1. Sketch of the social roles in a traditional university*

| | |
|---|---|
| **I.** | *I* go to class, *we* do a class project, *the various aspects of which* are things I can add to my portfolio or work-record; and *fundamentally it's all about* gaining a skill. |
| **II.** | *I* lead a class, *we* plan and implement the curriculum, *my work involves* giving lectures and feedback and, more infrequently, meetings with my colleagues; and *fundamentally it's all about* helping my students. |
| **III.** | *I* ask a thought-provoking question, *we* discuss or experiment, *the results* are written up in papers; and *fundamentally it's all about* generating new knowledge. |
| **IV.** | *I* transform ideas into code or policies, *we* collectively manage a body of work, *the pieces* are the components of a functioning system; and *fundamentally it's all about* creating a workflow that works. |
| **V.** | *I* engage in dialog, where I promote a certain position, *we* try to find common ground, *the results of* various interactions and transactions are assembled into strategies; and *fundamentally it's all about* creating a distinctive organizational identity and strong partnerships. |
| **VI.** | *I* endeavor to discern societal needs, *we* work to achieve a rough consensus with a larger body of stakeholders, *the results* describe a certain clearly-defined skill set; and *fundamentally it's all* about knowing the appropriateness and relevance of a certain training process. |

## BACKGROUND: EDUCATION AND ONLINE COMMUNITIES

We begin with a look at how educational communities are built. A traditional university, for example, is populated by students, teachers, researchers, administrators and staff, and possesses a certain legal status by maintaining a relationship with accreditation bodies and government. We get a sense of the dynamics of the university when we look at the actions that comprise these roles (Table 1).

These sketches shouldn't be taken to be definitive, but rather, as paradigmatic. It seems reasonable to say that any social setting that supports actions sufficiently like these is an "educational context". A setting that intersects only a few of them (e.g. an academic publishing house) nevertheless forms part of the broader social context in which education sits.

This leads to the idea that a given educational context can be distributed in space and time in various different ways. Many of the support functions related to infrastructure can be subcontracted or otherwise outsourced. For example, instead of giving lectures in person, an instructor may deliver lectures via podcast.

It is in this distributed setting that Open Educational Resources (OER) arise and become relevant to the future of education. We are concerned in Table 1 with *social* roles, and various social contexts arise to support these as well, ranging from study groups, to tutoring services, to "virtual colleges", to open source software developer communities, to professional associations, all the way through to UNESCO.

In many cases it is not suitable to view such settings as only "ancillary", as they are significant communities in their own right. For example, Wikipedia is more than just an encyclopedia: it is effectively the flagship of a social movement.

### Educational Communities, Online

Two well-known sources of educational content are Connexions[3] and MIT OpenCourseWare[4]. But while both Connexions and MIT OpenCourseWare contribute to the broader commons of OER as production communities, they appear to use commons-based peer production (CBPP) in a limited way. To illustrate the point: at the time of this writing, although thousands of people visit the Connexions website each day, discussion forums are not present on this site. Instead, the (essentially broadcast-based) Connexions blog is the central "community" feature[5].

*Table 2. We are in between several familiar institutions and a couple others that are less familiar.*

| | Solo consumer | Mediated sociality | Social consumer |
|---|---|---|---|
| **Highly modular** | Library | Gutenberg.org 2.0[6] | University |
| **Lightly integrated modules** | Encyclopedia | OER communities | Course |
| **Highly integrated** | Book | Interactive Hypertext | Tutor |

In Table 2, we suggest that consumers of OER exist in a realm in between that of the "solo consumer" and that of the "social consumer"; and that OER resources typically lie in between "highly integrated" systems and "highly modular" systems. This, together with a suitably broad understanding of openness, can make peer production of OER a natural choice.

In the next two sections of the chapter, we will examine two distinct educational communities, Peer 2 Peer University (P2PU), and PlanetMath.org. The former is building a social platform, together with a collection of best practices related to running peer-based courses online and a collection of course materials from previous courses. The latter is building a mathematics encyclopedia with integrated discussion forums, and, at present, has essentially nothing to do with courses or course materials per se.

We can imagine a team-up in which PlanetMath provides content and supports learner interactions, whereas P2PU provides training training to course organizers in the important non-mathematical skills for running an effective course, as well as a certification layer for course participants. The following sections will examine these two cases using an approach similar to the role-based analysis of Table 1, but they will go into much greater depth.

## P2PU CASE STUDY: "ONLINE BOOK CLUBS FOR OER"

### Overview, Objectives, Challenges

This case study is based on the experiences of the present first author as the facilitator of an online course called "DIY Math" that ran through the Peer 2 Peer University in Autumn of 2010. The interactions in and surrounding this course provide a lens on P2PU as a whole.

We will now present some background information on P2PU and the design of DIY Math. From the P2PU website:

*The Peer 2 Peer University is a grassroots open education project that organizes learning outside of institutional walls and gives learners recognition for their achievements. P2PU creates a model for lifelong learning alongside traditional formal higher education. Leveraging the internet and educational materials openly available online, P2PU enables high-quality low-cost education opportunities.*[7]

The Shuttleworth Foundation, who sponsored P2PU early on with a $30K fellowship, provides the following description of the organization:

*The Peer 2 Peer University (P2PU) is an online community of open study groups for short university-level courses. Think of it as online book clubs for open educational resources. The P2PU helps you navigate the wealth of open education materials that are out there, creates small groups of motivated learners, and supports the design and facilitation of courses. Students and tutors get recognition for their work, and we are building pathways to formal credit as well.*[8]

Although P2PU has only been online since 2009, they have gathered significant media attention and interest. They describe themselves as an

online community of about 1000 persons. Many of P2PU's core volunteers are also affiliated with other organizations in the open online education space. The P2PU approach is to be "open" to nearly any constructive form of participation. They are in the process of making their "horizontal governance" structure more clear, at the same time as they pursue formal incorporation and non-profit status. They are presently experimenting with a range of learning and communication technologies. They were enthusiastic about adding a mathematical course to their range of offerings.

The DIY Math course itself was designed to deal with the question *how should we build a course that supports both independent and peer-based mathematics learning?* By being expressly open to "mathematics learners at all levels", DIY Math followed a very atypical approach to the teaching and learning of mathematics. The rationale behind this was that learners would eventually self-organize a support system. Accordingly, the syllabus did not include specific mathematics topics or exercises, but rather, certain general suggestions regarding collaboration:

- be ready to introduce yourself and your interests in the mailing list by the time the course starts;
- reply to at least one other person's self-introduction to say how it relates to your own interests;
- come up with some candidate discussion guidelines or objectives;
- update the group when you start looking at a new mathematical resource;
- give a review saying whether you found that particular resource helpful or not, and why;
- post short summaries of any study strategies you're using;
- post reviews of your strategies;
- identify one or more study buddies who are interested in topics similar to the ones you're most interested in;
- post in response to one of the two prompts that follow: "Tell us about something you learned."; and/or "Tell us about something you're having difficulty with."

Twenty-two people initially signed up for the course, and their applications alone were enough to generate some useful reflections. In particular, they provided a first impression of the mathematical interests of the would-be participants. Despite the initial show of (wide-ranging) interest, less than half of these people made it past the hurdle of "self introductions", which were to take place in the course's Google Group.[9]

By the second week, things had already gotten pretty quiet. Although the essence of "DIY Math" had been made reasonably clear to the participants, this did not translate into anything resembling a serious commitment. To generalize from this experience:

1. *When organizing an event or setting up a context for any given social activity, it is a really good idea to ask people what they would like to do there*; and
2. *It is equally important to get the participants to commit to follow through on actually doing*
3. *what they said they are interested in* (if their commitment is insufficient, the description of the activity should be revised and commitments renegotiated accordingly).

The experience with DIY Math shows that the idea of self-organization alone will not create a successful peer-to-peer learning environment. In a traditional educational setting, there are a variety of factors (e.g. money, credentialization, informally enforced norms) that could help create a sense of commitment on the part of learners: by default, these are wholly missing in informal learning.

In P2PU at present, a lot of the "pull" comes from facilitators who not only decide what courses to run, but who also are charged with keeping the peer learners engaged, and contributing to a body

of knowledge about how this can work best. In this organization that places a strong value on non-hierarchical peer relationships, at present, the importance of the complex role of course facilitator is "not to be under-estimated".

## Role-Based Analysis

Tables 3-7 present several different roles associated with DIY Math and P2PU more generally. These are:

- The role of an an "ideal" participant in the DIY Math course (i.e. based on how things were supposed to work, according to the syllabus);
- The role of a more realistically-conceptualized participant in the DIY Math course (how things *actually* worked for most people);
- The role of a facilitator at P2PU;
- The possible future role of a peer learner in a structured "Short Calculus" course taught at P2PU;
- The possible future role of an "arbitrary participant" in P2PU where the organizational structure has become flatter.

## Lessons Learned

The flattening of roles in Table 7 is potentially surprising. The insight embodied in that gesture is as follows.

If P2PU was taking an ongoing survey of the "wished for" course topics, a future course organizer would presumably be able to create courses specifically tailored to the interests of pre-self-selected participants. Alternatively, interests could simply be listed ("social network style") on user profile pages and aggregated in an intuitive way. Levels of commitment could similarly be specified in advance, ranging from "I'd like to put 1 hour a week on this topic" to "I'm available to work on this full time." Detailed qualitative aspects of commitments could also be specified (e.g. "I'm willing to answer questions about Calculus but keep in mind I haven't taken it for a few years").

The key point is that the system could then be set up so that courses only would only run when sufficient interest had gathered. This design would use the idea of "shared context in motion" to help relationships form organically, and to emphasize the importance of appropriate commitments for building trust.

Another lesson is that a participant in a structured course like Short Calculus (Table 6) might benefit from the existence of an unstructured support system, something like DIY Math. This approach is consistent with the idea that a peer-based community doesn't easily arrange itself into explicitly delineated and wholly "visible" structures, but may instead have a lot going on in the background (Engeström, 2007).

Recalling the vision of "online book clubs for open educational resources", the possibility that a facilitator need only be very slightly involved again suggests itself. There appears to be a certain degree of trade-off between the time and energy spent preparing course materials, and the time and energy spent facilitating the course. (From the point of view of Table 1, part II, this is the difference between what "we" do together as

*Table 3. The role of an "ideal" participant in DIY Math course*

| | |
|---|---|
| I | say what I want to study |
| We | talk about difficulties and successes |
| Its | discussions on a shared mailing list |
| It | helps me learn mathematics (and improve my skills at being a self-directed and peer-to-peer learner) |

*Table 4. The role of a more realistically-conceptualized participant in the DIY Math course*

| | |
|---|---|
| I | try to figure out what to study |
| We | sometimes give or get advice that isn't always so helpful (and most of it isn't for me, anyway) |
| Its | a bunch of good intentions that lead nowhere |
| It | confirms my sense of the difficulty of learning anything in a self-directed fashion (and the difficulty of mathematics in particular) |

*Table 5. The current role of a facilitator at P2PU*

| | |
|---|---|
| I | come up with a course I'd like to facilitate, and then facilitate it |
| We | discuss ideas about how our courses might work and what "facilitation" means (e.g. as opposed to "teaching") |
| Its | discussions on a community mailing list and other settings (including discussions with participants in the course as it runs) |
| It | helps me improve my skills at a course designer and facilitator (and it's fun talking about and practicing this stuff!) |

*Table 6. The possible future role of a peer learner in a structured "Short Calculus" course at P2PU*

| | |
|---|---|
| I | work on calculus problems from the syllabus and ask questions in an appropriate venue when I get stuck |
| We | talk about difficulties and successes and examples |
| Its | discussions on one or more shared mailing lists or forums |
| It | helps me learn calculus (and develop some good study habits along the way) |

*Table 7. The possible future role of an "arbitrary participant" in P2PU*

| | |
|---|---|
| I | describe what I'd like to learn or teach |
| We | constitute a class when we have gathered a enough interested peers with the right mix of skills; in the mean time, and in general, we constitute a broader learning community |
| Its | a marketplace of interests, skills, and ideas on the one hand, and efforts, attempts, and assistance on the other |
| It | helps me learn what I want to learn and have fun in the process |

teachers, and what "I" do as a teacher.) It is conceivable that the best facilitator would not be an expert in the course's subject matter, but an expert in peer-supported learning per se.

In Winter 2011, the first author facilitated three more courses at P2PU, with mixed results, which nevertheless appear to strongly support the perspectives outlined above. One useful outcome of these experiences is that we have significantly improved our theory of peer-to-peer learning (summarized in the section entitled "Paragogy", below).[10]

## PLANETMATH CASE STUDY: "MATH FOR THE PEOPLE, BY THE PEOPLE"

### Overview, Objectives, Challenges

The scope of this case study is quite different from the one in the previous Section, since here we will look in overview at the last 10 years of PlanetMath.org, instead of the last few months.

PlanetMath came into being as the core of the work done by Aaron Krowne for his Masters degree at Virgina Tech (Krowne, 2003). On

PlanetMath, one finds the following statement by Krowne:

*The idea for the project was hatched by [Nathan] Egge, sometime around fall 2000, in response to the unfortunate removal of the great resource, "Eric Weisstein's World of Math" (or "MathWorld") from the internet. This incident created a void for a useful, comprehensive math encyclopedia freely available online, which we wanted to fill as quickly as possible. So Nathan and Aaron began brainstorming the concept and working out many of the problems inherent in a system that would do in real-time what had been static on MathWorld. We also wanted to make certain that the users of PlanetMath would never have to worry about the removal of the content they had contributed in favour of commercial sale.*[11]

Since its inception in the year 2000, over 11K encyclopedia articles have been contributed to PlanetMath, and over 14K posts have been made in the discussion forums. In recent years, the website has received 12-20K hits per day, the larger number when school is in session. It is worth noting that PlanetMath has many "competitors", the most obvious being Wikipedia, which went online in 2001, and also MathWorld, which has been *back* online since 2001. It is also worth noting that in fact a considerable amount of content has been *shared* in both directions between PlanetMath and Wikipedia under the terms of their common license, so in this sense they can be seen more as "collaborators" than as competitors. Another site that could be seen to compete-while-collaborating with PlanetMath is MathOverflow.net, a question and answer site hosted on StackExchange, and which, again, uses the Creative Commons Attribution-ShareAlike (CC-By-SA) license for its content.

At this point, while PlanetMath's content collection continues to grow, the growing number of competitors of various sorts, as well as growing opportunities for collaboration, serves to suggest that infrastructural development will be vital if PlanetMath is to have any significant success in its "next ten years".

Luckily, over the course of the past decade, there has also been a significant maturation of tools for dealing with mathematics, and a collection of such tools, developed by the KWARC research group at Jacobs University, in Bremen, Germany, now form the basis for a new software platform for PlanetMath that we call "Planetary" (David, Ginev, Kohlhase & Corneli, 2010). This technological leg up gives us a realistic opportunity to ask *what should PlanetMath develop into?*

For example, we could inject P2PU-like (or P2PU-hosted) courses into PlanetMath; assignments might include "write or improve a PlanetMath article" or "post questions and answers in the forum". PlanetMath could also draw more on other resources created elsewhere, as well as do more to share or mash up its resources and services with others (e.g. Wikipedia and MathOverflow.net; also with the popular mathematics preprint server, ArXiv).

All of these things should happen, and they will certainly change the context at PlanetMath. However, it is not entirely clear that such improvements will directly change the *roles* of PlanetMath users. By contrast, introducing *problem sets* into PlanetMath would create a whole new set of roles centered on creating, solving, and marking problem sets online. The persons engaged with such activities might be mathematics learners, volunteers, or paid tutors.

There are certainly other challenges we could take up, and many questions to answer along the way. In order to do this effectively, PlanetMath should round out its "open content" and "open source" communities with improved methodologies for doing *open governance*. Some important issues to decide will be our approach to *open data* (just what data will be open?) and more generally how PlanetMath will relate to the emerging field of *open science*. Our own involvement over the next couple of years will focus on the issue

of problem sets, but the other ideas mentioned here are likely to find other champions within the community.

## Role-Based Analysis

Tables 8-13 present several different roles associated with PlanetMath. These are:

- The role of a contributor to the PlanetMath encyclopedia
- Possible future role of a contributor to the PlanetMath's collection of problems.
- Possible future role of a person using PlanetMath as a place to solve problems
- Possible future role of a tutor on PlanetMath
- Possible future role of a "benevolent technocrat" on PlanetMath
- Possible future role of a "citizen programmer/hacktivist" on PlanetMath

Note that a "casual browser" or someone who only posts a few questions in the forum will have a profile that's very different from any of these. In particular, such a user's sense of "We" will likely have to do mainly with off-site relationships.

In light of this, it would be tempting to imagine that "producer" and "consumer" roles are quite distinct. But at least in some parts of PlanetMath, they tend to be closely combined. For example, one must actively engage with problems in order to solve them, and a consumer of problems tends to be a producer of problem-solutions.

Is a problem contributor (Table 9) significantly different from a contributor to the encyclopedia (Table 8)? At the very least, different motivations are likely to be near the surface (e.g. "altruistic" motivations centered on helping others learn; or perhaps an interest in the way the organization as a whole learns), but it also seems that the style of working is very different (e.g. creating a good problem, in this context, has more to do with connections between system objects than it has to do with "exposition").

## Lessons Learned

If there is one clear lesson from the past 10 years of PlanetMath, it is that there is no shortage of great ideas out there. Unfortunately, there *is* a shortage of idea-implementers. The difference between Table 12 and Table 13 is meant to provide a view on that. These two roles differ only rather subtly, but the main idea is that we could in theory have many "citizen programmers", whereas at any given point in time, we are likely to have only a few "benevolent technocrats".

By building much of the new Planetary System in the form of modular extensions to the popular open source Vanilla Forums, we hope to make Planetary itself relatively easy to extend. To actually get people involved in the programming effort may require us to go quite a bit further, possibly even so far as to create a "PlanetComputing" where the Planetary System's code can be discussed and improved.

We hope that by making the development process more visible within future implementations of PlanetMath – and easier to get involved with – that we will be able to help people translate "ideas" into "incentives", and channel user input to a variety of increasingly useful ends. Some things that are difficult or impossible to achieve now should be fairly easy to deal with once more people are involved in the development effort. This will require careful ongoing analysis and design of social roles and contexts.

# GENERAL RECOMMENDATIONS FOR BUILDING COMMUNITIES AND CROWDSOURCING EDUCATION

## Comparing Crowdsourced and Traditional Education

One important question is to ask what differences occur between Tables 3-13 and Table 1, and in particular, whether crowdsourced

*Table 8. The role of a contributor to the PlanetMath encyclopedia*

| | |
|---|---|
| I | write about things I'm interested in, in the form of encyclopedia articles |
| We | give each other feedback on the things that have been written so far |
| Its | a collection of articles, forum posts, and metadata |
| It | helps me learn mathematics (by giving me the chance to practice expressing myself clearly) |

*Table 9. Possible future role of a contributor to the PlanetMath's collection of problems*

| | |
|---|---|
| I | write or find and contribute problems that link together with other problems and with encyclopedia articles |
| We | help each other determine the context that is best-suited to a given problem |
| Its | a collection of problems and semantic links |
| It | helps me share mathematical understanding with others and helps me understand mathematics more deeply myself |

*Table 10. Possible future role of a person using PlanetMath as a place to solve problems*

| | |
|---|---|
| I | solve problems online and get help from encyclopedia articles, peers, or, if I want, a tutor |
| We | turn to each other for help when we see we have common interests |
| Its | a collection of exercises, articles, solutions, and metadata that's intended to support independent and peer-based learning |
| It | helps me learn mathematics much as I would in a traditional classroom (but I can go at my own pace and pick my own topics) |

*Table 11. Possible future role of a tutor on PlanetMath*

| | |
|---|---|
| I | get paid to answer questions |
| We | work together to create the best collection of resources for helping tutors answer questions (and helping learners work on their own) |
| Its | a collection of previous tutoring sessions, augmented with annotations and links created by us or others |
| It | lets me use my mathematical proficiency to make money and help other people |

*Table 12. Possible future role of a "benevolent technocrat" on PlanetMath*

| | |
|---|---|
| I | review usage data from the system, integrate this with feedback from the community, and implement systems that serve their interests |
| We | meet along the boundary between content and code |
| Its | a collection of policies and programs maintained on behalf of the user community |
| It | keeps the site alive (and, thanks to me, growing) |

*Table 13. Possible future role of a "citizen programmer/hacktivist" on PlanetMath*

| | |
|---|---|
| I | contribute to code development as part of my regular form of interaction with the site and the corpus |
| We | make decisions horizontally and have engineered out systems and policies so that no one person has much more power than any other (in particular, we've endeavored to widen out organizational bottlenecks) |
| Its | a collection of policies embodied in code |
| It | not only keeps our site running, but allows us to make further inroads into online education more generally |

education is very different from traditional education or not.

The "student" and "teacher" roles are well represented in the model of crowdsourced education that we've developed. "Research" is *not* explicitly mentioned in any of our tables, but it certainly takes place, i.e. both research *about* crowdsourcing and peer-to-peer education, and also crowdsourced, peer-based, research on other topics, e.g. "Density Hales-Jewett and Moser numbers" by D.H.J. Polymath.[12] Building better support for this kind of massive collaboration – or even just better support for more tame computer-mediated research collaborations – is on PlanetMath's equally massive todo list. "Developer/administrator" roles are represented in Table 12, and an interesting blended "developer/advocate" role is described in Table 12.

In short, it seems that the only role from Table 1 that we haven't encountered in our analysis of P2PU and PlanetMath is the "accreditation body" role. Of course, it would be somewhat rare to find such persons embedded in a traditional university, since accreditation is importantly an "outside" role. P2PU has some innovative ideas about "peer assessment" and other strategies for measuring when learning is taking place, but these are quite a ways off from offering actual credits or diplomas.[13] They are also considering some team-ups with traditional institutions, where the outside institution grants credit to their enrolled students for work on P2PU.

In the case of PlanetMath, sufficiently rich and robust online problem sets should provide a clear sense of a learner's current state of knowledge; such measurements may in some cases be substitutable for a degree. At least according to our sketch in Table 1, it all depends on "knowing the appropriateness and relevance" of the training process. In some cases, this could be measured by learner performance on outside exams (e.g. AP tests, GRE subject tests, or perhaps even the Mathematical Tripos).

It would therefore appear that crowdsourced educational models at least have a chance of being as complete as the traditional model. Do they come with any additional benefits or constraints? The idea of a "a marketplace of interests, skills, and ideas" from Table 7 seems to have no ready correspondent in the world of traditional universities, except perhaps in a limited form, in the course catalog. At the same time, despite the broad popularity of social networks and the integration of some social networking functionality in both P2PU and PlanetMath, social networking on these sites is currently quite primative (simple questions like "find me a PlanetMath user who is interested in pedagogy" don't have easy answers).

Another major difference is in the issues surrounding commitment. Whereas students in traditional universities are presumed to be motivated by the prospect of earning a degree or certification, in informal education, a learner's prospects are entirely related to the skills they acquire, and to any enjoyment that comes from the learning process itself. If people need to be cajoled into committing, that isn't a terribly good sign. It seems it would be better if the learning environment itself facilitated involvement from people in the ways they feel suits them best. Perhaps incentives don't need to be created so much as "exposed" or "acknowledged".

## General Recommendations

Different sites seem to exist precisely so that different modes of engagement will be supported. They certainly shouldn't be homogenized, and probably it's not even sensible to ask for them to be "rationalized". Rather, it seems important to design for a variety of different modes of engagement and patterns of use.

Because of the way we've framed the analysis, we've seen that there is a social dimension to every resource. These don't always have to be "open" in every sense to be successful. However,

it seems likely that one way to open things up more, when desired, is to continue to do analyses of the kind we've done here: every organization can be mapped in this way, and can be examined for its potential *to support learning*.

It is useful to recall that a social context is defined by a variety of roles, and that roles often cross-organizational boundaries. Further, from the examples we examined, it appears that mixed approaches (a social layer *plus* preexisting resources; or, independent learning *plus* tutoring) have particularly strong potential.

At the same time, we should remark that a certain degree of cohesion in the learning platform ought to help make the *sharedness* of context more clearly felt. The question "how much cohesion is good?" seems to depend on the two factors most relevant to CBPP, integration, and modularity. Depending on what seems useful for the purpose at hand, a group of learners might be organized as a closed and seconded group, or as an open group embedded within a broader commons.

Licensing is one way in which groups become cohesive (and keep others out). Initiatives like MIT Open Courseware and OpenLearn[14] use a Creative Commons Attribution-NonCommercial-ShareAlike 2.0 Licence; Wikipedia, PlanetMath, and P2PU use a Creative Commons Attribution-ShareAlike 2.0, and Connexions uses a Creative Commons Attribution 3.0 license. Licensing can powerfully change the way people relate to resources, especially when it comes to future re-use and/or modification.

An educational community may or may not seek to become sustainable. Sustainability could apply to features like growth, innovation, longevity, communication, and more, and it seems clear that a diversity of approaches must combine gracefully if prospects of sustainability are to have much hope. The style of analysis we have employed could help an organization define and create new roles to adapt to a shifting context.

## Paragogy

One outcome of thinking about how to improve on the DIY Math course while building on the strengths of the P2PU organization as a whole is a new theory of peer-based teaching-and-learning, termed "paragogy" (Corneli & Danoff, 2011). Paragogy uses the "five principles of andragogy" stated by Malcolm Knowles as a jumping off point (Knowles, 1968, 1980). In super-succinct form, Knowles's five principles are: (1) that adults are self-directed learners, (2) that adult learners bring a wealth of experience to the educational setting, (3) that adults enter educational settings ready to learn, (4) that adults are problem-centered in their learning, and (5) that adults are best motivated by internal factors. A useful review and critique that goes beyond the soundbytes above is given by Laurie C. Blondy (2007). The five paragogical principles are:

1. *Context as a decentered center.* For learning design in a peer-to-peer context, understanding the learner's self-concept – in particular, whether they see themselves as self-directed or not – may be less important than understanding the concept of shared context in motion.
2. *Meta-learning as a font of knowledge.* We all have a lot to learn about learning.
3. *Peers are equals, but different.* The learner mustn't seek only to confirm what they already know, and must therefor confront and make sense of difference as part of the learning experience.
4. *Learning is distributed and nonlinear.* Sidetracking is OK, but dissipation isn't likely to work. Part of paragogy is learning how to find one's way around a given social field.
5. *Realize the dream, then wake up!* Paragogy is the art of fulfilling motivations when this is possible, and then going on to the next thing.

Paragogy can be fruitfully compared with another andragogy spin-off, heutagogy, a theory of self-directed learning (Hase & Kenyon, 2000). Because of its focus on the entire learning context, paragogy appears to intersect andragogy at 90 degrees, whereas heutagogy, with its focus centered on the learner, is 180 degrees away from paragogy. However, these theories do not seem to intrinsically conflict with one another, even though they offer very different views on education.

We feel that paragogy provides a useful philosophy for the advocate and practitioner of crowdsourced education, and that, together with role-based analysis proposed in this paper, it can be valuable addition to the world of OER and open online education. More generally, paragogy and its concept of the peer production of the learning context can open a new window on the ideas of organizational learning coming from the classical SECI model.

## CONCLUSION AND OUTLOOK

The chapter describes the various social roles of the individuals involved with P2PU and PlanetMath.org, and envisions future roles that could be created at these organizations. We hope that our analysis will help inform the relevant design, implementation, and governance decisions.

In addition, we compared crowdsourced and traditional education and found that the crowdsourcing model has room for most, if not all, of the roles found in the traditional setting, accreditation and assessment being a key area where there are no definitive answers as yet. It appears that the crowdsourced model can offer some additional richness, thanks to the role played by social networking tools, though social networking that works well for education is still in development. Perhaps new approaches here will help understand the various styles of commitment that manifest schematically in the "long tail" phenomenon that fuels crowdsourcing.

We expect that the role-based style of analysis we employed throughout this chapter, and the idea of paragogy, introduced above, can be applied fruitfully in both offline and online contexts. We look forward to developing these techniques further in our research, teaching, and development work, and invite the OER community at large to use and improve them (as always!).

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## KEY TERMS AND DEFINITIONS

**Andragogy:** Andragogy refers to a learning strategy focused on engaging and motivating adult learners. Andragogy makes the following assumptions about learning design: (i) adults need to know

why they need to learn something, (ii) adults need to learn experientially, (iii) adults approach learning as problem-solving, and (iv) adults learn best when the topic is of immediate value (Knowles, 1968, 1980).

**Crowdsourcing Education:** Crowdsourcing is commonly known as the process of assimilating many small contributions into resources of high quality. Crowdsourcing has emerged as a vital aspect of education on the web, as it promotes the openness and reusability of online educational resources contributed by communities of learners and educators.

**Heutagogy:** Heutagogy is a theory of self-determined and self-directed learning (Hase & Kenyon, 2000). Heutagogy builds on humanistic theories described in the 1950s and adapts them to the needs of today's learners, particularly targeting the development of individual capability.

**Online Community:** An online community is a community the members of which use online tools and services in order to engage into social interactions. These tools and services vary from instant messaging (IM) tools and forums to social networks. The members of these communities normally share common interests and goals and use the web in order to communicate with their peers, participate into discussions, exchange ideas, and work collaboratively towards a common goal.

**Paragogy:** Paragogy is a theory of peer-based learning, built on top of the following principles: (i) context is a decentered center, (ii) meta-learning is a font of knowledge, (iii) peers are equals, but different, (iv) learning is distributed and nonlinear, and (v) realize the dream, then wake up! (Corneli & Danoff, 2011).

**Peer-Based Learning:** Peer-based learning is driven by peer-mentoring instead of a central mentoring authority. Learners are mentoring each other and engage into a collaborative learning journey. A peer-based course functions similarly to a book club, whose members take turns into recommending books that are studied by the group, followed by an exchange of reviews and opinions.

**SECI Model:** SECI stands for Socialisation, Externalisation, Combination, and Internalisation. These are the 4 phases of knowledge engineering according to (Nonaka et al., 2000, Nonaka et al., 2003). More specifically, Nonaka and Toyama suggest that knowledge is created as people interact in a shared context, through a process that can be broken up into repeated phases. The SECI model takes into account the range of different behaviours and modes of interaction, involving activities that are both individual and collective, and forms of engagement that are embodied or virtual.

## ENDNOTES

1 http://p2pu.org
2 http://planetmath.org
3 http://cnx.org/
4 http://ocw.mit.edu/index.htm
5 http://blog.cnx.org/
6 Although it doesn't exist yet, we can imagine an ongoing hypertext-based discussion of the classic public domain texts that are available on Project Gutenberg (http://gutenberg.org). Such discussions could take place in an attached layer of annotations, and would be "modular" in roughly the same way a library or a university is modular. Such annotations could subsequently be refactored into more integrated works of literary analysis.
7 http://p2pu.org/about
8 http://www.shuttleworthfoundation.org/projects/p2pu/
9 http://groups.google.com/group/diy-math
10 http://en.wikiversity.org/wiki/User:Arided/ImplementingParagogy
11 http://planetmath.org/?op=about
12 See http://bit.ly/polymath-intro.
13 http://wiki.p2pu.org/w/page/12427267/accreditation
14 http://www.open.ac.uk/openlearn/

# The PlanetMath Encyclopedia

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**Abstract**

The history of PlanetMath.org is discussed, tracing its inception, stabilization, and some defining challenges. Research and outreach efforts that have been conducted in the course of work on the PlanetMath project are reviewed, and the scope and reach of the resource are discussed. Recent developments are indicated briefly. Some remarks evaluating PlanetMath's trajectory and content conclude the paper.

**Keywords:** online communities, mathematics, collaboration, encyclopedias, Commons-Based Peer Production, PlanetMath

## 1 Introduction

From PlanetMath.org's landing page[1]:

> *PlanetMath is a virtual community which aims to help make mathematical knowledge more accessible. PlanetMath's content is created collaboratively: the main feature is the mathematics encyclopedia with entries written and reviewed by members.*

This short paper describes the history of the PlanetMath encyclopedia. The history of this resource cannot be easily separated from a history of the PlanetMath community and the technology behind the site, though the presentation here is not especially technical. The reader who is interested in a succinct overview of the *current* characteristics of the encyclopedia will find what they seek in Sections 6 (quantitative aspects) and 8 (qualitative aspects).

## 2 Beginnings

The early history of PlanetMath is wrapped up with that of the similarly-named website, MathWorld.[2]

Eric Weisstein began collecting the material now found in MathWorld as a high school student, and continued the project as a college student in the late 1980s. "Eric's Treasure Trove of Mathematics," went online in 1995, when Weisstein was a graduate student in astronomy at the California Institute of Technology.[3] In November 1998, Weisstein made a deal with the CRC Press to publish his encyclopedia in book format, as the *CRC Concise Encyclopedia of Mathematics*. One year later, Weisstein accepted the position of encyclopedist at Wolfram Research, Inc., and the renamed "MathWorld" site was unveiled in December 1999.[4] In March 2000, CRC Press sued Weisstein and Wolfram Research for copyright violation, forcing MathWorld off of the internet.[5]

[1] `http://planetmath.org`

[2] `http://mathworld.wolfram.com`

[3] `http://www.echarcha.com/forum/archive/index.php/t-19516.html` a copy of the now-defunct `http://mathworld.wolfram.com/about/erics_commentary.html`

[4] `http://en.wikipedia.org/wiki/Eric_W._Weisstein`

[5] 57 U.S.P.Q.2d 1220 (C.D. Ill. 2000)

In the words of Eric Weisstein: "if you ever assemble a body of knowledge that you want to share with others, you don't want to go through what I have just gone through."[6] So it came to pass that in the Fall of 2000, Nathan Egge and Aaron Krowne, at that time both undergraduates at Virginia Tech, came up with the idea for PlanetMath: a collaboratively created mathematics reference work that would have resistance to copyright threats built in, in the form of an open content license. By the summer of 2001, the basic infrastructure for creating an encyclopedia was complete, and a fledgling community had grown up around the resource.

The CRC lawsuit was settled for an undisclosed sum in late 2001, and on November 6, 2001, MathWorld returned to the internet.[7] But in the mean time, a new online community had been born – with some very different principles and practices. Whereas MathWorld's terms of use disallow archival copies, PlanetMath regularly publishes snapshots of the content for download. Moreover, users are permitted (and, indeed, encouraged) to copy, mirror, redistribute, print, remix, and reuse PlanetMath content for commercial or any other purpose – so long as all such works are published under the same license as PlanetMath, granting downstream users the same rights.

## 3 Stabilization

The key reference for PlanetMath is Aaron Krowne's 2003 Master's Thesis, written at Virginia Tech [6] under the supervision of Ed Fox. In this thesis, Krowne describes how the early design and development of the site benefited from continuous feedback in the #math IRC channel on Undernet.[8] He also details the key technical and community features of the site as they developed in this period:

- A state-of-the-art system for displaying mathematical notation on the web, starting from LaTeX sources.
- A flexible authority model that can support both wiki-style articles (that anyone can edit), and a more academic style, where articles are owned by one person, who may, if they wish, grant co-authorship permissions to chosen others, and who must respond to separate commentary from peer reviewers (cf. [7]).
- A discussion forum attached to every encyclopedia article, which helps give the resource a "pedagogical slant".
- An autolinking service that helps integrate content into the site, by enabling authors to focus on the contents of one article at a time.
- Workflow built around corrections and watches, including a feature whereby articles are "orphaned" if a correction is not responded to after a given period of time.
- A scoring feature that provides a rough estimate of how much value each user has contributed to the site.

In 2003, PlanetMath incorporated, and in 2005, obtained non-profit status, so that it could accept tax-deductible donations (in the US). Together with a small stream of ad revenue, this has covered hosting and other maintenance costs.

[6] See Footnote 3.
[7] `http://mathworld.wolfram.com/about/faq.html#history`
[8] `irc://irc.undernet.org/math`

# 4 Pushing the limits

In 2003, the present author was enrolled as a graduate student in mathematics at the University of Texas in Austin, and in possession of a large and growing personal collection of very tersely-written definitions and proofs relevant to the department's prelim exams.[9] In fact, this work had as much to do with the tradition of computer mathematics in the air in Austin (QED, Maxima, ACL2, AM) as it had to with exams. A representative example:

(lebesgue outer measure: fact: lebesgue outer measure is infimum of lebesgue outer measures of open supersets)

1: $X \subset \mathbf{R}^n$
2: $L = \{O \circledcirc \mathbf{R}^n : O \supset X\}$
3: $|X|_e = \inf_{O \in L}(\{|O|_e\})$

After discovering PlanetMath and striking up a correspondence with Krowne, one night we uploaded the contents of the "Austin Problems in Mathematics – Cross-Index" (styled APM-$\xi$) into the PlanetMath encyclopedia as world-editable "seed entries". This turned out to be a first-rate disaster.[10]

The primary complaints from community members were:

(1) the entries could not be understood without reading an accompanying FAQ;

(2) a casual visitor to the PlanetMath website might get the wrong impression about the nature of the encyclopedia when looking at "apmxi" entries; and,

(3) nearly 600 entries had been introduced into PlanetMath by the site's administrator in one big batch, circumventing, at least in outward appearances, the site's usual model of careful review and collaborative editing of entries.

Subsequent to a poll, it was decided that the apmxi entries would be "orphaned", and any that were not adopted by community members after a week would be deleted from the encyclopedia. This was the fate that befell most.

The event was a defining moment in the history of PlanetMath. In the first place, it was a testament to the strength of the community's norms. Secondly, it showed that the specific affordances of computers, e.g. for mass processing of data, or for dealing with hypertextual complexity associated with alternate related treatments of a given topic, needed to be tempered to work well for the *people* involved. These issues would set much of the research and development agenda around PlanetMath for the following decade.

# 5 Research, outreach, and some critiques

In 2005, several established PlanetMath contributors met in person at a Symposium on *Free Culture and the Digital Library* at Emory University, where Krowne was then based. Contributed papers looked at

- an adaptation of PlanetMath's software for collaborative creation of course notes in a graduate course on ordinary differential equations [10];
- experiments with a novel hypertext system based on the idea that everything is annotatable [1]; and,

[9] `http://metameso.org/~joe/math/Xi.pdf`
[10] `http://wiki.planetmath.org/cgi-bin/wiki.pl/one_week_in_october`

- the dynamic tension between the non-copyrightability of ideas, and the necessity of conveying ideas in copyrightable expressions, and the ramifications for mathematics [12].

These reflections on copyright (and copyleft) were subsequently expanded in an article for *First Monday*, which looked at the drawbacks of current copyleft licenses, particularly "license lock" [8].

We presented talks about PlanetMath in the *Math on the web pavilion* at two Joint Mathematics Meetings (San Antonio, 2006[11]; New Orleans, 2007[12]), and in a session on *The Role of Open Source Math Projects in the Mathematics Community* at MathFest (Madison, 2008)[13]; and at more specialized workshops: *The Evolution of Mathematical Communication in the Age of Digital Libraries* (Minneapolis, 2006)[14], and *Mathematical Knowledge Management: Sustainability, Scalability and Interoperability* (Halifax, 2007)[15].

We also made efforts to create a print version of the PlanetMath encyclopedia (retitled the "Free Encyclopedia of Mathematics"). The 2004 attempt, in two volumes[16], and a 2005 attempt in one much nicer-looking volume[17], thanks to Ross Moore's contribution of `multinclude.sty` and other tweaks.[18] Still, the resulting 1971 page PDF was more a proof of concept than a printer's proof.

On the development side, PlanetMath was thrice supported by Google's Summer of Code (2006–2008). The best outcome of this was that PlanetMath's autolinking subsystem was turned into a modular piece of code, NNexus[19], as written up in [5]. PlanetMath's software improved further under contract with Springer, pursuant to the creation of StatProb.com.[20] PlanetMath's sister site PlanetPhysics.org[21] is currently in the process of switching over to this platform, termed Noosphere 1.5.

However, the development effort wasn't particularly able to keep pace with the feature requests generated by the user community.[22] Nor did the Noosphere codebase present a particularly compelling resource for capable computer mathematics developers like Claus Zinn [14] and Christoph Lange [9], to jump into and improve. Zinn wrote:

> *The rapid growth of math resources on the web, which is further pushed by wiki-based communities, is both a threat and an opportunity for intelligent math learning environments [...] If we could harness the collaborative authoring process and encourage and guide wiki authors to continually provide content and metadata, then intelligent services could unleash their true potential.*

## 6 Scope and reach

At the time of this writing, PlanetMath contains 8945 entries, dealing with 15655 concepts. 298 people have contributed an entry in the encyclopedia, and 2742 have contributed something (perhaps just one forum post).

[11] `www.jointmathematicsmeetings.org/meetings/national/jmm/san-prog.pdf`
[12] `http://www.dessci.com/en/company/shows/jmm/mow2007.htm`
[13] `http://www.maa.org/abstracts/mf2008-program.pdf`
[14] `http://www.ima.umn.edu/2006-2007/SW12.8-9.06/`
[15] `http://projects.cs.dal.ca/ddrive/seminars/mkm.shtml`
[16] `http://www.scribd.com/doc/9691966/`, `http://www.scribd.com/doc/9692058/`
[17] `http://metameso.org/~joe/docs/book.pdf`
[18] `http://metameso.org/~joe/math/fem-2005.tar.gz`
[19] `http://code.google.com/p/nnexus/`
[20] `http://statprob.com`
[21] `http://planetphysics.org`
[22] `http://wiki.planetmath.org/cgi-bin/wiki.pl/Feature_Requests`

Out of these, an exceptional group of 24 authors have produced more than 100 encyclopedia articles apiece. Their contributions comprise 74% of the total number of articles. About 71% of this core group joined before 2004 (in the "early days" for the site).

130 users have a score of 1000 points or more, which would correspond to contributing 10 or more new encyclopedia articles, but actually, a significant fraction of this value has been contributed through things like corrections, revisions to existing objects, and posting in the forums. All told, this group has contributed 96% of the total number of articles. About 58% of this group joined before 2004.

According to Alexa.com, PlanetMath.org is currently the 165,011$^{\text{th}}$ most popular website in the world, comparable to the website of the Mathematical Association of America[23] (119,267$^{\text{th}}$), or relative newcomer MathOverflow.net[24] (184,818$^{\text{th}}$).

A far cry from competing with Wikipedia, but in fact we like to think of the two projects as mutually supporting. PlanetMath content is reused under the terms of the shared CC-By-SA license in hundreds of Wikipedia articles, and cited in over 1500.

## 7 A new era

In 2010, a new project to completely rebuild PlanetMath's software began. *Planetary* is based at Jacobs University, Bremen, and led by Michael Kohlhase, with major contributions from most members of his research group (along with the present author).[25] The Planetary system is described in [4]. Planetary is considerably easier to extend than Noosphere: it is currently comprised of around 20 plugins for the popular open source platform, Vanilla Forums[26], many of which integrate sophisticated software tools previously developed by the KWARC group. Notably, the system now includes support for semantic authoring and flexible metadata interaction, addressing the critiques mentioned in Section 5.

2011 will see the publication of a book chapter discussing the future use of PlanetMath as the core of a problem-based learning system [2], the focus of the author's doctoral studies [3]. With this as a basis for a showcase of innovative uses for the PlanetMath content, and with a well-documented and easy to extend software platform supporting the system, we hope to see PlanetMath become a central integration platform for free software projects working with freely licensed math on the web.

## 8 Conclusion

PlanetMath has been successful as an online community: the software stabilized early on and has required little upkeep, while the site has continued to grow. However, there has been a danger that with the software system running more or less on "autopilot", new features would not be developed. With any luck, this threat is in the process of being eliminated, heralding in the opportunity to build one or more "new" online communities in close relationship to PlanetMath (e.g. a developer community working on sophisticated tools for scientific communication; a learning community using the PlanetMath encyclopedia as part of a remix-driven interactive textbook). We should do a careful evaluation of what has worked and what could be improved for next time. There is not room in the current paper to conduct this discussion, but framework proposed by Resnick *et al.* would be an excellent place to begin [13].

For all of its potential as a software showcase and its possible future role as a player in a larger landscape of related interlinked online communities, PlanetMath should, at least for the moment, be evaluated first and foremost as an encyclopedia. This too would be best as an ongoing task. For now,

[23] `http://maa.org`
[24] `http://mathoverflow.net`
[25] `http://trac.mathweb.org/planetary`
[26] `http://vanillaforums.org`

| | |
|---|---|
| **Coverage** | The median entry would be an advanced undergraduate or beginning graduate topic. PlanetMath is generally considered to have more in-depth treatment of technical issues, e.g. of proofs, than Wikipedia. |
| **References** | Present in many articles, although there is not yet a unified database of references or style of presenting them (this is planned). |
| **History** | 315 items, mostly 20th Century or later.[27] |
| **Audience** | Consistent with the coverage, there have been 4127 posts in the "Graduate/Advanced" forum, 4261 posts in the "University/Tertiary" forum, and 1199 posts in the "High School/Secondary" forum. |
| **Clarity** | There is no hard and fast rule. Some articles are minimalistic, but precise (e.g. from strong contributors who have have English as a second language). Other articles may be verbose and vague. In any case, debates over clarity of presentation are intense, and a high standard is generally maintained (see Section 4). |
| **Pictures** | There are over 600 images, but this is only about 7% of all entries. A unified database/gallery of pictures might encourage more submissions. |
| **Accuracy** | At the time of this writing there are 48 outstanding corrections[28]; more than 14080 corrections have been filed since the site began, though 2337 are classified as "addenda", meaning that no mistake is implied, and 9182 are classified as "meta/minor", meaning that in the entire history of PlanetMath some 2561 real errors have been found through the peer review process (and all but a few fixed!). Note that these numbers do not take into account direct changes by authors (and would tend to under-represent error fixing in world-editable articles). |
| **Unusual** | PlanetMath provides "math for the people, by the people". |
| **Weight** | PlanetMath can be used from any browser, and comfortably edited from a lightweight laptop or netbook, weighing about 1kg. |

Table 1: Succinct review of the PlanetMath Encyclopedia

a quick summary following the outline used by Emma Previato in her review [11] of the *CRC Concise* gives us a look at how PlanetMath measures up (Table 1).

Because the technology that supports the site is special-purpose, we have been able to hone in on what works best for commons-based peer production in mathematics. Features like the autolinker facilitate integration of content, and the corrections system helps avoid messy battles. There is much more work to be done, but at the close of its first decade of life, PlanetMath may be poised to become a encyclopedia in the literal classical sense of a "complete instruction".

[27] `http://planetmath.org/browse/objects/01Axx/`
[28] `http://planetmath.org/?op=globalcors`

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# The Planetary System: Web 3.0 & Active Documents for STEM

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**Abstract**

In this paper we present the Active Documents Paradigm (semantically annotated documents associated with a content commons that holds the corresponding background ontologies) and the Planetary system (as an active document player). We show that the current Planetary system gives a solid foundation and can be extended modularly to address most of the criteria of the Executable Papers Challenge.

*Keywords:* active documents, executable paper challenge, content commons, semantic annotation, ontologies

## 1. Introduction

The Executable Papers Challenge (EPChallenge) calls for a framework that allows papers to be *executable*, give access to *provenance* and *copyright/licensing* information, and support various forms of *validation*. We contend that these challenges can be realized based on *semantically annotated documents* together with semantic background ontologies, which we call the **content commons**. This information can then be used by user-visible, interactive services like program execution, computation, visualization, navigation, information aggregation and information retrieval. Finally a document player application can embed these services to make documents executable. We call this framework the **Active Documents Paradigm** (ADP), since documents can also actively adapt to user preferences and environment rather than only executing services upon user request. In this paper, we present the ADP with a focus on the Planetary system as a document player on three distinct annotation levels, "presentation structure", "semantic", and "formal".

The Planetary system (see [1] for an introduction) is a Web 3.0 system[1] for semantically annotated document collections in Science, Technology, Engineering and Mathematics (STEM). In our approach, *documents published in the* Planetary *system become flexible, adaptive interfaces to a content commons* of domain objects, context, and their relations. The system achieves this by providing embedded user assistance through an extended set of user interactions with technical documents, based on an extensible set of client- and server-side services that draw on explicit (and thus machine-understandable) representations in the content commons.

## 2. The Planetary System: Current State

Development on the Planetary system began in summer 2010 based on an established set of knowledge management components; we have tested it on a variety of paradigmatic document collections, which we will describe

[1] We adopt the nomenclature where Web 3.0 stands for extension of the Social Web with Semantic Web/Linked Open Data technologies.

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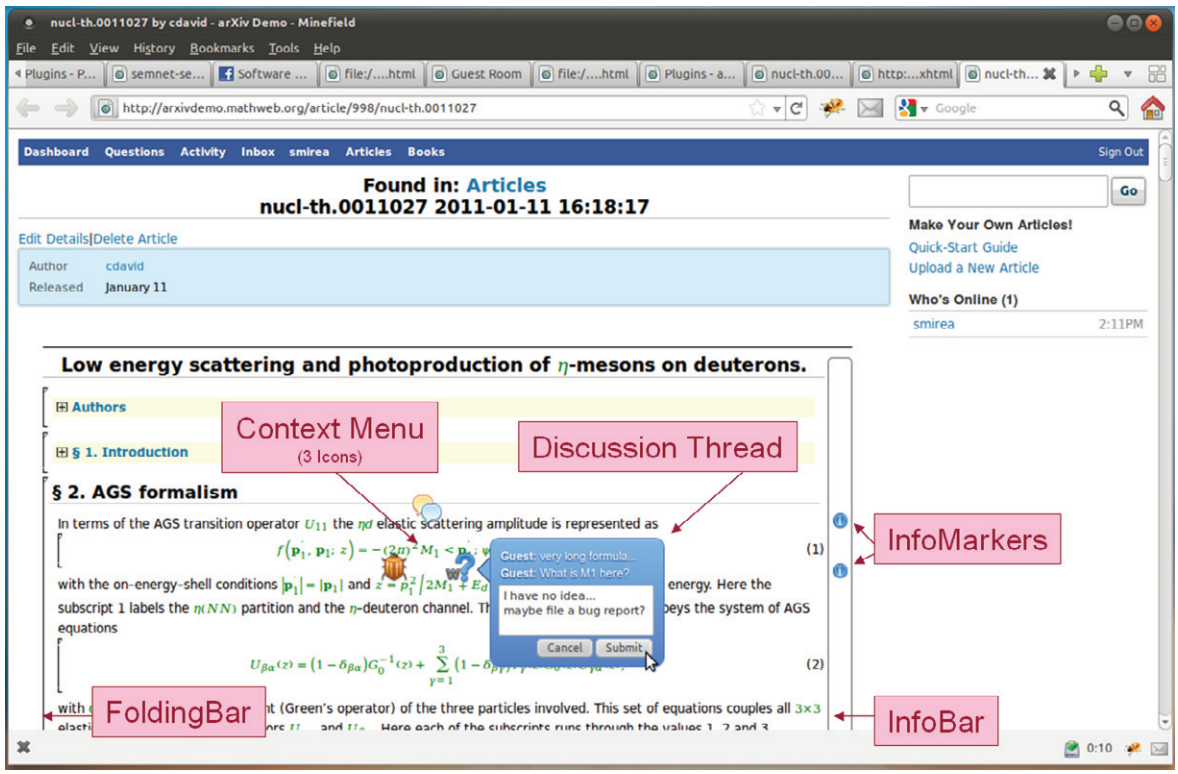



Figure 1: Interacting with an arXiv Article via `FoldingBar`, `InfoBar`, and Discussion Thread

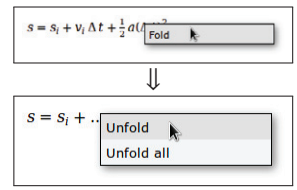
now. We group these by the level of semantic markup involved: generally, more semantic markup translates into more active documents, but requires a higher investment of the authors (but see Section 3.1). Note that PLANETARY gracefully adapts to the level of markup available: even though it was designed for the "semantic level" (see Section 2.2), it can also function well with a level of markup that can be extracted from pre-existing document formats, e.g. LaTeX (we call this the "presentation structure level"). The study of different levels of semantic saturation of individual documents is complemented by looking at libraries and collections of interlinked documents; Section 2.4 covers this aspect.

### *2.1. The Presentation Structure Level:* PLANETARY *arXiv and PlanetMath*

The importance of the presentation structure level is that PLANETARY can turn legacy documents into active documents by transforming them into XHTML+MathML+SVG-encoded documents with semantic annotations in RDFa. We have transformed over half a million articles from the Cornell ePrint arXiv to XHTML+MathML with LaTeXML [2], preserving properties like document and formula structures and embedded them into an instance of the PLANETARY system [3].

The document structure can then be exploited for a `FoldingBar` service (see on the left in Figure 1) and for localizing discussions about document content to document structures and subformulae – e.g. for questions/answers, or reviewers' comments. In the situation in Figure 1 we have clicked on formula (1), which pops up the `IconMenu` with three options: reporting errors in the content (bug icon), asking/answering a question (question mark icon), and accessing the discussion threads of this element (balloons icon). Here, a click on the question mark icon allowed us to pose a question and hope for an anser by other users in the forum. Figure 1 also shows the PLANETARY `InfoBar` with information markers on the right, which indicate the availability and state of the discussion threads pertaining to information objects in the line they are horizontally aligned with. Clicking them will highlight all items that have discussions. Localized discussions have proven a very valuable tool for community-based validation of papers, especially if they are coupled with a discussion subscription/trackback system for readers and personal notification system for authors.

The formula structure allows us to provide semantic services like subformula folding. The image on the right shows a formula from [3] where we right-clicked on the formula $s = s_i + v_i\Delta t + \frac{1}{2}a(\Delta t)^2$ for the law of uniformly accelerated motion to get the PLANETARY context menu. Folding a subformula (i.e., replacing it by an ellipsis marker yielding $s = s_i + \ldots$) yields the lower (simplified) formula, which in turn can be unfolded again whenever wanted.




### 2.2. The Semantic Level: Semiformal Digital Libraries

We can considerably improve the user experience by extending the depth of semantic annotations. For this we employ OMDoc [4] (Open Mathematical Documents), an XML-based content-oriented, semiformal representation format for scientific and technical documents. It builds on the OpenMath/MathML3 semantic representation format for mathematical formulae [5, 6]. OMDoc extends OpenMath with an infrastructure for context and domain models from Formal Methods, as well as a generic document infrastructure. At the semantic level PLANETARY is based on STEX documents [7], which can be transformed to OMDoc and via a user-adaptive and context-based presentation process further to XHTML+MathML+SVG+RDFa. The generated OMDoc documents are committed to an instance of the versioned XML database TNTBase [8] that indexes them by semantic functional criteria, and can then perform server-side semantic services via user-defined XQuery queries.

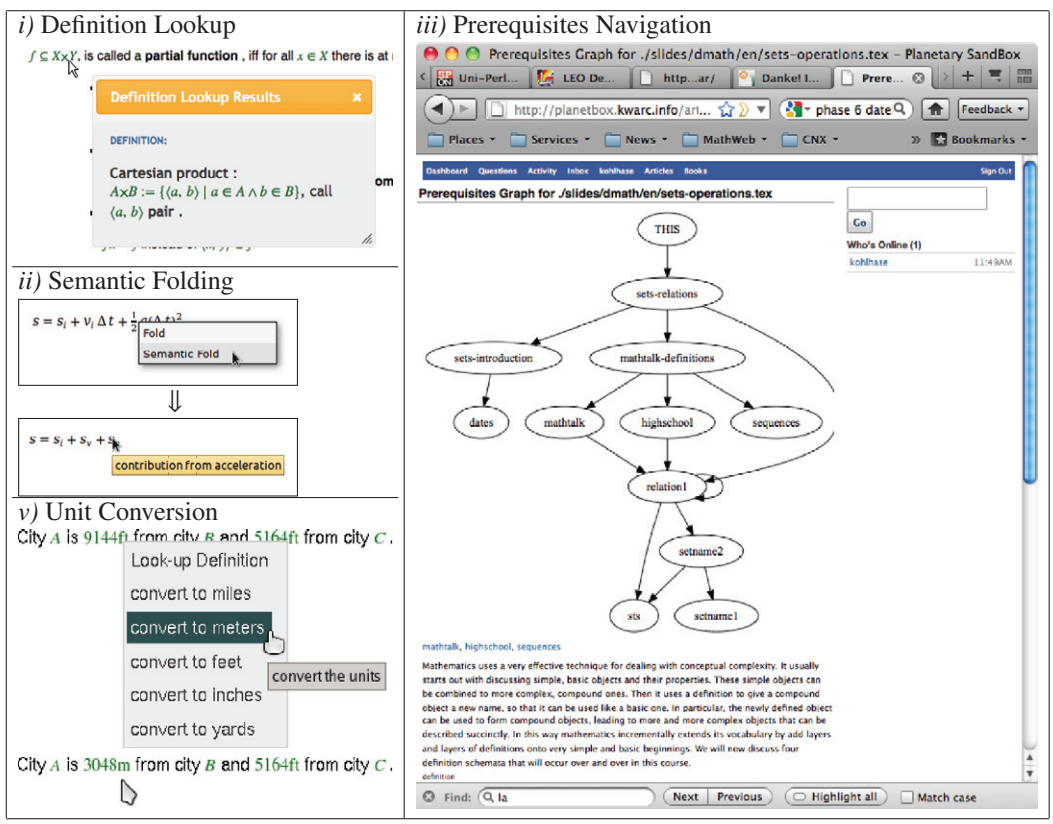
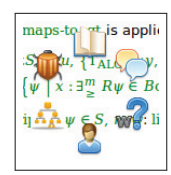



Figure 2: User Services at the Semantic Level

TNTBase thus becomes a source of the user-adaptive, custom-generated documents forming PLANETARY's content commons. Many semantic services can directly be derived from this setup. At the semantic level, the `IconMenu` we know from Figure 1 can be extended, depending on the services available for the semantic item in focus. The book icon triggers definition lookup (see item *i)* below) and the graph and portrait icons prerequisites navigation (see *iii)*). Figure 2 shows results of these and other services.



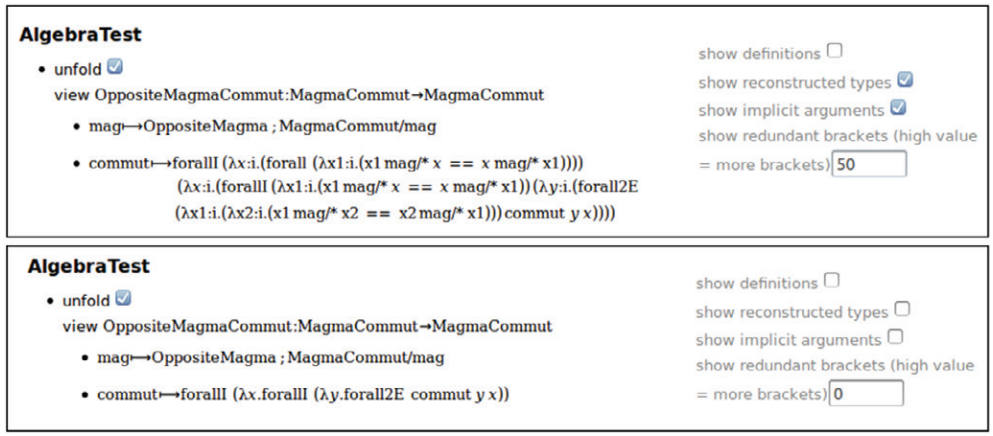



Figure 3: Formal Representations Adapted to Distinct User Settings (Customized via the `DocDash` Widget on the Right)

*i) Definition Lookup*: All technical terms and symbols in formulae presented in PLANETARY are linked to their semantic counterparts in the content commons, which in turn are linked to their definitions [9].

*ii) Semantic Folding*: If any explanations of the meanings of subformulae have been added as annotations, folding can use these instead of "...". In Figure 2/*ii)* the motion law above is semantically folded to $s_0 + s_v + s_a$ and the abbreviations $s_*$ are explained via flyover help.

*iii) Prerequisites Navigation*: As the content commons has an inherent notion of semantic dependency, we can use that to show prerequisites leading to a concept. Currently PLANETARY supports two ways of dealing with prerequisites: *i)* a *concept graph view*, where the required concepts can be navigated on demand by clicking on concept nodes, and *ii) guided tours*, where the necessary content is generated in a coherent narrative [10].

*iv) Executable Formulae*: As the formulae in the documents are generated from OpenMath objects, we can export them to a computer algebra system like Mathematica (or in our case the open-source system GAP [11], which has been made available as an OpenMath-based Web service [12]) for evaluation, graphing, or experimentation.

*v) Unit Conversion*: In scientific papers, formulae often contain expressions for measurable quantities; these can be automatically converted to other unit systems [9].

Note that the underlying OMDoc format and the services based on it address a peculiarity of documents in the STEM disciplines. Here, documents are *dynamic* in the sense that they declare new concepts, definitions, model assumptions, terminology, notations, etc., as they go along, or else they explicitly (or implicitly) import such items from other documents. As a consequence, all knowledge items in STEM documents have a non-trivial context of declarations. This must be managed explicitly in our representations of these documents, in the content commons, and within user interaction in order for semantic services to be effective. This effect is especially pronounced in mathematical sciences (including Computer Science, Physics, etc.), and somewhat less so in Chemistry and the Life Sciences, where a global context in the form of external terminology and notation databases often suffices.

### *2.3. The Formal Level: Specification and Verification*

Finally, we can use PLANETARY as a frontend system for completely formal content. OMDoc is also a foundation-agnostic integration format for mathematical knowledge that can express web ontologies [13, 14], program specifications and verifications [15], and even representations of logics in logical frameworks like LF [16]. In formal systems, documents are dominated by complex formulae, and users need support in navigating, abstracting, and evaluating them in order to cope with this complexity. Figure 3 shows how fully formalized formulae can be adapted to user preferences, about, e.g., the level of brackets and the availability of inferred arguments or definitions. Further semantic services at the fully formalized level include access to automated theorem provers via the HETS system [17] and argument reconstruction via the Twelf system [18].

It is important to note that programs (and program fragments) are also knowledge items at the fully formal level, as all their semantics can be recovered by parsing. Program fragments are not yet supported by PLANETARY (but see section 3.4).

### 2.4. The Network Layer: Library Integration of Document Metadata

We have concerned ourselves with individual active documents at various levels of semantic saturation so far. The PLANETARY system adds infrastructure for document collections and libraries as well (see Figure 4 and [19] for details). The library organizational level is realized as individual PLANETARY installations and provides the base URI of the system and its content. Examples include our PLANETARY-based scientific encyclopedia portal [20] or a course system. In these examples, the collection level corresponds to individual disciplinary encyclopedias such as PlanetMath [21] or individual courses in the course system. The contribution of the library and collection levels consists mostly in an inter-document, or network, layer of semantics via the aggregation of document metadata annotations. We represent metadata and cross-links between documents and their fragments as RDF graphs, reusing existing metadata vocabularies/ontologies[2] wherever appropriate. PLANETARY makes metadata and links editable to authors via LaTeX convenience macros, whereas some administrative metadata (e.g. the date of last modification) are recorded automatically. For services that interact with an active document, we employ *semantics-preserving transformations* that embed the metadata and links as user-invisible but machine-readable RDFa annotations.

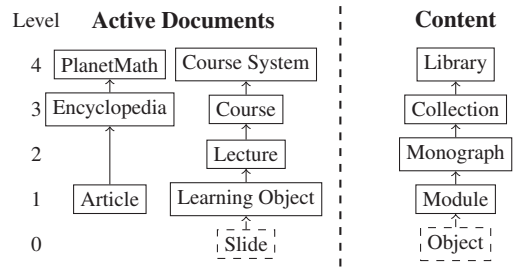


Figure 4: Levels of Organisation of Content

We keep metadata separate from the analysis of document semantics above to emphasize its independence of the document content, going one meta level up. This is also the justification behind its primary role as a connector (or least common denominator) between collections or entire libraries of documents, as opposed to the intra-document aspects outlined previously.[3] To enable this functionality, we add an RDF triple store as a system component and integrate it via its SPARQL endpoint API.

The utility of such annotations comes from (i) their reintegration into access methods or identifiers to collection and library subsets, and from (ii) the way PLANETARY makes the *vocabularies* of these annotations customizable and extensible within the system. Regarding (i), a generic mechanism of a "view" on a collection of documents can be instantiated to both simple and complex realizations, such as viewing the encyclopedia of PlanetMath alphabetically, by the Math Subject Classification (MSC [23]) categories of its articles or by posing the query "top rated new articles on Abelian groups, which were revised in the last three months (i.e. since I last read about such groups)". By making such views accessible by their own URLs, PLANETARY leverages the power of the resource locators in order to point to dynamic pages, which resolve an encoded metadata query. This new use of URLs generalizes the classic idea of what a resource is, in order to provide a more expressive way to communicate for users, for example in discussions about refactoring content, citing prior work, or pinpointing the next steps of a future effort. Regarding (ii), PLANETARY treats the annotation vocabularies used for metadata and links – their informal description, their formalization, and their documentation – exactly like any other of its STEM documents. In previous work (cf. [13, 24]), we have demonstrated that the OMDoc and STEX languages subsume common semantic web ontology languages, such as OWL, and go beyond them in their support for logical heterogeneity (cf. Section 2.3), integrated documentation up to literate programming (cf. Section 3.4), and modularity and reuse. By making the vocabularies editable inside PLANETARY, we enable the community to customize them. This is particularly powerful when a vocabulary immediately defines the functionality of a *service*, as is the case with our semantic discussion and refereeing facility explained in Section 3.3. Users (with sufficient permissions) can thus enhance the functionality of the service by introducing new vocabulary, they can refine the functionality by extending the formalization of existing vocabulary, and they can enable better user assistance by elaborating the documentation of those vocabulary terms that system functionality is based on. In that sense, *the ontology is the API* of the PLANETARY system.

We thus have the potential to envision metadata-enhanced social-semantic features such as subscribing to custom news feeds, social bookmarking, interoperability with external libraries and more.

[2] Here, we use the terms "vocabularies" and "ontologies" synonymously. Ontologies are the usual means of formalizing the semantics of vocabularies on the Semantic Web.

[3] Note that this separation is not total as there are interactions between the metadata level and the document semantics level. For instance, following a practice from the PlanetMath encyclopedia we use the NNexus tool [22] to semi-automatically discover concepts within documents (a document feature) and add links to their defining encyclopedia articles (a network feature).

### 2.5. The Active Documents Paradigm and the EPChallenge

We see the main impact of the Active Documents Paradigm and the Planetary system to the EPChallenge concentrated on the semantic level described in Section 2.2. At the presentation structure level the embeddable semantic services are largely limited by the semantic information that can be recovered from the LaTeX sources[4]. In the future some semantic annotations can be automatically discovered by heuristic or linguistic analysis methods [25], enabling services from the semantic level to be deployed on legacy documents. These analysis methods are subject to active research in our group, but beyond the scope of this paper.

Importantly, the Planetary system supports the incremental semantic upgrade of documents: any semantic annotation is directly made available to the semantic services, making documents more interactive, and thus contributes to the user experience of the enhanced active documents. This constitutes an opportunity for crowd-sourcing the semantic annotation of documents: Communities may propose annotations to documents which the document owner can then accept or reject. We can also think of alternative methods for quality control of semantic annotation, e.g. reputation/karma-based ones. A prerequisite for this will be tools that allow other users to express the semantic annotations, collect them, and incorporate them into the Planetary content modules as if they were author-supplied.

## 3. The Planetary EPChallenge Version

In this section we sketch how the Planetary system can be extended to answer most of the EPChallenge criteria before ICCS in June 2011. Details can be found on [26].

### 3.1. Semantic Editing Support

One of our priorities is to make the authoring process of semantic documents/discussions (to reach the semantic level) as easy as possible. We do that by providing a web-based authoring environment which lifts in-browser editing of sTeX documents to the level of popular LaTeX editors. Moreover, the environment is aware of sTeX semantics and can highlight fragments according to their functions (semantic syntax highlighting) as well as suggest autocompletions based on the context at request location. Crucially, the editor supports the creation of semantic links between documents, which is very tedious without this extension. For example, we implemented an instant search feature which helps the author in finding semantic concepts defined in other documents and subsequently reuse them without leaving the editing environment. Also, hovering over a concept shows its definition (if available), that is a definition lookup in the *source* document.

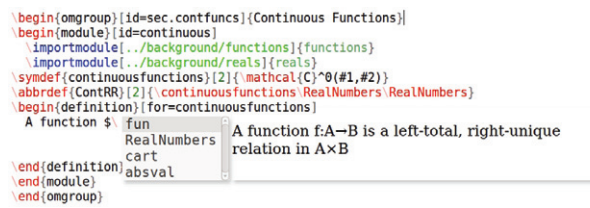



Figure 5: Autocompletion with Definition Suggestion

### 3.2. Integrated Semantic/Formula Search

Currently, the Planetary system only supports standard text-based search of the LaTeX/sTeX sources of articles and forum posts. We plan to integrate our currently independent semantic formula search engine MathWebSearch [27] for semantic content, using semantic copy-and-paste to alleviate/circumvent the remaining "math input problem". With such a formula search engine, a user can search mathematical content by its structure, e.g. the recollection that the energy of a signal $f(t)$ is given by the square of the signal integrated over time which can be expressed by a schema $\int_{\boxed{l}}^{\boxed{h}} \boxed{\mathrm{f}}(x)^2 dx$, where the boxed variables are search variables. This would find e.g. Parseval's Theorem $\frac{1}{T}\int_0^T s^2(t)dt = \sum_{k=-\infty}^{\infty} \|c_k\|^2$ (the search also took into account renaming the bound variable) which allows to compute the energy. Note that the definition lookup feature shown above comes in very handy here to look up the meaning of $c_k$: they are Fourier coefficients of $s$. The main problem to be solved here is to automatically infer the context of semantic elements in the documents. Formula search at the presentation structure level can be realized via heuristic transformations from presentation to content markup.

[4]Note that even though we have mainly studied documents that are automatically generated from LaTeX sources, initial experiments show that the Planetary system is equally applicable to other sources. In particular, the XML+MathML sources of Proceria Computer Science provided by Elsevier to the EPChallenge finalists could be integrated into the system after an afternoon's work.

### 3.3. Semantically Supported Discussions and Refereeing

We are semantically enhancing discussions with our previously developed argumentation ontology (which includes extensions to STEM-specific argumentative primitives) [28, 14]: Users can indicate categories for forum posts and fine-grained relations between forum posts and documents, or between posts and other posts. In particular, the argumentation ontology defines what types of *problems* apply to what types of STEM knowledge items (e.g. a declaration of a mathematical symbol can be ill-formed, but not invalid, as a proof could be), and what types of *solutions* apply to these problems (e.g. renaming a definition that a reader does not understand is less likely to solve the problem than providing an example for it).

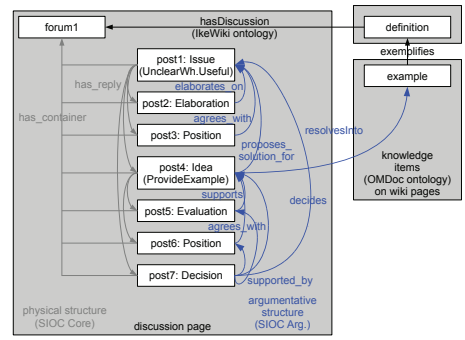



Figure 6: RDF Graph of a Discussion Thread about a Mathematical Definition (using our STEM-specific argumentation ontology)

The discussion facility is an application of our "ontology as API" approach introduced in Section 2.4, in that the discussion forum user interface uses such information from the ontology to guide users who want to write or reply to a post, and additionally reuses the documentation of the argumentation ontology as context-sensitive online help for the discussion forum user interface. As the argumentation ontology is editable inside PLANETARY, any community running a PLANETARY site can adapt it to their preferences, e.g. by defining new types of problems and solutions from their particular STEM subdomain, or refining the existing ones.

Adding versioned references [29] (facilitated by the TNTBase storage) allows incremental document/content commons development and commenting workflows to be managed in distributed teams. Building on the PLANETARY system's user management, we can extend this into a localized refereeing feature.

### 3.4. Program Fragments and Semantic Literate Programming

OMDoc and STEX have been used to deal with semiformal [15] and formal software engineering, but support for programming in the PLANETARY system is still lacking. The main problem to solve is that of modeling the scope of program fragments relative to background theory, e.g. in Computer Science papers. When that is solved, we will have a basis for "semantic literate programming", which can be supported by semantic services. To this end, we are currently working on integrating the notion of "biform theories" [30] into OMDoc for a more principled support of program fragments at the formal and semantic levels. Given this, we can make program fragments in a programming language $\mathcal{L}$ executable via an "universal OpenMath $\mathcal{L}$-Machine", a web service which accepts an $\mathcal{L}$-fragment $f$ and arguments $a_1, \ldots, a_n$ encoded as OpenMath objects (i.e. in the form in which they occur in the documents that appear in the content commons), executes $f(a_1, \ldots, a_n)$, and returns the result as an OpenMath object. With this web service, program fragments will be executable in analogy to formulae (see the services discussed in Section 2.2).

### 3.5. Management of Change

Keeping document collections consistent in the face of changes induced by development is a grand challenge in the STEM disciplines. Even though published documents are considered immutable, changes do occur especially in the development of STEM knowledge. It is a long-standing assumption that the semantic annotations in documents that drive the machine-supported interaction can support automated change impact analyses at the same time. Generally, any change in a semantic object (say a definition) induces an impact on all objects that depend on it. In [31] we have taken first steps of integrating a change impact analysis into the editing and management workflow of the PLANETARY system by establishing a flexible interface that allows to deal with impacts. One limitation of the current integration that we want to alleviate in the near future is that our integration currently assumes a single-user mode of operation. The main conceptual problem to be solved here is how to deal with propagating changes by "other authors". For that we plan to build in the notion of versioned links proposed in [29] in addition to our TNTBase versioned storage.

### 3.6. Integration of Scientific Data and Provenance

So far we have been mainly concerned with infrastructures for abstract and theoretical sciences. Papers in the natural sciences need an infrastructure for dealing with *experimental and measurement data*. We have already dealt with foundational aspects of experiments in [32]. For dealing with data aspects at the semantic level, we plan to build on the SACHS system [33, 34], our semantic user assistance system for spreadsheets, which understands spreadsheets as active documents with respect to a background ontology in the content commons. We plan to integrate the "semantic illustration" technology [35] originally developed for SACHS into PLANETARY, directly implementing its user assistance [36] into PLANETARY functionality, and generalizing the data handling and provenance functionality for scientific data. We conjecture that this will yield an integrative enabling technology for transparent, semantic embedding of data. When we additionally combine this with the executable programs feature from 3.4, we can document the computation steps that aggregated the data reported in papers from their experimental foundations and open them to experimentation for the user.

### 3.7. PLANETARY *Pit Stop*

Finally, users need to manage interface complexity (not all services are interesting to all users) and want to contribute their own services. We will build an "app store" for semantic services that hides the complexities of installation, shadowing, and removal.

## 4. Conclusion

We have presented the PLANETARY system as an example for the Active Document Paradigm (ADP) – semantically annotated documents connected to a content commons accessed through an adaptive document player – and as an answer to the challenge of creating "executable papers". We have shown the initial feasibility of the concept in a variety of publicly available case studies [3, 21, 37, 38][5]. Details on the system are available in the PLANETARY developer forum [39]. In particular, see the EPCsystem page [26] for more information and [40] for a sandbox installation of the system.

The PLANETARY system addresses all of the challenge categories. Concretely:

**Executability** is achieved by an extensible and configurable collection of semantic services that can be embedded into documents. In particular, code can be made executable via external computation machines, and computational experiments can be repeated and varied (via the same mechanism). Where the functionality of a service depends on ontologies, the user community can customize the service by customizing the ontology inside PLANETARY.

**Short and long-term compatibility** is guaranteed by usage of open standards in representation formats and protocols (XML, RDF/RDFa, OpenMath, MathML, XQuery, SPARQL, XHTML, SVG) supporting a web service framework, and hence operating-system-independent. Of course, PLANETARY can export monographs, collections and even entire libraries both as PDF (*inactive* documents) or in the EPUB eBook format.

**Validation** and in particular human refereeing and scientific validation can be facilitated via an in-text discussion feature. Moreover, documents can be automatically validated via semantic services, e.g. automated SI-dimensionality checking.

**Copyright/licensing** is represented by fine-grained RDFa-based metadata annotations in STEX and OMDoc, which are maintained over the presentation process. So they can be used for filtering or attribution either on the backend storage level (TNTBase, RDF triple store) or in the frontend PLANETARY system. Together with the user management and permission system, PLANETARY can be extended to enforce compliance. In fact, as documents are assembled for the user at view-time they can be adapted to the license status of the user (e.g. it is possible to make a document license conforming by leaving out examples that are not licensed to her specific institution).

**Systems** As the PLANETARY system is entirely based on web standards and communicates via RESTful interfaces, it is simple to wrap external systems into web services, if we can equip them with OMDoc, OpenMath, or RDF interfaces. In fact, that is exactly what runs the unit conversion service in Section 2.2/*v)*.

[5] Note that all of these are research systems under constant development, so your mileage may vary.

**Size** Even though individual human-written documents are modest in size, journals and encyclopedias can get big – consider e.g. the Wikipedia or the arXiv. The PLANETARY system has been tested on the latter; the underlying data stores scale sufficiently for large document collections. Furthermore, the modular and semantic document formats accommodate the integration of external data stores via 'special' links, which the PLANETARY player can interpret on view, hence keeping the storage minimal and the experience optimal. To the best of our knowledge, the semantically transparent integration of data into a document player application is a new feature of the PLANETARY system.

**Provenance** comes in various aspects. Data provenance can be specified by the techniques for semantic integration of data fields. For instance we can specify units (as OpenMath objects) and computations to obtain the displayed data from raw data, etc. As all of these are content representations in the documents or the content commons, they can be handled with semantic services. The system state provenance (i.e., what actions of the user led to the current state of interaction in the PLANETARY system), can be handled by recording system data ("who did what when") in the metadata store of PLANETARY. This can be opened to querying the system ontologies we have developed for semantically transparent system self-documentation (see Section 2.4).

An encouraging aspect of this work is that document authors only need expertise in their own domain. In particular, no system-level programming is necessary for authors: the semantic representation formats involved act as a high-level conceptual interface between content authors and system/service/interface developers. We are convinced that without such a separation of concerns, "the next generation of publishing" will not scale enough to become practical.

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# Live Annotation and Content Discovery in Personal Learning Environments

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**Abstract.** Personal Learning Environments (PLEs) are gradually gaining ground over traditional Learning Management Systems (LMS) by facilitating the lone or collaborative study of user-chosen blends of content and courses from heterogeneous sources, including Open Educational Resources (OER). Our work is focused on the social aspect of PLEs, and particularly on live interactions between learners, real-time annotation of OER and content discovery. This paper describes Gravpad, a service for sharing live annotations, and showcases how it can contribute to social learning within a PLE.

**Keywords:** personal learning environment, open educational resource, live annotation, content discovery

## 1 Introduction

Personal Learning Environments (PLEs) have recently emerged as an answer to the growing need for ubiquitous and flexible learning environments. PLEs follow a learner-centric approach, allowing the use of lightweight services and tools that can be accessed and customised from a variety of devices, such as PCs, netbooks, smartphones, etc. Rather than integrating different services into a centralised system, PLEs provide the learner with a variety of services and hand over control to her to select and use these services the way she deems fit [1].

In the context of the European project ROLE (Responsive Open Learning Environments[1]) we are working towards the transformation of OpenLearn[2], which is currently a public, course-centered LMS, into a PLE. The goal is to enable learners to easily construct and maintain their own learning environment, consisting of mash-ups of their preferred learning tools, services and resources, including:

- an e-portfolio describing learner goals and achievements,
- tools for collaborative authoring,
- tools for real-time meetings, and
- portals/aggregators connecting each learner to other learners and to additional external resources.

1 http://www.role-project.eu
2 http://openlearn.open.ac.uk/

There are some thematic critical questions to ask at the outset. The answers will frame our approach in the following sections.

*Question 1: What makes a "personal learning environment" special?*

The idea is that extra value can be delivered to the student by observing the way they interact in the PLE, and adapting the system's behaviour based on these observations.

*Question 2: Where will the aggregated content in the PLE come from, and how can we be sure it is useful?*

To the greatest extent possible, the PLE should be truly personalized by its users: not only should they be empowered to share content and ideas, but also to change the defining features of the system itself.

*Question 3: What is different about providing a collection of community-curated content on the one hand, and simply advising learners to use the internet as a whole on the other?*

The difference lies in the social aspect of working on topics of common interest. Search engines exist to provide access to information, but typically they do much less in the other direction, i.e. when it comes to sharing information with the world. This gap is partially filled by blogs, mailing lists, and so forth. However, we believe we can do better, by creating a service that adapts not only to the preferences and behaviour of individuals, but also to the behaviour of groups who share common interests.

In this paper, we will expand on these ideas through the particular example of Gravpad, a modified version of Etherpad[3] that can be used to make and share annotations to web content in real time. We will discuss the ways in which Gravpad relates to the other PLE elements mentioned in the bullet points above. First, we will review the PLE idea by looking at a particular learning scenario.

## 2 Personal Learning Environments: The OpenLearn scenario

The following scenario showcases the use of widget mash-ups in the context of collaborative authoring between OpenLearn users. Let us consider Ann, a student at the Open University, who is taking a course on sustainable energy (T206 - Energy for a sustainable future[4]). One of her group assignments is on the 10:10 climate change campaign. In order to assist the class in this assignment, her tutor has setup a special page in OpenLearn, with introductory OER from Open University learning units, iTunesU albums, and YouTube videos[5].

3 http://code.google.com/p/etherpad/
4 http://www3.open.ac.uk/study/undergraduate/course/t206.htm
5 http://labspace.open.ac.uk/course/view.php?id=5177

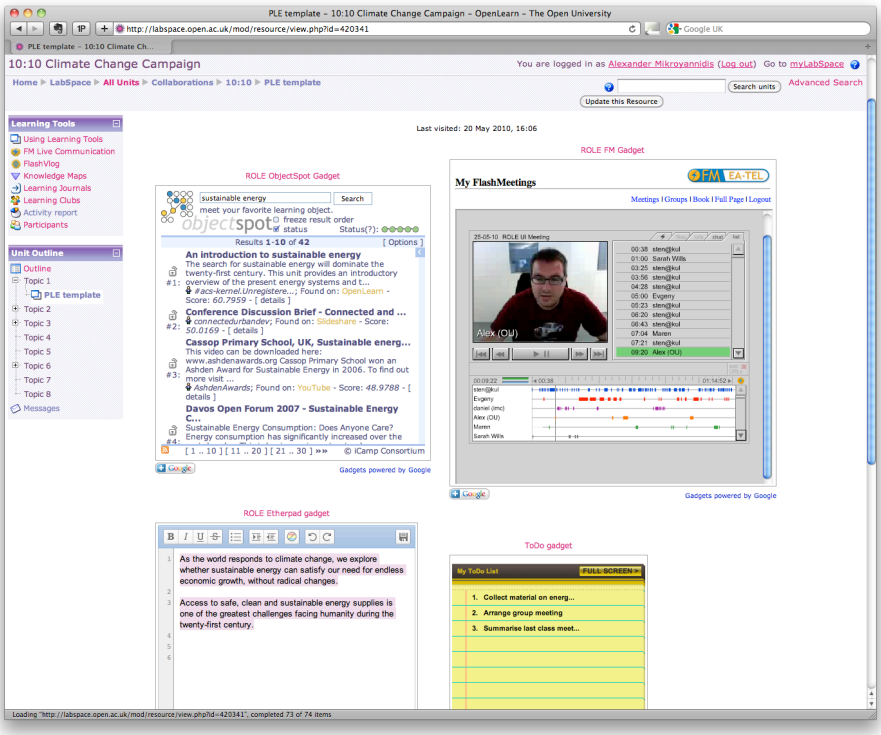



Figure 1. An OpenLearn PLE template for the climate change student[6]

Ann studies this material first, and then decides to explore some more. Her tutor has suggested a set of widgets within the OpenLearn PLE template shown in Figure 1. Ann starts off with the ObjectSpot search widget to discover more OER on sustainable energy, coming from OpenLearn courses, iTunesU, Wikipedia, SlideShare, and YouTube. She takes notes on the EtherPad widget, which she shares with her group members. She then has a videoconference with them, through the FlashMeeting widget, revising their joint notes on the EtherPad widget at the same time. Ann adds some figures about carbon emissions from the carbon emissions gadget into the group's EtherPad notes, and she and others in her group look for references to web pages about airlines that offer carbon offset services. Finally, Ann's group is ready to create their report for the assignment, by expanding and formatting their notes.

6 http://labspace.open.ac.uk/mod/resource/view.php?id=420341&direct=1

### 2.1 Swarms: A sense of co-presence

Suppose Stan is also on the same course, but in another group. Because the Etherpad widget is set up with Gravpad extensions, he is able to follow changes from other students in real time. For example, the URLs that Ann's group found about carbon offset programmes are drawn into a link aggregator shared by all course students. This is useful to Stan because his group has decided to focus on the impact of national polices around climate change, and he notices from Ann's notes in the link aggregator that carbon offset services tend to conform to the requirements of various national programmes. His group uses Gravpad to make further annotations attached to the pages Ann's group found, as shown in Figure 2.

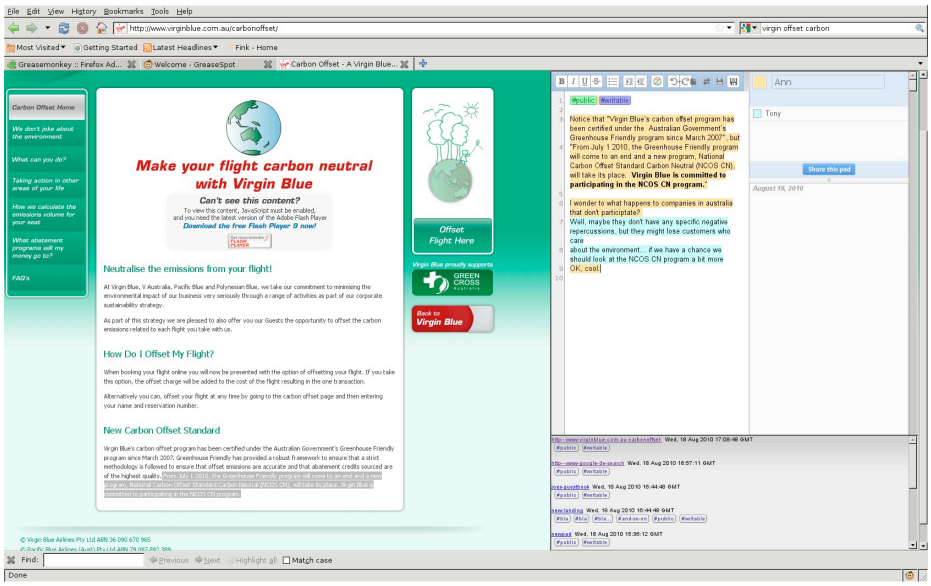



Figure 2. Group annotations and discussions via Gravpad

In the mean time, Ann's group has finished a first draft of their report. Before submitting, they check for activity in the course and find several key points they hadn't considered have been discussed in the notes attached to the pages that comprised their key references. They decide to make another round of revisions in light of the new things they've learned.

## 3 Implementing Gravpad

We saw in the previous example how social interactions can shape a learning landscape. Indeed, the central idea for Gravpad is precisely to "bend" a collection of existing content to one's purpose.

In a way, this idea is already present in the PLE concept. Annotations are just one example of a paradigm of social-semantic knowledge management. Gravpad operates as a recommender system applied in a real-time context; a variety of data streams in addition to web annotations could be used. For example, in [2], Wolpers et al. sketch out a broad scheme for making use of Contextualized Attention Metadata (CAM). This scheme resolves at a level of detail that includes things like the links followed in a given browsing session, or the programs used together in a given editing session. Gravpad should allow us to dive into interactions at a very fine-grained level, e.g. to make interventions on a word-by-word basis.

The current Gravpad implementation [3] has been built on top of the Greasemonkey[7] plugin for Firefox. This plugin allows the Javascript code of a modified Etherpad to be executed in the selected web page, offering a real-time, shared pad alongside the web page, as shown in Figure 2. The idea of using Etherpad to offer real-time writing advice is part of another contemporary research project (Dana Chandler, personal communication). Future implementation work around Gravpad will develop its usefulness in resource discovery, real-time context awareness, and social networking.

### 3.1 Free culture as motivator

Aaron Krowne writes in [4]:

> *"I contend that free culture is so compelling that users will be drawn to virtual places that allow them to work with it, use it, manipulate it, and bring it into the context of their lives and their interests."*

Krowne cites the work of Brusilovsky *et al*. on web annotation (cf. [5], [6]) as one example of a "free culture" trend.

Comparing Etherpad and the Web, we see the difference between a radically open/free production model (given the URL of a pad, anyone can contribute) and a more closed, ownership-based model (I control what happens on my domain, my blog, my course, etc.).

We also see how these two models fade into one another (e.g. Etherpad's "pro" setting allows the user to create a password-protected collection of pads, and blogs are typically open for comments). Even so, Etherpad seems paradigmatic for today's "open web". Gravpad mashes up this new radically open model and the more staid model of Web 1.0.

The resulting ability to make comments on anyone's web page without their explicit or implied permission, combined with the ability to filter out comments and pages based on various user-definable and extensible criteria, including content-based, time-based and social metadata, all serves to suggest that Gravpad may be "web annotation done right" – at least from the point of view of free culture. If user freedom is indeed a strong incentive for participation, will be what counts.

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[7] https://addons.mozilla.org/en-US/firefox/addon/748/

### 3.2 Buy-in and scalability

Now that Etherpad is available as an easy-to-install open source tool, it can be deployed on any GNU/Linux-based server, and mixed with any collection of tools to gather and manage data about interactions.

Accordingly, Gravpad could be used "internet-wide" – or on a very small scale. New features in Etherpad and Gravpad allow for fine-grained management of access and event notifications. As mentioned above, Etherpad is only one of many potential sources of content that could be exploited by a Gravpad server in order to make high-quality recommendations.

Of course, syndicating operational transform data is more intensive than syndicating lightweight status changes. In the current demonstration version of Gravpad, recent changes are just published as JSON data. In principle, a distributed collection of Etherpad instances could be polled for changes in a similar manner, and their recent changes lists aggregated. The Salmon Protocol[8] provides a way for "annotations to swim upstream to original update sources", which could remove the need for active polling in favor of a push-based mechanism.

Future challenges in include federating Gravpad servers (perhaps taking the Wave federation protocol as a starting point[9]); becoming increasingly proficient at noticing implicit connections, in order to create explicit annotations; and developing additional tools to measure the "closeness" and mutual attraction of resources and activities, and the appropriateness of recommendations using these measures.

## 4 Related work

Google Wave is the most well known related example. We can learn several things from the course of development taken by Wave.

1. *Don't try to do everything on one centralized server*. As noted, Wave was designed with federation in mind, but Etherpad is perhaps already ahead of the game simply by being easy to install on your own server.

2. *Do one thing and do it well*. Etherpad seems to get perfect marks here. Wave may have suffered from being too complex for both users and developers.

3. *Be compatible with the way people already work*. For example, Jan Moringen and Joe Corneli have been working to integrate Emacs and Etherpad in a branch of the Rudel project[10]. At the opposite extreme, Wave wasn't directly connected to more popular Google tools like Gmail, which seems like a missed opportunity.

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8 http://www.salmon-protocol.org

9 http://www.waveprotocol.org/

10 http://www.emacswiki.org/emacs/Rudel

At the same time, despite being discontinued as a stand-alone project, Wave's legacy may be with us for a long time, either in the form of Google API-based products (like social widgets), or the Novell's Wave-like and Wave Federation-compatible Pulse[11].

In any case, real-time collaboration is here to stay. People have used the idea of operational transform since 1989 [7] to synchronize content ranging from digital designs to raw data [8]. The literature on personalization and recommendations is too large to review here, but one can begin with a recent book [9]. Finally, new standard for syndicating semantic information about real-time interactions has been coming into its own recently in the form of "Activity Streams"[12].

Current Web 2.0 practices in user-generated annotations can also provide us with useful directions. A popular way for organizing content in Web 2.0, is tagging it with descriptive terms. This bottom-up collaborative process results in social classifications known as folksonomies [10]. An important aspect of a folksonomy is that it does not contain a hierarchy, or any directly specified parent-child or sibling relationship between its tags. Nevertheless, there exist relations that cluster tags referring to common resources. Compared to ontologies, folksonomies offer greater flexibility and adaptability in organizing information. A similar approach can be adopted in this project for organizing the annotations users produce in real time.

## 5 Conclusions and Future Work

We have seen that Gravpad gives people the opportunity to share context, whether temporal (co-editing a document in real time) or spatial (comments appear in-place, attached to the objects of interest). We also saw in an example how Gravpad can spread interactions out over time, bringing new resources to light and new creating opportunities for multimodal interaction in a shared PLE. Our future efforts will be directed towards integrating Etherpad with other social gadgets in the PLE, and on refining our recommendation algorithms accordingly. Finally, a key theme of this paper has been that interaction tools should be tailored to their users (and not the other way around), and we will endeavor to make Gravpad, like Etherpad, easy to customize and extend.

## Acknowledgements

The research work described in this paper is partially funded through the ROLE Integrated Project, part of the Seventh Framework Programme for Research and Technological Development (FP7) of the European Union in Information and Communication Technologies.

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[12] http://activitystrea.ms/

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# GravPad

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## ABSTRACT

GravPad is the next major step in the evolution of the EtherPad, a real-time collaborative editor that was first developed at AppJet, Inc. (`http://appjet.com/`) and recently open-sourced when AppJet was acquired by Google. EtherPad's developer community, headed up by Egil Möller and John McLear, has made a plugin engine and added standard wiki features to the editor (`http://github.com/ether/pad`). My GravPad demonstration will show a hacked EtherPad running in a "sidewiki" format – the basis of a platform for live web annotation and content discovery.

## Categories and Subject Descriptors

H.3.5 [**Information Storage and Retrieval**]: Online Information Services—*Data sharing, Web-based services*; H.3.1 [**Information Storage and Retrieval**]: Content Analysis and Indexing

## General Terms

Design

## Keywords

Wikis, Annotation, Content discovery

## 1. A NEW WAY TO COLLABORATE

In addition to facilitating live, *in situ*, interaction with web content (Fig. 1), GravPad users can receive real-time notifications about activity taking place anywhere on the web, filtered according to settings in user profiles (Fig. 2). Users will be able to follow people (like on Twitter, indeed, 'following' lists can be imported easily), pages (like using a Watch List on a wiki), or concepts (something new).

Combinations of these effects can be used to "bend" the web around one's interests. For example, GravPad together with `http://gutenberg.org` would create a massive world-wide book group, which a user can then narrow to focus on topics like "science" or "detective fiction". User profiles can also be linked together to facilitate swarming behavior, e.g. when working on a class project.

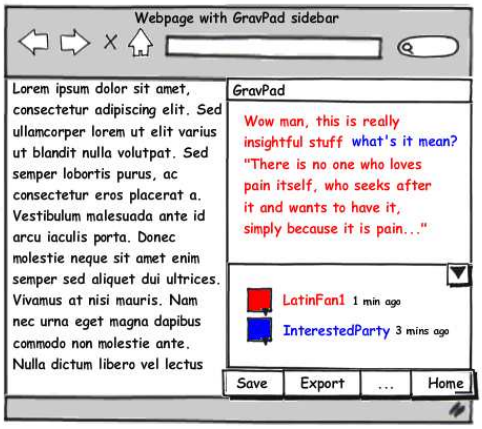



**Figure 1: Live interaction**

As it matures, GravPad will be great a place to experiment with the temporal and social dimensions of PageRank-like algorithms. In my presentation, I hope to attract interested parties to collaborate on future development.

Eventually, GravPad should have plugin, web portal, and drop-in Javascript interfaces, in order useful to many different people and organizations "right out of the box". Thanks to the idea of bending, web annotation will be positioned as a key productivity tool for the next decade.

## 2. THE DEMONSTRATION VERSION

I will begin with a brief tour of the new UI features that have been added to EtherPad since its open source release; features that transform it from being a real time collaborative editor, to being a real time wiki (i.e. links, tags, and a live recent changes page).

I will then show how this real-time wiki can be integrated seamlessly into the browsing experience as a plug-in in Firefox. This plugin will allow the user to open a sidebar containing a "live wiki page" hosting a discussion about the current URL, and a scrolling "live recent changes" frame showing any activities taking place on the Web that the user has

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WikiSym '10, July 7-9, 2010, Gdańsk, Poland Copyright ©2010 ACM 978-1-4503-0056-8/10/07... $10.00

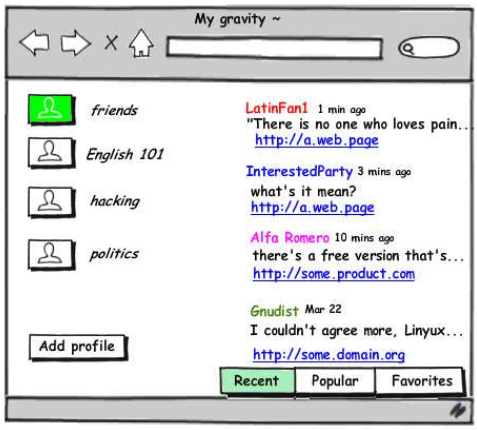



**Figure 2: Receiving notifications**

asked to be notified about.

The plugin's configuration will allow the user to create several profiles, each of which can specify pages, people, and concepts to subscribe to. This version of GravPad matches concepts using the "concept forest" algorithm introduced in [2].

## 3. CONCLUSION

Although "really real-time" document-based collaboration has been possible for some time using programs like Gobby [1], spreading such interactions across the web seems to be something new. Scaling up presents a variety challenges, but already we see in this demo how GravPad's two key features – live interaction and intelligent content discovery – make it a truly social tool, and an important step towards the "read-write web".

## 4. ACKNOWLEDGMENTS

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I would like to thank Craig Ugoretz for preliminary discussions about the GravPad backend. Many thanks also to the EtherPad team!

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# eMath 3.0: Building Blocks for a Social and Semantic Web for Online Mathematics & eLearning

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October 27, 2010

**Abstract**

In this paper we present recent developments in content markup for mathematics, and a corresponding software stack that functions as an enabling technology for a social and semantic web for the STEM disciplines. We show the potential of this technology in two eMath 3.0 applications: PlanetMathRedux, a re-implementation of the mathematical encyclopedia PlanetMath.org, and PantaRheiRedux, a community reader for course materials. These applications indicate both present and potential uses for this software as a basis for eLearning applications in Science, Technology, Engineering and Mathematics through the addition of suitable pedagogies.

## 1 Introduction

The Internet has revolutionized our access to information: much of what we need to know is available online, and can be found via search engines. In the last decade, this trend has been accelerated by the advent of the social and semantic web.

The *social web* (also called Web 2.0) fills various online commons with user-generated content and peer-based interactions. The social web has greatly extended the material available on the Internet; for instance, Wikipedia has accumulated more than 16 million articles in almost all the world's languages over the last 10 years.

The *Semantic Web* (by convention, referred to with capital letters) adds formal descriptions to web resources, so that the information they contain becomes machine-understandable. Semantic information retrieval can be used to combine different information sources to obtain facts and functionalities that are entailed, but nowhere explicitly represented, in the original sources themselves.

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The emerging combination of social and semantic web techniques has sometimes been called Web 3.0. Applications like semantic wikis (see e.g. [Lan+10] and earlier workshops) and DBPedia [Dbp] (a semantic query engine based on the content of Wikipedia) are showing the considerable promise of this mixed approach.

Perhaps surprisingly, the Semantic Web has not taken off for the Science, Technology, Engineering & Mathematics (STEM) disciplines, even though STEM documents are ostensibly more rigorous, have more overt formal structure, and should therefore be amenable to formal, programmatic, treatment via Semantic Web technologies. Three possible impediments seem to be likely explanations for the slow uptake:

1. Both the STEM and Semantic Web user communities are small compared to the social web as a whole.
2. There is significant inherent difficulty and complexity in STEM subject matter.
3. Special representation languages are used by both practitioners (e.g. mathematical and chemical formulae) and technologists (e.g. XML and ontologies).

We believe that the first two factors are in fact offset by the dedication of STEM practitioners, and we claim that the third one – which is the main problem – has been solved, for the mathematical sciences, by recent technological developments. We will survey these broadly in the next section, and describe in detail a specific set of related technologies developed at Jacobs University Bremen over the last decade in section 3. In section 4 we will then show how these building blocks can be assembled into an *eMath 3.0 system* system for developing and exploiting social and semantic mathematics on the Web. Section 5 concludes the paper, with an emphasis on planned future work.

# 2 Mathematics on the Web: MathML, OpenMath and RDFa

Technologies that provide support for Mathematics on the Web have been increasing in prevalence over the last decade, and have reached a state, where they will become universally usable out of the box. This has been especially driven forward by the continued development of MathML and OpenMath as representation formats for mathematical formulae, and by the integration of Semantic Web technologies into widely-accepted web standards.

## 2.1 Mathematical Formulae on the Web: State of the Art

For mathematical formulae, the relevant developments are:

**M1:** MathML3, the upcoming W3C recommendation [Aus+10] extends MathML with linebreaking support (important e.g. for mobile applications), markup for elementary mathematics (for high school eLearning), and completely reworks the integration of content markup into MathML. Furthermore, the improved integration of MathML into host languages and environments gives a more solid basis for Math-enabled web applications.

**M2:** MathML is in the main browsers: natively in Firefox, via the MathPlayer plugin in Internet Explorer, via CSS in Opera, and native MathML support in Webkit has recently been



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announced, so we can expect it in the Webkit-based browsers (Chrome, Safari and Konqueror) in the near future. Additionally, Math support for all browsers can be enabled via MathJax [Mat], a JavaScript-based solution for displaying MathML or LaTeX in the browser.

**M3:** MathML is part of the upcoming HTML5 standard [Hic10], which is expected to drive the application-centered Web in the future: MathML elements (both content and presentation) are placed in the DOM, in the MathML namespace.

**M4:** TEX/LATEX documents can be transformed to XHTML+MathML automatically by systems like Tralics [Tra] or LaTeXML [Mil]; see [Sta+09] for an overview.

**M5:** More and more word processors and office applications include support for mathematical formulae (most prominently MS Word since Office 2007), and allow an export to MathML [Car07].

**M6:** As a consequence, very large collections of mathematical documents have become available online marked up with standard web formatting, most prominently, Zentralblatt Math [Zbl] and a web-enabled version of the Cornell ePrint arXiv [Sta+10; Arx].

## 2.2 Semantic Web Technologies for Mathematical Documents

Semantic web technologies are able to describe relationships within and between mathematical documents above the formula level.

**S1:** Text fragments can be classified by their mathematical role and related to each other via RDF triples [MM04], which can be efficiently queried with RDF query systems like SPARQL [PS08].

**S2:** RDF triples can be embedded into XHTML documents via RDFa; the upcoming version RDFa 1.1 [Adi+10] extends this to arbitrary XML languages.

**S3:** HTML5 embeds similar functionality in web pages via "microformats"; see [Ten09] for an analysis.

# 3 The KWARC Technology Stack

In the last five years, the KWARC research group at Jacobs University has developed a stack of technologies for content-oriented representation of mathematics in Web contexts together with tools for processing and interacting with these representations in sophisticated ways (see Figure 1). They have arisen as generalizations of system components in the course of developing systems like the SWiM semantic Wiki for mathematics [Lan08], Panta Rhei, a semantic community reader [MK08], and the ActiveMath eLearning System [Mel+03a]. We will review them before we show in the next section how they can be re-combined in novel applications.

## 3.1 Representation: OMDoc & sTeX

OMDoc is an XML-based content-oriented representation format for scientific documents, which is now used in a large set of projects in Automated Theorem Proving [Mül06], eLearning [Mel+03b; KK08b; Koh07], eScience [HKS06], Document Retrieval [KŞ06], User Assistance [KK08a;

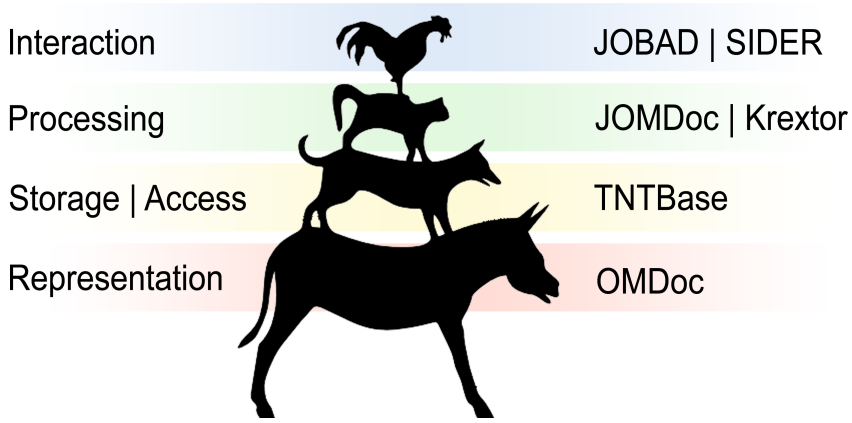



Figure 1: The KWARC Software Stack

KK09], and in Formal Digital Libraries [Url]. The OMDoc format builds on existing semantic representation formats for mathematical formulae (OpenMath [Bus+04] objects and Content MathML [Aus+03] representations), and extends them by an infrastructure for context and domain models from Formal Methods for important structural properties as well as for semi-formal content. Work on the OMDoc format shows that many added-value services in Knowledge Management do not need tedious formalization, but can be based at the structural/semantic level.

We have developed two approaches to ease the non-trivial task of authoring OMDoc documents. The first is an OMDoc-based semantic Wiki, which integrates server-based editing with user-adaptive and context-based presentation [LK08; Lan07]. The second approach we call an "invasive technology" [Koh05], since we build OMDoc-aware editing facilities into existing editing frame-works to make the most of existing functionalities and get around the learning curve involved with a new editor. We have evaluated this approach for MS PowerPoint [KK04; Koh05] and sTEX [Koh08; KKL10], a semantic variant of LaTeX; in both formats we can embed OMDoc markup as well as generate pure OMDoc as an export format.

## 3.2 Storage: TNTBase

Large scale collaborative authoring of mathematical documents requires versioned storage. On the language end, OMDoc supports this by making all identifiers URIs so that OMDoc documents can be distributed among authors and networks and reference each other. On the storage end, we use the TNTBase system [ZK09], a versioned XML-database with a client-server architecture. It integrates Berkeley DB XML with a Subversion server [Apa]. DB XML stores HEAD revisions of XML files; non-XML content like PDF, images or LaTeX source files, as well as differences between revisions, directory entry lists and other repository information are retained in a usual SVN back-end storage (Berkeley DB, in our case). Keeping XML documents in DB XML allows access to files via not only an SVN client, but also through the DB XML API, which supports efficient querying of XML content via XQuery, as well as (versioned) modification of the XML content via XQuery Update.

Versioning and distribution can also be realized with a plain SVN server, but for mathematics,

it is additionally important that the storage backend is aware of at least some aspects of the mathematical semantics. For example, in large-scale authoring processes, an important requirement is to guarantee consistency, i.e., it should be possible to reject commits of invalid documents. TNTBase can support format-specific validation of language-specific constraints and invariants that cannot be expressed in the XML schema languages.

For document management, TNTBase provides Virtual Documents. The author writes a Virtual Document skeleton document that embeds XQueries which are materialized by TNTBase when the content is served. This is particularly useful in eLearning applications, since Virtual Documents make it easy to generate aggregated- and user-adapted documents, as well as other kinds of document variants. TNTBase permits committing back changed Virtual Documents; the changes are distributed to the original files the Virtual Document was assembled from; see [ZK10] for details.

## 3.3 Processing: JOMDoc

JOMDoc [Jom] is a Java API for OMDoc documents, which facilitates parsing OMDoc XML documents into an internal Java data structure, allowing for convenient manipulation, and ultimately, serialization of this internal representation back to XML. JOMDoc has been integrated into TNTBase via the latter's plugin architecture for document format-specific customizations [ZKR10]. This makes made TNTBase OMDoc-aware so that data-intensive JOMDoc algorithms can be executed within the database, alleviating the need of sending the contents over the network for processing. Additionally, JOMDoc is used as a presentation framework for OMDoc. With its notation service [KMR08], it allows for context-sensitive rendering of XML documents containing mathematical formulae in content markup (Content MathML or OpenMath) into Presentation MathML. Optionally this can be presented as parallel markup, i.e., interlinked with the original content markup. Transformation of OMDoc documents to XHTML is supported by bundled XSLT stylesheets. In particular, this presentation service can be used to serve OMDoc documents in a human-readable presentation.

## 3.4 Interaction: JOBAD

Our JOBAD architecture embeds interactive mathematical services into XHTML+MathML documents. JOBAD is a modular JavaScript framework for interactive services such as term folding or definition lookup.

Our vision of a document is that it should be something the user can adapt according to his or her preferences and interests while reading it. This goes beyond customizing the display of the rendered document in the browser, to include, for example, changing notations (which requires re-rendering at least portions of the document), or retrieving additional information from services on the web to enhance the document with annotations. Consider a student reading lecture notes: whenever an unfamiliar mathematical symbol occurs in some formula, JOBAD enables the look up its definition without opening another document, but adds an explanation right into the current reading context. Converting between physical units (e.g. imperial and SI) can also be effected automatically, in-place and on the fly.



planetary.tex 301 2010-10-27 06:04:43Z kohlhase

# 4 The Planetary System: Assembling Applications

In August 2010, the authors started using the building blocks described in the last section to build a new front-end system for eMath 3.0 applications: the Planetary System. The starting point of this development project was our aim to make PlanetMath.org [Plab], one of the original eMath2.0 systems, more semantic, by integrating it with the KWARC technologies described in the last section.

PlanetMath.org is a relatively well-known online community devoted to mathematics. At present, its central feature is a mathematics encyclopedia with around 9K entries, which has been built and peer reviewed through effort of several hundred of volunteers since the site went online in 2001. PlanetMath also includes several general-purpose discussion forums which have received around 15K posts to date. Its most popular forums, containing about half of these posts, are devoted to Q&A about mathematics at the university, post-graduate, and research levels. Notably, each encyclopedia article also has its own attached discussion forum.

While the PlanetMath concept, community and vision remain alive and active, the Noösphere web application that underlies the site is showing its age. In particular, PlanetMath does not currently make significant use of any of the state of the art technologies described in section 2, other than using LaTeXas an input syntax.

To bring PlanetMath up to date, and simultaneously make its software easier to extend in the future, we decided to recreate the relevant functionality of Noösphere by integrating contemporary mathematical communication features into the existing open source web platform, Vanilla Forums. Vanilla offers a general-purpose online infrastructure, including user management and discussion forums, together with a plugin system that makes it relatively easy to adapt different components to a given special-purpose use.

Soon after we began to carry out this plan, we realized that Vanilla's plugin architecture would allow us to build a system that could be configured into multiple different eMath3.0 applications. This gave rise of the concept of the Planetary System. We are currently exploring its possibilities in two main applications: PlanetMathRedux [Plaa] (a new PlanetMath.org based on the Planetary System), and PantaRheiRedux [Pan].

## 4.1 The Software Base: Vanilla Forums

Vanilla Forums is an open-source, standards-compliant discussion forum platform with a very large user base (around 390K communities). Using this "off the shelf" forum software gave us a lot "for free", including user and permission management, and an extensive set of existing plugins to enhance the content and display. Most importantly, Vanilla's plugin architecture nicely complements the software stack that we already have which can be integrated into the system with plugins and applications, as described in the following subsections. The only significant problem we encountered with Vanilla was that while it is advertised to be XHTML-compatible, it seems to be only served with the `text/html` media type (as tag soup) in practice. Thus we had to correct numerous XHML validity errors when changing to the `application/xhtml+xml` media type, a prerequisite for embedding MathML into Vanilla.

## 4.2 Adding Math to Vanilla

Vanilla Forums does not provide authoring tools for mathematical formulae, hence we had to extend the forum functionality by adding a LaTeX-editor plug-in via the LaTeXML LaTeX-to-XHTML+MathML converter [Mil]. There were many alternatives to this choice (see [Sta+09]), but none of them scale to the full expressivity of LATEX, which is the input format used by PlanetMath.org. We were able to make use of our extensions to the batch-mode LaTeXML converter that turned it into a daemon web-service, to decreasing startup latency. This allows high-throughput conversion of arbitrary LaTeX fragments – ranging from simple expressions, to entire chapters or books.

We developed a Vanilla plugin that integrates the LaTeXML daemon with forum posts in the following manner: if a post is categorized as LaTeX then instead of letting Vanilla embed it into HTML directly, we first have it transformed by the resident LaTeXML daemon and integrate the XHTML+MathML result instead. Due to the swiftness of the conversion, it is possible to create "on-the-fly" editors like [Sta], in which the author can see the produced content as they type.

Using the LaTeX syntax significantly increases the expressivity of the authoring process when compared with Rich Text and Wiki syntax. LaTeX supports mathematical formulas, creating graphics and charts, easy fine-tuning of tables, complex page partitioning, custom commands, preambles, abbreviations, invasive preloading of semantics and more. Probably even more importantly, LaTeX continues to be deeply ingrained in contemporary mathematical publication and communication processes.

## 4.3 Encyclopedia Articles

The probably largest difference between standard Vanilla forums and PlanetMath.org is that the main content of PlanetMath is organized in and around "encyclopedia articles". Each of these is a versioned, LaTeX-encoded description of a particular mathematical object or topic, with an attached discussion, as we mentioned above. We added functionality for "encyclopedia articles" to Vanilla forums via a new "application" (Vanilla's term for a complex style of plugin that hooks into the same core, but adds new features, instead of merely changing old the way old features work). In contrast to unversioned forum posts, articles are stored in an associated Subversion repository, whose versioning functionalities are then exposed in the new interface. Vanilla's own database is used as a cache for efficient web publishing.

The default mode of navigating a content collection in Vanilla (based on listing articles by name) is unsuitable for the 9K articles in PlanetMath.org. Access via the Math Subject Classification (MSC [Msc]), as in the original version of PlanetMath.org, can be realized via the Vanilla Metadata scheme; we can even make these metadata properties versioned by encoding them into Subversion properties.

We have also added a new navigation method, called a "virtual bookshelf" which contains "books" created by users (here, authors, aggregators, or readers) according to their interest. The hierarchical document structure of a "book‘' is encoded by allowing inclusion primitives (a variant of "`\input`" from LaTeX) in articles, which then become sectioning nodes (e.g. chapters, sections), which can be mixed with transitional text. A narrative structure can be represented

simply as a "next" relation that links such articles. In a course setting, our Articles application can be used to highlight important books together with the forum interface (e.g. these may be portions of the course notes or other required readings).

With the JOBAD system we can already add a very nice feature to PlanetMathRedux: fine-grained forum posts. As JOBAD has access to the document object model (including that of the mathematical formulae in MathML), we can use it to attach forum posts to arbitrary document sub-structures, so that a user can e.g. ask a question about a particular definition, make a line-level proposal for an alternative proof, or offer a correction to a subformula. This fine-grained embedding of the forum into the documents allows Planetary System to be used as a "community reader", supporting discussion, document inspection, and refereeing.

The Articles application, together with the math editing plugin described above (and other some off-the-shelf Vanilla plugins) are enough to replicate most of the relevant parts of the Noösphere functionality, and they form the core of the PlanetMathRedux system. Note that PlanetMathRedux does not yet make any essential use of the semantics implicit in the articles.

## 4.4 Semantic Interaction

We consider mere "reading" of an article to be a deeply eMath2.0 activity and note that mathematics consumers will want to engage with the content of the article in more interactive ways, for more efficient and enjoyable learning experience. In mathematics, a dialogue with the expert is often a better way to settle an uncertain matter than reading a book, or searching through the library or the internet. Reading without thinking, computing and proving intermediate results is generally not the way to learn mathematics.

A prerequisite for the Planetary System to offer solid support for mathematical interaction is for it to have the content markup for articles available (and maybe eventually the corresponding content markup for forum posts as well). To support this, we have extended the articles application to handle sTeX. In this extension, we use TNTBase instead of Subversion and transform sTeX articles to OMDoc, which is then managed in TNTBase.

Our goal is to create "active documents" which adapt to the environment and can richly support user interaction. Some interactions only depend on information that is only related to the document at hand, e.g. the elision of formulae parts like brackets, types or inferable arguments. These can be implemented in the browser: JOMDoc exports the respective semantic information from the OMDoc representation into the XHTML+MathML documents as RDFa annotations [Adi+08], where they can be picked up via the document-embedded JOBAD services. For interactions that depend on larger amounts of data from outside the document itself, JOBAD implements call-backs to TNTBase. For instance, this is used for definition lookup, and for the generation of a concept graph of an exercise problem; see [Dav+10] for details on other services. In some cases, we need mixed computation models for semantic services, e.g. where some information is only present on the client (personal information which cannot be transferred to the server due to privacy concerns).

# 5 Conclusion & Future work

We have presented a set of content-based technologies (the KWARC stack) for building semantically and socially enabled math-aware online applications (eMath3.0 applications). We have shown how this can be done using our new Planetary System as an example. Originally planned as a re-implementation of the software underlying PlanetMath.org, it quickly grew more general and is now also used as the basis of PantaRheiRedux, a semantic eLearning platform in active use at Jacobs University.

We do not view the Planetary System (or the KWARC stack for that matter) as an eLearning system in its own right, but rather as an enabling technology for eLearning in the STEM disciplines. An eLearning platform additionally requires pedagogy. In PantaRheiRedux, the pedagogical aspects are minimal: the system gives access to the course materials, allows students to discuss them, and gives access to semantic services. It would be possible to add more pedagogy, e.g. formative assessments, learner modeling, adaptiveness and instant feedback based on the learner models.

We are planning additional applications of the KWARC stack and the Planetary System: in the arXMLiv project [Sta+10; Arx], we have transformed a large corpus of scientific papers to XHTML+MathML. We are currently working towards extracting a subset of the OMDoc format automatically from these, which would allow us to use the Planetary System as a lightweight community reading platform for the arXiv, where readers can discuss scientific questions, annotate semantic relations in the papers, and interact with the content of these papers more fully than in the current PDF-based system.

Finally, we are working on a version of the Planetary System for Formal Methods: we want to use the Planetary System as a front-end for a knowledge base of modular logic representations and logic transformations represented in an upcoming version of OMDoc; see [KMR] for details. The formal documents can support very powerful semantic services like borrowing automated theorem provers or the automated translation between formalizations in different logics. The formal setting provides a compelling angle on the useful range of this system, which we have seen includes support for both lightweight social to very intense semantic interaction.

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# DOUBLE BUBBLES IN GAUSS SPACE AND SPHERES

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ABSTRACT. We prove that a standard Y is an area-minimizing partition of Gauss space into three given volumes, provided that the standard double bubble is an area-minimizing partition of high-dimensional spheres. We prove that the standard double bubble is the area-minimizing partition of spheres of any dimension where the volumes differ by at most 4%.

## 1. INTRODUCTION

In 2002, Hutchings, Morgan, Ritoré, and Ros ([HMRR], or see [Mo1]) proved the Double Bubble Conjecture, which says that the area-minimizing way to enclose and separate two given volumes in $R^3$ is a standard double bubble, defined as three spherical caps meeting in threes at 120 degrees. There have been partial extensions to $R^n$, the sphere $S^3$, and hyperbolic space $H^3$ ([Rei],[CF], [CH]). We provide extensions to $S^n$ and Gauss space $G^m$ for the case where each volume is approximately 1/3 of the total volume of the space.

1.1. **Single and Double Bubbles in Gauss Space.** Gauss space $G^m$, of much interest to probabilists (see e.g. [McK] and [Str]), is $R^m$ endowed with density

$$(2\pi)^{-m/2}e^{-x^2/2},$$

with total volume 1. Given any three positive volumes $V_1 + V_2 + V_3 = 1$, there exists a standard Y, unique up to symmetries of $G^m$, consisting of three half-hyperplanes meeting at 120 degrees, that partitions $G^m$ into three regions of those prescribed volumes (Proposition 2.11). Note that the center $m-2$ dimensional meeting plane of a Y passes through the origin if and only if the three volumes are equal. Our main theorem proves that the standard Y is area minimizing for certain cases:

> **Theorem 2.14**: *The standard Y is an area-minimizing partition of $G^m$ for three given positive volumes $V_1 + V_2 + V_3 = 1$, under the hypothesis that the standard double bubble for the same volume fractions is area minimizing in $S^n$ for infinitely many $n$.*

Section 3 proves that the standard double bubble is area minimizing in $S^n$ for all $n$ for nearly equal volumes. Using Brakke's Surface Evolver, Section 2.3 provides corroborating computer evidence that a standard Y is area minimizing for volumes $(1/3, 1/3, 1/3)$ (as we prove) and for volumes $(1/3, 1/10, 17/30)$ (which is outside our region of proof). This evidence makes it likely that the standard Y is minimizing for all volumes (see Section 1.2 below). It is worthwhile to note that while Section 3 proves a *unique* area-minimizing partition for certain volumes in $S^n$, the uniqueness of the corresponding partition in $G^m$ is an open question.

**The Proof.** The main idea comes from Mehler's 1856 observation that Gauss space is weakly the limit of projections of suitably normalized high-dimensional spheres (Proposition 2.1) as used by Borell [Bo] and Sudakov-Tsirel'son [ST] to prove hyperplanes optimal single bubbles in $G^m$. The proof requires stronger convergence of perimeter for fixed volumes (Proposition 2.12), including a technical Lemma 2.13 on how to repair small volume discrepancies at low area cost.

1.2. **Double Bubbles in $S^n$.** We prove the Double Bubble Conjecture in the sphere $S^n$ for volume fractions near 1/3:

> **Theorem 3.7**: *For all $n$, the $n-$dimensional standard double bubble is the unique area-minimizing partition of $S^n$ into any three volume fractions within .04 of 1/3.*

This, in conjunction with Theorem 2.14, proves that the standard Y is an area minimizer in $G^m$ if all three volumes are within .04 of 1/3. Section 3.3 shows how to extend this result to other volumes where the Hutchings function is positive. To extend to regions where the Hutchings function is negative, a different method of proof would be necessary. Reichardt et al. [Rei] proved that in $R^n$ it suffices to show that one region is connected. Extending this to $S^n$ would be a big step towards proving our result for all volumes.

**The Proof.** By Hutchings [Hu] and Cotton and Freeman [CF], it suffices to show that the Hutchings function $F_{S^n}(V_1, V_2, V_3)$ (Definition 4) of the prescribed volumes is positive. We consider more tractable lower bounds $\tilde{F}$ on the Hutchings function (Definition 5). By a monotonicity result of Barthe, it suffices to prove $\tilde{F}$ positive on high-dimensional spheres (Lemma 3.6). At this moment, we reuse the fact that Gauss space $G^2$ is the limit of projections of high-dimensional spheres,

so it suffices to prove $\tilde{F}$ positive in $G^2$ (Lemma 3.2), which we prove analytically (Lemma 3.3).

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Lectures by Antonio Ros at the Clay Mathematics Institute Summer School [Ros] provided the initial inspiration for the investigation presented in this paper, which was first pursued by Corneli and others in the 2003 SMALL Geometry Group. The computer experiments of Section 2.3 stemmed from conversations with and help from Ken Brakke in Paris.

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## 2. The Double Bubble Conjecture in Gauss Space

Our main result, Theorem 2.14, proves that the standard Y is an area-minimizing partition of Gauss space $G^m$ into three prescribed volumes, as long as the standard double bubble is area minimizing in high-dimensional spheres for the same volume fractions (as shown for certain volume fractions in Section 3). Proposition 2.1 provides the main idea: that Gauss space is the limit of projections of high-dimensional spheres. Section 2.2 discusses the standard Y, and Proposition 2.12 proves the convergence of standard double bubbles to the standard Y.

2.1. **Area and Volume Convergence in Gauss Space.** Gauss space $G^m$ is $R^m$ endowed with Gaussian density $f(x) = (2\pi)^{-m/2}e^{-x^2/2}$. We recall that volume and area in $G^m$ and in manifolds with density in general, are given by integrating the density with respect to the Euclidean volume and area (Hausdorff measure). Propositions 2.1 and 2.2 show that this density can be obtained by projecting high-dimensional spheres $S^n(\sqrt{n})$ of radius $\sqrt{n}$ into $R^m$. Proposition 2.4 shows that the areas of the inverse projections in $S^n$ of a hypersurface in $G^m$ converge to the area of the hypersurface as $n$ approaches infinity.

The following proposition, often attributed to Poincaré, is due to Mehler (see Stroock [Str], Exercise 2.1.40, page 76 and footnote page 77).

**Proposition 2.1** (Mehler, 1856)**.** *Gaussian measure on $R^m$ is obtained as the limit as $n$ approaches infinity of (orthogonal) projections $P_n$ of uniform probability density on $S^n(\sqrt{n}) \subset R^{n+1}$ to $R^m$.*

PROOF. Observe that $G^m$ is the projection of $G^n$ onto its first $m$ coordinates. The coordinates $X_1, \ldots, X_n$ of $G^n$ are Gaussian random variables. By the Central Limit Theorem, the distance to the origin $\sqrt{X_1^2 + \cdots + X_n^2}$ converges to $\sqrt{n}$ as $n \to \infty$. By the spherical symmetry of $G^n$, the volume of $G^n$ concentrates uniformly on $S^{n-1}(\sqrt{n})$ for large $n$. Therefore, $G^m$ can be obtained as the limit as $n \to \infty$ of projections $P_n$ of the uniform probability density on $S^{n-1}(\sqrt{n})$ or $S^n(\sqrt{n})$ to $R^m$. □

Furthermore, the induced density converges pointwise, not just in measure, to the Gaussian density.

**Proposition 2.2.** *The projections $P_n$ of the uniform probability density on $S^n(\sqrt{n})$ to $R^m$ for fixed $m$ converge pointwise to Gaussian density on $R^m$.*

PROOF. For a fixed $m$, let $f_n(x)$ be the density function resulting from projecting the uniform probability measure on $S^n(\sqrt{n})$ to $R^m$. By symmetry, we may assume that $x$ lies on the nonnegative first coordinate axis. Let $\theta$ be the angle between $S^n(\sqrt{n})$ and $R^m$; then $\sin\theta = x/\sqrt{n}$. A volume element $dV$ of $S^n$ is projected to a weighted volume element $dV\cos\theta$ on $R^m$. Therefore, a volume element $dV$ on $R^m$ has weighted volume $dV/(\cos\theta)$. Observe that $P_n^{-1}(x)$ is the $(n-m)$-sphere with radius $(\cos\theta)\sqrt{n}$ and thus has area proportional to $(\cos\theta)^{n-m-1}$. Therefore,

$$f_n(x) = c_n(2\pi)^{-m/2}(\cos\theta)^{n-m-2}$$

where $c_n$ is some normalization constant. Since $\cos^2\theta = 1 - \sin^2\theta = 1 - x^2/n$,

$$f_n(x) = c_n(2\pi)^{-m/2}(1 - x^2/n)^{(n-m-2)/2}.$$

Since $(1 - x^2/n)^{(n-m-2)/2}$ converges to $e^{-x^2/2}$ as $n \to \infty$, $c_n \to 1$, because the $f_n(x)$ converge to $f(x)$, the Gaussian density, in measure by Proposition 2.1. □

**Lemma 2.3.** *Let $g_n(x) = (1+x/n)^n$ and $g(x) = e^x$. Then $g_1(x) \le g_2(x) \le \cdots \le g(x)$ for all $x \in R$.*

PROOF. Note that $g_1(x) = 1 + x \le e^x$ for all $x$ by comparison of the derivatives.

Let $g'$ denote the derivative of $g_n(x)$ in $n$; we wish to show that $g' \ge 0$ for all $x$. Differentiating,

$$g' = \exp\left(n\log\left(1+\frac{x}{n}\right)\right)\left(\log\left(1+\frac{x}{n}\right) - \frac{x}{n+x}\right).$$

Since $\exp(n\log(1+x/n)) > 0$, we need only show that $\log(1+x/n)-x/(n+x) \geq 0$.

When $x$ is nonnegative,

$$\log\left(1+\frac{x}{n}\right) = \log(x+n) - \log(n) = \int_n^{x+n} \frac{dt}{t} \geq \int_n^{x+n} \frac{dt}{x+n} = \frac{x}{x+n}$$

and therefore $\log(1+x/n) - x/(n+x) \geq 0$.

When $x$ is negative, $n+x < n$, so

$$-\log\left(1+\frac{x}{n}\right) = \log(n) - \log(x+n) = \int_{n+x}^{n} \frac{dt}{t} \leq \int_{n+x}^{n} \frac{dt}{x+n} = -\frac{x}{x+n}$$

and therefore $\log(1+x/n) - x/(n+x) \geq 0$.

Thus, the $g_n(x)$ are nondecreasing in $n$. □

**Proposition 2.4.** *The areas of the inverse projections to $S^n(\sqrt{n})$ of a measurable hypersurface $\Sigma \subset G^m$ converge to the area of $\Sigma$ as $n$ approaches infinity.*

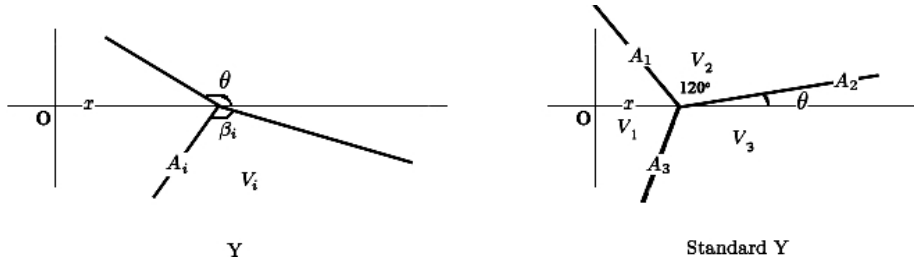
PROOF. Define asymptotically equivalent density functions

$$F_n(x) = \frac{f_n(x)}{c_n} = (2\pi)^{-m/2}\left(1-\frac{x^2}{n}\right)^{n-m-2}$$

which differ from $f_n(x)$ by proportional constants $c_n$, which approach 1 as $n$ approaches $\infty$ (see proof of Proposition 2.2). By Proposition 2.2 and Lemma 2.3, $F_n(x)$ converges to $f(x)$ monotonically from below. By Lebesgue's Monotone Convergence Theorem, $\int_\Sigma F_n(x)$ converges to $\int_\Sigma f(x)$. Therefore, the areas of the inverse projections, $\int_\Sigma F_n(x)$, converge to the area of $\Sigma$, $\int_\Sigma f(x)$. □

2.2. **The Y and the Standard Y.** This section introduces the standard Y and its properties. Proposition 2.12 proves that the area of the standard double bubble on $S^n(\sqrt{n})$ converges to the area of the standard Y in $G^m$ of the same volume fractions as $n$ grows large.

**Definition 1.** A $Y$ in $G^m$ consists of three half-hyperplanes meeting along an $(m-2)$-dimensional plane. Let $\beta_i$ denote the angles between the hyperplanes. A *standard $Y$* is a Y with each $\beta_i = 2\pi/3$. Let $\theta$ denote the angle with the horizontal as in Figure 1. By $O(m)$ symmetry, we may assume that the $(m-2)$-dimensional plane lies normal to the $x$-$y$ plane and contains the nonnegative $x$-axis. The Y partitions $G^m$ into three volumes $V_1$, $V_2$, and $V_3$. The volumes partitioned by the standard Y satisfy $V_1 \geq V_2 \geq V_3$ as in Figure 1. Let $\mathcal{V}_i(x,\theta,\beta_1,\beta_2)$ and $\mathcal{A}_i(x,\theta,\beta_1,\beta_2)$ denote the volume and area, respectively, of the $i^{th}$ region as functions of $x$, $\theta$, $\beta_1$, and $\beta_2$.




Y Standard Y

FIGURE 1. We may assume that the Y is centered along the nonnegative $x$-axis. The standard Y partitions $G^m$ into volumes $V_1 \geq V_2 \geq V_3$.

**Lemma 2.5.** *In $G^m$, the values of $\mathcal{V}_i$ and $\mathcal{A}_i$ are independent of $m$ for $m \geq 2$.*

PROOF. This follows from the fact that Gauss space is a product measure and a Y has translational symmetry along an $(m-2)$-dimensional hyperplane. □

We state the following five minor lemmas without proof, the last two of which are used in Section 3.3. The proofs of these lemmas follow from simple geometric arguments.

**Lemma 2.6.** *In $G^m$, $\mathcal{V}_i(x, \theta, \beta_1, \beta_2)$ is continuous in $x$, $\theta$, $\beta_1$, and $\beta_2$, and for compact regions of $x$ and $\theta$, $\mathcal{V}_i$ is uniformly continuous in $\beta_1$ and $\beta_2$.*

**Lemma 2.7.** *In $G^m$, $\mathcal{A}_i(x, \theta, \beta_1, \beta_2)$ is continuous in $x$, $\theta$, $\beta_1$, and $\beta_2$, and for compact regions of $x$ and $\theta$, $\mathcal{A}_i$ is uniformly continuous in $\beta_1$ and $\beta_2$.*

Note that for a standard Y, we need only consider $0 \leq \theta \leq \pi/3$ due to $O(m)$ symmetry. Let $\mathcal{V}_i(x, \theta)$ denote $\mathcal{V}_i(x, \theta, 2\pi/3, 2\pi/3)$.

**Lemma 2.8.** *For a standard Y in $G^m$ with $x > 0$, $\mathcal{V}_1$ is strictly increasing in $x$ and strictly decreasing in $\theta$; $\mathcal{V}_3$ is strictly decreasing in $x$ and strictly decreasing in $\theta$.*

**Lemma 2.9.** *For standard Ys in $G^m$ with $V_1$ fixed, increasing $V_2$ increases the signed distance from the origin to the line between the regions of volume $V_2$ and $V_3$.*

**Lemma 2.10.** *For standard Ys in $G^m$ with $V_2 = V_3$, increasing $V_1$ increases the signed distance from the origin to the lines bounding the region of volume $V_1$.*

**Proposition 2.11.** *Given three volumes $V_1$, $V_2$ and $V_3$ such that $V_1+V_2+V_3 = 1$ and $V_1 \geq V_2 \geq V_3$, there exists a unique standard Y, up to $O(m)$ symmetry, which partitions $G^m$ into volumes $V_1$, $V_2$ and $V_3$.*

PROOF. By Lemma 2.5, we need consider only the case $m = 2$. We exclude the trivial case $V_1 = V_2 = V_3$. Since $\mathcal{V}_3(0, \pi/3) = 1/3$ and $\lim_{x\to\infty} \mathcal{V}_3(x, \pi/3) = 0$, by Lemma 2.6 there is a unique $x_0 > 0$ such that $\mathcal{V}_3(x_0, \pi/3) = V_3$. Similarly, there is a unique $x_1 > x_0$ such that $\mathcal{V}_3(x_1, 0) = V_3$.

For every $x_0 \leq x \leq x_1$, there is a unique $\theta(x)$ such that $\mathcal{V}_3(x, \theta(x)) = V_3$. Also, for $x < x_0$ or $x > x_1$, there exists no $\theta(x)$ for which $\mathcal{V}_3(x, \theta(x)) = V_3$. By Lemma 2.6 and the fact that $\theta(x)$ is continuous and strictly decreasing in $x$, $\mathcal{V}_1(x, \theta(x))$ is continuous and strictly increasing in $x$. Since $\mathcal{V}_1(x_0, \pi/3) \leq V_1 \leq \mathcal{V}_1(x_1, 0)$, there is a unique $x_0 \leq x \leq x_1$ such that $\mathcal{V}_1(x, \theta(x)) = V_1$ and, consequently, $\mathcal{V}_2(x, \theta(x)) = V_2$, proving the uniqueness of Y (Figure 2). □

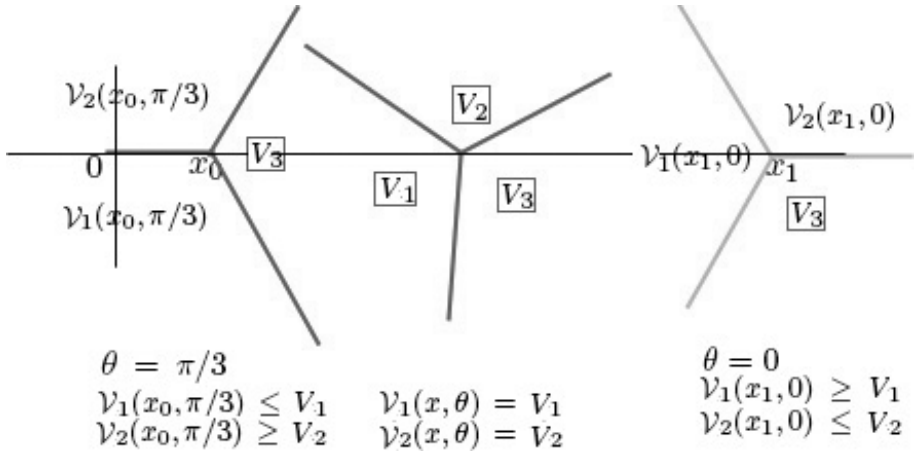



FIGURE 2. By the Intermediate Value Theorem, there exists a unique standard Y that partitions $G^2$ into given volumes.

**Definition 2.** A *standard double bubble* partitioning $S^n$ into three prescribed volumes consists of three $(n-1)$-dimensional spherical caps meeting along an $(n-2)$-sphere at 120 degrees. There exists a standard double bubble for any given set of prescribed volumes ([CF], Proposition 2.6.), unique up to isometries of $S^n$.

**Proposition 2.12.** *The area of the standard double bubble on $S^n(\sqrt{n})$ converges to the area of the standard Y in $G^m$ as $n$ approaches infinity for the same volume fractions.*

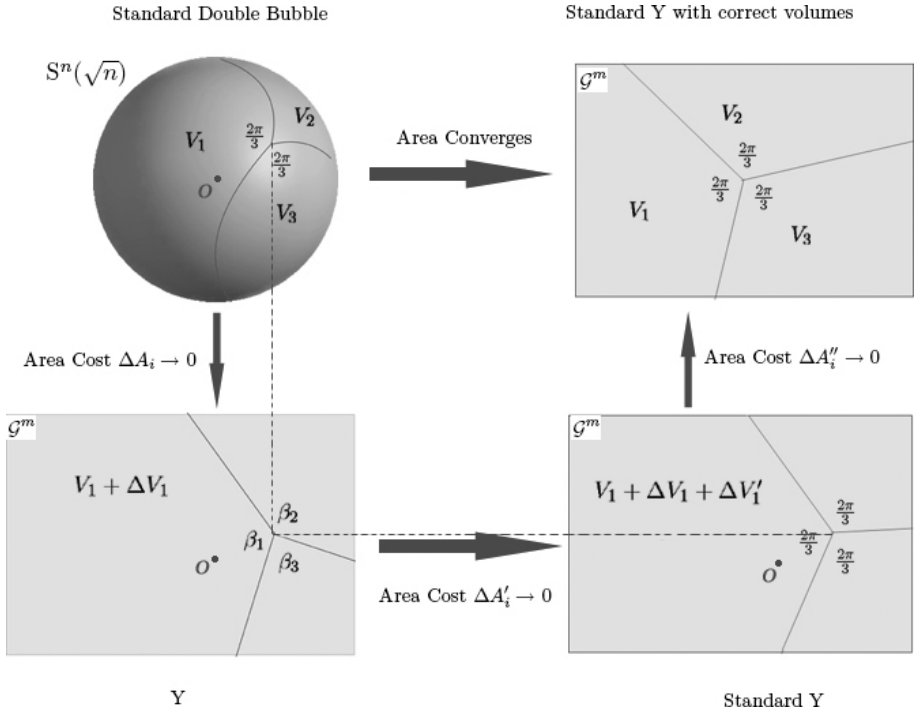



FIGURE 3. The Y projected from the standard double bubble in $S^n(\sqrt{n})$ can be transformed to the standard Y of the correct volume fractions with an area difference that approaches 0 as $n$ approaches $\infty$.

PROOF. Since the inverse image by the projection of a plane in $G^m$ is a sphere in $S^n$, the inverse image of a Y (three half-hyperplanes meeting along an $(m-2)$-plane) is three spherical caps meeting along an $(n-2)$-sphere at some angles. Given volumes $V_1$, $V_2$ and $V_3$, choose $\mathrm{Y}_n$ in $G^m$ (not necessarily with angles $2\pi/3$) with inverse image the standard double bubble (with angles $2\pi/3$) in $S^n$ with the given volumes.

The distance of the vertex of the $\mathrm{Y}_n$ from the origin is bounded above as $n \to \infty$, since too great a distance implies that $V_3$ is too small. As $n$ increases,

the radius of $S^n(\sqrt{n})$ increases, and the section of the sphere above the vertex of $\mathrm{Y}_n$ grows increasingly flat. Thus, the angles of the $\mathrm{Y}_n$ approach the angles of the preimage—namely, $2\pi/3$.

Let the volume and area differences between the standard double bubble in $S^n$ and the $\mathrm{Y}_n$ be denoted $\Delta V_n$ and $\Delta A_n$ as in Figure 3. By Propositions 2.1 and 2.4, the $\Delta V_n$ and $\Delta A_n$ approach 0 as $n \to \infty$. Rotate two rays of the $\mathrm{Y}_n$ so that the $\beta_i$ are all $2\pi/3$ (so that it is a standard Y, though not necessarily of prescribed volume), and denote the area and volume changes in the Y as $\Delta V_n'$ and $\Delta A_n'$. Since the changes in $\beta_i$ go to 0 as $n \to \infty$ and $x$ and $\theta$ are restricted to a compact region, Lemmas 2.6 and 2.7 imply that $\Delta V_n'$ and $\Delta A_n'$ both go to 0 as $n \to \infty$. By Proposition 2.11, this standard Y can then be rotated and translated so that it has the correct volumes $V_1$, $V_2$ and $V_3$. From Lemma 2.8, $V_1$ and $V_3$ are locally invertible in $x$ and $\theta$. Therefore, for the small volume change which goes to 0 as $n \to \infty$, Lemma 2.7 implies that there is a correspondingly small area change $\Delta A_n''$ which similarly goes to 0 as $n \to \infty$. Therefore, for prescribed volumes $V_1$, $V_2$ and $V_3$, the total area difference between the standard double bubble on $S^n$ and the standard Y is thus $\Delta A_n + \Delta A_n' + \Delta A_n''$ which goes to 0 as $n \to \infty$.

□

*Remark.* The proof above can be easily modified to show that the area of the single bubble of volume $V_1$ on $S^n(\sqrt{n})$ converges, as $n$ approaches infinity, to the area of the hyperplane partitioning $G^m$ into volumes $V_1$ and $1 - V_1$.

2.3. **An Area-Minimizing Partition of Gauss Space.** Theorem 2.14 proves that a standard Y is area minimizing in $G^m$ under the assumption that the standard double bubble is area minimizing in $S^n(\sqrt{n})$ for infinitely many $n$. The proof uses Lemma 2.13 to adjust volumes with controlled area cost. As in Proposition 2.2, let $f_n$ be the density of $R^m$ from the projection of $S^n(\sqrt{n})$:

$$f_n = c_n(2\pi)^{-m/2}(1 - x^2/n)^{(n-m-2)/2}$$

**Lemma 2.13** (Volume balancing at low area cost)**.** *Given a compact region $R \subset R^m$ and an $\epsilon > 0$, there exists $\delta > 0$ and $N$ such that for all $n > N$ and any $\Delta V < \delta$, a ball with $f_n$-weighted volume $\Delta V$ which is split by two hyperplanes has $f_n$-weighted total surface area (including the area of the hyperplanes inside the ball) $\Delta A < \epsilon$.*

Proof. Lemma 2.3 implies that $F_n = (2\pi)^{-m/2}(1 - x^2/n)^{(n-m-2)/2}$ converges monotonically from below to $(2\pi)^{-m/2}e^{-x^2/2}$ and assumes positive values for all

$x < \sqrt{n}$. By Proposition 2.1, the $c_n$ converge monotonically from above to 1. Hence, for $n \geq N$, $f_n$ is bounded below by $\min_R\{F_N\} > 0$ and bounded above by $\max_R(c_N(2\pi)^{-m/2}e^{-x^2/2}) \leq c_N(2\pi)^{-m/2}$. Therefore the surface area of the weighted ball goes to 0 with the volume. □

**Theorem 2.14** (An area-minimizing partition of Gauss space)**.** *The standard Y is an area-minimizing partition of $G^m$ for three given positive volumes $V_1 + V_2 + V_3 = 1$, under the hypothesis that the standard double bubble for the same volume fractions is area minimizing in $S^n$ for infinitely many $n$.*

*Remark.* Theorem 3.7 proves that the standard double bubble enclosing nearly equal volumes is area minimizing in $S^n(\sqrt{n})$ for all $n$. Corollary 3.8 combines that theorem with Theorem 2.14 above to prove that the standard Y is an area-minimizing partition of $G^m$ for nearly equal volumes.

Proof. Note that from scaling, an equivalent hypothesis to that of the theorem is that the standard double bubble for the same volume fractions is area minimizing in $S^n(\sqrt{n})$ for infinitely many $n$.

Let Y denote the standard Y with area $A$ and assume, to obtain a contradiction, that there exists a Y′ with area $A' = A - \alpha$ where $\alpha > 0$ (Figure 4).

By Proposition 2.12, for large $n$, the difference in area between the standard double bubble in $S^n(\sqrt{n})$ with the same volume fractions and the Y is less than $\alpha/2$.

Observe that there exists a compact region $R \subset R^m$ which contains distinct "Lebesgue" points of density 1 of each of the three regions. For the $i^{th}$ region, choose $\delta > 0$ as in Lemma 2.13 for $\epsilon = \alpha/12$ such that the balls are disjoint and the volume of $V_i$ in the ball exceeds half of the volume of the ball. The total surface area cost of constructing three such balls is less than $\alpha/4$.

By Propositions 2.1 and 2.4, for large $n$, the area of Y′ and the area of the preimage of Y′ on $S^n(\sqrt{n})$ differ by at most $\alpha/4$ while the volumes differ by at most $\Delta V'/2$. By Lemma 2.13, we can enclose in our balls at least $\Delta V'/2$ of each volume partition at area change of at most $\alpha/4$. Delete the volume from these balls and observe that every region now has less volume than it did in Y′. By splitting the balls with hyperplanes, we reassign the appropriate amount of volume to each region so that the total volume of each region equals the volume in each region of Y′.

We have just constructed a competitor on $S^n(\sqrt{n})$ to the standard double bubble. The total area difference between the competitor and the standard double

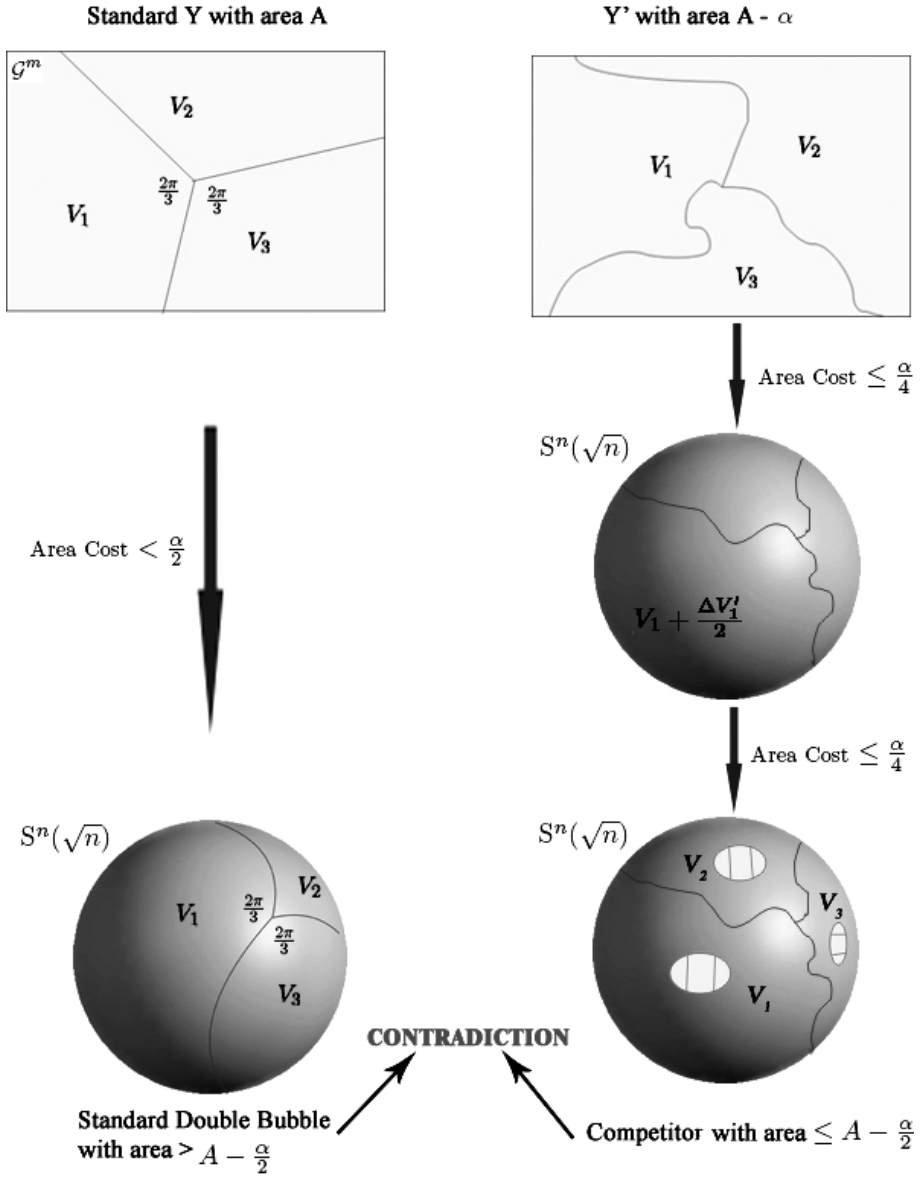



FIGURE 4. Assume there exists a Y′ better than our standard Y. This implies that the preimage of Y′ in $S^n(\sqrt{n})$ beats the standard double bubble, which is a contradiction.

bubble on $S^n(\sqrt{n})$ is less than $\alpha/2$ (area difference between the Y and the standard double bubble) plus $\alpha/4$ (difference between Y′ and preimage on $S^n(\sqrt{n})$) plus $\alpha/4$ (balls to balance volume), and hence less than $\alpha$. Y′ therefore partitions $S^n(\sqrt{n})$ into the three given volumes and uses less area than the standard double bubble, contradicting our hypothesis. □

*Remark.* Brakke's Surface Evolver [Br], available online at no cost, is a program expressly designed for the modeling of soap bubbles, foams, liquid solder, capillary shapes, and other liquid surfaces shaped by minimizing energy subject to various constraints. It inspired and helped us to simulate the perimeter-minimizing partition of three areas in $G^2$. It is a good way to see how the double bubble in Euclidean space turns into a standard Y in Gauss space.

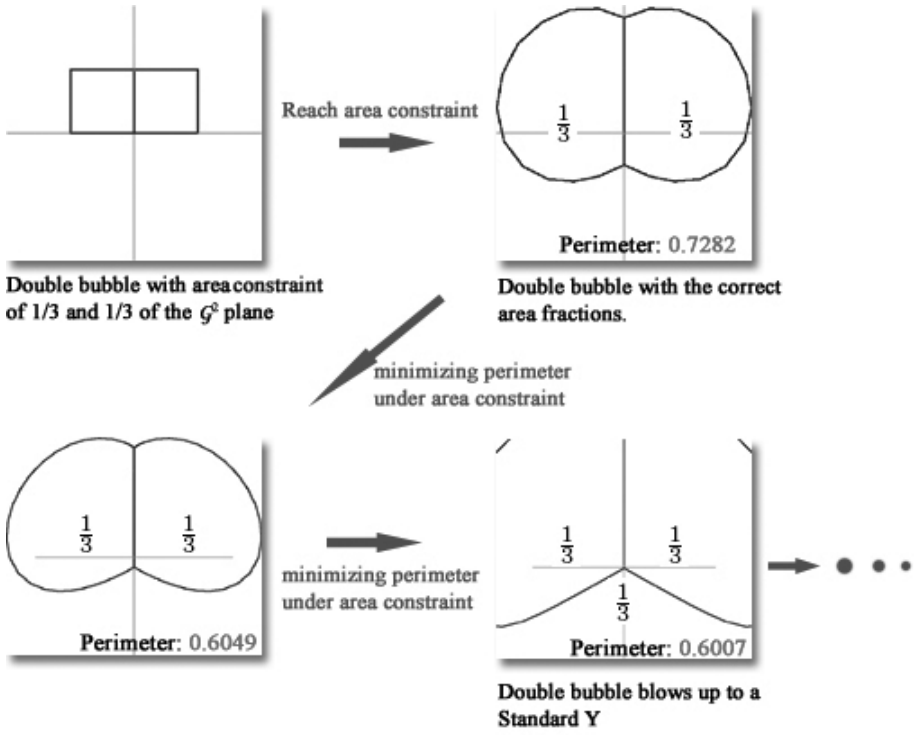



FIGURE 5. The evolution of an equal-area double bubble in $G^2$ of limiting perimeter 0.5984.

We first evolved the equal-area double bubble with area fractions $(1/3, 1/3, 1/3)$, as in Figure 5. Notice that the double bubble blows up and one vertex moves to the origin with three rays spaced approximately at angles of $2\pi/3$. The evolved shape has perimeter approaching $3/(2\sqrt{2\pi}) = 0.5984$, which is the perimeter of the standard Y at the origin.

We then evolved a double bubble with un-equal area fractions $(1/10, 1/3, 17/30)$. Notice that this bubble also blows up, forming a shifted standard Y (Figure 6).

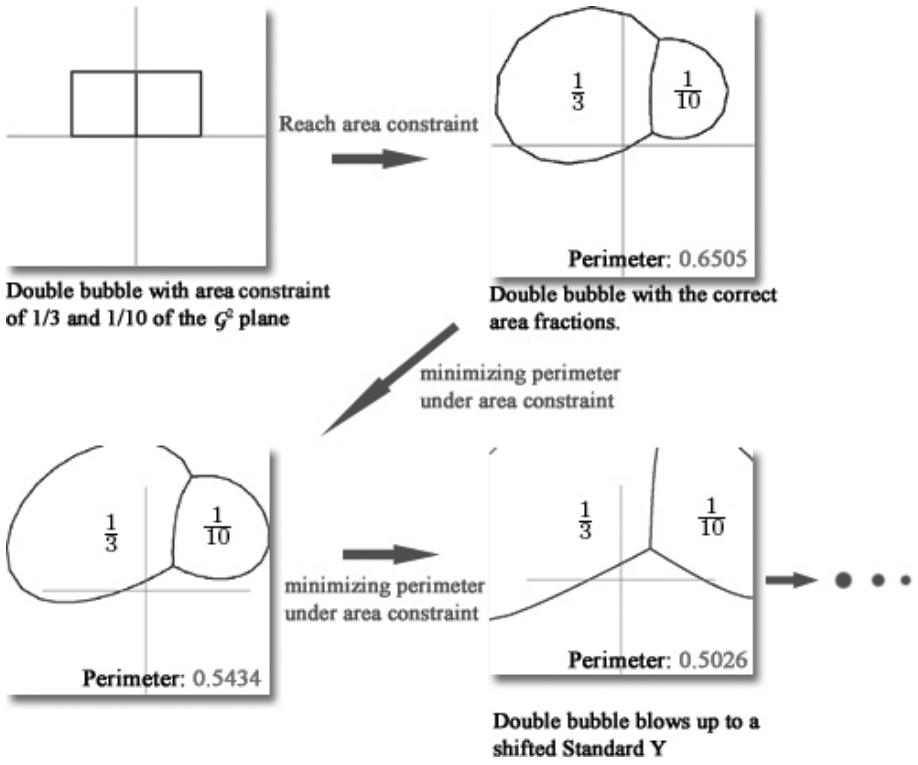



FIGURE 6. The evolution of an unequal-area double bubble in $G^2$.

*Remark.* Extending our work we propose the following conjecture.

**Conjecture 2.1.** *The perimeter-minimizing partition of $G^2$ into $n$ volumes $V_1 + V_2 + \ldots + V_n = 1$ consists of $n$ halflines and $n-3$ line segments, meeting at* 120 *degrees, as in Figure 7.*

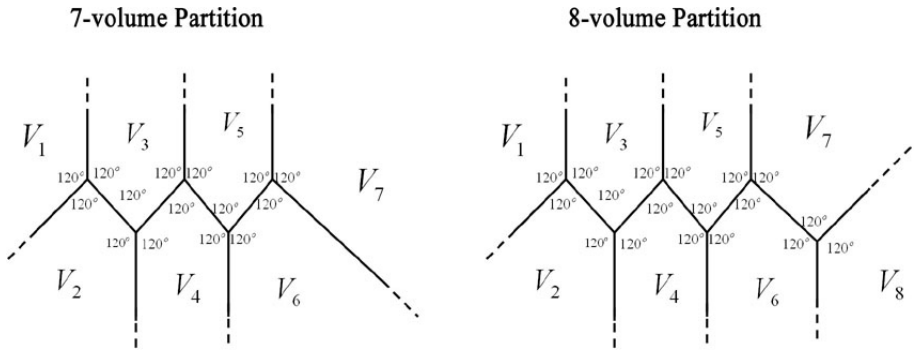



FIGURE 7. Our conjectured area-minimizing partition of $G^2$ into 7 and 8 volumes.

## 3. The Double Bubble Conjecture in $S^n$

Our main Theorem 3.7 shows that the unique area-minimizing partition of the sphere $S^n$ into three nearly equal volumes is a standard double bubble (see Definition 2). The main Lemmas 3.3–3.6 prove the Hutchings function positive in $G^m$, from which Theorem 3.7 follows. Section 3.3 extends Theorem 3.7 to a larger range of volumes.

A partial solution to the double bubble conjecture in $S^3$ obtained in [CF] states:

**Theorem 1.1.** *A least-area enclosure of two equal volumes in $S^3$ which add up to at most 90 percent of the total volume of $S^3$ must be the (unique) standard double bubble.*

Theorem 3.7 stems from this result.

### 3.1. The Hutchings Function in $G^m$ and $S^n$.

**Definition 3.** Define $T$ to be the triangular region in $R^3$ of positive volumes $(V_1, V_2, V_3)$ such that $V_1 + V_2 + V_3 = 1$. Throughout this section, unless otherwise noted, we assume that volumes $(V_1, V_2, V_3)$ are in $T$. Figures 8 and 9 illustrate this barycentric coordinate system.

**Definition 4.** The Hutchings function on the sphere $S^n$ is defined as

$$F_{S^n}(V_1, V_2, V_3) = 2A_{S^n}(V_1/2) + A_{S^n}(V_2) + A_{S^n}(V_3) - 2A_{S^n}(V_1, V_2, V_3),$$

where $A_{S^n}(v)$ denotes the surface area of the ball of volume fraction $v$ in $S^n(\sqrt{n})$ and $A_{S^n}(V_1, V_2, V_3)$ denotes the surface area of the standard double bubble which

partitions $S^n(\sqrt{n})$ into volume fractions $V_1, V_2, V_3$. Similarly, the Hutchings function in Gauss space $G^m$ is defined as

$$F_{G^m}(V_1, V_2, V_3) = 2A_{G^m}(V_1/2) + A_{G^m}(V_2) + A_{G^m}(V_3) - 2A_{G^m}(V_1, V_2, V_3),$$

where $A_{G^m}(v)$ denotes the surface area of a halfspace of $G^m$ of volume $v$ and $A_{G^m}(V_1, V_2, V_3)$ denotes the surface area of the standard Y which partitions $G^m$ into volumes $V_1, V_2, V_3$.

*Remark.* Hutchings ([Hu], Theorem 4.2) shows that if

$$F_{S^n}(V_1, V_2, V_3) > 0,$$

then the region of volume fraction $V_1$ is connected in an area-minimizing double bubble enclosing volume fractions $V_1, V_2, V_3$ in the sphere $S^n$. Hutchings actually uses a theoretically larger function, taking $A_{S^n}(V_1, V_2, V_3)$ to be the minimum area, so its positivity follows from the positivity of our $F_{S^n}$.

Note that the positivity of the Hutchings function is independent of the radius of $S^n$, since changing the radius scales the Hutchings function by a positive factor. The Hutchings theory does not apply directly to $G^m$, for lack of translational symmetry.

Cotton and Freeman ([CF], Proposition 7.3) show that for a minimizer in $S^n$, if each region and the exterior is connected, then the minimizer is a standard double bubble.

**Lemma 3.1.** *As $n$ approaches infinity, $F_{S^n}(V_1, V_2, V_3)$ approaches $F_{G^m}(V_1, V_2, V_3)$.*

PROOF. By Remark 2.2, $\lim_{n\to\infty} A_{S^n}(v) = A_{G^1}(v)$. By Proposition 2.12, $\lim_{n\to\infty} A_{S^n}(V_1, V_2, V_3) = A_{G^m}(V_1, V_2, V_3)$. Therefore,

$$F_G(V_1, V_2, V_3) = \lim_{n\to\infty} F_{S^n}(V_1, V_2, V_3).$$

□

*Remark.* Recall that $m$-dimensional Gauss space $G^m$ is $R^m$ endowed with density $(2\pi)^{-m/2}e^{-x^2/2}$. Let $(-\infty, x]$ be an interval of weighted length $v$ in $G^1$. Then

$$A_{G^1}(v) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

where $x$ is defined by

$$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = v.$$

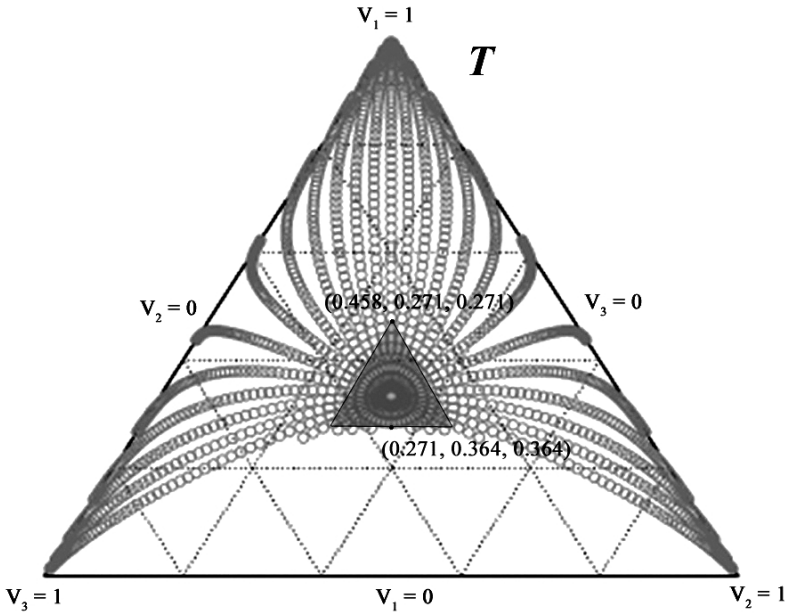



FIGURE 8. Small triangular region in $T$ where all three permutations of the Hutchings function are positive. The larger shaded region is where a particular one of them is positive; the small triangle is the intersection of three such regions.

Writing $A_{G^1}(v)$ as $e^{\psi(x)}$, observe that

$$\frac{dA_{G^1}(v)}{dv} = \frac{dA_{G^1}/dx}{dv/dx} = \psi'(x) = -x. \tag{1}$$

Therefore $A_{G^1}(v)$ is concave with maximum $A_{G^1}(1/2) = 1/\sqrt{2\pi}$.

**Definition 5.** Define the function $\tilde{F}_{S^n}(V_1, V_2, V_3)$, a lower bound of the Hutchings function $F_{S^n}(V_1, V_2, V_3)$, as

$$\tilde{F}_{S^n}(V_1, V_2, V_3) \quad = \quad 2A_{S^n}(V_1/2) + A_{S^n}(V_2) + A_{S^n}(V_3) - 3A_{S^n}\left(\frac{1}{2}\right).$$

Here $3A_{S^n}(1/2)$ arises as an overestimate for $2A_{S^n}(V_1, V_2, V_3)$ since three great $(n-1)$-hemispheres (each of which has area $A_{S^n}(1/2)/2$) meeting along a great $(n-2)$-sphere partition $S^n(\sqrt{n})$ into volume fractions $V_1, V_2, V_3$.

Similarly, define the function $\tilde{F}_G(V_1, V_2, V_3)$, a lower bound (for any $m$) of the Hutchings function $F_{G^m}(V_1, V_2, V_3)$, as

$$\tilde{F}_G(V_1, V_2, V_3) \quad = \quad 2A_{G^1}(V_1/2) + A_{G^1}(V_2) + A_{G^1}(V_3) - \frac{3}{\sqrt{2\pi}}.$$

Here $3/\sqrt{2\pi} = 3A_{G^1}(1/2)$ (as computed in Remark 3.1) arises as an overestimate of $2A_{G^m}(V_1, V_2, V_3)$. Indeed, except for the trivial case $m = 1$ three half-hyperplanes (each of which has area $A_{G^m}(1/2)/2$) meeting along an $(m-2)$-dimensional plane passing through the origin can partition $G^m$ into volumes $V_1, V_2, V_3$. Since the Gauss measure is a spherically symmetric product measure with total measure one, $A_{G^m}(v)$ is independent of $m$, so we can take $m = 1$.

**Lemma 3.2.** *As $n$ approaches infinity, $\tilde{F}_{S^n}(V_1, V_2, V_3)$ approaches $\tilde{F}_G(V_1, V_2, V_3)$.*

PROOF. By Remark 2.2, $\lim_{n\to\infty} A_{S^n}(v) = A_{G^1}(v)$. Therefore,

$$\tilde{F}_G(V_1, V_2, V_3) = \lim_{n\to\infty} \tilde{F}_{S^n}(V_1, V_2, V_3).$$

□

3.2. **Region of Positivity of the Hutchings Function.** Theorem 3.7 shows that the standard double bubble is an area minimizer for nearly equal volumes in $S^n$ for all $n$. The proof depends on the positivity of the Hutchings function $F_{S^n}$, which follows from the positivity of $F_{G^m}(V_1, V_2, V_3)$ and its permutations $F_{G^m}(V_2, V_3, V_1)$ and $F_{G^m}(V_3, V_1, V_2)$.

**Lemma 3.3.** *For equal volume fractions, $\tilde{F}_G$ is greater than* 0.029.

PROOF. Using $-100$ as an approximation for $-\infty$, we use numerical approximation to find $x$-values of $-.44$ and $-.97$ for terms of the Hutchings function applied to equal areas. Since

$$\int_{-100}^{-.97} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt < .16666 < \frac{1}{6}$$

and

$$\int_{-100}^{-.44} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt < .33333 < \frac{1}{3},$$

we have that

$$A_{G^1}(\frac{1}{6}) > \frac{1}{\sqrt{2\pi}} e^{-(-.97)^2/2} > .24922$$

and

$$A_{G^1}(\frac{1}{3}) > \frac{1}{\sqrt{2\pi}} e^{-(-.44)^2/2} > .36214.$$

Combining these inequalities yields

$$\begin{aligned} \tilde{F}_G\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) &= 2A_{G^1}\left(\frac{1}{6}\right) + A_{G^1}\left(\frac{1}{3}\right) + A_{G^1}\left(\frac{2}{3}\right) - \frac{3}{\sqrt{2\pi}} \\ &> 2(.24922) + 2(.36214) - \frac{3}{\sqrt{2\pi}} > .029. \end{aligned}$$

□

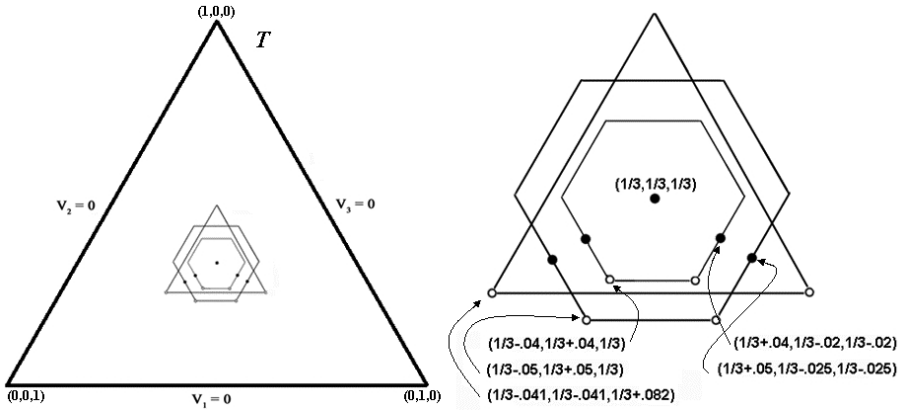



FIGURE 9. Small hexagons and triangle about $(1/3, 1/3, 1/3)$ where we prove the Hutchings function positive.

**Lemma 3.4.** *For a hexagon* $\{|V_i - 1/3| \le \delta \le 1/9\}$, $\tilde{F}_G(V_1, V_2, V_3)$ *is smallest at its two bottom vertices (with minimal* $V_1$ *as shown with the open circles in Figure 9). Similarly on a triangle* $\{1/3 - V_i \le \delta \le 1/18\}$ *the same result holds.*

PROOF. By Remark 3.1, $A_{G^1}$ is concave, hence so is $\tilde{F}_G$. Therefore $\tilde{F}_G$ attains its minimal value on the vertices of the hexagon and triangle.

We temporarily relax the constraint that $(V_1, V_2, V_3) \in T$. Then the partial derivatives of $\tilde{F}_G(V_1, V_2, V_3)$ are given by

$$\begin{aligned}\frac{\partial \tilde{F}_G}{\partial V_1} &= A'\left(\frac{V_1}{2}\right)\\ \frac{\partial \tilde{F}_G}{\partial V_2} &= A'(V_2)\\ \frac{\partial \tilde{F}_G}{\partial V_3} &= A'(V_3).\end{aligned}$$

By (1), $A'(V_1/2) = -x_1$, $A'(V_2) = -x_2$ and $A'(V_3) = -x_3$. Since $V_1/2 \leq 1/2$, $x_1$ is negative, and since $V_1/2$ is less than or equal to $V_2$ and $V_3$, therefore $x_1$ is less than or equal to $x_2$ and $x_3$. Thus for all $V_1, V_2, V_3$ in the hexagon,

$$\frac{\partial \tilde{F}_G}{\partial V_1} \geq \frac{\partial \tilde{F}_G}{\partial V_2}, \tag{2}$$

$$\frac{\partial \tilde{F}_G}{\partial V_1} \geq \frac{\partial \tilde{F}_G}{\partial V_3}. \tag{3}$$

Returning to $T$, for motion downward along the upper-righthand edge of the hexagon (as illustrated in Figure 9), $V_3$ is constant while $V_1$ decreases at the same rate that $V_2$ increases. Therefore by (2) $\tilde{F}_G$ is non-increasing. Similar analysis applies to the other non-horizontal edges. Therefore $\tilde{F}_G$ is smallest at the bottom vertices. For the triangle, the same argument applies. □

**Lemma 3.5.** *In the hexagon* $\{|V_i - 1/3| \leq .04\}$, $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$ *for large* $n$.

Proof. By Lemma 3.4, $\tilde{F}_G(V_1, V_2, V_3)$ achieves its minimum at $(1/3, 1/3, 1/3) + .04(-1, 1, 0)$. Computations show that

$$A_{G^1}\left(\frac{1}{6} - .02\right) > .24964, \quad A_{G^1}\left(\tfrac{1}{3} + .04\right) > .37866, \quad A_{G^1}\left(\frac{2}{3}\right) > .36359.$$

Hence,

$$\tilde{F}_G\left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + .04(-1, 1, 0)\right) =$$

$$= 2A_{G^1}\left(\frac{1}{6} - .02\right) + A_{G^1}\left(\frac{1}{3} + .04\right) + A_{G^1}\left(\frac{2}{3}\right) - \frac{3}{\sqrt{2\pi}} > .004 > 0.$$

Thus in the hexagon $\{|V_i - 1/3| \leq .04\}$, $\tilde{F}_G(V_1, V_2, V_3) > 0$ and by Lemma 3.2, $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$ for large $n$. □

**Lemma 3.6.** *If $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$ for infinitely many n, then $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$ for all n.*

PROOF. By Barthe ([Bar], Proposition 11; see [Ros], Theorem 21),

$$\frac{A_{S^{(n+1)}}(v)}{A_{S^{(n+1)}}(\frac{1}{2})} \leq \frac{A_{S^n}(v)}{A_{S^n}(\frac{1}{2})}. \tag{4}$$

Dividing the modified Hutchings function by $A_{S^n}(1/2)$ yields

$$\frac{\tilde{F}_{S^n}(V_1, V_2, V_3)}{A_{S^n}(\frac{1}{2})} = \frac{A_{S^n}(V_1/2)}{A_{S^n}(\frac{1}{2})} + \frac{A_{S^n}(V_2)}{A_{S^n}(\frac{1}{2})} + \frac{A_{S^n}(V_3)}{A_{S^n}(\frac{1}{2})} - 3$$

By (4), $\tilde{F}_{S^n}(V_1, V_2, V_3)/A_{S^n}(1/2)$ is nonincreasing in n. The result follows. □

**Theorem 3.7.** *For all n, the $n-$dimensional standard double bubble is the unique area-minimizing partition of $S^n$ into any three volume fractions within .04 of 1/3.*

PROOF. By Lemma 3.5, for large $n$, $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$. By Lemma 3.6, for all $n$, $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$ and therefore $F_{S^n}(V_1, V_2, V_3) > 0$. By Hutchings (Remark 3.1) and the symmetry of permuting the $V_i$, each region is connected. By Cotton and Freeman (Remark 3.1), the regions must form a standard double bubble. □

**Corollary 3.8.** *In $G^m$ the standard Y is an area-minimizing partition for any three volume fractions within .04 of 1/3.*

PROOF. This follows immediately from Theorems 3.7 and 2.14. In fact, this result can be reached independent of Lemma 3.6 since it only depends on standard double bubbles with large $n$. □

3.3. **Extending the known region of positivity for the Hutchings function.** Using computational lower bounds on the Hutchings function we enlarge the known region of positivity for the Hutchings function in Gauss space and hence in high-dimensional spheres from the hexagon $\{|V_i - 1/3| \leq .04\}$ to $\{|V_i - 1/3| \leq .05\}$ and finally to the triangle $\{1/3 - V_i \leq .041\}$. (We do not know how to generalize Lemma 3.6 on lower-dimensional spheres.) Figure 9 shows the inner (.04) and outer (.05) hexagons along with the .041 triangle. An improved overestimate for $A_{G^m}(V_1, V_2, V_3)$ is necessary in order to extend the region of positivity . To this end we prove two lemmas about the concave nature of $A_{G^m}(V_1, V_2, V_3)$.

*Remark.* The concept of mean curvature can be generalized to hypersurfaces in manifolds with density, such as Gauss space ([Gr], [CHSX], [Mo2]). A first variation formula shows that the curvature $\kappa$ of a curve in an equilibrium configuration

equals minus the rate of change of perimeter or "area" with respect to volume transfer across the curve. A line in Gauss space has constant curvature equal to its signed distance from the origin (Remark 3.1).

**Lemma 3.9.** *In $T$, outside of the hexagon $\{|V_i - 1/3| < \delta\}$, $A_{G^m}(V_1, V_2, V_3)$ attains its maximum value at the middle of an edge.*

PROOF. Every candidate maximum point lies on the boundary of an expanded hexagon $\{|V_i - 1/3| \leq \delta_1\}$, for some $\delta_1 \geq \delta$. Consider moving along an edge of the hexagon. For example, hold $V_1$ constant while $V_2$ changes. By Remark 3.3, $-dA_{G^m}/dV_2$ is a signed distance which is strictly increasing by Lemma 2.9. Thus $dA_{G^m}/dV_2$ is strictly decreasing and hence $A_{G^m}$ is concave in $V_2$. By this concavity and symmetry we may assume that the candidate maximum point lies at the middle of an edge. Now consider holding $V_2 = V_3$ and changing $V_1$. By Remark 3.3, $-dA_{G^m}/dV_1$ is a signed distance which is strictly increasing by Lemma 2.10. Thus $dA_{G^m}/dV_1$ is strictly decreasing and hence $A_{G^m}$ is concave in $V_1$. By this concavity and symmetry, we may assume that $\delta_1 = \delta$. □

**Theorem 3.10.** *In $G^m$ the standard $Y$ is an area-minimizing partition for any three volume fractions within .05 of 1/3.*

PROOF. By Corollary 3.8, we need consider only the region $R$ between the outer hexagon (.05) and the inner hexagon (.04) of Figure 9. By Lemma 3.9 and the symmetry of the hexagon, we need consider only two edge midpoints

$$\begin{aligned} p_1 &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + .04\left(-1, \frac{1}{2}, \frac{1}{2}\right), \\ p_2 &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + .04\left(1, -\frac{1}{2}, -\frac{1}{2}\right). \end{aligned}$$

By computation $A_{G^m}(p_1)$ is approximately 0.596374, and $A_{G^m}(p_2)$ is approximately 0.596436. Therefore $A_{G^m}(V_1, V_2, V_3) \leq 0.596437$. Using this overestimate for $A_{G^m}(V_1, V_2, V_3)$ in region $R$, we define a modified $\tilde{F}_G$ (a new lower bound for $F_{G^m}$ in $R$), changing the constant term from $3/\sqrt{2\pi}$ to 0.596437. By Lemma 3.4, the lower bound for the $\tilde{F}_G$ in the outer hexagon occurs at $(1/3, 1/3, 1/3) + .05(-1, 1, 0)$. By computation this lower bound is approximately 0.0012288, and therefore $F_{G^m}$ is positive in region $R$.

By Lemma 3.1, $F_{S^n}(V_1, V_2, V_3)$ converges to $F_{G^m}(V_1, V_2, V_3)$. Therefore, for large n $F_{S^n}(V_1, V_2, V_3) > 0$. By symmetry of the hexagon, each $F_{S^n}(V_i, V_j, V_k)$ is positive. By Hutchings (Remark 3.1), each region, as well as its complement,

is connected. By Cotton and Freeman (Remark 3.1), the regions must form a standard double bubble in high-dimensional spheres. By Theorem 2.14, the minimizer in Gauss space is a standard Y. □

**Theorem 3.11.** *In $G^m$ the standard Y is an area-minimizing partition for any three volume fractions greater than or equal to* $1/3 - .041$.

Proof. By Corollary 3.8, we need consider only the region $R$ between the (.041) triangle and the outer hexagon (.05) of Figure 9. By Lemma 3.9 and the symmetry of the hexagon, we need consider only two edge midpoints

$$\begin{aligned} p_1 &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + .05\left(-1, \frac{1}{2}, \frac{1}{2}\right), \\ p_2 &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + .05\left(1, -\frac{1}{2}, -\frac{1}{2}\right). \end{aligned}$$

By computation $A_{G^m}(p_1)$ is approximately 0.595213, and $A_{G^m}(p_2)$ is approximately 0.595334. Therefore $A_{G^m}(V_1, V_2, V_3) \leq 0.595335$. Using this overestimate for $A_{G^m}(V_1, V_2, V_3)$ in region $R$, we define a modified $\tilde{F}_G$ (a new lower bound for $F_{G^m}$ in $R$), changing the constant term from $3/\sqrt{2\pi}$ to 0.595335. By Lemma 3.4, the lower bound for the $\tilde{F}_G$ in the triangle occurs at $(1/3, 1/3, 1/3) + .041(-1, 2, -1)$. By computation this lower bound is approximately 0.0011504, and therefore $F_{G^m}$ is positive in region $R$.

As in the proof of Theorem 3.10, the minimizer in Gauss space is a standard Y. □

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# Double Bubbles in $\mathbf{S}^3$ and $\mathbf{H}^3$

By *Joseph Corneli, Neil Hoffman, Paul Holt, George Lee, Nicholas Leger, Stephen Moseley, and Eric Schoenfeld*

*ABSTRACT. We prove the double bubble conjecture in the three-sphere* $\mathbf{S}^3$ *and hyperbolic three-space* $\mathbf{H}^3$ *in the cases where we can apply Hutchings theory:*

- *In* $\mathbf{S}^3$*, when each enclosed volume and the complement occupy at least 10% of the volume of* $\mathbf{S}^3$*;*
- *in* $\mathbf{H}^3$*, when the smaller volume is at least 85% that of the larger.*

*A balancing argument and asymptotic analysis reduce the problem in* $\mathbf{S}^3$ *and* $\mathbf{H}^3$ *to some computer checking. The computer analysis has been designed and fully implemented for both spaces.*

## 1. Introduction

In March of 2002, Hutchings, Morgan, Ritoré, and Ros [11] proved that the area-minimizing way to enclose and separate two given volumes in $\mathbf{R}^3$ is by a standard double bubble, defined as three spherical caps meeting in threes at 120 degree angles. In 2003, Cotton and Freeman [7] partially extended these methods to $\mathbf{S}^3$ and $\mathbf{H}^3$, proving that the standard double bubble is the most efficient way to enclose and separate two equal volumes with the added condition that in $\mathbf{S}^3$ the exterior of the double bubble takes up at least 10% of the volume of $\mathbf{S}^3$. More recently, Corneli et al. [3] have proved the double bubble conjecture in spheres of all dimensions provided the double bubble partitions the sphere into the nearly equal volumes.

Our main theorem in $\mathbf{S}^3$ (Theorem 6.3) improves upon Cotton and Freeman's results when $v$, $w$ are sufficiently large, showing that the standard double bubble is the least-area way to enclose and separate two regions of prescribed unequal volumes, whenever each region and the exterior contains at least ten percent of the total volume. In $\mathbf{H}^3$, our Main Theorem 6.4 improves upon the results of Cotton and Freeman, showing double bubbles are standard whenever the smallest region has volume at least 85% that of the larger region. Computer plots, Figures 1 and 2, show that this is approximately the largest range on which current methods of proof using the positivity of the Hutchings function can work. Indeed, the main focus of this article is rigorously proving that the Hutchings function is positive on most of the regions where it appears positive in these plots. The plots suggest that $F(v, w)$ is always positive when $v > w$. This fact is proved in Remark 2.11, answering the open question of Cotton and Freeman [7, Open Question 1.1].

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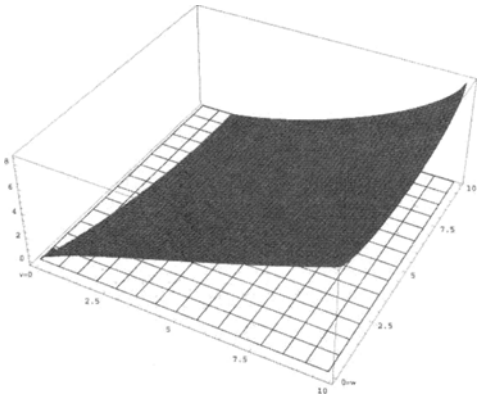


FIGURE 1 Plot of the Hutchings function $F(v, w)$ where it is positive in $\mathbf{S}^3$.

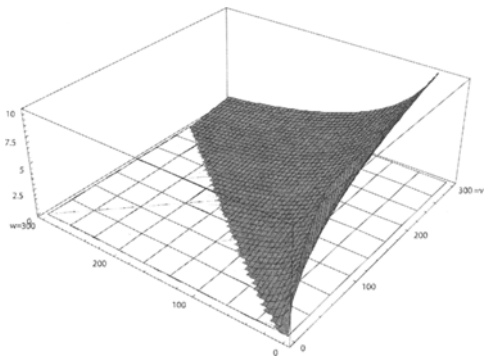


FIGURE 2 Plot of the Hutchings function $F(v, w)$where it is positive in $\mathbf{H}^3$.

**The proof.** The main difficulty is showing that both regions of a minimizing double bubble and the complement all have just one component each. Given such connectedness, Cotton and Freeman [7] extend the instability argument of [11] to show that a minimizing double bubble must be standard. By work of Hutchings [10], the first region in a minimizer of volumes $v$, $w$ is known to be connected if a certain "Hutchings function" $F(v, w)$, determined implicitly by integral formulas for volumes of spheres and standard double bubbles (see Definition 2.2) is positive. In their article, Cotton and Freeman use asymptotic analysis and intensive computation to verify this result along most of the line $v = w$ in $\mathbf{S}^3$ and all of the line $v = w$ in $\mathbf{H}^3$, thus proving that the standard double bubble is area-minimizing for equal volumes. For the two-dimensional domain of volumes $(v, w)$ in our unequal volume conjecture, the proof that $F(v, w) > 0$ is considerably harder. Sections 3 and 4 contain the requisite asymptotic analysis for the noncompact space $\mathbf{H}^3$. For small volumes, we use a Euclidean approximation (Section 3), and for large volumes we obtain the interesting result (Proposition 4.12) that

$$\lim_{w\to\infty} F(\psi w, w) = 2\pi \ln \frac{4(\psi+1)}{e^2},$$

which is positive if and only if $\psi > \lambda = e^2/4 - 1 < .85$. Along the line $v = \lambda w$, F is *decreasing* in $w$ when $w \geq 300$. Moreover, once $w > 150$, $\partial F/\partial v > 0$. This asymptotic analysis reduces the problem in $\mathbf{H}^3$ to a bounded domain. We are able to use a computer to show that the Hutchings function is positive on this bounded domain. In $\mathbf{S}^3$, which is a space of finite volume, our result requires each region to contain at least 10% of the total volume of $\mathbf{S}^3$, and we are again examining a bounded domain. Next we use a new balancing argument (Proposition 2.8), which says that if $F(\frac{v+w}{2}, \frac{v+w}{2})$ is positive, so is $F(v, w)$ for $v > 2w$. This argument, together with symmetry

(used only for $\mathbf{S}^3$) and a balancing argument due to Hutchings (Lemma 2.9), reduces the size of the domain to check. We then break up the domain into small rectangles (and triangles). Finally, we decompose the Hutchings function as the difference of a concave function and an increasing function, so that it suffices to consider the values on (or beyond) the corners of the rectangles (see Lemma 5.1). We find values of the parameters to locate such points and verify that $F(v, w) > 0$, allowing a safe margin of error. If necessary, we subdivide the rectangles and repeat the process.

Though using a computer introduces error, we bound this computation error and account for it in Section 5. For a complete version of the computer code, the reader is referred to [4].

## 2. The Hutchings function in $\mathbf{S}^3$ and $\mathbf{H}^3$

In $\mathbf{S}^3$ and $\mathbf{H}^3$, given volume $v$, a round ball has least boundary area $A(v)$ [14], [12, p. 127]. Similarly, given volumes $v$, $w$, there is an area-minimizing double bubble that encloses and separates these two volumes; furthermore, this double bubble is comprised of smooth constant-mean-curvature hypersurfaces, except possibly for a set of measure zero [12, Theorem 13.4, Remark before Proposition 13.8], [7, Proposition 2.3], and it is symmetric about some geodesic [10, Lemma 2.9, Remark 3.8]. Of course, in $\mathbf{S}^3$, we assume that $v, w$ satisfy $v + w < |\mathbf{S}^3|$, where $|\mathbf{S}^3|$ denotes the volume of $\mathbf{S}^3$. Also, we put $\bar{v}, \bar{w}, \bar{u}$ to be $v/|\mathbf{S}^3|$, $w/|\mathbf{S}^3|$, $u/|\mathbf{S}^3|$, respectively. Though the area of the area-minimizing double bubble is sometimes denoted $A(v, w)$ we reserve this notation for the area of the standard double bubble. The following argument shows $A(v, w)$ is well defined.

**Lemma 2.1** (Unique standard double bubble [7, Proposition 2.6] ). *Given volumes $v$ and $w$:*

- *Up to isometries of the space, there is a unique double bubble in $\mathbf{S}^3$ consisting of three spherical caps meeting at 120° that encloses and separates a region of volume $v$ and a region of volume $w$ (whenever $\bar{v} + \bar{w} < 1$).*
- *Up to isometries of the space, there is a unique double bubble in $\mathbf{H}^3$ consisting of two spheres (the outer caps) and a sphere, hyposphere, horosphere or geodesic plane (the inner cap) meeting at 120° that encloses and separates a region of volume $v$ and a region of volume $w$.*

***Definition 2.2.*** The *Hutchings function* in $\mathbf{S}^3$ is $F_{\mathbf{S}^3} : \{(v, w) : \bar{v} + \bar{w} < 1\} \to \mathbf{R}$, where

$$F_{\mathbf{S}^3}(v, w) = A_{\mathbf{S}^3}\left(\frac{v}{2}\right) + A_{\mathbf{S}^3}(w) + A_{\mathbf{S}^3}(v + w) - A_{\mathbf{S}^3}(v, w)\,, \tag{2.1}$$

and $A_{\mathbf{S}^3}(v, w)$ is the area of the standard double bubble.

The definition of $F_{\mathbf{H}^3}$ is similar.

We recall for the reader the basic property of the Hutchings function, namely that it can be used to limit the number of components in an area-minimizing double bubble.

**Proposition 2.3.** *In $\mathbf{S}^3$ or $\mathbf{H}^3$, if $F(v, w)$ is positive, then in an area-minimizing double bubble enclosing and separating volumes $v$ and $w$ the region of volume $v$ is connected.*

***Proof.*** This is a direct consequence of [7, Proposition 4.8] since the area of the standard double bubble has at least as much area as the area-minimizing double bubble. □

The argument given by Cotton and Freeman is essentially the same as the argument in $\mathbf{R}^n$ [12, p. 147], [10, Theorem 4.2].

Since each standard double bubble in $\mathbf{S}^3$ and $\mathbf{H}^3$ is symmetric about a geodesic line, it is convenient to compute area and volume of a standard double bubble by considering the revolution of some generating curve consisting of three circular arcs meeting at 120°. Figure 3 represents an arbitrary generating curve labeled with relevant measurements.

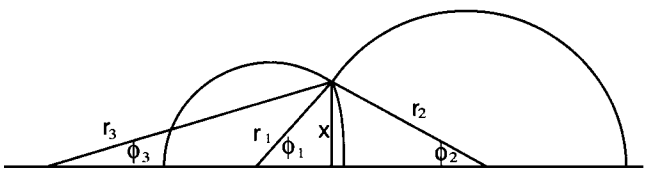



FIGURE 3 The generating curve for a standard double bubble.

Here we present the standard results on volume and surface area of spheres and double bubbles in $\mathbf{S}^3$ and $\mathbf{H}^3$.

***Remark 2.4.*** The formulas for surface area of a sphere of radius $r$ in $\mathbf{S}^3$ and $\mathbf{H}^3$ are

$$A_{\mathbf{S}^3} = 4\pi \sin^2 r \text{ and } A_{\mathbf{H}^3} = 4\pi \sinh^2 r \,.$$

The volume formulas for a ball of radius $r$ in $\mathbf{S}^3$ and $\mathbf{H}^3$ are

$$V_{\mathbf{S}^3} = \pi(2r - \sin 2r) \text{ and } V_{\mathbf{H}^3} = \pi(\sinh 2r - 2r) \,.$$

In $\mathbf{S}^3$ the area of a spherical cap subtended by an angle $2\phi_0$ is

$$\int_0^{2\pi} \int_0^{\phi_0} \sin^2 r \sin\phi \, d\phi \, d\theta = 2\pi \sin^2 r(1 - \cos\phi_0) \,. \tag{2.2}$$

The volume of a spherical cap that is subtended by an angle $2\phi_0$ is

$$-\pi\big(\tan^{-1}(\cos\phi_0 \tan r) - r\cos\phi_0 + (-1 + \cos\phi_0)(r - \cos r \sin r)\big) \,. \tag{2.3}$$

In $\mathbf{H}^3$ the area of a spherical cap in a sphere of radius $r$ subtended by an angle $2\phi_0$ is

$$\int_0^{2\pi} \int_0^{\phi_0} \sinh^2 r \sin\phi \, d\phi \, d\theta = 2\pi \sinh^2 r(1 - \cos\phi_0) \,. \tag{2.4}$$

The volume of a spherical cap in a sphere of radius $r$ subtended by an angle $2\phi_0$ is

$$\pi\big(-r + \tanh^{-1}(\cos\phi_0 \tanh r) - \cos\phi_0 \cosh r \sinh r + \cosh r \sinh r)\big)\big) \,. \tag{2.5}$$

Now we can compute the volume and areas of standard double bubbles.

For each bubble component, we compute the volume and area of the spherical cap which separates it from the exterior. To find the area of the double bubble, we simply add the area of the three spherical caps.

To compute the volume of the bubble components we add the volume contained by the separating cap to that contained by the smaller of the two outer cap to obtain the volume of the smaller region, and subtract the volume contained by the separating cap from that contained by the larger of the two outer cap to obtain the volume of the larger region. (Of course, when the two outer caps are the same size, the separating cap contains no volume so there is nothing to add or subtract.)

Examining Figure 3, we observe that the angle $\phi_1$ has a different orientation (relative to $r_2$ and $r_3$) depending on the relative size of the two bubbles. When $\phi_1$ is 90°, $r_2$ and $r_3$ are equal causing the volumes enclosed by both regions of the double bubble to be equal. Tracing back through the equations, we notice that this happens when

$$\frac{\tan(r_2) - \tan(r_1)}{\tan(r_2) + \tan(r_1)} = 1/3 \text{ (in } \mathbf{S}^3\text{) and } \frac{k_1 - k_2}{k_1 + k_2} = 1/3 \text{ (in } \mathbf{H}^3\text{)}. \tag{2.6}$$

We make adjustments to $\phi_1$ accordingly, and call the adjusted angle $\hat{\phi}_1$. Using this angle in the formulas for area and volumes of spherical caps yields the desired result, i.e., if $\hat{\phi}_1 > \frac{\pi}{2}$, the formulas return values corresponding to a cap smaller than a hemisphere, otherwise the values returned describe a cap that is larger than a hemisphere.

The concavity of $A(v)$ is well-known in both $\mathbf{S}^3$ and $\mathbf{H}^3$. Hutchings shows that the area of the area-minimizing double bubble enclosing volumes $v$ and $w$ is a concave function in both variables [10, Remark 3.8, Theorem 3.9]. For our computations, the concavity of the standard double bubble is relevant.

**Proposition 2.5.** *In* $\mathbf{S}^n$ *and* $\mathbf{H}^n$ *($n \geq 2$), $A(v, w)$ is a strictly concave function.*

***Proof.*** By uniqueness of standard double bubble (Lemma 2.1), it is the only equilibrium and hence the minimizer among spherical surfaces meeting along a sphere. Given $v_0$, $w_0$, and the standard double bubble, consider the area $A_1(v, w)$ of spherical double bubbles for nearby values of $v$ and $w$ obtained by varying the curvatures of the two bubbles, but leaving the curvature of the interface unchanged and giving up the 120-degree-angle condition. Along any line in $(v, w)$ space, if $v$ is strictly increasing, then the mean curvature $H_1$ of the first spherical cap is strictly decreasing, and similarly for $w$ and $H_2$. Hence, $A_1' = H_1\, dv/dt + H_2\, dw/dt$ is strictly decreasing and $A_1$ is strictly concave. Since $A(v, w) \leq A_1(v, w)$, with equality at $(v_0, w_0)$, $A(v, w)$ is strictly concave. □

The following two propositions follow directly from Proposition 2.5.

**Proposition 2.6.** *$A(v,w)$ in $\mathbf{S}^3$ is strictly increasing in $v$, $w$ on the closed triangular domain defined by $(0, 0)$, $(\frac{|\mathbf{S}^3|}{3}, \frac{|\mathbf{S}^3|}{3})$, $(0, \frac{|\mathbf{S}^3|}{2})$. Furthermore, $A(v, w)$ is also increasing on the line segment with end points $(0, \frac{|\mathbf{S}^3|}{2})$, $(\frac{|\mathbf{S}^3|}{3}, \frac{|\mathbf{S}^3|}{3})$.*

**Proposition 2.7.** *$A_{\mathbf{H}^3}(v, w)$ is strictly increasing in each variable.*

We call the following useful proposition "S-balancing" because it relies on the concavity of area of the standard double bubble, in contrast to Hutchings balancing (Lemma 2.9) which relies on the concavity of area of the minimizing double bubble.

**Proposition 2.8** (S-Balancing). *In $\mathbf{R}^n$, $\mathbf{H}^n$, or $\mathbf{S}^n$, for $n \geq 3$, if $w < v \leq 2w$ and $F(\frac{v+w}{2}, \frac{v+w}{2}) > 0$, then $F(v, w) > 0$.*

***Proof.*** By the concavity and symmetry of $A(x, y)$, we know

$$F(v, w) > 2A\left(\frac{v}{2}\right) + A(w) + A(v + w) - 2A\left(\frac{v + w}{2}, \frac{v + w}{2}\right). \tag{2.7}$$

Since $F\left(\frac{v+w}{2}, \frac{v+w}{2}\right) > 0$, to prove $F(v, w) > 0$ it suffices to show that

$$2A\left(\frac{v}{2}\right) + A(w) > 2A\left(\frac{v + w}{4}\right) + A\left(\frac{v + w}{2}\right). \tag{2.8}$$

But the concavity of $A(v)$ gives us that $f(x) = 2A(\frac{x}{2}) + A(v + w - x)$ is strictly increasing if $v + w$ is fixed and $x \in [\frac{v+w}{2}, \frac{2\cdot(v+w)}{3}]$. This shows Inequality (2.8), completing the proof. □

**Lemma 2.9** (Hutchings Balancing). ([10, Remark 3.8, Theorem 3.5, Corollary 3.9]). *In $\mathbf{R}^n$, $\mathbf{H}^n$, or $\mathbf{S}^n$, for $n \geq 3$, if $v > 2w$ (or in $\mathbf{S}^3$ if $v > 2u$), then the region of volume $v$ is connected.*

Permutation utilizes symmetries that exist with the Hutchings function when applied in compact spaces. Though this argument is applied in $\mathbf{S}^3$, it can be used in any space with finite volume that has a Hutchings function.

**Proposition 2.10** (Permutation). *In $\mathbf{S}^3$, $F(v, w) = F(v, u)$ where $u$ is the volume of the complement.*

***Proof.*** Notice $A(v, w) = A(v, u)$, $A(v+w) = A(u)$, and $A(w) = A(v+u)$. Thus, $F(v, w) = F(v, u)$. □

***Remark 2.11.*** S-balancing (Proposition 2.8), Hutchings balancing (Lemma 2.9), and permutation (Proposition 2.10) yield a complete answer to a question of Cotton and Freeman [7, Open Question 1]. In particular, the larger region of an area-minimizing double bubble in $\mathbf{H}^3$ is always connected. If $v > 2w$, this follows from Hutchings balancing (Lemma 2.9). Otherwise, connectedness follows from the positivity of the Hutchings function $F(v, w)$ (Proposition 2.3). This in turn follows from S-balancing (Proposition 2.8) and the positivity of the Hutchings function $F(v, v)$ for equal volumes [7, Propositions 5.11, 5.14, 5.19]. In $\mathbf{S}^3$, Hutchings balancing (Lemma 2.9) shows connectedness when $v > 2w$ or $v > 2u$. By Proposition 2.3, it suffices to show that $F(v, w) > 0$ if $w < v \leq 2w$ or $u < v \leq 2u$. If $\bar{v} \leq .4$, $F(v, v) > 0$ [7, Propositions 5.5, 5.8]. S-balancing (Proposition 2.8) shows that $F(v, w) > 0$ if $w < v \leq 2w$. Finally, permutation (Proposition 2.10) shows $F(v, w) > 0$ if $u < v \leq 2u$.

## 3. Positivity of the Hutchings function for small volumes in $\mathbf{H}^3$

**Theorem 3.1** (Double Bubble Theorem in $\mathbf{R}^3$ [11]). *The unique least-area enclosure of prescribed volumes $v$, $w$ in $\mathbf{R}^3$ is a standard double bubble.*

***Remark 3.2.*** For small prescribed volumes in $\mathbf{H}^3$, the space around any standard double bubble is nearly flat and looks like a portion of $\mathbf{R}^3$. By estimating the amount of distortion carefully, we can use information about the Hutchings function in $\mathbf{R}^3$ to prove that the Hutchings function in $\mathbf{H}^3$ is positive for certain prescribed volumes near 0. In the following proof, $\lambda$ is a constant which captures the amount of distortion, and $r$ represents the radius of a small ball in $\mathbf{H}^3$ which is only slightly affected by the curvature of $\mathbf{H}^3$.

**Lemma 3.3.** *For any pair of prescribed volumes less than* $(\frac{2\pi}{3} + \frac{\pi\sqrt{3}}{2})(\frac{2}{2+\sqrt{3}}r)^3$, *some minimizing double bubble in* $\mathbf{R}^3$ *enclosing those volumes fits inside a ball of radius* $r$.

***Proof.*** Without loss of generality assume that $v \leq w$ and fix $w$. The exterior spherical cap enclosing the region of volume $w$ has a radius, $r_w$. Consider standard double bubbles where the radius of this spherical cap is fixed at $r_w$. In this case, the equal volumes double bubble has the greatest diameter. However, the region enclosed by the spherical cap with radius $r_w$ has volume less than $w$. By increasing the radius until this region has volume $w$, the diameter increases. Thus for any nonequal volume double bubble enclosing volumes $v$ and $w$, where $v \leq w$, there will be a double bubble enclosing two regions of volume $w$ with a bigger diameter.

If $r_w$ is the radius of the spherical cap that forms the exterior of $W$, then the diameter of the equal volume double bubble is $2r_w + r_w\sqrt{3}$. So, this double bubble will fit inside a ball of radius $r > r_w\frac{(2+\sqrt{3})}{2}$.

We also know that in $\mathbf{R}^3$ a spherical cap of radius $r_0$ subtended by an angle $2\phi$ has volume:

$$V(r) = \frac{2\pi}{3}(r_0)^3 + \pi r_0{}^3\cos\phi\,. \tag{3.1}$$

Since $\phi \in [0, \frac{\pi}{6}]$,

$$w \leq (r_w)^3\Big(\frac{2\pi}{3} + \frac{\pi\sqrt{3}}{2}\Big)\,. \tag{3.2}$$

Substituting $\frac{2}{2+\sqrt{3}}r$ for $r_w$ in the above inequality gives the desired result. □

**Lemma 3.4.** *If* $.84w \leq v \leq w$, *then*

$$2A_{\mathbb{R}^3}\Big(\frac{v}{2}\Big) + A_{\mathbb{R}^3}(w) + A_{\mathbb{R}^3}(v+w) > (2.02676)A_{\mathbb{R}^3}(v, w)\,. \tag{3.3}$$

***Proof.*** (Modification of [11, Proposition 6.2]). By rescaling, we may assume that $v = 1 - w$ and that $w \in [\frac{1}{2}, \frac{1}{1.84}]$. As $A_{\mathbb{R}^3}(v, w)$ is concave [10, Theorem 3.2], we have

$$A_{\mathbb{R}^3}(1-w, w) \leq A_{\mathbb{R}^3}\big(\tfrac{1}{2}, \tfrac{1}{2}\big) = \big(2^{-\frac{4}{3}}\cdot 3\big)A_{\mathbb{R}^3}(1)\,.$$

Hence, it suffices to prove that

$$2A_{\mathbb{R}^3}\big(\tfrac{1-w}{2}\big) + A_{\mathbb{R}^3}(w) + A_{\mathbb{R}^3}(1) \geq 2.02676\big(2^{-\frac{4}{3}}\cdot 3\big)A_{\mathbb{R}^3}(1)\,,$$

which follows from the concavity of the left-hand side. □

**Proposition 3.5.** *If* $0 < v, w < .002743$ *and* $.84w \leq v \leq w$, *then*

$$2A_{\mathbf{H}^3}\Big(\frac{v}{2}\Big) + A_{\mathbf{H}^3}(w) + A_{\mathbf{H}^3}(v+w) - 2A_{\mathbf{H}^3}(v, w) > 0\,. \tag{3.4}$$

***Proof.*** By Lemma 3.3, we can fit a double bubble in $\mathbf{R}^3$ with prescribed volumes $v, w < .002743$ in a ball $B$ of radius $r < .1547$.

For ease of notation let $\lambda = \frac{\sinh r}{r} < 1.003994$.

Notice that if this ball can enclose a standard double bubble of volumes $(\lambda^{-2}v, \lambda^{-2}w)$ then it can enclose spheres of volumes:
$\{\lambda^{-2}\frac{v}{2}, \lambda^{-2}w, \lambda^{-2}(v+w)\}$. We may estimate the surface area of any sphere in $\mathbf{R}^3$ which fits inside $B$. Suppose that a sphere enclosing volume $\lambda^{-2}\alpha$ fits inside $B$; in particular, this holds for $\alpha \in \{\frac{v}{2}, w, (v+w)\}$. The single bubble in $\mathbf{H}^3$ encloses a volume $\alpha' \leq \alpha$ and has surface area at least $A_{\mathbf{R}^3}(\lambda^{-2}\alpha)$. Thus,

$$A_{\mathbf{H}^3}(\alpha) \geq A_{\mathbf{R}^3}\left(\lambda^{-2}\alpha\right). \tag{3.5}$$

Therefore,

$$\begin{aligned} &2A_{\mathbf{H}^3}\left(\frac{v}{2}\right) + A_{\mathbf{H}^3}(w) + A_{\mathbf{H}^3}(v+w) \\ &\quad \geq 2A_{\mathbf{R}^3}\left(\lambda^{-2}v/2\right) + A_{\mathbf{R}^3}\left(\lambda^{-2}w\right) + A_{\mathbf{R}^3}\left(\lambda^{-2}(v+w)\right). \end{aligned} \tag{3.6}$$

By Lemma 3.4,

$$2A_{\mathbf{R}^3}\left(\frac{\lambda^{-2}v}{2}\right) + A_{\mathbf{R}^3}\left(\lambda^{-2}w\right) + A_{\mathbf{R}^3}\left(\lambda^{-2}(v+w)\right) \geq (2.02676)A_{\mathbf{R}^3}\left(\lambda^{-2}v, \lambda^{-2}w\right). \tag{3.7}$$

Scaling of double bubbles in $\mathbf{R}^3$ tells us

$$A_{\mathbf{R}^3}\left(\lambda^{-2}v, \lambda^{-2}w\right) = (\lambda)^{-4/3}A_{\mathbf{R}^3}(v, w). \tag{3.8}$$

Because $B$ contains some minimizing double bubble $\Sigma_1$ in $\mathbf{R}^3$ enclosing volumes $v, w$, we can estimate the surface area of the image of $\Sigma_1$ in $\mathbf{H}^3$. This double bubble $\Sigma_2$ encloses volumes $v_2 \geq v$ and $w_2 \geq w$, with surface area less than or equal to $A_{\mathbf{R}^3}(v_2, w_2)\lambda^2$. Therefore,

$$A_{\mathbf{R}^3}(v, w) \geq \lambda^{-2}A_{\mathbf{H}^3}(v_2, w_2). \tag{3.9}$$

By Proposition 2.7, $A_{\mathbf{H}^3}(x, y)$ is increasing in each variable, implying that

$$A_{\mathbf{H}^3}(v_2, w_2) \geq A_{\mathbf{H}^3}(v, w). \tag{3.10}$$

Combining inequalities (3.6), (3.7), (3.9), and (3.10) with equality (3.8) gives

$$2A_{\mathbf{H}^3}\left(\frac{v}{2}\right) + A_{\mathbf{H}^3}(w) + A_{\mathbf{H}^3}(v+w) \geq (2.0676)(\lambda)^{-10/3}A_{\mathbf{H}^3}(v, w). \tag{3.11}$$

The desired result follows from the inequality $(2.02676)\lambda^{-10/3} > 2$. □

***Remark 3.6*** (Notes on $\mathbf{S}^3$). Although we are only concerned with volumes greater than 10% of the volume $\mathbf{S}^3$, the above argument works in $\mathbf{S}^3$ with a distortion factor, $\lambda = \frac{\sin r}{r}$. Otherwise the proof follows with very little adjustment. Using Lemma 3.4, the Hutchings Function is positive for small volumes in $\mathbf{S}^3$ namely $v, w < 0.002738$ and $.84w \leq v \leq w$.

## 4. Positivity of the Hutchings function for large volumes in $\mathbf{H}^3$

Theorem 4.21 shows that for sufficiently large $v$ and $w$, the Hutchings function in $\mathbf{H}^3$ is positive whenever $\psi = v/w \geq \lambda$ and $w \geq 300$. (Throughout this section we set $\lambda = e^2/4 - 1$.) The proof of Theorem 4.21 uses three ancillary results.

Proposition 4.12. *For a fixed ratio $v/w$, the limit of the Hutchings function as $w \to \infty$ is nonnegative if and only if $\psi \geq \lambda < .85$.*

Proposition 4.18. *The Hutchings function strictly decreases if one travels outward along the line* $v = \lambda w$ *as long as* $w > 300$.

Lemma 4.20. *For any fixed* $w \geq 150$ *and* $v \geq \lambda w$, $\frac{\partial F}{\partial v} > 0$.

Together, these results show that the Hutchings function is positive for $\psi \geq \lambda$ and $w > 300$. In order to prove Proposition 4.12, we must describe the limits of various quantities for standard double bubbles enclosing volumes $v$, $w$ as $v$, $w$ grow large. Proving Proposition 4.18 and Lemma 4.20 requires a closer examination of how fast these quantities converge to their limits. So, the results about the limits appear alongside various inexact numerical estimates.

If $r = r(v)$ is the radius of a sphere with volume $v$, then

$$r > \frac{1}{2} \ln \frac{2v}{\pi} . \tag{4.1}$$

**Lemma 4.1.** *If* $r = r(v)$ *is the radius of a sphere with volume* $v$, *then*

$$\lim_{v \to \infty} \left( r - \frac{1}{2} \ln \frac{2v}{\pi} \right) = 0 . \tag{4.2}$$

***Proof.*** By manipulation of the volume formula,

$$\frac{2v}{\pi} \left( e^{2(r - \frac{1}{2} \ln \frac{2v}{\pi})} - 1 \right) = e^{-2r} + 4r . \tag{4.3}$$

Notice for any $\epsilon > 0$ we can find a sufficiently large $r$, such that,

$$e^{-2r} + 4r < 2(\sinh 2r - 2r)\left(e^{2\epsilon} - 1\right) . \tag{4.4}$$

Hence, $\frac{2v}{\pi}(e^{2(r - \frac{1}{2} \ln \frac{2v}{\pi})} - 1) < \frac{2v}{\pi}(e^{2\epsilon} - 1)$ and $2(r - \frac{1}{2} \ln \frac{2v}{\pi}) < 2\epsilon$, as needed. □

***Remark 4.2.*** When $v > 150\lambda > \pi(\sinh(4.4) - 4.4)$, we have $r > 2.2$ and Inequality (4.4) holds for $\epsilon = 0.06$. Thus, $r < \frac{1}{2} \ln \frac{2v}{\pi} + .06$ for such $v$. We obtain the following numerical estimate: If $v > 150\lambda < 127.09$, then $r < \frac{1}{2} \ln \frac{2v}{\pi} + .06$.

**Lemma 4.3.** *As* $v \to \infty$, $A(v)$ *tends to* $2v + 2\pi \ln v - 2\pi(1 - \ln(\pi/2))$.

***Proof.*** Let $r = r(v)$ denote the radius of a sphere with volume $v$. Since

$$r = \frac{1}{2} \ln \left( \frac{2v}{\pi} + e^{-2r} + 4r \right) , \tag{4.5}$$

we see that,

$$A(v) = 2v - 2\pi + 2\pi e^{-2r} + 4\pi r . \tag{4.6}$$

By Lemma 4.1,

$$\lim_{v \to \infty} r - \frac{1}{2} \ln \frac{2v}{\pi} = \lim_{v \to \infty} r - \frac{1}{2} \ln v + \frac{1}{2} \ln \frac{\pi}{2} = 0 . \tag{4.7}$$

Also, $\lim_{v \to \infty} e^{-2r} = 0$. The desired result follows easily. □

**Lemma 4.4.** *The curvature of a sphere with volume $v$ and radius $r$ is*

$$2 + \frac{2\pi}{v + 2\pi r + \frac{\pi}{2}e^{-2r} - \pi/2}\,. \tag{4.8}$$

***Proof.*** This holds since the curvature is $2\coth r$ and $2v/\pi = e^{2r} - e^{-2r} - 4r$. □

**Lemma 4.5.** $A'(x) < 2 + \frac{2\pi}{x+\pi\ln x - 3}$.

***Proof.*** From (4.1) we have the following bound on the radius $r$ of a sphere containing volume $x$:

$$r > \frac{1}{2}\ln(2x/\pi) = \frac{1}{2}\ln x + (1/2)\ln(2/\pi) \tag{4.9}$$

$$> \frac{1}{2}\ln x - 3/2\pi + 1/4\,. \tag{4.10}$$

By Lemma 4.4,

$$A'(x) = 2 + \frac{2\pi}{x + 2\pi r + (\pi/2)e^{-2r} - \pi/2}\,. \tag{4.11}$$

Also, $e^{-2r} > 0$. Applying these inequalities in the above expression for $A'(x)$ yields the desired result. □

**Lemma 4.6.** *If $x > 150\lambda$, then $A'(x) > 2 + \frac{2\pi}{x+\pi\ln x - 1.041}$.*

***Proof.*** By Lemma 4.4,

$$A'(v) = 2 + \frac{2\pi}{v + 2\pi r + \frac{\pi}{2}e^{-2r} - \frac{\pi}{2}}\,. \tag{4.12}$$

Notice, $\frac{\pi}{2}e^{-2r} - \frac{\pi}{2} < 0$. Also, we have $r < \frac{1}{2}\ln\frac{2v}{\pi} + 0.06$ because $v > 150\lambda$ (Remark 4.2). Hence,

$$2\pi r < \pi\ln\frac{2v}{\pi} + 0.12\pi = \pi\ln v + \pi\ln\frac{2}{\pi} + 0.12\pi < \pi\ln v - 1.041\,. \tag{4.13}$$

With these approximations, we find

$$A'(v) > 2 + \frac{2\pi}{v + \pi\ln v - 1.041}\,. \tag{4.14}$$ □

**Corollary 4.7.** *If $x > 300\lambda$ then $A'(x+3) > 2 + \frac{2\pi}{x+\pi\ln x + 2}$.*

***Proof.*** By Lemma 4.6, $A'(x) > 2 + \frac{2\pi}{x+\pi\ln x - 1.041}$, if $x > 150\lambda$.

Consider the denominator of the fraction in (4.8). Because $e^{-2r} < 1$, the third term minus $\pi/2$ is less than 0. We now approximate the second term. By Remark 4.2, we have $r < \frac{1}{2}\ln\frac{2v}{\pi} + 0.06$ because $v > 150\lambda$. Hence, the second term satisfies

$$2\pi r < \pi\ln\frac{2v}{\pi} + 0.12\pi = \pi\ln v + \pi\ln\frac{2}{\pi} + 0.12\pi < \pi\ln v - 1.041\,. \tag{4.15}$$

With these approximations, we find

$$A'(v) > 2 + \frac{2\pi}{v + \pi \ln v - 1.041} . \tag{4.16}$$

For $x > 300\lambda$, we have

$$\pi \ln \frac{x+3}{x} - 1.041 < -1.004 . \tag{4.17}$$

Hence,

$$A'(x+3) > 2 + \frac{2\pi}{x + 3 + \pi \ln x - 1.004} > 2 + \frac{2\pi}{x + \pi \ln x + 2} . \qquad \square \tag{4.18}$$

**Lemma 4.8.** *Given a standard double bubble, the radius of the circular interface where the three caps meet is less than* $\cosh^{-1}(2)$.

***Proof.*** Let $D$ be the disk whose boundary is the circular interface. Let $y$ be the radius of $D$. $D$ meets the bigger of the outer caps of the double bubble at an angle $\alpha \geq 2\pi/3$. The radii of this outer cap and $D$ come together at an angle of $\beta \geq \pi/6$ along the circular interface. Let $2\phi_o$ be the angle that subtends this cap. By the formula for the area of a spherical cap in $\mathbf{H}^3$ [Equation (2.4)] and the hyperbolic laws of cosines and sines the surface area of the outer cap is

$$\frac{2\pi \sinh^2 y}{1 - \cosh y \sin \beta} . \tag{4.19}$$

The surface area of this cap is positive and finite, implying that $\cosh y < \csc \beta \leq \csc(\pi/6) = 2$. Because $t \mapsto \cosh t$ is increasing for $t > 0$, $y < \cosh^{-1}(2)$. $\square$

**Lemma 4.9.** *Given a spherical cap subtended by an angle of* $2\phi$ *on a sphere of radius* $r$, *let* $\theta$ *be the angle the sphere makes with the disk of radius* $y$ *that shares a boundary with the spherical cap. The volume of this cap can be expressed as,*

$$\pi \left( \frac{\sinh y \sin\theta}{sech\, y + \cos\theta} - \tanh^{-1}\left( \frac{\sinh y \sin\theta}{\cosh y + \cos\theta} \right) \right) . \tag{4.20}$$

***Proof.*** This follows from using the identities:

$$\begin{aligned}
\tanh(a+b) &= \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b} \text{ where } (a+b) = \left( -r + \tanh^{-1}(\tanh r \cos\phi) \right) , \\
1 - \tanh^2 x &= \operatorname{sech}^2 x \text{ and } \cosh b = \frac{\sin\phi}{\cos\gamma} , \\
\cosh c &= \cosh y \cosh b , \\
\cos\gamma &= \sin\phi \cosh b , \\
\tanh r \cos\gamma &= \tanh y, \cosh y \sin\gamma = \cos\phi , \\
\sinh y &= \sin\phi \sinh r \text{ and } \cos\gamma = \frac{\tanh y}{\tanh r}, \text{ and} \\
\sin\phi &= \frac{\sinh y}{\sinh r} .
\end{aligned}$$

$\square$

**Lemma 4.10.** *Given a standard double bubble enclosing volumes $v, w$, let $\theta$ be the angle between the separating cap and the disk, D, with the same boundary, and let $y$ be the radius of D. As $v, w$ grow large, $\theta \to 0$ and $y \to \cosh^{-1}(2)$.*

***Proof.*** By Lemma 4.8, $y < \cosh^{-1}(2)$. The two caps of the region containing D meet D at angles $\theta \in [0, \pi/3)$ and $\alpha = 2\pi/3 - \theta$. Put $x$ to be the volume of this region. Using the volume formulas we see

$$x < \pi \left( \frac{\sinh y \sin \alpha}{\operatorname{sech} y + \cos \alpha} + \frac{\sinh y \sin \theta}{\operatorname{sech} y + \cos \theta} \right) . \tag{4.21}$$

Since $\operatorname{sech} y > 1/2$ and $\cos\theta, \cos\alpha > -1/2$,

$$\begin{aligned} x &< \pi \left( \frac{\sinh\left(\cosh^{-1}(2)\right) \sin(\pi/2)}{\operatorname{sech} y + \cos \theta} + \frac{\sinh\left(\cosh^{-1}(2)\right) \sin(\pi/2)}{\operatorname{sech} y + \cos \alpha} \right) \\ &< \frac{2\pi\sqrt{3}}{\operatorname{sech} y + \cos(2\pi/3 - \theta)} . \end{aligned}$$

As $v, w$ approach infinity, $x \geq \min\{v, w\}$ must approach infinity as well, implying that the positive quantity $\operatorname{sech} y + \cos(2\pi/3 - \theta)$ must approach 0—that is, $y$ must approach $\cosh^{-1}(2)$, and $\cos(2\pi/3 - \theta)$ must approach $-\frac{1}{2}$. Because $2\pi/3 - \theta \in (\pi/3, 2\pi/3]$, $\theta$ must approach 0. In particular, when $v, w > x_0 = 300\lambda$, we have

$$x_0 < \frac{2\pi\sqrt{3}}{\operatorname{sech} y + \cos(2\pi/3 - \theta)} < \frac{2\pi\sqrt{3}}{1/2 + \cos(2\pi/3 - \theta)} \tag{4.22}$$

or

$$\cos(2\pi/3 - \theta) < \frac{2\pi\sqrt{3}}{x_0} - 1/2 < -.457 . \tag{4.23}$$

For $\theta \in [0, \pi/3)$, this is true only if $\theta < 1/20$. □

Note, the following numerical estimate: If $v, w > 300\lambda \approx 254.18$, then $\theta < 1/20$.

**Lemma 4.11.** *As $v, w$ grow large, the surface area of a standard double bubble enclosing volumes $v, w$ approaches*

$$A_{\mathbf{H}^3}(v + v_\infty) + A_{\mathbf{H}^3}(w + v_\infty) - 2a_\infty + c_\infty \tag{4.24}$$

*where $v_\infty = \pi(3/2 - \ln 2)$, $a_\infty = 3\pi$, and $c_\infty = 2\pi$.*

***Proof.*** Given a standard double bubble enclosing volumes $v, w$, consider the region enclosing volume $v$. Its outer cap lies on a sphere; let the volume of this sphere be $v + v^{(1)}_{v,w}$ and let the surface area of the sphere exceed that of the outer cap by $a^{(1)}_{v,w}$. Define $v^{(2)}_{v,w}$ and $a^{(2)}_{v,w}$ similarly with respect to the region enclosing volume $w$. Finally, let $c_{v,w}$ denote the surface area of the separating cap. Then the surface area of the standard double bubble is

$$A_{\mathbf{H}^3}\left(v + v^{(1)}_{v,w}\right) + A_{\mathbf{H}^3}\left(w + v^{(2)}_{v,w}\right) - a^{(1)}_{v,w} - a^{(2)}_{v,w} + c_{v,w} . \tag{4.25}$$

Suppose that $\theta$ is the angle between the separating cap and the disk with the same boundary, and let $y$ be the radius of the disk. By Lemma 4.10, as $v$ and $w$ grow large, $\theta \to 0$ and

$y \to y_0 = \cosh^{-1}(2)$. Using formulas for volume and surface area, one can easily check that

$$\lim_{v,w\to\infty} v^{(1)}_{v,w} = \lim_{v,w\to\infty} v^{(2)}_{v,w} = \pi\left(\frac{\sinh y_0 \sin(\pi/3)}{\operatorname{sech} y + \cos(\pi/3)} - \tanh^{-1}\left(\frac{\sinh y_0 \sin(\pi/3)}{\cos y + \cos(\pi/3)}\right)\right) = v_\infty$$

$$\lim_{v,w\to\infty} a^{(1)}_{v,w} = \lim_{v,w\to\infty} a^{(2)}_{v,w} = \frac{2\pi \sinh^2 y_0}{1+\cos(\pi/3)\cosh y_0} = a_\infty$$

and

$$\lim_{v,w\to\infty} c_{v,w} = \frac{2\pi \sinh^2 y_0}{1+\cos 0 \cosh y_0} = c_\infty .$$

Hence,

$$\left|A\left(v + v^{(1)}_{v,w}\right) - A(v+v_\infty)\right| < A'(v)\left|v^{(1)}_{v,w} - v_\infty\right|$$

approaches 0 as $v$, $w$ grow large. So too does $|A(w+v^{(2)}_{v,w}) - A(w+v_\infty)|$. From these computed limits, it follows that the surface area of the standard double bubble enclosing volumes $v$, $w$—as calculated in Equation 4.25—approaches $A(v+v_\infty)+A(w+v_\infty)-2a_\infty+c_\infty$ as $v$, $w$ get large. □

**Proposition 4.12.** *For each fixed* $\psi > 0$, $\lim_{w\to\infty} F(\psi w, w) = 2\pi \ln \frac{4(\psi+1)}{e^2}$.

***Proof.*** We write $v = v(w) = \lambda w$. Let $v_\infty = \pi(3/2 - \ln 2)$, $a_\infty = 3\pi$, and $c_\infty = 2\pi$. By Lemma 4.11, the given limit equals

$$\lim_{w\to\infty} 2A\left(\frac{v}{2}\right) + A(w) + A(v+w) - 2(A(v+v_\infty) + A(w+v_\infty) - 2a_\infty + c_\infty) \tag{4.26}$$

provided that the latter limit exists. By Lemma 4.3,

$$\lim_{x\to\infty} A(x) = 2x + 2\pi \ln x - 2\pi(1 - \ln(\pi/2)) \tag{4.27}$$

for $x \in \{\frac{v}{2}, w, v+w, 2v+v_\infty, w+v_\infty\}$. Substituting these expressions into (4.26) and simplifying gives

$$\lim_{w\to\infty}\left(2\pi\left(2\ln\left(\frac{v}{2}\right)+\ln w+\ln(v+w)-2\ln(v+v_\infty)-2\ln(w+v_\infty)\right) - 8v_\infty+4a_\infty-2c_\infty\right)$$

provided that this new limit exists. Indeed, this new limit does exist and equals

$$\begin{aligned}
&\lim_{w\to\infty}\left(2\pi \ln\left(\frac{v^2 w(v+w)}{4(v+v_\infty)^2(w+v_\infty)^2}\right) - 8v_\infty + 4a_\infty - 2c_\infty\right)\\
&= \lim_{w\to\infty}\left(2\pi\ln\left(\frac{v+w}{4w}\right) - 8v_\infty + 4a_\infty - 2c_\infty\right)\\
&= 2\pi \ln\left(\frac{4(\psi+1)}{e^2}\right).
\end{aligned}$$

Hence, the limit in Equation (4.26) evaluates to $2\pi \ln(\frac{4(\psi+1)}{e^2})$. □

(Note that this limit is nonnegative if and only if $\psi \geq \lambda < 0.85$.)

**Lemma 4.13.** *Given a standard double bubble enclosing volumes* $v$, $w$, *the outer cap of the region of volume* $v$ *lies on a sphere, say with volume* $v + v_1$ *(resp. the outer cap of the region of volume* $w$ *lies on a sphere with volume* $w + w_1$*). If* $v, w > 300\lambda$ *then* $v_1, w_1 < 3$.

***Proof.*** Let $\mathrm{vol}(y, \alpha)$ denote the volume of a spherical cap with associated disk of radius $y$, where the cap and disk meet at an angle $\alpha \in [0, \pi]$. This volume is increasing in both $y$ and $\alpha$. Without loss of generality, assume that $v \leq w$. The three caps of the standard double bubble meet along a circular interface with radius $y_0$; let the disk with this circle as its boundary meet the separating cap of the double bubble at angle $\theta$. By Lemma 4.8, $y_0 < \cosh^{-1}(2)$. By Lemma 4.10, $\theta < 1/20$. Both $v_1 = \mathrm{vol}(y_0, \pi/3 + \theta) - \mathrm{vol}(y_0, \theta)$ and $w_1 = \mathrm{vol}(y_0, \pi/3 - \theta) + \mathrm{vol}(y_0, \theta)$ are less than $\mathrm{vol}(y_0, \pi/3 + \theta) + \mathrm{vol}(y_0, \theta)$, which in turn is less than $\mathrm{vol}(\cosh^{-1}(2), \pi/3 + 1/20) + \mathrm{vol}(\cosh^{-1}(2), 1/20) < 3$. □

We now prove some algebraic lemmas that will be used in the proof of Proposition 4.18 and later in Proposition 4.19.

**Lemma 4.14.** *For positive numbers $\mu$, $w$ with $\ln \mu + a > 0$*

$$\frac{\mu}{\mu w + \pi \ln(\mu w) + a} > \frac{1}{w} - \frac{1}{w^2}\left(\frac{\pi \ln w}{\mu} + \frac{\pi \ln \mu}{\mu} - \frac{a}{\mu}\right). \tag{4.28}$$

***Proof.*** If $\pi \ln x + a > 0$, then

$$\frac{1}{x + \pi \ln x + a} > \frac{1}{x} - \frac{\pi \ln x + a}{x^2}. \tag{4.29}$$

Making the substitution $x = \mu w$ yields the desired result. □

**Lemma 4.15.** *For positive $\mu$, $w$ , with $\mu w > 150$*

$$\frac{\mu}{\mu w + \pi \ln(\mu w) - 3} < \frac{1}{w} - \frac{1}{w^2}\left(\frac{\pi \ln w}{1.1\mu} + \frac{\pi \ln \mu}{1.1\mu} - \frac{3}{1.1\mu}\right). \tag{4.30}$$

***Proof.*** If $x > 150\lambda$, we have $1.1x > x + \pi \ln x - 3 > 0$. Hence,

$$\frac{1}{x + \pi \ln x - 3} = \frac{1}{x} - \frac{\pi \ln x - 3}{x(x + \pi \ln x - 3)} < \frac{1}{x} - \frac{\pi \ln x - 3}{1.1x^2}. \tag{4.31}$$

Making the substitution $x = \mu w$ yields the desired result. □

**Lemma 4.16.** *For all $w \geq 300$,*

$$\begin{aligned} &2\left(\frac{\pi \ln w}{\lambda} + \frac{\pi \ln \lambda}{\lambda} - \frac{2}{\lambda} + \pi \ln w - 2\right) \\ &\quad < \frac{2\pi \ln w}{1.1\frac{\lambda}{2}} + \frac{2\pi \ln\left(\frac{\lambda}{2}\right)}{1.1\frac{\lambda}{2}} - \frac{6}{1.1\frac{\lambda}{2}} + \frac{\pi \ln w}{1.1} - \frac{3}{1.1} \\ &\qquad + \frac{\pi \ln w}{1.1(\lambda + 1)} + \frac{\pi \ln(\lambda + 1)}{1.1(\lambda + 1)} - \frac{3}{1.1(\lambda + 1)}. \end{aligned}$$

***Proof.*** For all $w \geq 300$,

$$\begin{aligned} &\pi\left(2\frac{\ln \lambda}{\lambda} - 2\frac{\ln \lambda/2}{1.1\lambda/2} - \frac{\ln(\lambda + 1)}{1.1(\lambda + 1)}\right) \\ &\qquad + \left(2\frac{3}{1.1\lambda/2} + \frac{3}{1.1} + \frac{3}{1.1(\lambda + 1)} - \frac{4}{\lambda} - 4\right) \\ &\quad < \left(\frac{2}{1.1\lambda/2} + \frac{1}{1.1} + \frac{1}{1.1(\lambda + 1)} - \frac{2}{\lambda} - 2\right)\pi \ln w. \end{aligned}$$

Simple algebraic manipulation gives the desired result. □

**Lemma 4.17.** *If* $w \geq 300$, *then*

$$\left(\frac{2\lambda}{\lambda w + \pi \ln(\lambda w) + 2} + \frac{2}{w + \pi \ln(w) + 2}\right) > \left(\frac{\lambda}{\lambda\frac{w}{2} + \pi \ln\left(\lambda\frac{w}{2}\right) - 3} + \frac{1}{w + \pi \ln(w) - 3} + \frac{\lambda + 1}{(\lambda + 1)w + \pi \ln((\lambda + 1)w) - 3}\right).$$

***Proof.*** By Lemma 4.14,

$$\begin{aligned}
&\frac{2\lambda}{\lambda w + \pi \ln(\lambda w) + 2} + \frac{2}{w + \pi \ln(w) + 2} \\
&\quad > 2\left(\frac{2}{w} - \frac{1}{w^2}\left(\frac{\pi \ln w}{\lambda} + \frac{\pi \ln(\lambda)}{\lambda} - \frac{2}{\lambda}\right) - \frac{1}{w^2}(\pi \ln w - 2)\right) \\
&\quad > \frac{4}{w} - \frac{1}{w^2}\left(\frac{2\pi \ln w}{1.1\frac{\lambda}{2}} + \frac{2\pi \ln\left(\frac{\lambda}{2}\right)}{1.1\frac{\lambda}{2}} - \frac{6}{1.1\frac{\lambda}{2}} + \frac{\pi \ln w}{1.1} - \frac{3}{1.1}\right. \\
&\qquad \left. + \frac{\pi \ln w}{1.1(\lambda + 1)} + \frac{\pi \ln(\lambda + 1)}{1.1(\lambda + 1)} - \frac{3}{1.1(\lambda + 1)}\right)
\end{aligned}$$

(by Lemma 4.16)

$$> \frac{\lambda}{\frac{\lambda w}{2} + \pi \ln\frac{\lambda w}{2} - 3} + \frac{1}{w + \pi \ln w - 3} + \frac{(1 + \lambda)}{(\lambda + 1)w + \pi \ln((\lambda + 1)w) - 3}$$

(by Lemma 4.15). □

**Proposition 4.18.** *If* $w \geq 300$, *then* $\frac{d}{dw}F(\lambda w, w) < 0$.

***Proof.*** Consider a standard double bubble enclosing volumes $\lambda w$ and $w$. Let $v_1, w_1$, be the volumes needed to complete the two outer caps of the double bubble which enclose volumes $\lambda w$ and $w$. Then,

$$\frac{dA}{dw}(\lambda w, w) = \lambda A'(\lambda w + v_1) + A'(w + w_1)\,. \tag{4.32}$$

The bound on $A'(x)$ given in Lemma 4.5 tells us that

$$2\lambda + 2\pi\left(\frac{\lambda}{\lambda\frac{w}{2} + \pi \ln\left(\lambda\frac{w}{2}\right) - 3}\right) > \lambda A'\left(\frac{\lambda w}{2}\right), \tag{4.33}$$

$$2 + 2\pi\left(\frac{1}{w + \pi \ln(w) - 3}\right) > A'(w), \text{ and} \tag{4.34}$$

$$2 + 2\pi\left(\frac{\lambda + 1}{(\lambda + 1)w + \pi \ln((\lambda + 1)w) - 3}\right) > A'((\lambda + 1)w)\,. \tag{4.35}$$

By Corollary 4.7,

$$\lambda A'(\lambda w + 3) + A'(w + 3) > \lambda\left(2 + \frac{2\pi}{\lambda w + \pi \ln(\lambda w) + 2}\right) + \left(2 + \frac{2\pi}{w + \pi \ln w + 2}\right). \tag{4.36}$$

Since $\lambda w$ and $w$ are greater than $300\lambda$, we can apply Lemma 4.13 to see that $v_1, w_1 < 3$. Hence,

$$\lambda A'(\lambda w + v_1) + A'(w + w_1) > \lambda A'(\lambda w + 3) + A'(w + 3)\,. \tag{4.37}$$

Therefore,

$$\lambda A'(\lambda w+v_1)+A'(w+w_1) > \lambda\left(2+\frac{2\pi}{\lambda w+\pi\ln\lambda w+2}\right)+\left(2+\frac{2\pi}{w+\pi\ln w+2}\right)\,. \tag{4.38}$$

By inequalities (4.38), (4.33), (4.34), (4.35), and Lemma 4.17, we can see for all $\lambda w, w > 300\lambda$,

$$2\lambda A'(\lambda w + v_1) + 2A'(w + w_1) > \lambda A'\left(\frac{\lambda w}{2}\right) + A'(w) + A'(\lambda w + w)\,. \tag{4.39}$$

Thus, we get the desired result,

$$2\frac{d}{dw}A(\lambda w, w) > \lambda A'\left(\frac{\lambda w}{2}\right) + A'(w) + A'(\lambda w + w)\,. \qquad \square \tag{4.40}$$

**Proposition 4.19.** *For $v \geq \lambda w$, $w \geq 150$,*

$$\frac{1}{\frac{v}{2}+\pi\ln\frac{v}{2}-1.041}+\frac{1}{v+w+\pi\ln(v+w)-1.041} > 2\frac{1}{v\pi\ln v-3}\,. \tag{4.41}$$

***Proof.*** Since $v \geq \lambda w$, $w \geq 150$,

$$\frac{v^2}{2} > 4(v+w)\left(\pi\ln\frac{v}{2}-1.041\right) \text{ and } \frac{v+w}{2} > \pi\ln(v+w). \tag{4.42}$$

Hence,

$$v^2(v+w) > (v+w)^2\frac{\pi\ln\frac{v}{2}-1.041}{\frac{1}{4}}+v^2\pi\ln(v+w)-1.041v^2-(v+w)^2\frac{\pi\ln v-3}{1.1}\,. \tag{4.43}$$

So,

$$\frac{2}{v}-\frac{\pi\ln\frac{v}{2}-1.041}{\left(\frac{v}{2}\right)^2}+\frac{1}{v+w}-\frac{\pi\ln(v+w)-1.041}{(v+w)^2} > 2\left(\frac{1}{v}-\frac{\pi\ln v-3}{1.1v^2}\right)\,. \tag{4.44}$$

Applying Lemmas 4.14 and 4.15 to the above equation yields,

$$\frac{1}{\frac{v}{2}+\pi\ln\frac{v}{2}-1.041}+\frac{1}{v+w+\pi\ln(v+w)-1.041} > 2\frac{1}{v+\pi\ln v-3} \tag{4.45}$$

as desired. $\square$

**Lemma 4.20.** *For any fixed $w \geq 150$ and $v \geq \lambda w$, $\frac{\partial F}{\partial v} > 0$.*

***Proof.*** Differentiating the Hutchings function with respect to $v$ yields

$$\frac{\partial F}{\partial v} = A'\left(\frac{v}{2}\right) + A'(v+w) - 2\frac{\partial}{\partial v}A(v,w) > A'\left(\frac{v}{2}\right) + A'(v+w) - 2A'(v)\,. \tag{4.46}$$

By Lemmas 4.6 and 4.5,

$$\frac{\partial F}{\partial v} > \frac{1}{\frac{v}{2}+\pi\ln\frac{v}{2}-1.041} + \frac{1}{v+w+\pi\ln(v+w)-1.041} - 2\frac{1}{v+\pi\ln v-3}\,. \tag{4.47}$$

The desired inequality follows from Proposition 4.19. □

**Theorem 4.21** (Hutchings function is positive for large volumes). *For all $v \le w$ such that $w > 300$ and $v > .85w$ (or indeed $v \ge \lambda w \approx .841w$), the Hutchings function $F(v,w) = 2A(\frac{v}{2}) + A(w) + A(v+w) - 2A(v,w)$ is positive.*

***Proof.*** Consider $(v,w) = (v_0, w_0)$, where $v_0/w_0 > .85 > \lambda$ and $w_0 > 300$. By Proposition 4.20 the derivative of $F$ with respect to its first argument is positive for the volume pairs under consideration, so $F(v_0, w_0) > F(\lambda w_0, w_0)$.

By Proposition 4.18 $F(\lambda w, w)$ decreases as $w$ increases and by Proposition 4.12,

$$F(\lambda w_0, w_0) > \lim_{w\to\infty} F(\lambda w, w) = 0\,.$$

Hence, $F(v_0, w_0) > 0$, as claimed. □

***Remark 4.22.*** The ratio $\lambda = e^2/4 - 1$ is sharp in the following sense: For $\psi < \lambda$, there exist arbitrarily large $v$, $w$ with $v/w = \psi$ such that the Hutchings function is negative. This is a result of Proposition 4.12.

## 5. The computer proof of the positivity of the Hutchings function in $\mathbf{S}^3$ and $\mathbf{H}^3$

Rigorously showing positivity of the Hutchings function is equivalent to showing that $g(v,w) = 2A(\frac{v}{2}) + A(w) + A(v+w)$ is strictly greater than $h(v,w) = 2A(v,w)$. The functions $g$ and $h$ have very similar properties in $\mathbf{S}^3$ and $\mathbf{H}^3$, but the proofs vary slightly because the areas of single bubbles and standard double bubbles are not increasing for all volumes in $\mathbf{S}^3$ and the $\mathbf{H}^3$ code has to efficiently calculate the curvatures of double bubbles.

The computer proof relies on the facts that $g$ is concave and $h$ is increasing on relevant domains (Propositions 2.6 and 2.7) in both spaces and that $g$ is increasing on all of $\mathbf{H}^3$. If $g$ and $h$ were each concave increasing functions of one variable, we could show that $g$ is bigger than $h$ on the interval $[v, v+b]$, by showing $g(v) > h(v+a)$ and $g(v+a) > h(v+b)$ (see Figure 4). The computational lemma (Lemma 5.1) extends this idea into two dimensions showing that $g > h$ on a polygonal domain in the $vw$-plane if the minimum value of $g$ on the vertex set is bigger than the maximum value of $h$.

Using a computer does introduce error, which must be accounted for. A computer uses approximations for $g$ and $h$, which we will call $g_{\text{comp}}$ and $h_{\text{comp}}$. However, if $g$ is underestimated and $h$ is overestimated then the computer test will show $g_{\text{comp}} > h_{\text{comp}}$ only if $g > h$.

The following lemma shows that checking a finite number of points can give a global inequality.

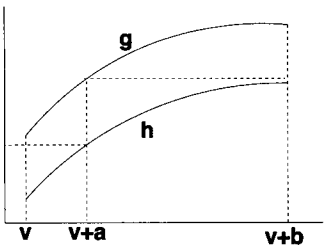



FIGURE 4 The steps to show that $g > h$ if both are concave increasing functions of one variable.

**Lemma 5.1.** *Given a polygonal domain $D$ with vertices identified by ordered pairs $(v_i, w_i)$, a concave function $g(v, w)$, and another function $h(v, w)$ with a maximum at $(v_0, w_0)$, if $\min(g(v_i, w_i)) > h(v_0, w_0)$ then $g > h$ on $D$.*

***Proof.*** Since $g$ is concave it has a minimum on $D$ at a vertex of the polygon. Thus, $\min(g(v_i, w_i))$ is a lower bound for $g$ on $D$ and $h(v_0, w_0)$ is an upper bound for $h$ on $D$. Hence, $g > h$ follows directly. □

**The computer proof in $S^3$.** In $S^3$, rectangular and triangular regions are considered. In $S^3$, there is additional computational complexity because $A(v)$ and $A(v, w)$ are not increasing for all volume pairs $(v, w)$. For rectangles where $h(v, w)$ is strictly increasing in a neighborhood of the rectangle, the area of the standard double bubble is over-approximated by using radii of the outer spherical caps corresponding to volumes $v_4 > v$ and $w_4 > w$ [see Figure 5(a)].

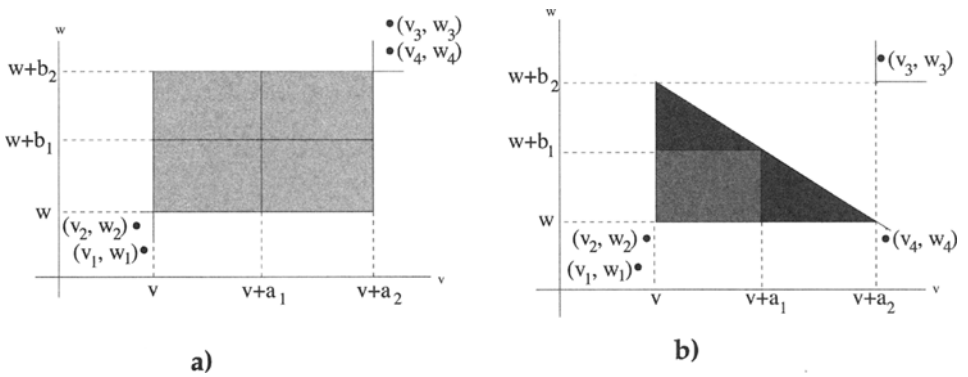



FIGURE 5 The steps needed to show that $g > h$ on (a) a rectangular region (b) a triangular region with A(v,w) increasing along the hypotenuse.

Let $g_1(v, w) = A\left(\frac{v}{2}\right) + A(w)$ and $g_2(v, w) = A(v + w)$. For the area of the single bubble each part of $g(v, w)$ is under-approximated. $A(\frac{v}{2})$ and $A(w)$ are always under-approximated by taking radii that correspond to spheres enclosing volumes less than $\frac{v}{2}$ and $w$. Evaluating $g_1$ at $(v_1, w_1)$ in Figure 5(a) and (b) gives a lower bound on $g_1(v, w)$ on the whole polygon. $A(v + w)$ is under-approximated by considering the minimum of the value of $A$ at a point to the lower left and a point to the upper right of the rectangle. This is the minimum of $g_2(v_2, w_2)$ and $g_2(v_3, w_3)$ in Figure 5(a) and (b).

If $g(v, w) \leq h(v+a_2, w+b_2)$ (where $v+a_2$ and $w+b_2$ correspond to Figure 5(a)), then the inequality is checked on the four rectangles with lower left corners of $(v, w)$, $(v+a_1, w)$, $(v, w+b_1)$ and $(v + a_1, w + b_1)$.

Next, we consider the entire region where numerical analysis is necessary (Figure 6). The

triangular domain is first broken into parts: The right triangle $(\bar{v}, \bar{w}) = (.1, .1), (.1, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3})$ and the triangle $(\bar{v}, \bar{w}) = (.1, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3})$ and $(.1, .45)$. The latter, upper triangle is difficult to deal with because $h$ is increasing up to the hypotenuse and decreasing above the hypotenuse. We can assure that we land exactly on this line by setting the two parameters $r_1$ and $r_3$ to be equal. Getting an over-approximation for area on this line just involves overestimating $r_1$. In addition, $A(\frac{1}{3}, \frac{1}{3})$ must be hard coded into the program as it is the global maximum. If we want to over-approximate $h(v, w)$ we move the point $(v, w)$ southeast along the line where $w = u$ to $(v_4, w_4)$ and use $h(v_4, w_4)$ as an over-approximation [see Figure 5(a)]. For $g$, we still just take the appropriate corners of the rectangle containing the triangle. If not, then we break the triangle into a rectangle and two smaller right triangles by connecting a point on the hypotenuse to the two legs of the triangle via perpendicular lines. The desired inequality is then shown to be true on the rectangle and the two smaller triangles. Similarly for the rectangle, if the desired inequality cannot be shown on the large rectangle, the rectangle is split into four smaller rectangles and the inequality is shown to hold true on them [see Figure 5(a)]. Both subdivisions repeat until the desired inequality can be shown on each subdivided rectangle or triangle. In the case where the upper right corner of a rectangle is on the line where $w = u$, we over-approximate $v, w$ using the same methods as the triangle. The lower triangle can be replaced by the rectangle containing it, which can then be subdivided into smaller rectangles by the argument above.

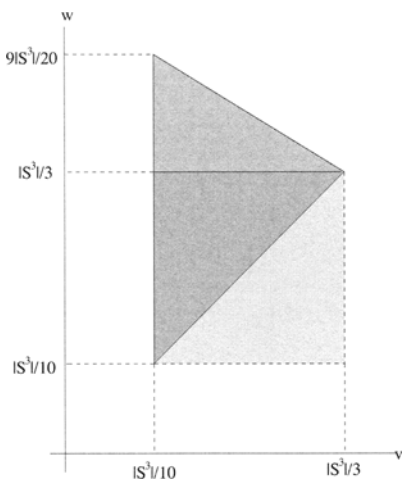



FIGURE 6 The triangular domain of the $(v, w)$ where numerical analysis is implemented.

Mathematica uses computer algebra (with infinite precision variables). When checking inequalities, it reports true, false, or null if it cannot tell for sure [15].

**The computer proof in $\mathbf{H^3}$.** In $\mathbf{H^3}$, rectangular regions that tile the area between two lines through the origin are considered (see Figure 7). The function $g$ is strictly increasing, so $g(v_0, w_0)$ is less than any value of $g(v, w)$ on a rectangular region with $v_0 \leq v$ and $w_0 \leq w$. We can assure that these two conditions hold if we take a guess at the curvature that we know to be too low (corresponding to a volume that is too high). Then, we increase the curvature (decreasing the volume) of the sphere by multiplying it by some constant. This process is done first with a relatively large constant until the curvature corresponds to volume that is too low; then the curvature is divided by the constant and the curvature is increased by a smaller constant until it corresponds to a volume that is less then the volume being approximated. Using this curvature for our surface area calculations under-approximates the surface area of single bubbles.

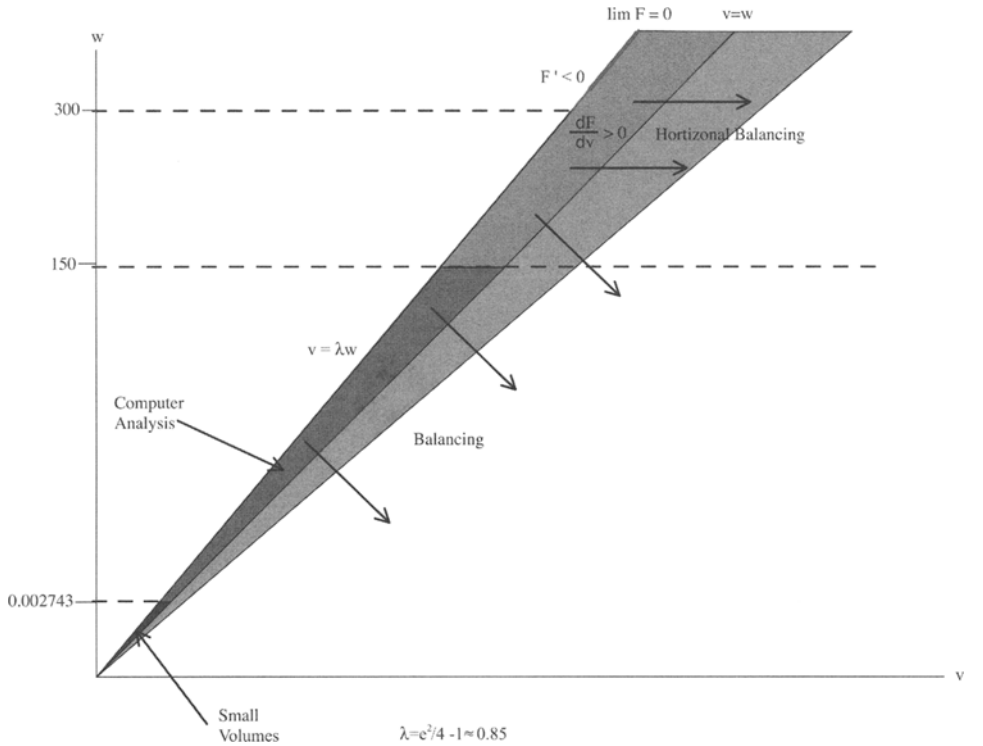



FIGURE 7 The basic steps of the proof that F is positive for $v \geq .85w$ and $w \geq .85v$ in $\mathbf{H}^3$.

For $A(v, w)$, the double bubble is completely determined by the mean curvatures of the outer caps. Thus, under-approximating the curvatures of these caps (by an analogous method to the single bubble case) corresponds to a double bubble with larger volumes and hence more surface area.

We use Mathematica's built-in *SetAccuracy* function, which assures accuracy on 25 binary digits right of the decimal point in our important computations. When we calculate curvature, we check that the calculated associated volume $VolSphere[k]$ is less than $v - 2^{-24}$, which guarantees that the associated volume is less that $v$. Hence, the associated area will be less than A. Finally, we subtract $2^{-24}$ from our calculated area to be sure that we have a lower bound.

We check that the calculated associated volumes $VolBubV[k1, k2]$ and $VolBubW[k1, k2]$ exceed v and w by $2^{-23} = 2 \cdot 2^{-24}$, because sometimes there are two numbers summed in the calculation. Hence, the associated area will be greater than A. We add $3 \cdot 2^{-24}$ to the calculated area to be sure.

When $w \geq 150$, the proof-function is restricted to showing that the Hutchings function is positive only on rectangles that include the line $v = .85w$.

***Remark 5.2.*** In $\mathbf{S}^3$, the proof-function determines the size of each polygon via a recursive algorithm. Both proof-functions (Claims 5.5 and 5.6) complete in under six hours each depending on computer speed. In $\mathbf{H}^3$, the proof-functions take about 150 hours to complete. The function is broken into pieces in order to maximize efficiency by choosing the rectangle size to be relatively large on a given region. Finally, the claim dealing with the ray $v = .85w$ completes in under three hours.

This section discusses the code in [4]. The computer proof in $\mathbf{S}^3$ uses two proof functions. The first is called ProofFunctionTriangle. The other is called ProofFunctionRectangle. The following claims can be proved by examining and running the code in the Appendix of [4]. The proof-functions returns a 1.

**Claim 5.3.** *The computed function A(v_, error_) returns a lower bound on the area of a sphere in* $\mathbf{S}^3$ *with given volume v.*

**Claim 5.4.** *The function A(v_, w_, VError_, WError_, changeToV_, changeToW_) gives an upper bound on A(v,w).*

**Claim 5.5.** *If ProofFunctionRectangle returns a 1 on a region, then the Hutchings Function is positive on this region.*

**Claim 5.6.** *If ProofFunctionTriangle returns a 1 on a region, then the Hutchings Function is positive on this region.*

**Implementation in** $\mathbf{H}^3$**.** The following claim can be proved by examining and running the code in the Appendix. The program stops if it finds a rectangle where $g \leq h$. Therefore, if the proof-function completes, the Hutchings function is positive on the region it was called on (Claim 5.7).

**Claim 5.7.** *If ArrayFillingProof completes then the Hutchings function is positive on the region it was called on.*

The following propositions consolidate the claims that the computer code runs showing the Hutchings function is positive on the desired domains. In order to decrease the number of computations needed, [4] uses a modified version of the code that checks only that the Hutchings function is positive on a small domain that includes the line $v = .85w$.

**Proposition 5.8.** *If an area-minimizing double bubble in* $\mathbf{S}^3$ *encloses volumes* $.1 \leq \bar{v} \leq \min\{\bar{w}, 1 - 2\bar{w}\}$, *then the Hutchings function* $F_{\mathbf{S}^3}(v, w) > 0$.

***Proof.*** This result can be verified by implementing ProofFunctionRectangle and ProofFunctionTriangle in the $\mathbf{S}^3$ section of [4]. □

**Proposition 5.9.** *If an area-minimizing double bubble in* $\mathbf{H}^3$ *encloses volumes* $v$, $w$ *such that* $.002743 \leq v \leq w \leq 150$ *and* $w \geq .85v$ *or such that* $w = .85v$ *and* $150 \leq w \leq 300$, *then the Hutchings function* $F_{\mathbf{H}^3}(v, w) > 0$.

***Proof.*** This result can be verified by implementing the ArrayFillingProof functions at the end of [4]. In the case where $w = .85v$, ArrayFillingProof is slightly modified so that the positivity of the Hutchings function is checked only on a neighborhood of $w = .85v$. □

## 6. Double bubbles in $\mathbf{S}^3$ and $\mathbf{H}^3$

Theorems 6.3 and 6.4 provide our main results on double bubbles in $\mathbf{S}^3$ and $\mathbf{H}^3$. They depend on Propositions 6.1 and 6.2 of Cotton and Freeman [7], after Hutchings et al., which reduce the proofs to our connectivity results of Sections 3–5 via the Hutchings Function.

**Proposition 6.1** ([7, Proposition 7.3]). *An area-minimizing double bubble in* $\mathbf{S}^3$ *for which both enclosed regions and the exterior are connected must be standard.*

**Proposition 6.2** ([7, Proposition 7.7]). *An area-minimizing double bubble in* $\mathbf{H}^3$ *for which both enclosed regions are connected must be standard.*

**Theorem 6.3** (Area-minimizing double bubbles in $\mathbf{S}^3$). *An area-minimizing double bubble in* $\mathbf{S}^3$ *is standard if the volume of each of the three regions is larger than 10% of the total volume of* $\mathbf{S}^3$.

***Proof.*** By Proposition 6.1, it suffices to show that all regions are connected. By Lemma 2.9, we know that the region of volume $v$ is connected whenever $v > 2w$ or $v > 2u$ [Figure 8(a)]. This reduces the problem to showing that the region of volume $v$ is connected on the pentagonal domain of Figure 8(e) defined by the points: $(\bar{v}, \bar{w}, \bar{u}) = (.1, .1, .8), (.1, .8, .1), (.2, .7, .1), (.5, .25, .25)$, and $(.2, .1, .7)$. By Proposition 2.3, it suffices to show that the Hutchings function $F(v, w)$ is positive on this domain. When $.1 \leq \bar{v} \leq \bar{w}$ and $\bar{v} \leq 1 - 2\bar{w}$, computer analysis [Proposition 5.8, see Figure 8(b)] tells us that the Hutchings function $F(v, w)$ is positive. In particular, for $\bar{v} \leq \frac{1}{3}$, $F(v, v)$ is positive. When $\frac{1}{3} \geq \bar{v} > \bar{w}$ and $v \leq 2w$, by S-balancing [Proposition 2.8, see Figure 8(c)] $F(v, w) > 0$ on the quadrilateral domain with endpoints: $(\bar{v}, \bar{w}, \bar{u}) = (.1, .1, .8), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{4}{9}, \frac{2}{9}, \frac{1}{3})$, and $(.2, .1, .7)$. Thus $F(v, w) > 0$ on the pentagonal domain gained by S-balancing in addition to the region where the computer analysis was done. Hence, by permutation [Proposition 2.10, see Figure 8(d)], $F(v, w) > 0$ on the pentagonal domain with endpoints: $(\bar{v}, \bar{w}, \bar{u}) = (.2, .7, .1), (\frac{4}{9}, \frac{1}{3}, \frac{2}{9}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (.1, .45, .45)$, and $(.1, .8, .1)$. In particular, if $\bar{v} \leq .45$, $F(v, v) > 0$. By S-balancing [Proposition 2.8, see Figure 8(e)] $F(v, w) > 0$ whenever $\bar{w} \leq \bar{v} \leq 2\bar{w}$. Thus, $F(v, w) > 0$ on the pentagonal domain $(\bar{v}, \bar{w}, \bar{u}) = (.1, .1, .8), (.1, .8, .1), (.2, .7, .1), (.5, .25, .25)$, and $(.2, .1, .7)$. □

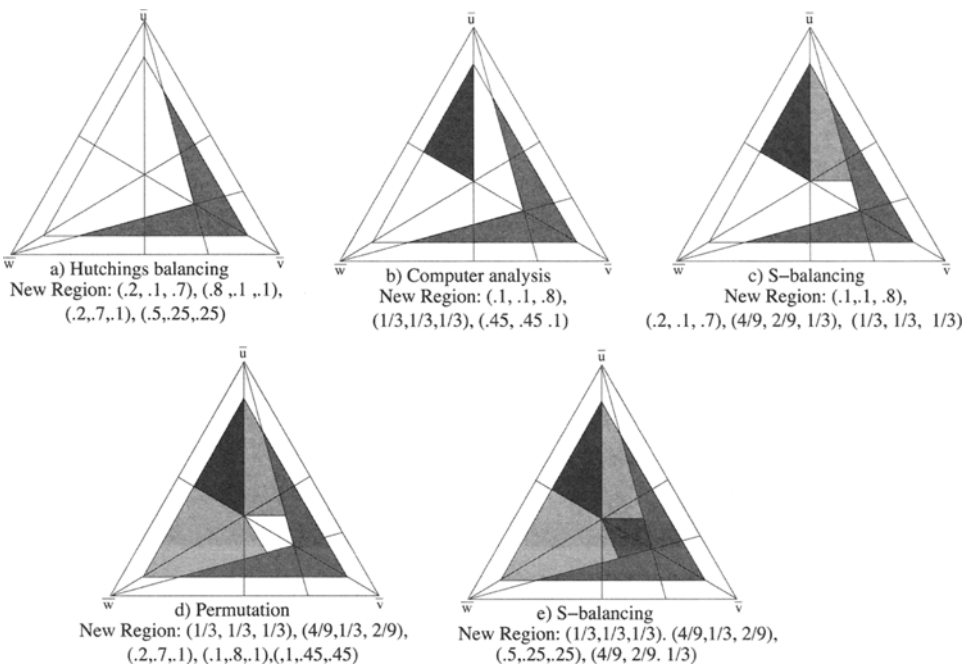



FIGURE 8 An illustration of the steps taken to cover the whole domain of volumes on $\mathbf{S}^3$.

**Theorem 6.4** (Area-minimizing double bubbles in $\mathbf{H}^3$). *An area-minimizing double bubble in* $\mathbf{H}^3$ *is standard if the volume of the smaller region is at least* 85% *of the volume of the larger region.*

***Proof.*** By Proposition 6.2 it suffices to show that the two enclosed regions are connected. By Proposition 2.3, it suffices to show that the Hutchings function $F(v, w)$ is positive for $(.85)^{-1}w \geq v \geq .85w$ (see Figure 7). Computer analysis (Proposition 5.9) shows that when $.002743 \leq v \leq w \leq 150$ and $v \geq .85w$, the Hutchings function $F(v, w)$ is positive. It also shows that the Hutchings function is positive along the line segment $v = .85w$, $w \in [150, 300]$. Since the partial derivative of the Hutchings function with respect to v is positive for $w \geq 150$ (Lemma 4.20), $F(v, w) > 0$ for $w \in [150, 300]$. Asymptotic analysis (Theorems 3.5 and 4.21) shows that $F(v, w) > 0$ when $v \leq w < .002743$ or $v/.85 \geq w \geq 300$. In particular, $F(v, v) > 0$ is positive. When $(.85)^{-1}w \geq v \geq w$, Proposition 2.8 shows that $F(v, w) > 0$. □

***Remark 6.5.*** In hyperbolic quotient spaces, Theorem 6.4 and [1, Remark 4.4] prove that the standard double bubble is area minimizing for some small volume pairs where the smaller volume is at least 85 percent that of the larger.

**Future work.** It follows from Theorem 6.4 and balancing (Lemma 2.9, Proposition 2.8) that the largest region in $\mathbf{H}^3$ is always connected. Connectivity of the largest region is the key hypothesis for the even more general instability argument used in $\mathbf{R}^4$ [13, Section 8]; future work might produce an analog of this result for $\mathbf{H}^3$.

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# A Scholia-based Document Model for Commons-based Peer Production

Joseph Corneli and Aaron Krowne

**Abstract:** Commons-based peer production is a term that describes authorship of shared information resources. In this article we examine the technical aspects of writing-in-common. We begin with a simple model: that of text and commentary. This scholia-based model emphasizes *ownership of speech* and *freedom of speech*. We then consider what happens when the *freedom to create derivative versions* is added to the mix. The resulting model proves to be quite sophisticated, and flexible enough to describe many different commons-based peer production systems. We provide an overview of our implementation of this model, and suggest some ideas for subsequent work. We conclude by discussing the implications of our model for distributed authorship and writing.

## INTRODUCTION

The simplest model of a document is a list of characters; a string, file, or buffer. This model is not diachronic; a diachronic model keeps track of editing operations, or summarizes them as a sequence of differences between document versions.

Many texts have features which can not be modeled adequately without still more information: markup or metadata. Compilations, hypertext, and collaboratively written documents are examples. In this essay, we advance the idea that it can be useful to treat such documents and document ensembles as collections of *scholia*.

### The Digital Library and the Document

For our purposes, the document and the library are essentially the same. In other words, the traditional library-document dichotomy can be viewed as a smooth spectrum, which we consider as a whole. Towards one end of the spectrum, the number of authors decreases and the topics under discussion become more integrated, and the information artifacts look more document-like. Towards the other end, the number of authors grows and the semantic gaps

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between topics increase, and the information artifacts become more library-like. To be clear, when we wish to refer to elements of a given information artifact in a generic way, we will call them *articles*. We will use the term "scholium" to describe an article that is about another article. While a given article need not be about another article in general, our view is that it is about *something*, or perhaps about many things. An article that does not express these relationships explicitly is a degenerate scholium.

**Scholia**

According to Webster,[1] a *scholium* (pluralized *scholia*) is,

1. A marginal annotation; an explanatory remark or comment; specifically, an explanatory comment on the text of a classic author by an early grammarian;

2. A remark or observation subjoined to a demonstration or a train of reasoning.

The Talmud is an excellent example of a document that is comprised of scholia. This document is a collection of interconnected commentaries and reflections on Jewish law (the *Torah*), composed by generations of Jewish religious scholars. There is an obvious and oftnoted comparison between the internet and the Talmud: we readily see ancient scholia-based documents as primitive hypertext. However, a deeper comparison is not simply technological but psycho-social.[2] There are similarities between our experiences of "culturally comprehensive" documents, whether secular or religious in nature. Of course, if you subscribe to the McLuhan view of media ("the medium is the message"), you would expect to see similarities (Federman 2004; McLuhan 1964).

Scholia also appear in the writings of mathematical scientists (*e.g.* Euclid, Galileo, and Newton) or philosophers writing in a similar style (*e.g.* Spinoza). Today, mathematicians typically call peripheral observations (frequently, of secondary importance) "corollaries" or "remarks." Mathematicians also mark up parts of their texts as "axioms," "definitions," "propositions," "theorems," "lemmas," or "proofs." Markup, in general, can be thought of as scholia-based, *i.e.*, as commentary that instructs the typesetter or the reader to treat a given piece of content in a certain way.

## Free culture and the scholia-based document

This paper, and the system it describes, are a response to certain concerns that face us when we work with and think about shared information resources. These concerns have to do with the issues of *intersubjectivity*, *ownership*, and *freedom*. Content in a shared document is typically of non-trivial intersubjective importance. However, it is more than likely that different parties will have some different ideas about the information that is presented in a given text (it is no coincidence that the root of *intersubjective* is *subjective*)!

Free content (*i.e.* free as in freedom, or *libre* content) is one way to nurture difference. Free content can be modified and redistributed (and, in particular, *forked*) without permission or apology. Nevertheless, free content typically manifests aspects of a common resource as well as an open access resource; while anyone can do essentially whatever they wish with the content offline, in its online life, the content is managed in a socially-mediated way. In particular, rights to *in situ* modification tend to be strictly controlled. The details differ from ownership model to ownership model (*e.g.*, PlanetMath has article owners and access control lists, software projects like Emacs have a limited number of developers with commit privileges, members of the Wikipedia community enforce rules about what kinds of content are allowed, and so forth). In these instances and generally, unwillingness to cooperate comes at a cost of valuable support from the community, which must be balanced against the limitations the community imposes on the individual.

By finding new ways to support freedom of speech within CBPP documents, we embrace subjectivity as a way to enhance the content of an intersubjectively valued corpus. In the context of "hackable" media and maintenance protocols, the semantics with which scholia are handled can be improved upon indefinitely on a user-by-user basis and a resource-wide basis. This is free culture in action.

## Hyperreal Texts

Our interest in scholia-based documents largely derives from an interest in helping assemble a human-friendly, AI-enriched mathematics learning and communication interface. The immediate goal is a system in which text and code are both well-supported, as two sides of the same coin. Qualitatively, we want want to be able to represent and work with complex ontological

relationships between entities that are encoded in the system. The system should be useful for both humans and programs—balance along the human/artificial axis is as important toward this end as balance along the freedom/ownership axis or the intersubjective/subjective axis. In short, we find it natural to use a scholia-based platform as the foundation of our *Hyperreal Dictionary of Mathematics* project.[3] Applications in other areas of intellectual inquiry (including digital libraries), as well as in business, government, and day-to-day life also seem to hold great promise.

## Inspiration

We have already seen glimpses of the ideology informing this paper (in particular, we've touched on free software, commons-based peer production, the Talmud, and mathematics). We'll come to more of this later on. Here, we would like to describe certain key technological inspirations. These should serve to illustrate the naturality of a scholiabased document model, as well as to contextualize our implementation of the model. Usenet, Slashdot, the World Wide Web and WikiWikiWeb are all inspirations. In the first two, the most obvious type of scholium is the *followup*. In the latter two, it is the *link*.

PlanetMath[4] uses several types of scholia. Discussion fora are attached to pretty much every "substantial" object in the system; there are auto-generated invocation links between articles; and attachment relationships can be asserted to apply to between objects. Objects can themselves be distinguished as being one of several different types and belonging to one of a number of different subject categories. Also, and most importantly for the current discussion, PlanetMath employs an explicit article ownership model.

Emacs text properties provide a facility for unlimited markup of strings, buffers, and files.[5] However, text properties are hierarchical (treelike), whereas we are looking for something web-like. Of course, locally, text properties are great (and, indeed, we use them in our implementation).

The semantic web project is exploring ways in which to make metadata about web pages available to computers with reasoning capabilities.[6] Our scholia-based documents are small "semantic webs" with a particular flavor, though it may be more appropriate to use the older and more generic term "semantic network" (Quillian 1968) to describe them. As a data structure, a scholia-

based document is a network with arbitrary text (and metadata) on its nodes and arbitrary annotations on the connections between nodes. Graphically, this closely resembles the notion of a *concept map* (Novak 1998), albeit with significant text content attached to each node.

Semantic networks and concept maps are just two of the many different strategies for knowledge representation. Cyc[7] and KM[8] are two other systems we've studied. Our goal is a system with similar expressive power, but which is more human-friendly (with *reasoning* capabilities to come eventually).

The first author's earlier hypertext experiments were heavily inspired by LISP, and the results ended up resembling Gopher (Karger 2000). The current system is meant to be much richer, but it is still LISP-inspired, particularly in terms of support for self-modification. Indeed, one of the key inspirations for the current system are the text-based "game" environments (MUDs,[9] etc.), in which people are able to "hack" the text-based virtual worlds that they are interacting in and with, in real time.[10] Scholia can include actionable features, and with appropriate support, contributing authors can modify the medium in which the scholia reside, not just its content.

**Survey**

The scholium-based document is not a new idea. Here, we are referring not to classical works, but to Ted Nelson's project and document model entitled *Xanadu*, which is described in his book *Literary Machines* (Nelson 1990), originally published in 1982. His development strategy was quite different from the one taken here, however—in particular, while he and his cohorts were focused on developing powerful server-side technology, we focus on implementing user-ware with a simple p2p extension. Importantly, while at least some of Nelson's ideas have been implemented and released under a free software license, the code didn't compile for us.[11] A biography of sorts together with a history of the Xanadu project has been published in *Wired*.[12] Nelson is admired by FSF General Counsel, Eben Moglen, for "identifying the predicament of information ownership in the digital age" (Williams 2002).

One rather important difference between Nelson's vision (as expressed in *Literary Machines*) and the present one is that he deprioritizes artificial intelligence in favor of human intelligence. As remarked above, actionable features and artificial intelligence

occupy an important place in the scholium system. Our situation with regard to artificial intelligence is similar to the one described by Minsky in *The Society of Mind* (Minsky 1985). The connection between ideas and agents is not so dissimilar; only the translation to actionable form is missing to make an idea into an agent. Meaning clusters translate to complex agencies. As time goes by, we expect to find actionable and non-actionable features paralleling one another along some dimensions, intertwining along some, and diverging on others.

"Superimposed information" is a subject of current research in the field of digital libraries.[13] Note that while this model is *locally* similar to the scholia-based document model, this branch of research focuses on one superimposed layer. This makes a certain amount of sense for traditional libraries, which hold a specific, administratively-controlled collection of information. The superimposed information model emphasizes making this primary artifact more useful via value-added "attachments" (annotations and so forth). However, for us, neither library nor document is static, and annotations are an integral part of both. Thus, scholia-based documents are as much "community" as they are "collection."

We find it compelling that, in the context of a digital library, marginal conversations within a text provide a chance for readers to interact with primary authors and with each other, and to become primary authors themselves, all at once. While marginalia are considered to be *vandalism* in physical library books, in a digital library, there is no reason to fear them—they can easily be hidden away. The scholia-based document model reflects the standard postmodern pun which says that writing on a text or subject (*i.e.* criticism, discourse) is writing *on* the text (making ones mark). We expect meaning to accumulate in the "margins" of texts, and for meaning complexes to grow by stitching documents together along their margins.

## IMPLEMENTATION OVERVIEW

In this section we give a tour of our implementation of the scholium system. The critical elements of the system are described in general terms here. Details, including code (and literate markup of the code) may be found at (Corneli 2004).

## Articles

The foundation of the scholium system is a catalog of articles (list or hash table). Adding or updating the reference for an article to the catalog is the fundamental operation in the system. This operation stores

- the article's name;
- its text or a pointer to its text;
- a record of what it is about—nothing if it is degenerate, otherwise, some article(s) or passage(s) of articles in the collection;
- the designation of a type (link, followup, forum, action to take if a certain event happens at the parent, etc.);
- and bookkeeping information to keep track of ownership information and editing history.

In Figure 1, we illustrate the conceptual model of the scholium system with a hypothetical content instance. Shown are the key entities of *people*, *articles*, and *references*.

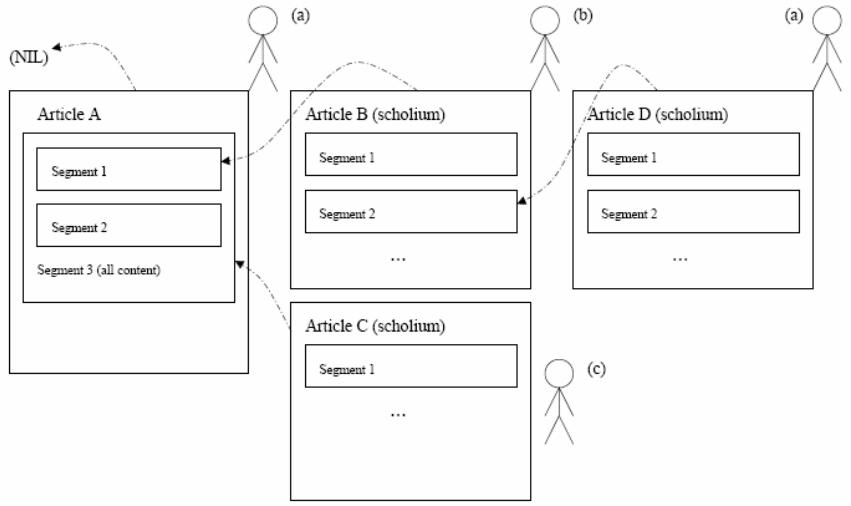



Figure 1: The key elements of the scholium system, for a hypothetical small document/library. There are three people (a, b, and c), and four articles (A, B, C, and D). All articles are scholia (i.e. refer to another article), except A. The contents of each article have been broken into "segments" for clarity. Links are shown as dashed arrows from articles to article segments.

## Environment

The current prototype system runs under GNU Emacs. We try to conform to the Emacs Way to the greatest extent possible. Typically, scholia are written about files and buffers, and are displayed alongside them. The default display uses color to associate displayed scholia with regions that are being commented on, but delimiters can be used for this purpose as well (for the benefit of those working without fontlock). Color can also be used to distinguish between different types of scholia (*e.g.* comments by different authors).

Our ability to render articles is limited to things Emacs can display: plain text, code, and pictures all work, but special proprietary formats are not supported in this implementation, nor do we have support for making scholia about specific pieces of a rendered diagram, for example.

The user can navigate the display in various ways, e.g., by scrolling between marked-up regions of the main article, or finding the region(s) associated with a given scholium, or the scholia associated with a given region. Additional browsing methods are described below.

## Displaying Scholia

When an article is displayed, the system finds attached scholia and displays them too. Selective displays are possible (*e.g.* just show all the *links* attached to a given article). Finding scholia in the catalog requires search; sometimes we can limit the search to make it faster (this will be described further below). In Figure 2, we show a number of ways in which the underlying content from Figure 1 can be displayed.

## Adding to the library

We support various convenient ways to add articles. Different kinds of articles need different treatment. For example, the content of a buffer is lost if the buffer is killed, so a backup of the buffer's contents should be made immediately. In addition to specifying the article's text and stating what (if anything) it is about, one can specify the article's type. We provide built-in functions for creating scholia about the current buffer, and for creating scholia about mixed collections of other articles and passages.

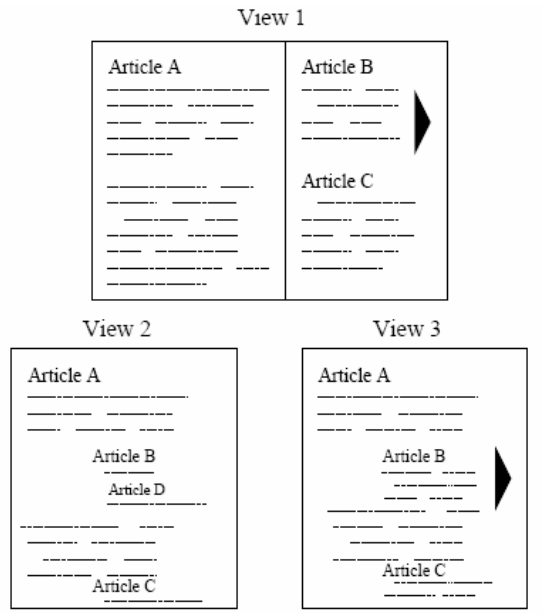



Figure 2: Different possible views of the articles from Figure 1. Three views are shown. View 1 is a two-pane style, with attached scholia on the right and the main article on the left. View 2 includes only title and linkage information for attached scholia, unfolding recursively to a user-specified depth. View 3 is a single-frame view, with one level of attached scholia inclusion (deeper levels are indicated with an arrow).

## Browsing

We provide several different browsing mechanisms. Simple local navigation features (scrolling etc.) were mentioned above. “Following” a scholium to make it into the new current article is supported. Browsing by catalog (*i.e.*, by list) is supported; we use a generic menu mechanism that make it easy to select different collections of articles matching different criteria, and perform various actions on them. Finally, web-browser-like history-based navigation is provided.

## Editing, deleting, saving, restoring

Articles are versioned and entire document versions or, alternatively, a collection of patches sufficient to move between versions, is maintained for each article. When articles change, it is typically necessary to propagate data to attached scholia in order to maintain coherence of references. Propagated changes need not be accepted, in which case the attachment relationship is fixed at the previous version.

Similarly, permission is typically requested whenever a portion of an article that has been *transcluded* changes. ("Transclusion" is Nelson's term for "included by reference"; both transclusion and inclusion of pieces of other articles are supported.)

Changes that take place to articles outside of the system (*e.g.* moving a file from the shell) are of course nearly impossible to deal with. However, scholia can be saved in explicit, invariant forms. (CVS is natively supported).[14] Saved articles can be read back in selectively.

Currently, name conflicts pose a bit of a problem; if every scholium has a unique name, then the problem goes away. In general, we can approximate a solution by asking the user to uniquify names when conflicts are encountered.

### Namespaces

Another solution to the problem of name conflicts is provided by namespaces. For example, if article *A* has type "sublibrary" and we read in a scholium of type "member of sublibrary *A*," then we don't have to worry about name conflicts with articles not in sublibrary *A*. Namespaces can be used to make search convenient. For example, we can store all assertions of type *Z* in a given namespace; then if we are building a display that only relates to objects of type *Z*. In other words, we need only search one namespace instead of the whole library.

## DISTRIBUTED AUTHORSHIP

Distributed authorship is actually very easy using this system. Each contributing author posts her or his contributions to the document at their own chosen location. Then the other authors download the articles stored in all of the locations they want to use. (All authors don't necessarily need to have all of the articles.) In the case of document revision, changed versions are simply posted, and other authors learn of the changes whenever they sync.

### Derivative versions

In general, the author may be the only person with permission to create modified versions, but we can also distribute this permission over a wider group of people. As we've already mentioned, derivative versions can be put together using two principles, inclusion and transclusion. Inclusion tends to reduce search but increase storage.

Here we illustrate *tracking* of derivative versions with a scenario. Suppose that a series of definitions was quoted in a textbook-style entry. The author of the textbook might receive a question from a reader and then adjust one of these definitions to include more expository text. If the original author was tracking derivative versions, the new expository text could be added as a scholium attached directly to the original, or the original could be modified.

These sorts of exchanges should still be possible even if one of the agents is working outside of the scholia-based system. In order to make *bi-directional updating* work in this case, both content-sharing parties need to be able to read a stream of diffs generated by edits taking place in another system and decide how to incorporate the modifications.

These issues should be of interest to anyone maintaining a collaborative digital library; information-sharing between such entities typically needs to support content that can change on both ends of the pipeline.

## FUTURE WORK

Some ideas for explorations to undertake with the scholium system and related concepts follow. Note that some of these could be considered feasibility or proof-of-utility experiments:

- Import a wiki and build a wiki-like interface to the scholium system;
- Use the scholium system to write a synchronous or asynchronous multiplayer game;
- Use the scholium system to maintain a text-based forum or set of fora, as found on PlanetMath or Slashdot;
- Implement a Slashdot-like scoring system for quality control;
- Use the scholium system to manage an evolving codebase (*i.e.*, take advantage of the functionality which subsumes a system like CVS);
- Use the scholium system to manage TODO lists;
- Port the GNU Collaborative Dictionary of English (GCIDE) to Emacs and give instant access to definitions as scholia;
- Implement code to make an index or do autolinking as you type;
- Port WordNet15 to the scholium system and use the system to collaboratively improve the database;

- Implement semi-automated content sharing between two collaborative digital libraries, for instance, PlanetMath, and Wikipedia.

This list should give some idea of the range of capabilities the scholium system, in theory, encompasses. Long term investigations under conditions of fairly wide uptake would shed light on broader social implications of the system. We imagine that social institutions—from peer review to popular science, and from online shopping to participatory government—would tend to be transformed by widespread use of these systems. In fact, some of this social transformation is already apparent—in specialized implementations of the scholia concept (*e.g.* weblogs, wikis, online forums, PlanetMath). Currently we can only speculate as to how the general system outlined here would interact with these trends.

One thing we can say at this point is that the model seems to provide a useful basis from which to explore the design and implications of social contracts in online communities. We hope future work will take up this issue.

## CONCLUSION

In this article, we have described a scholia-based document model and outlined an implementation of a system that supports this model. We have discussed some ramifications of scholia-based documents and libraries. We have have also shown how the model can be used to facilitate powerful collaboration dynamics in a wide array of scenarios and social settings—by fostering and managing alternative perspectives, encouraging responsible maintainership, and enabling readers to routinely make useful contributions.

The scholia-based document model corresponds to a culture with empowering conceptualizations of freedom and ownership. Accordingly, this paper has been a description of a model as well as something of a social manifesto. We hope to see the ideas presented here take off, as we and others work to push the limits of the model.

### Acknowledgments

Thanks to Ray Puzio for helpful comments and encouragement as the scholium system developed. Thanks also to Thien-Thi Nguyen and Sacha Chua for their comments on free software and hypertext, including comments on an earlier prototype of the system by the first author.

## ENDNOTES

1. http://www.ibiblio.org/webster/.
2. See “The Talmud and the Internet” (Rosen, 2000). You can also find the Talmud on the internet, at http://www.sacred-texts.com/jud/talmud.htm.
3. See http://planetx.cc.vt.edu/AsteroidMeta/HDM.
4. http://planetmath.org/.
5. (info "elisp(Text Properties)")
6. http://www.w3.org/DesignIssues/Semantic.html.
7. http://www.cyc.com/cyc/technology/whatiscyc.
8. http://www.cs.utexas.edu/users/mfkb/RKF/km.html.
9. Marshal McLuhan’s theory ties in very nicely with MUD-like systems. Those with a user-hackable infrastructure embody a distinctly free “message.” For cultural materialists, the upshot is hackable superstructure. The fact that the MUD is virtual to begin with adds an interesting twist in this analysis.
10. It is interesting to compare the experience of these immersive worlds to the experience of internet or Talmudic scholars, mentioned previously. See (Rheingold 1993).
11. Xanadu development seems to be going slowly at present; see http://udanax.com/ and the mailing list at http://xanadu.com.au/mail/ .
12. See http://www.wired.com/wired/archive/3.06/xanadu_pr.html, but note that this work is not endorsed by Nelson, who writes at http://ted.hyperland.com/whatsay/ “I believe the piece is a study in cunning and deliberate dishonesty, the most dastardly piece of dirty journalism I have ever seen.”
13. For example, see (Maier and Delcambre 1999), http://nsdl.org/community/projects.php?this_sort=start_date&keyword=&project_id=0435496, or http://datalab.cs.pdx.edu/sparce/.
14. Code Versioning System, an extremely popular free software program to collaborative manage software codebases. See http://www.gnu.org/software/cvs/.
15. WordNet is a “lexical database for the English language”—essentially a semantic network of words and relationships, which can be used as a dictionary or thesaurus. See http://wordnet.princeton.edu/.

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# THE DOUBLE BUBBLE PROBLEM ON THE FLAT TWO-TORUS

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Abstract. We characterize the perimeter-minimizing double bubbles on all flat two-tori and, as corollaries, on the flat infinite cylinder and the flat infinite strip with free boundary. Specifically, we show that there are five distinct types of minimizers on flat two-tori, depending on the areas to be enclosed.

## 1. Introduction

Our Main Theorem 6.1 shows that on any flat two-torus, the least-perimeter way to enclose and separate two prescribed areas is a double bubble of one of the five types shown in Figure 1: the standard double bubble, the band lens, a standard chain, the double band, or the standard hexagon tiling. Figure 2 gives "phase portraits," computational plots of the minimizing type for given areas, for four different tori. (Section 8 describes the creation of these phase portraits, using formulas for the perimeter and area of the five minimizers derived in Section 7.)

The strategy of the proof is to separate all candidates into five classes, as described in Proposition 4.6. Proposition 4.6 says that a minimizer must either be the double band or (possibly after relabelling the two interior regions and the exterior) belong to one of four other topological classes: Section 5.1 deals with contractible double bubbles; Section 5.2 deals with double bubbles whose components taken together wrap around the torus, and whose complement is not contractible; Section 5.3 deals with double bubbles with one band adjacent to a contractible set of components; and Section 5.4 deals with double bubbles for which both enclosed regions and the exterior region are contractible.

The last of these topological classes (with both regions and the exterior contractible) we refer to as "tilings" because the three regions taken together lift to a tiling of the plane. Despite escaping our initial conjectures, this class turned out to be one of the most interesting. Whereas the nontiling types tie only for prescribed areas in the transitions between phases (Figure 2), the hexagon tiling and the double band provide distinct minimizers over an open set of areas (Figure 2(b)) — the first example of such a phenomenon in double bubble history. Another candidate tiling, the octagon square, with one region consisting of two four-sided components (Figure 23), was the last candidate to be eliminated (Proposition 5.4.22).

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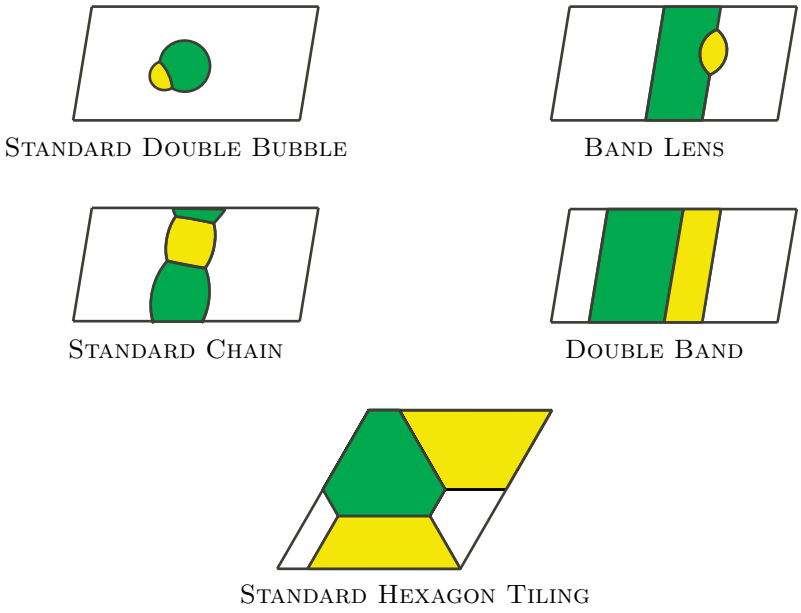



FIGURE 1. There are five types of perimeter-minimizing double bubbles on a flat two-torus. The standard hexagon tiling occurs only on the hexagonal torus.

We also consider the double bubble problem on the flat infinite cylinder, which is simpler because there are no candidate tilings. Corollary 6.3 provides a complete characterization of the minimizers in this space as a trivial consequence of the torus result, and the cylinder result is used in Corollary 6.5 to characterize all minimizers on the flat infinite strip with free boundary.

The torus and cylinder lack the useful symmetries used in other spaces to establish connectedness of the exterior and bounds on the number of components. Luckily, the recent work of Wichiramala [MW] on the triple bubble problem in the plane gave us a very useful component bound (Proposition 3.7).

**History and recent developments.** The double bubble problem is a generalization of the isoperimetric problem. Given a geometric space, the latter seeks the least-area way to enclose a single volume. (A solution to the isoperimetric problem on the flat two-torus can be found in [HHM].) The double bubble problem seeks the least-area way to enclose and separate two volumes. In recent years, this problem has been solved in a number of spaces, including $\mathbf{R}^2$ [F], $\mathbf{S}^2$ [Ma2], and still more recently $\mathbf{R}^3$ [HMRR] and $\mathbf{R}^4$ [RHLS]. In these cases the minimizers are all standard double bubbles, in contrast to the multiplicity of types on the torus. Some of the major contributions were due to undergraduates affiliated with Williams College and the Williams "SMALL" REU program. Underlying all of these results is geometric measure theory as developed by Federer, Fleming, Almgren, and others. For a more complete discussion of the history of the double bubble problem as well as many related problems on films, foams, and other efficient surfaces, see [M1].

The triple bubble problem in the plane has recently been solved by Wichiramala [W]. On the other hand, even the single bubble problem remains unsolved on a torus

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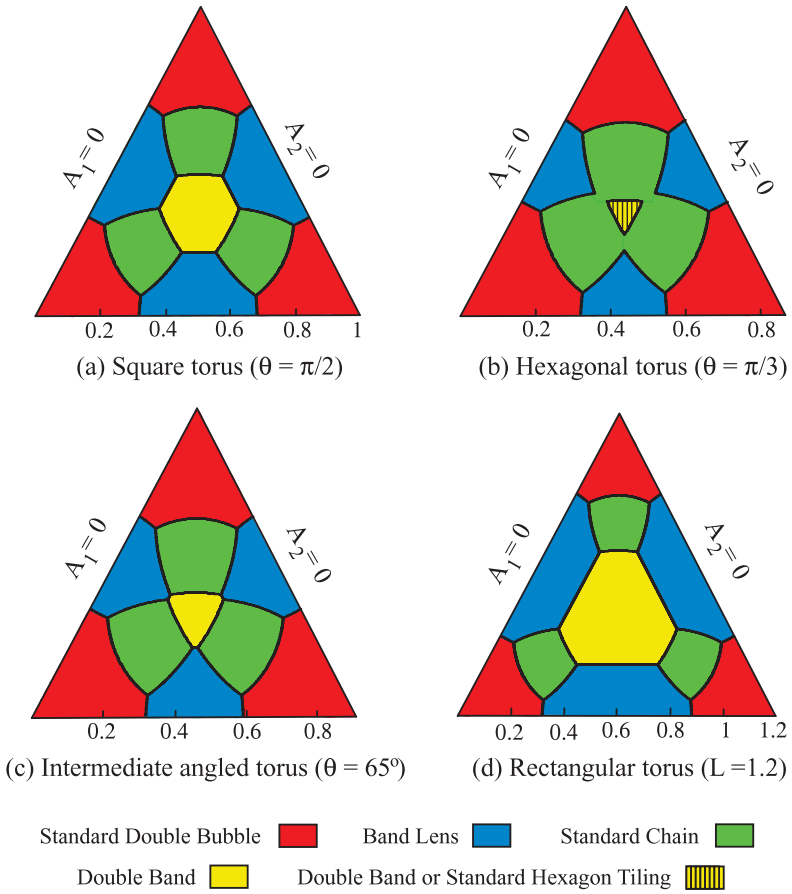



FIGURE 2. These phase portraits show which type of double bubble is minimizing given prescribed area pairs $(A_1, A_2)$ on four specific flat tori. Each torus for (a)–(c) has a rhombic fundamental domain with interior angle $\theta$; the torus for (d) has a rectangular fundamental domain with side lengths 1 and $L = 1.2$.

of revolution. We have contributed to a paper that gives experimental evidence on the solutions to the double bubble problem in a cubic flat three-torus [CCWB], another space where the single bubble problem is still open.

## 2. Definitions

This section contains definitions for, among other things, the classes of potential minimizers mentioned above.

### 2.1. **Tori and the cylinder.**

**Definition 1.** A *flat two-dimensional torus* can be represented by a planar parallelogram with opposite sides identified. Throughout the paper the torus is normalized so that each shortest closed geodesic has unit length.

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*Remark* 2.1.1. The parallelogram can be chosen such that one shortest side is a shortest closed geodesic and no interior angle is less than sixty degrees. Then the area of the parallelogram, and hence the area of the torus, is at least $\sqrt{3}/2$ with equality precisely for the hexagonal torus (see Definition 2).

**Definition 2.** The *hexagonal torus* can be represented either by a regular hexagon with opposite sides identified with the same orientation, or by a 60-degree rhombus with opposite sides identified with the same orientation.

**Definition 3.** We say that each shortest closed geodesic lies along a *short direction* of the torus. On certain nonhexagonal tori (those that can be represented by a parallelogram with four sides of unit length), there are two short directions; on the hexagonal torus, there are three.

**Definition 4.** The *flat infinite cylinder* can be represented as the surface contained between two identified infinite parallel lines in the plane, where each segment perpendicular to and connecting the lines is a closed geodesic. The *flat infinite strip with free boundary* is the surface between the same two lines, except that the lines are not identified; portions of the lines may be included in the boundary of a double bubble without contributing to the perimeter of the double bubble.

2.2. **Topological definitions.**

**Definition 5.** A *double bubble* on a smooth Riemannian surface consists of two disjoint regions (i.e., nonempty open sets) bounded by piecewise smooth curves, such that the exterior of these regions has nonzero area. The boundary of the double bubble divides the surface into three regions: the two regions belonging to the double bubble, which we call the *enclosed* regions, and the complement of the closure of the double bubble, which we call the *exterior* region. Any of the three regions may have multiple components.

*Remark* 2.2.1. On the torus, by relabelling which of the three regions in a double bubble is the exterior, we can obtain two other double bubbles with the same boundary. On the infinite cylinder, the exterior region has infinite area.

**Definition 6.** A *tiling* is a double bubble on the torus for which each component, including each component of the exterior, is contractible, such as the standard hexagon tiling of Figure 1. It is called a tiling because the two enclosed regions plus the exterior region lift to a tiling of the plane.

**Definition 7.** An *octagon-square tiling* is a tiling in which one of the three regions consists of two curvilinear quadrilaterals, and each of the other two regions consists of one curvilinear octagon (see Figure 23).

**Definition 8.** A *band* is a noncontractible annulus.

**Definition 9.** A *swath* is a set of contractible components with noncontractible complement and whose union has noncontractible and connected closure, such as the standard chain of Figure 1.

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**Definition 10.** A *chain*, such as the standard chain of Figure 1 or the chain of Figure 3, is a minimal swath — that is, a swath without a proper subset of components that themselves form a swath. Note that every swath contains at least one chain and that every chain is itself a swath.

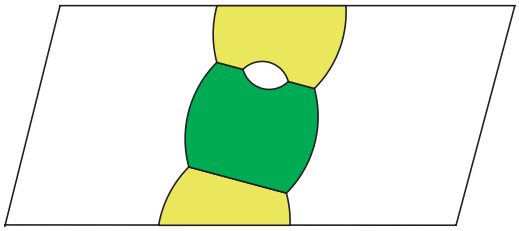


FIGURE 3. The two shaded six-sided components form a *chain*, even though there is a curvilinear digon embedded between them. The two shaded components plus the curvilinear digon do not form a chain, because the swath they form is not minimal.

**Definition 11.** Two closed curves lie in the same *homology class*, or have the same homology, if they can be smoothly deformed into each other along the surface of the torus. Given two different oriented directions of the torus (i.e., directions of closed geodesics on the torus), the homology class of a closed curve is given by an ordered pair $(p, q)$, where the curve wraps around the torus $p$ times in the first direction and $q$ times in the other direction. (The ordered pair is well defined up to negating both $p$ and $q$.) We will refer to the homology of a band or chain $K$, by which we mean the homology of any noncontractible component of the boundary of $\overline{K}$.

2.3. **Geometric definitions.**

**Definition 12.** The *standard double bubble* is comprised of three circular arcs meeting in threes at 120°, such that the curvature of the arc separating the contractible components is the difference of the curvatures of the outer caps. For prescribed areas, there is a standard double bubble in the Euclidean plane, unique up to congruence [M1, Proposition 14.1]; it may or may not fit on the torus.

**Definition 13.** The *double band* consists of two adjacent bands, bounded by three shortest closed geodesics.

**Definition 14.** A *symmetric chain* is a chain that is symmetric about a closed geodesic with the same homology as the chain.

**Definition 15.** A *standard chain* is a symmetric chain enclosing two four-sided components whose boundary is comprised of six circular arcs that satisfy the regularity conditions in Proposition 3.1. That is, the arcs meet in threes at four common vertices at 120° such that the curvature of each arc separating the two components in the chain is the difference of the curvatures of the outer arcs (see Figure 1). The *axis-length* of a standard chain is the length of the geodesic with the same homology as the chain.

*Remark* 2.3.1. Our parameterization of the standard chain (Section 7.2, see Remark 7.2.3) proves existence of some family of standard chains. Furthermore, there exists

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graphical evidence that the only chains that are minimizing are those with the same homology as a shortest closed geodesic (Remark 5.2.16).

We believe the standard chain is also unique up to isometries of the torus for given areas and homology class. Indeed, a parametric plot in *Mathematica* (see Figure 4) suggests that different standard chains with the same homology enclose different area pairs.

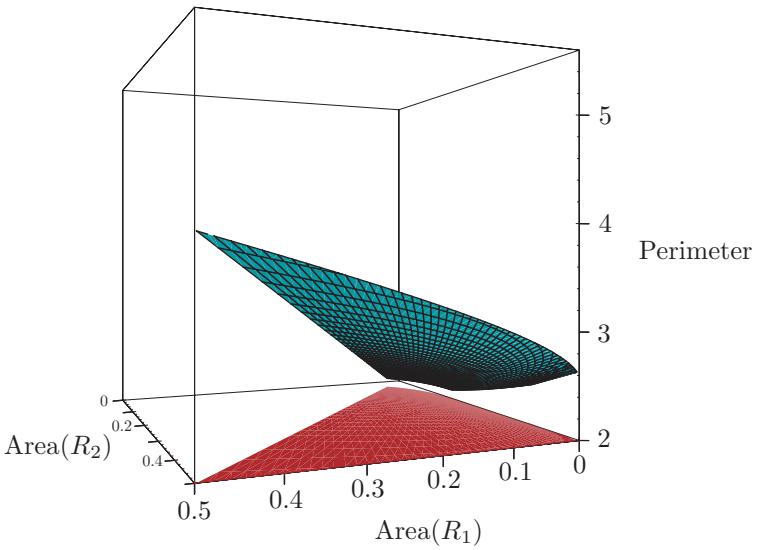



FIGURE 4. This parametric plot from *Mathematica* of perimeter as a function of areas for standard chains with axis-length 1. The plot has no folds in it and contains only one sheet, suggesting uniqueness for prescribed area pairs $A_1 \leq A_2$.

**Definition 16.** A *lens* is a pair of congruent circular arcs meeting at two vertices at 120°.

*Remark* 2.3.2. It is easily shown that if the distance between the two vertices is less than or equal to 1, then the lens can be embedded on any torus with its axis along any short direction.

**Definition 17.** A *band lens* consists of a band and a lens, such that the boundary curves meet in threes at two vertices at 120°, as in Figure 1.

*Remark* 2.3.3. By Lemma 5.3.6, a minimizing band lens on the torus has the same homology as a shortest closed geodesic (that is, the closed geodesic bordering the band has length one). It is easy to see that when a minimizing band lens enclosing two given areas exists, it is essentially unique for those areas (up to isometries of the torus).

**Definition 18.** A *hexagon tiling* is a tiling in which every component of the double bubble and exterior is a hexagon with interior angles of 120°.

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**Definition 19.** The *standard hexagon tiling* is a hexagon tiling on the hexagonal torus, which partitions the torus into three hexagonal regions, whose boundary edges lie parallel to the three short directions of the torus. By Corollary 7.4.3 and Lemma 7.4.4, the standard hexagon tiling exists and is unique for a particular range of area pairs.

## 3. Existence, regularity, and basic properties

This section provides existence and regularity of perimeter-minimizing double bubbles (Proposition 3.1), a bound on the number of components for minimizers (Proposition 3.7), and an easy but important perimeter bound (Proposition 3.8).

**Proposition 3.1** (Existence and Regularity Theorem [M2, 2.3 and 2.4])**.** *In a smooth Riemannian surface $S$ with compact quotient by its isometry group, for any two areas $A_1$ and $A_2$ (whose sum is less than the area of $S$), there exists a least-perimeter enclosure of the two areas. This enclosure consists of finitely many smooth constant-curvature curves meeting in threes at* 120° *angles. No boundary curve separates components of the same region, and all curves separating a specific pair of the three regions have the same (signed) curvature. Moreover, the enclosure satisfies the cocycle condition: the sum of the signed curvatures around any closed path is zero.*

*Remark* 3.2. In particular, given any minimizing double bubble on the torus, each of the three regions contains finitely many components.

*Remark* 3.3. It follows that in a minimizing double bubble on the torus, the union of the closures of any two regions is connected. Otherwise, two disjoint pieces of this union can be translated to be made tangent, forming a new minimizer that violates regularity, a contradiction. In particular, each component (in either of the two enclosed regions or the exterior region) is adjacent to components of both other regions.

*Remark* 3.4. It also follows that alternating sides of a component must border different regions. Hence, any contractible component in a minimizing double bubble or its exterior has an even number of sides.

*Remark* 3.5. A perimeter-minimizing double bubble is in *equilibrium*: under any smooth variation that preserves areas, the derivative of perimeter with respect to time is initially zero. (Here by "variation" we mean a perturbation of the double bubble over time.) For a piecewise smooth double bubble, being in equilibrium is equivalent to having boundary curves that satisfy the given curvature conditions, satisfy the cocycle condition, and meet in threes at 120° (see, for instance, [HMRR], Lemma 3.1).

*Remark* 3.6. The curvature and cocycle conditions imply the existence of a well-defined *pressure* for each region with the following property: the difference between the pressures of any two regions in an equilibrium double bubble is the curvature of the separating arcs between those regions, signed so that a curve separating two regions bows into the region of lower pressure. (For instance, we could set the pressure of the exterior region to equal zero and set the pressure of each enclosed region to equal the curvature of the interface separating that region from the exterior.) A least-pressure nonpolygonal contractible component must have more than six sides;

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and a highest-pressure nonpolygonal contractible component must have fewer than six sides.

**Proposition 3.7** (Wichiramala component bound [MW, Proposition 3.1])**.** *Unless all three regions have equal pressure, each region of highest pressure in a perimeter-minimizing double bubble in the flat two-torus has at most two contractible components.*

*Proof.* The proof in [MW], which we sketch here, uses variations that shrink or expand a high-pressure nonpolygonal component at a constant rate along its boundary. If there are at least three such components in some region, then some nontrivial linear combination of the corresponding "shrink and expand" variations will preserve the areas of each region to first order. Each of the three variations will contribute a negative term to the second variation of the combined variation, so that the new variation has negative second variation (i.e., negative second derivative of perimeter with respect to time). In this case, we call the double bubble "unstable," and no such double bubble is perimeter minimizing. The topology of the torus does not affect this proof. □

**Proposition 3.8** (Perimeter bound)**.** *The perimeter of a perimeter-minimizing double bubble in the flat two-torus is at most* 3*.*

*Proof.* A double band of perimeter three can enclose any pair of prescribed areas, so no minimizer can have greater perimeter. □

## 4. Topological classification of minimizers

This section proves two important propositions. Proposition 4.5 says that if the three regions in a perimeter-minimizing double bubble all have equal pressure, then it is a hexagon tiling or the double band. Proposition 4.6 is the backbone for the proof of our Main Theorem 6.1. It shows that any minimizing double bubble must be a tiling, a contractible double bubble, a swath, a single band adjacent to a contractible set of components, or the double band. Section 5 will treat in turn each of these topological classes.

**Lemma 4.1.** *A noncontractible component $C$ in a minimizing double bubble must either be a band or have contractible complement.*

*Proof.* If every component of the boundary of $C$ is contractible, then the complement of $C$ is contractible.

Otherwise, some component $\sigma$ of the boundary of $C$ is noncontractible, and any other noncontractible component of the boundary of $C$ has the same homology as $\sigma$. Thus, $C$ must be a band with that homology, except possibly with some collection of contractible components in its interior. But such contractible components never exist in a minimizing double bubble (Remark 3.3), so $C$ must indeed be a band. □

**Lemma 4.2.** *Consider a fundamental domain of the torus, a parallelogram with sides* $1, L$ *and interior angle* $\theta \in [\pi/3, \pi/2]$*. Suppose that a chain or band is part of a minimizing double bubble and has homology* $(p, q)$*, defined with respect to two directions along the sides of the parallelogram. Then* $|p|, |q| \leq 1$*.*

*Proof.* Each band or chain has least two disjoint, noncontractible boundary curves. If $|p| \geq 2$ or $|q| \geq 2$, then the length of each of these boundary curves is at least

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$\sqrt{3}$, and the total perimeter is at least $2\sqrt{3}$. By the perimeter bound (Proposition 3.8), the double bubble cannot be a minimizer. □

**Lemma 4.3.** *A perimeter-minimizing double bubble on the torus, together with its exterior, has at most three disjoint bands or chains. If there are three, then the double bubble must be the double band.*

*Proof.* Each band or chain has at least two noncontractible boundary curves, and each such boundary curve borders at most two bands or chains. Hence, if there are $n$ bands and chains, then there will be at least $n$ such noncontractible boundary curves, each with length at least one. Then by Proposition 3.8, $n \leq 3$. If $n = 3$, then the boundary must consist entirely of three parallel closed geodesics, each of length one. In this case, the double bubble must be the double band. □

**Lemma 4.4.** *In a perimeter-minimizing double bubble with three equal pressures, every component in the double bubble plus exterior is either a vertex-free band or a hexagon.*

*Proof.* Consider any component in the double bubble plus exterior, and let $\mathcal{C}$ be the boundary of this component.

Because the double bubble has three equal pressures, any vertex-free component of $\mathcal{C}$ is linear and thus must be a closed geodesic.

Suppose that a component of $\mathcal{C}$ has at least one vertex. Then it has positive total turning angle, whereas any noncontractible closed curve has total turning angle 0. Hence, if a component of $\mathcal{C}$ has at least one vertex, then it is contractible. Also, it consists of straight edges meeting at interior angles measuring $2\pi/3$ — precisely the configuration of a hexagon.

Therefore, every component of $\mathcal{C}$ is a closed geodesic or a contractible hexagon, and the desired result follows easily. □

**Proposition 4.5.** *A perimeter-minimizing double bubble with three equal pressures is either a hexagon tiling or the double band.*

*Proof.* By Lemma 4.4, each component in the double bubble plus exterior is a vertex-free band or a hexagon. If every component is a hexagon, then the double bubble is a hexagon tiling. If every component is a band, then because there are at least three components, by Lemma 4.3 the double bubble is a double band. Finally, the double bubble cannot contain both bands and hexagons, because no band can be adjacent to a hexagon without violating regularity (Proposition 3.1). □

**Proposition 4.6.** *A minimizing double bubble on the torus must be a contractible double bubble, a tiling, a swath, a band adjacent to a set of components such that the closure of their union is contractible, or the double band (after perhaps relabelling the two regions and the exterior).*

*Proof.* We will categorize the minimizing double bubble by the number and type of noncontractible components in the double bubble plus its exterior. By Lemma 4.1 any noncontractible component must either be a band or have contractible complement. If the double bubble plus its exterior has in any region (say, in the exterior) a component with contractible complement, then the double bubble is contractible. Thus, for our remaining cases we can assume that all noncontractible components are bands.

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If the double bubble plus its exterior has no bands, then every component (of the double bubble plus its exterior) is contractible, and by definition the double bubble is a tiling.

If the double bubble plus its exterior has exactly one band, say in the exterior, then the double bubble consists of contractible components. But the closure of the union of this set of components is noncontractible, because it shares a noncontractible boundary curve with the band. Furthermore, any minimizing double bubble is connected (Remark 3.3). Then by definition the double bubble is a swath.

If the double bubble plus its exterior has exactly two bands (say, with at least one band in the exterior), then consider the set $C$ of the contractible components — that is, the set of components different from the bands. Suppose some subset of these components has noncontractible and connected closure. Then $C$ contains a swath, and therefore a chain. Now the double bubble plus its exterior has two bands and at least one chain, so by Lemma 4.3 it is not perimeter minimizing. Therefore, $C$ must be a contractible set of components. The two bands must then be adjacent to each other, and hence lie in different regions. Then the double bubble contains exactly one band, together with a subset of $C$ — that is, the double bubble is a band adjacent to contractible set of components.

If the double bubble plus its exterior has three or more bands, then by Lemma 4.3, the double bubble must be the double band. □

## 5. The four classes with contractible components

By Proposition 4.6, upon relabelling, every minimizing double bubble on the torus must be a contractible double bubble, a tiling, a swath, a band adjacent to a set of components whose union is contractible, or the double band. While the double band describes a specific geometric configuration, each of the remaining possibilities is merely a topological class of configurations. This section identifies the potential minimizer in each of these four remaining topological classes: the standard double bubble among contractible double bubbles (Proposition 5.1.2), a standard chain among swaths (Proposition 5.2.15), the band lens among double bubbles with a single band adjacent to a contractible set of components (Proposition 5.3.5), and the standard hexagon tiling among tilings (Proposition 5.4.23).

5.1. **Contractible double bubbles.** Proposition 5.1.2 uses the result from $\mathbf{R}^2$ that perimeter-minimizing double bubbles are standard [F], to show that the only potential minimizer among contractible double bubbles is the standard double bubble.

**Proposition 5.1.1.** *The perimeter of any planar standard double bubble is greater than $\pi$ times its diameter: $P > \pi D$.*

*Proof.* We obtain a formula for $P/D$ as a function $f(\theta)$, where $\theta \in [0, \pi/3)$ is the angle between the interior arc of the standard double bubble and the chord that connects the endpoints of the arc (see Figure 5). Then we show $f(\theta) > \pi$ for the entire domain $0 < \theta < \pi/3$. (A separate computation easily verifies the inequality $P > \pi D$ when $\theta = 0$.)

Let $r_1, r_2, c_1, c_2$ be the radii and centers of the longer and shorter exterior arcs of the standard double bubble, respectively. Then the diameter is the sum of $r_1$, $r_2$, and the distance between $c_1$ and $c_2$:

$$D = r_1 + r_2 + r_1 \cos(\pi/3 - \theta) + r_2 \cos(\pi/3 + \theta).$$

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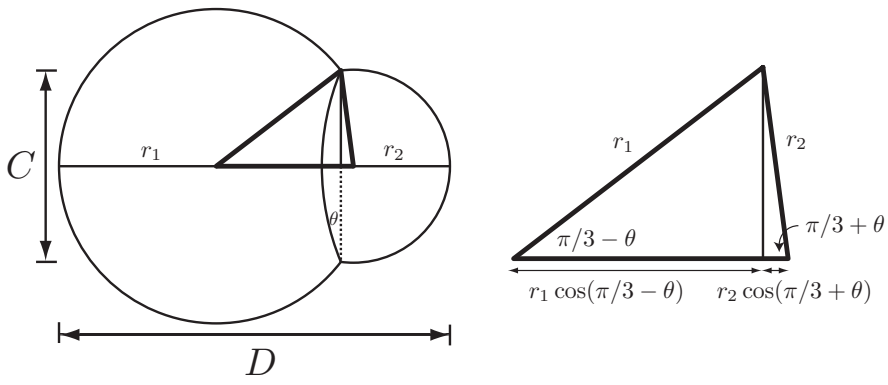



FIGURE 5. The perimeter of the standard double bubble is greater than $\pi$ times its diameter.

Applying the formula for radii (see (7.3)), we obtain an expression that can be solved for $C$, the length of the chord connecting the two vertices:

$$D = C\left(\frac{1+\cos(\pi/3-\theta)}{2\sin(2\pi/3+\theta)} + \frac{1+\cos(\pi/3+\theta)}{2\sin(2\pi/3-\theta)}\right).$$

Substituting for $C$ in the formula for perimeter (see (7.2)) yields after some simplification the desired function in $\theta$:

$$f(\theta) = P/D = \frac{8\pi\sin\theta\cos\theta + 3\sqrt{3}\theta}{6\sin\theta\cos\theta + 3\sin\theta}.$$

Now, showing $f(\theta) > \pi$ is equivalent to showing

$$8\pi\sin\theta\cos\theta + 3\sqrt{3}\theta > \pi(6\sin\theta\cos\theta + 3\sin\theta).$$

Moving the terms to the left-hand side and using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ gives the equivalent inequality

$$g(\theta) = \pi\sin(2\theta) + 3\sqrt{3}\theta - 3\pi\sin\theta > 0$$

with

$$g''(\theta) = -4\pi\sin(2\theta) + 3\pi\sin\theta = \pi\sin\theta(-8\cos\theta + 3).$$

For $\theta$ in $(0, \pi/3)$, $\sin\theta > 0$ while $\cos\theta > \cos(\pi/3) > 3/8$. Therefore, $g''(\theta)$ is strictly negative. Because $g(\theta)$ equals 0 at both endpoints of this interval, it must be positive along the interior, as desired. □

**Proposition 5.1.2** (Suggested by Masters [Ma1])**.** *The standard double bubble is the only contractible double bubble that may be a minimizer.*

*Proof.* Assume that there is a perimeter-minimizing contractible double bubble $\Sigma$ enclosing areas $A_1$ and $A_2$ that is not the standard double bubble. By [F], the unique perimeter-minimizing solution in $\mathbf{R}^2$ for the same prescribed areas is a standard double bubble, $\Theta$. By the perimeter bound (Proposition 3.8), $\Sigma$ has perimeter at most three. Since $\Theta$ has less perimeter than $\Sigma$, the perimeter $P$ of $\Theta$ satisfies $P < 3$.

By Proposition 5.1.1, the diameter $D$ of $\Theta$ satisfies $D < P/\pi < 1$. Hence, it fits on the torus with its axis of symmetry in any direction (Figure 6). Therefore, $\Theta$ is

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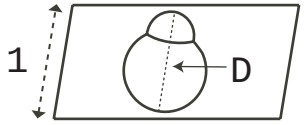



FIGURE 6. A standard double bubble with minimizing perimeter has diameter $D \leq 1$ and hence fits on the torus.

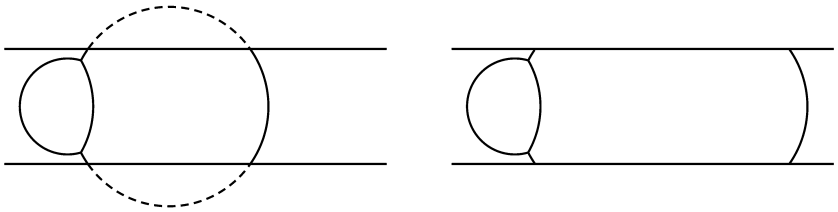


FIGURE 7. If a standard double bubble does not fit on the cylinder, we can easily construct a double bubble with even less perimeter.

a double bubble on the torus enclosing areas $A_1$ and $A_2$ with less perimeter than $\Sigma$, a contradiction. □

*Remark* 5.1.3. There is an easier argument on the infinite cylinder: when $\Theta$ does not fit with its diameter parallel to the sides of the cylinder, we can modify its boundary to create a new double bubble with less perimeter than $\Theta$ and enclosing the same areas, contradicting the assumption that there is a nonstandard contractible minimizer (see Figure 7).

5.2. **Swaths of contractible components.** Recall that by Definition 9 a swath is a set of contractible components with noncontractible and connected closure, and with noncontractible complement, such as the standard chain of Figure 1. This section shows that minimizing chains are standard chains (Definition 15), exactly as in Figure 1. Lemma 5.2.11 shows that minimizing swaths must be chains consisting of curvilinear quadrilateral components. Lemma 5.2.13 shows that asymmetric chains satisfying regularity cannot be minimizers since they have perimeter greater than three. Finally, Proposition 5.2.15 shows that minimizing symmetric chains are standard.

*Remark* 5.2.1. In this section, we often speak about contractible four-sided components as if they sat in the plane rather than a torus, because it is easier to talk about lines, reflections, and circles in the plane. (Lines can wrap around a torus infinitely many times, reflections across lines on a torus do not always exist, and circles on a torus may overlap themselves.)

**Lemma 5.2.2** ([W, Lemma 5.31])**.** *Given a contractible four-sided component in a regularity-satisfying double bubble or its exterior, the vertices of the component lie on a circle or a line. They lie clockwise on the circle or line in the same or opposite order as they appear clockwise on the boundary of the four-sided component.*

*Proof.* [W, Lemma 5.31] actually considers four-sided components in regularity-satisfying triple bubbles in the plane. However, the proof looks at each contractible

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four-sided component in isolation from the rest of the triple bubble, using only the facts that each side has constant curvature, that the sides do not intersect each other, and that the sides meet at angles of $2\pi/3$. Thus, the proof also applies to contractible four-sided components of double bubbles in the torus.

[W, Lemma 5.31] is also slightly more limited than our result, not proving that if the vertices of the component lie on a line, then they do so in the same order that they appear on the boundary of the component. However, it does offer a proof of the analogous fact about the vertices when they lie on a circle, and this proof can be easily modified to give our complete result. □

**Lemma 5.2.3.** *Given a contractible four-sided component with vertices $A, B, C, D$ in a regularity-satisfying double bubble or its exterior, $AB = CD$ unless sides $AB$ and $CD$ of the component lie on the same circle or line.*

*Proof.* By Lemma 5.2.2, vertices $A, B, C, D$ lie on a circle or line $\omega$. Suppose that sides $AB$ and $CD$ do not lie on the same circle or line. By regularity in double bubbles, sides $AB$ and $CD$ have constant curvature and lie on some circles or lines $\omega_1$ and $\omega_2$, respectively. Also by regularity, sides $AB$ and $CD$ have the same curvature, and they meet side $BC$ at equal angles (namely, $2\pi/3$). Hence, $\omega_1$ and $\omega_2$ are reflections of each other across the perpendicular bisector of $\overline{BC}$. Thus, $\omega_1 \cap \omega = \{A, B\}$ and $\omega_2 \cap \omega = \{C, D\}$ are reflections of each other across the perpendicular bisector of $\overline{BC}$ as well, implying that $AB = CD$. □

**Lemma 5.2.4.** *Given a contractible four-sided component in a regularity-satisfying double bubble or its exterior, the vertices of the component do not lie on a line.*

*Proof.* Let the vertices of the component be $A, B, C, D$ in that order along the boundary of the component. Suppose, for sake of contradiction, that the vertices lie on a line $\ell$. By Lemma 5.2.2, $A, B, C, D$ lie in that order along $\ell$ — without loss of generality, $\ell$ is horizontal with $A$ at the left and $D$ at the right. Because $BC \neq DA$, Lemma 5.2.3 implies that sides $DA$ and $BC$ of the component lie on the same circle or line. The only circle or line that contains all four vertices $A, B, C, D$ is $\ell$, so that sides $DA$ and $BC$ of the component are linear and lie along $\ell$. However, this is impossible because then sides $DA$ and $BC$ would overlap, a contradiction. □

**Lemma 5.2.5** (Compare [W, proof of Lemma 5.3])**.** *Given a contractible four-sided component in a regularity-satisfying double bubble or its exterior, the vertices of the component form a nondegenerate isosceles trapezoid. The vertices appear clockwise on this trapezoid in the same order as they appear clockwise on the boundary of the component. If the trapezoid is not a rectangle, then the two boundary curves subtended by the parallel sides of the trapezoid lie on a circle circumscribing the polygonal quadrilateral.*

*Proof.* By Lemmas 5.2.2 and 5.2.4, the polygonal quadrilateral $ABCD$ formed by the vertices of the component is inscribed in some circle $\omega$; the vertices appear on the quadrilateral in the same order that they appear on the circle (although perhaps with different orientation).

By regularity, curvilinear sides $AB, BC, CD, DA$ cannot all lie along $\omega$. Without loss of generality, assume that sides $AB$ and $CD$ do not lie on the same circle. By Lemma 5.2.3, we have $AB = CD$. Hence, because quadrilateral $ABCD$ is inscribed in a circle, it must be an isosceles trapezoid with $\overline{BC} \parallel \overline{DA}$.

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If sides $BC$ and $DA$ of the curvilinear component do not lie on the same circle, then by the same analysis we have $BC = DA$, so that quadrilateral $ABCD$ is a rectangle (Figure 8(a)). Also, if trapezoid $ABCD$ is not a rectangle, say because $BC \neq DA$, then sides $BC$ and $DA$ of the curvilinear component *do* lie on the same circle (Figure 8(b)).

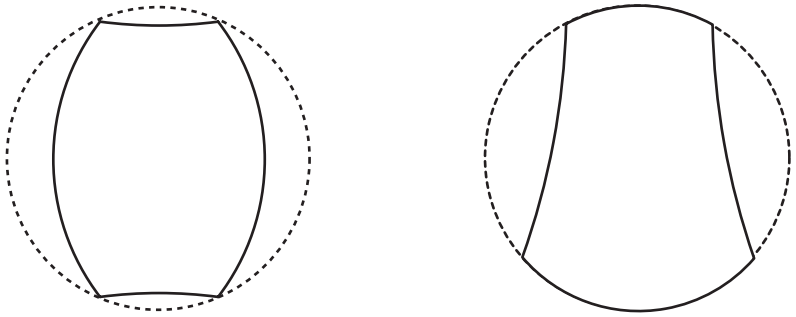



FIGURE 8. The vertices of any four-sided component lie on a circle. Either (a) the vertices form a rectangle, or (b) two opposite sides of the component lie along the same circle.

Finally, it remains to show that the vertices appear clockwise on the boundary of the component in the same order as they appear clockwise on the trapezoid. Because the vertices form a trapezoid (say, with $\overline{BC} \parallel \overline{DA}$), the four-sided component is symmetric about the perpendicular bisector $m$ of sides $BC$, $DA$ of the component. It follows easily that neither side $AB$ nor side $CD$ intersects $m$, implying the desired fact about the order of the vertices. □

**Lemma 5.2.6.** *In a contractible four-sided component of a highest-pressure region in a regularity satisfying double bubble or its exterior, the arcs bordering a least-pressure region are subtended by parallel chords, and the arcs bordering a medium-pressure region are subtended by congruent chords.*

*Proof.* By Lemma 5.2.5, the chords subtending the boundary curves of the component form an isosceles trapezoid inscribed in a circle $\omega$. If the trapezoid is a rectangle, we are done. Otherwise, the two boundary curves $\sigma, \sigma'$ subtended by parallel chords lie on $\omega$, and the other two boundary curves $\mu, \mu'$ are subtended by congruent chords. Because $\sigma, \sigma'$ meet the remainder of $\omega$ at angles *greater* than $2\pi/3$ (namely, $\pi$), the other boundary curves $\mu, \mu'$ of the component bow within $\omega$. In other words, the curvature of $\mu$ is less than the curvature of $\omega$, which equals the curvature of $\sigma$. It follows that $\sigma, \sigma'$ bound a region of less pressure than $\mu, \mu'$ do. □

**Lemma 5.2.7.** *For a minimizing tiling on a torus, the average number of edges per component is exactly six.*

*Proof.* The Euler characteristic on the torus is $v - e + c = 0$, where $v$ is the number of vertices, $e$ is the number of edges, and $c$ is the number of components. By regularity, (Proposition 3.1 and Remark 3.2), $v$, $e$ and $c$ are all finite. Now, let $a$ be the average number of edges per component. Then $c = \frac{2}{a}e$, since each edge is

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adjacent to two components. Similarly, $v = \frac{2}{3}e$, since each edge is adjacent to two vertices and (by regularity, Proposition 3.1) each vertex is adjacent to three edges. Substituting into the Euler formula, we obtain $(\frac{2}{3} - 1 + \frac{2}{a})e = 0$, from which we obtain $a = 6$. □

**Proposition 5.2.8.** *A minimizing double bubble with a contractible component in a region of least pressure must be either an octagon-square tiling or a hexagon tiling.*

*Proof.* If the pressures of the three regions are equal, then by Proposition 4.5 either the double bubble is a hexagon tiling or the double band; the latter case is impossible because the double bubble must have a contractible component by assumption. Hence, we may assume that the three regions do not all have the same pressure.

If any component in the double bubble plus exterior is a vertex-free band, then its boundary curves are closed geodesics. This either implies that both regions and the exterior have equal pressure, contradicting our previous assumption, or we can slide the band until it collides with some other boundary curve, for a violation of regularity (Remark 3.3). Thus, no component is a vertex-free band.

If any region has contractible complement, then by Proposition 5.1.2, the double bubble is the standard double bubble. Again, the least-pressure region consists solely of noncontractible components, a contradiction. Hence, we may assume that every region has noncontractible complement.

Let $R_1$ be a region of highest pressure. From the previous paragraph and Lemma 4.1, any noncontractible component of $R_1$ is a band. Because $R_1$ is of highest pressure, such a band cannot have any vertices; if it did, then each boundary edge of the band would have positive turning angle, a contradiction. However, we have already shown that no region contains a vertex-free band. Therefore, any component in $R_1$ is contractible. By Proposition 3.7, $R_1$ has at most two contractible components. Also, each component of $R_1$ contains at most four vertices (Remark 3.6). Hence, $R_1$ contains at most eight vertices.

Let $R_2$ be a region of lowest pressure, containing a contractible component with $n$ sides (and perhaps other contractible, or noncontractible, components). This component has least pressure but is not polygonal (because the three pressures are not all equal), so $n > 6$ (Remark 3.6). By Remark 3.4, $n$ is even, implying that $n \geq 8$.

Because every vertex of $R_1$ is a vertex of $R_2$ (and vice versa), and because neither $R_1$ nor $R_2$ contains a vertex-free band, it follows that $R_1$ consists of two contractible four-sided components and that $R_2$ consists of a single contractible curvilinear octagon.

Because no region has contractible complement, $R_2$ and some four-sided component $C$ in $R_1$ must form a chain that wraps around the torus. We now consider two cases.

*Case* 1: $C$ borders two opposite sides of $R_2$. Then just as $R_2$ and $C$ form a chain wrapping around the torus, so too do $R_2$ and the *other* four-sided component of $R_1$ — wrapping around the torus in a different direction. In this case, the double bubble is a tiling. Because $R_1$ and $R_2$ each have eight vertices, $R_3$ must have eight vertices as well, for a total count of 24 vertices. But by Lemma 5.2.7, the average number of vertices per component must be exactly six. Therefore, there are exactly four components, implying that $R_3$ consists of one curvilinear octagon. Therefore, as shown in Figure 9(a), the configuration is an octagon-square tiling.

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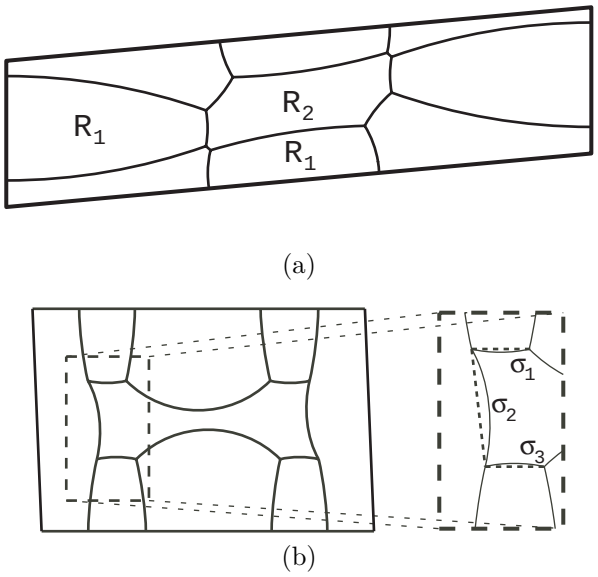



FIGURE 9. Two double bubbles with a purported contractible low-pressure component.

*Case* 2: $C$ borders two sides $\sigma_1, \sigma_3$ of $R_2$ as depicted in Figure 9(b). Let $\sigma_2$ be the side of the eight-sided component between them. By Lemma 5.2.6, the chords subtending $\sigma_1$ and $\sigma_3$ are parallel. Therefore, one of these chords must meet the chord subtending $\sigma_2$ at a nonobtuse angle. But then the corresponding arcs meet at an angle less than $2\pi/3$, a contradiction.

Therefore, if a least-pressure region contains a contractible component, then the double bubble is a hexagon tiling or an octagon-square tiling. □

**Corollary 5.2.9.** *A perimeter-minimizing tiling is either a hexagon tiling or an octagon-square tiling.*

**Lemma 5.2.10.** *Let $\gamma$ be a boundary edge separating two components $X_1$ and $X_2$ in a minimizing double bubble, where both $X_1$ and $X_2$ have different pressures from the third region of the double bubble. Let the two boundary arcs of $X_1$ adjacent to $\gamma$ be $\sigma_1$ and $\sigma_2$, and let the two boundary arcs of $X_2$ adjacent to $\gamma$ be $\mu_1$ and $\mu_2$. (The $\sigma_i$ coincide if $X_1$ has two sides, and the $\mu_i$ coincide if $X_2$ has two sides.) Suppose that $\sigma_1$ and $\sigma_2$ have the following property: they lie on a common arc, and this arc forms an immersed curvilinear digon with $\gamma$. Then $\mu_1$ and $\mu_2$ have this property as well.*

*Proof.* Consider the curvilinear digon bordered by $\gamma$ and the arc containing $\sigma_1$ and $\sigma_2$. We may add a third arc $\mu$ to the curvilinear digon to form an immersed standard double bubble. Because the cocycle condition holds for both the given double bubble and for the immersed standard double bubble, $\mu_1$, $\mu_2$, and $\mu$ have the same (signed) curvatures. Therefore, $\mu_1$ and $\mu_2$ lie on the same circular arc $\mu$, which forms a curvilinear digon with $\gamma$. □

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**Lemma 5.2.11.** *A minimizing swath must be a chain of four-sided components. The exterior is a band with less pressure than either enclosed region of the swath.*

*Proof.* By Proposition 5.2.8, every component of a region of least pressure is noncontractible. This implies two facts: first, because the swath contains contractible components from two regions, the exterior region is the only region of least pressure, as claimed. Second, because the exterior is a region of least pressure, each component of the exterior region is noncontractible. Also, the *complement* of each component in the exterior is noncontractible, because this complement contains the swath formed by the two enclosed regions. Thus, by Lemma 4.1, each component of the exterior is a band. Because the swath formed by the two enclosed regions is connected (Remark 3.3), the exterior region consists of exactly one band, as claimed. Also, the swath cannot contain a contractible cycle of components, because otherwise the cycle would encircle a contractible component of the exterior.

Consider a chain $K$ in the swath. The components in $K$ must alternate between two regions $R_1$ and $R_2$ with higher pressure than the exterior. We will show that the swath contains no additional component of $R_1$ or $R_2$ attached to the chain, as in Figure 10.

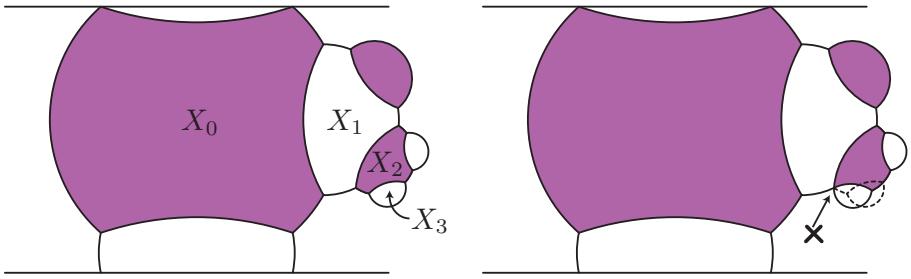



FIGURE 10. Extra components attached to a chain could be slid to create a double bubble violating regularity.

Suppose, for sake of contradiction, that the swath contained an additional component $X_1$ attached to some component $X_0$ in the chain. Either $X_1$ has two sides, or another component $X_2$ attaches to it. In this case, either $X_2$ has two sides, or another component $X_3$ attaches to it. Because the swath contains no contractible cycle, this process terminates after yielding some sequence of components $X_0, X_1, \ldots, X_k$, where $X_k$ has two sides. By Lemma 5.2.10, the boundary edges of $X_{k-1}$ meeting the curvilinear digon $X_k$ lie on the same circle. Therefore, $X_k$ can be slid along the boundary of this circle until it bumps into the boundary of another component, creating a minimizer with an illegal singularity (Proposition 3.1), a contradiction. Therefore, our original assumption was false, and no additional components of $R_1$ or $R_2$ attach to the chain.

Thus, the double bubble is a chain. Because the exterior consists of a single band, each component in the chain is enclosed by four boundary curves: two separating it from the other components of the chain, and two separating it from the exterior. Therefore, the double bubble is a chain of four-sided components. □

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**Lemma 5.2.12.** *Consider a chain of four-sided components in a perimeter-minimizing double bubble. If the arcs separating each component from the exterior of the chain lie on the same circle, then the perimeter of the chain is greater than* 3.

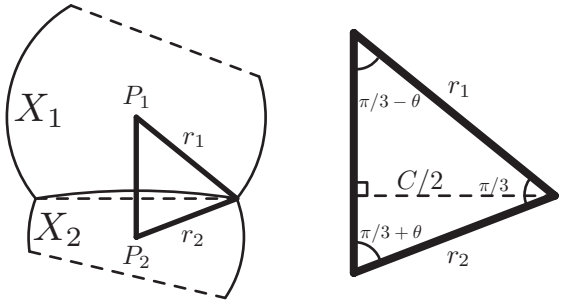



FIGURE 11. Triangulating chains in which opposite arcs are cocircular leads to a perimeter estimate.

*Proof.* Consider two adjacent components $X_1$ and $X_2$ of the chain, as drawn in Figure 11 so that the chord $C$ of the arc separating them is horizontal and lies below $X_1$. Consider the four sides of the polygonal quadrilateral inscribed on the vertices of $X_1$. By Lemma 5.2.11, the chain neighbors a least-pressure region; hence, by Lemma 5.2.6, $C$ and the side opposite it in the quadrilateral are congruent. It follows that $C$ lies below $P_1$, the center of the circle passing through the vertices of $X_1$. Similarly, $C$ lies above the center $P_2$ of the circle passing through the vertices of $X_2$.

We draw the triangle whose vertices are $P_1$, $P_2$, and an endpoint $P_3$ of the chord $C$. The triangle has angles $\pi/3-\theta, \pi/3, \pi/3+\theta$ and side lengths $r_1, r_2, P_1P_2$ for some $\theta, r_1, r_2$. Because $P_1$ and $P_2$ lie on opposite sides of $C$, angles $\angle P_1P_2P_3$ and $\angle P_2P_1P_3$ are acute, implying that $\theta \in (-\pi/6, \pi/6)$. Using the Law of Sines in this triangle, we have $r_1 = P_1P_2 \frac{\sin(\pi/3+\theta)}{\sin(\pi/3)}$. Also, $|C|/2 = r_1 \sin(\pi/3-\theta)$. Hence,

$$|C| = \frac{4}{\sqrt{3}} P_1P_2 \sin(\pi/3+\theta)\sin(\pi/3-\theta).$$

Applying the standard transformation $2\sin x \sin y = \cos(x-y) - \cos(x+y)$ shows that

$$|C| = \frac{2}{\sqrt{3}} P_1P_2(\cos(2\theta) - \cos(2\pi/3)).$$

With the restriction $\theta \in (-\pi/6, \pi/6)$, we see that $|C| \geq \frac{2}{\sqrt{3}} P_1P_2$.

As we vary over pairs of adjacent components, the segments analogous to $\overline{P_1P_2}$ form a closed noncontractible curve with total length greater than or equal to one. Hence, the total length of the chords analogous to $C$ is at least $\frac{2}{\sqrt{3}} > 1$, implying that the total length of the corresponding arcs is also greater than one. Furthermore, the arcs separating the exterior and the chain form two closed noncontractible curves with total length greater than or equal to two. Therefore, the total perimeter of the chain is greater than three. □

**Lemma 5.2.13.** *No asymmetric chain is perimeter minimizing.*

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*Proof.* Suppose, for sake of contradiction, that an asymmetric chain is minimizing. By Lemma 5.2.11, it consists of four-sided components, and the exterior is a least-pressure region consisting of a single band. By Lemma 5.2.5, each component of the chain has vertices that form an isosceles trapezoid. If the trapezoid corresponding to each component is a rectangle, then the rectangles (and hence the chain) are all symmetric about a closed geodesic (with the same homology as the chain), a contradiction. Therefore, one of the components has vertices that form a nonrectangular isosceles trapezoid. By Lemma 5.2.5 and Lemma 5.2.6, the arcs separating this component from the least-pressure exterior lie on a single circle. By Lemma 5.2.10, the same is true for all other components in the chain. Applying Lemma 5.2.12, the perimeter of the chain is greater than 3. By Proposition 3.8, the asymmetric chain cannot be minimizing, a contradiction. □

**Lemma 5.2.14.** *A minimizing symmetric chain must be a standard chain.*

*Proof.* By Lemma 5.2.11, such a chain consists of four-sided components, where alternate components lie in each of two regions. Thus, the chain contains an even number of components. By Lemma 5.2.5, each four-sided component is circumscribed about an isosceles trapezoid. Because the chain is symmetric, and using Lemma 5.2.6, we find that each four-sided component must be circumscribed about a rectangle. It easily follows that any pair of adjacent components in the chain is congruent to any other pair.

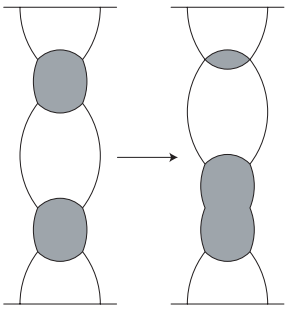


FIGURE 12. Rearranging the components of a symmetric chain with more than two components brings about a violation of regularity, proving that a minimizing chain has two components.

Now if the chain contains more than two components, then rearranging four adjacent components as in Figure 12 maintains area and perimeter while violating regularity, so that the original chain cannot be perimeter minimizing. Therefore, the chain contains exactly two components. No curvilinear quadrilateral lies in a least-pressure region, so the two components of the chain belong to the highest- and medium-pressure regions. Applying the cocycle condition (Proposition 3.1) shows that the chain is standard. □

**Proposition 5.2.15.** *Every perimeter-minimizing swath is a standard chain.*

*Proof.* By Lemma 5.2.11, a minimizing swath must be a chain of four-sided components. By Lemma 5.2.13 a minimizing chain must be symmetric. By Lemma 5.2.14, a minimizing symmetric chain must be a standard chain. □

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*Remark* 5.2.16. While we will show that a minimizing band lens must have the same homology as a shortest closed geodesic (Lemma 5.3.6), we do not have a proof of the analogous fact for minimizing standard chains. However, *Mathematica* plots for perimeter suggest that this property holds at least for the particular tori described by Figure 2.

5.3. **A single band adjacent to a contractible set of components.** Proposition 5.3.5 uses regularity (Proposition 3.1) and Proposition 5.2.8 to show that the only potential minimizer in the class of double bubbles with a single band adjacent to a contractible set of components is the band lens.

Throughout this section, we will consider a proposed minimizer with regions $R_1$, $R_2$ and the exterior, such that $R_1$ is contractible and $R_2$ contains a single band (and possibly some contractible components).

**Lemma 5.3.1.** *In a minimizing double bubble with some contractible region $R_1$, and another region $R_2$ containing a single band, the pressure difference between $R_2$ and the third (exterior) region must be zero, that is, all curves separating $R_2$ from the exterior must be straight line segments.*

*Proof.* Consider a curve separating the band in $R_2$ from the exterior. If it is a straight line segment, the proof is complete, since all curves separating $R_2$ from the exterior have the same curvature by regularity (Proposition 3.1).

If the curve has nonzero curvature, continuously straighten the curve to reduce perimeter, while sliding one component of the boundary of the band in $R_2$ in order to maintain area. Either we will be able to straighten all such pieces while reducing perimeter, or we will first cause a violation of regularity. □

**Lemma 5.3.2.** *In a minimizing double bubble with some contractible region $R_1$, and another region $R_2$ containing a single band, $R_1$ must have higher pressure than each of $R_2$ and the exterior region.*

*Proof.* By Lemma 5.3.1, $R_2$ and the exterior have the same pressure. Since the double bubble is neither a tiling nor the double band, $R_1$ cannot also have the same pressure (Proposition 4.5). Moreover, by Proposition 5.2.8, since $R_1$ is contractible it cannot be a region of least pressure. Thus, $R_1$ must have higher pressure than both $R_2$ and the exterior. □

**Corollary 5.3.3.** *All contractible components of the double bubble plus exterior belong to $R_1$, and $R_2$ consists of a single band.*

*Proof.* By Lemma 5.3.1 and Lemma 5.3.2, $R_2$ and the exterior have equal pressure, less than that of $R_1$. Thus, by Proposition 5.2.8, their components must be non-contractible. Hence only components of $R_1$ can be contractible, and $R_2$ consists of the only noncontractible component of the double bubble — a band. □

**Lemma 5.3.4.** *In a minimizing double bubble with some contractible region $R_1$, and another region $R_2$ containing a single band, all components of $R_1$ are lenses.*

*Proof.* By Lemma 5.3.2, $R_1$ has higher pressure than both $R_2$ and the exterior. By Remarks 3.6 and 3.4, all components of $R_1$ have either two or four sides. We will assume that some component of $R_1$ is a curvilinear quadrilateral $Z$, and show that this leads to a contradiction. Let the four sides of $Z$ be labelled clockwise $x_1$, $y_1$,

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$x_2$, $y_2$, where $x_1, x_2$ border $R_2$ (a single band, by Corollary 5.3.3) and $y_1, y_2$ border $Y_1, Y_2$ in the exterior.

If $x_1$ and $x_2$ lie on the same side of $R_2$, then either $y_1$ or $y_2$ lies along that same side of the band; the corresponding component $Y_1$ or $Y_2$ is contractible, contradicting Corollary 5.3.3. Otherwise, $x_1$ and $x_2$ lie on opposite sides of $R_2$. In this case, the closure of $R_2 \cup Z$ has contractible complement, so that both $Y_1$ and $Y_2$ are contractible — again contradicting Corollary 5.3.3.

Thus, all components of $R_1$ are curvilinear digons. Now, by Lemma 5.3.1, $R_2$ and the exterior have equal pressure. Therefore, all curves bounding components of $R_1$ have equal, nonzero curvature, and so all curvilinear digons in $R_1$ must be lenses. □

**Proposition 5.3.5.** *A perimeter-minimizing double bubble with a single band adjacent to a contractible set of components must be the band lens.*

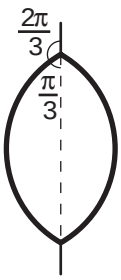



FIGURE 13. Any lens from $R_1$ must be embedded in a closed geodesic.

*Proof.* By Corollary 5.3.3 and Lemma 5.3.4, one region $R_1$ consists only of lenses and the other region $R_2$ consists of a single band. Corollary 5.3.3 also implies that the exterior contains no contractible components. It follows from Lemma 4.1 that the exterior consists solely of bands. Because the double bubble is connected (Remark 3.3), the exterior consists of a single band.

Each lens in $R_1$ has two vertices, each the endpoint of some curve separating $R_2$ and the exterior. By Lemma 5.3.1, these curves are straight line segments. Furthermore, both segments must lie on the line passing through the diameter of the lens, as depicted in Figure 13. Hence, the diameters of the lenses, together with the segments separating $R_2$ from the exterior, form closed geodesics. Since $R_2$ and the exterior each consist of exactly one band, the boundary of the double bubble consists of exactly two closed geodesics, with lenses from $R_1$ embedded in at least one of them. Each of these two closed geodesics has length $L' \geq 1$.

Let the lenses in $R_1$ have total perimeter $p$ and let their diameters have total length $d$. If $d \geq 1$, then the perimeter of the double bubble is at least $(2L' - d) + p > (2L' - d) + 2d \geq 3$, so by Proposition 3.8 the double bubble is not perimeter minimizing. Otherwise, we can move all the lenses so that they are adjacent to each other and embedded in the same closed geodesic, creating a new double bubble enclosing and separating the same areas. If there is more than one lens, then the new double bubble violates regularity (Figure 14), a contradiction. Therefore, the original minimizing double bubble contains exactly one lens, and so must be the band lens. □

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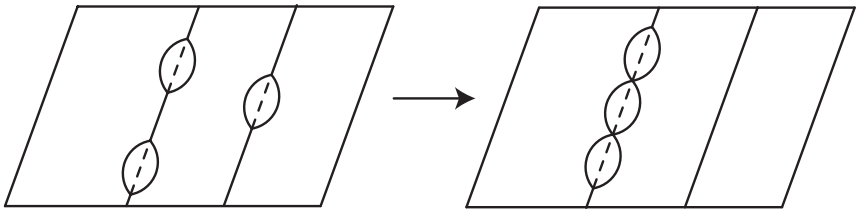


FIGURE 14. Multiple lenses can be slid together to violate regularity. Therefore, there can only be one lens.

**Lemma 5.3.6.** *In a minimizing band lens, the band must have the same homology as a shortest closed geodesic.*

*Proof.* We show that any pair of areas that can be enclosed by a band lens wrapping around a longer direction of the torus can be enclosed by an embedded band lens wrapping around a short direction. Suppose not. By Proposition 3.8, the perimeter of the original band lens is less than or equal to three, so the perimeter of the lens is less than two. Therefore, the diameter of the lens is less than one, and the lens can be embedded on the torus with its axis along a short direction (Remark 2.3.2). Thus, if the band lens wrapping around the short direction does not exist, the closed geodesic not meeting the lens cannot get close enough to make the band thin enough. But if this happens in the short direction, then it will certainly happen for the band lens wrapping around the longer direction, a contradiction.

Hence, if there is a minimizing band lens wrapping around a longer direction of the torus — say, parallel to a closed geodesic of length $L_0 > 1$ — there is a band lens in a short direction of the torus enclosing the same areas. However, the new band lens has perimeter $2(L_0 - 1)$ less than the initial band lens, a contradiction. Therefore, any minimizing band lens has the same homology as a shortest closed geodesic. □

5.4. **Tilings.** This section will show that every perimeter-minimizing tiling is a standard hexagon tiling (Proposition 5.4.23). Corollary 5.2.9 showed that if a tiling is minimizing, then it is a hexagon tiling or an octagon-square tiling. Proposition 5.4.13 shows that every minimizing hexagon tiling is standard, and Proposition 5.4.22 shows that no octagon-square tiling is perimeter minimizing.

**Hexagon tilings**. Lemmas 5.4.3, 5.4.5, and 5.4.6 show that a minimizing hexagon tiling divides the torus into exactly three hexagonal components. Lemma 5.4.9 shows that such a tiling can be transformed without changing perimeter to make the three hexagons congruent translations of each other. By Lemma 5.4.3, the perimeter of the new hexagon tiling is greater than 3 except on the hexagonal torus, implying that the original minimizing hexagon tiling lies on the hexagonal torus. Finally, Proposition 5.4.13 shows that any such hexagon tiling is the standard hexagon tiling.

**Lemma 5.4.1.** *The number of components in a perimeter-minimizing hexagon tiling plus exterior is a multiple of three.*

*Proof.* By Proposition 3.1, there are finitely many vertices in the tiling — say, $v$ — each bordering three distinct regions. Consider any of the three regions $R_i$. Each

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of its components borders six distinct vertices in the tiling, and each vertex in the tiling borders exactly one component of $R_i$. Hence, $R_i$ contains $v/6$ components. Therefore, $v/6$ is an integer, and there are $3(v/6)$ components in all. □

**Lemma 5.4.2.** *Given $n \geq 3$ real numbers $a_1, \ldots, a_n \in [0, A/3]$ such that $\sum_{i=1}^n a_i = A$, we have $\sum_{i=1}^n \sqrt{a_i} \geq 3\sqrt{A/3}$.*

*Proof.* Because $\sqrt{A}$ is strictly concave, $\sum_{i=1}^n \sqrt{a_i}$ attains its minimum at some vertex of the given domain. At each vertex, three of the $a_i$ equal $A/3$, and the other $a_i$ equal 0. The result follows immediately. □

**Lemma 5.4.3.** *In a hexagon tiling on a torus with area $A$ such that each component has area at most $A/3$, the perimeter of the tiling is greater than or equal to 3. Equality can hold only for a tiling on a hexagonal torus with three components in the tiling plus exterior.*

*Proof.* By Lemma 5.4.1, if there are $n$ components in the tiling plus exterior, then $n \geq 3$. Let $a_1, \ldots, a_n$ be the areas of the $n$ components $C_1, \ldots, C_n$, and let $p_1, \ldots, p_n$ be their perimeters. By the isoperimetric inequality for hexagons, $p_i \geq \sqrt{8\sqrt{3}}\sqrt{a_i}$ for each $i$, with equality if and only if $C_i$ is a regular hexagon.

The total perimeter of the tiling, $\frac{1}{2}\sum_{i=1}^n p_i$, is thus greater than or equal to $\sqrt{2\sqrt{3}}\sum_{i=1}^n \sqrt{a_i}$. By Lemma 5.4.2, $\sqrt{2\sqrt{3}}\sum_{i=1}^n \sqrt{a_i}$ is in turn greater than or equal to $\sqrt{2\sqrt{3}} \cdot 3\sqrt{A/3} = \sqrt{6\sqrt{3}A}$.

Because $A \geq \sqrt{3}/2$ with equality only for the hexagonal torus (Remark 2.1.1), the perimeter of the tiling is at least $\sqrt{6\sqrt{3}A} \geq \sqrt{6\sqrt{3}\frac{\sqrt{3}}{2}} = 3$. The perimeter is exactly 3 only if equality holds in all the intermediate inequalities. In other words, the torus must have area $\sqrt{3}/2$, which only holds for the hexagonal torus; each component must be a regular hexagon and there must be 3 components each with area $A/3$. □

*Remark* 5.4.4. This method of comparing hexagon tilings to tilings of three equal areas was inspired by the result of Hales that the regular hexagon is the most efficient way to tile the plane — not necessarily with polygons — into unit areas [H]. Earlier, Fejes-Tóth proved the result for polygonal tilings ([FT1, Chapter III, Section 9, p. 84] or [FT2, Section 26, Corollary, p. 183] after [FT3]), a result more comparable than Hales' to ours.

**Lemma 5.4.5.** *Consider a hexagon tiling with $n \geq 6$ components in the tiling plus exterior. For each component $H$ in any of the three regions, the perimeter of the tiling exceeds 1 plus the perimeter of $H$.*

*Proof.* We orient the torus so that the edges of the tiling form angles of 0 and $\pm\pi/3$ with the horizontal. Consider $H$ along with the hexagons neighboring it, labelled $H_1, \ldots, H_6$ as in Figure 15.

Although some of $H_1, \ldots, H_6$ may coincide (if $H$ and some component share more than one edge), we claim that $H_2$, $H_4$, and $H_6$ cannot *all* coincide. Assume not. $H_3$ borders $H_2$ from below, and $H_5$ borders $H_6$ from below; because $H_2 = H_6$, we have $H_3 = H_5$. We can likewise show that $H_1$, $H_3$, and $H_5$ are all the same component. The top edge, bottom-right, and bottom-left edges of this component border $H_2 =$

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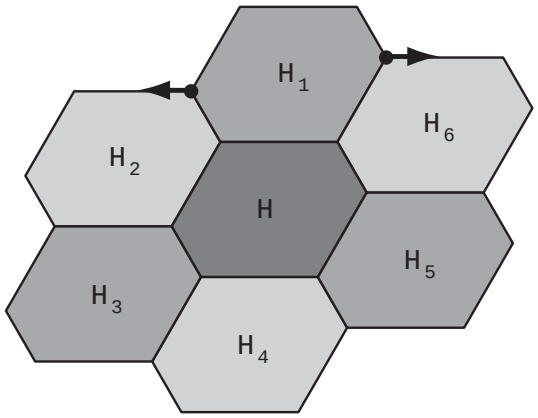



FIGURE 15. A hexagon $H$ on the torus and its six neighbors, shaded according to which regions they lie in. (Some of the six hexagons neighboring $H$ in this picture may actually represent the same component.) In order to find a noncontractible curve not intersecting $H$, we avoid travelling along a downward edge from either of the two marked vertices.

$H_4 = H_6$ (since, as shown in Figure 15, the top edge of $H_3$ borders $H_2$, the bottom-right edge of $H_1$ borders $H_6$, and the bottom-left edge of $H_5$ borders $H_4$); similarly, the other edges of this component border $H$. Thus, $H_1$ only neighbors two distinct components, $H$ and $H_2$. Likewise, $H$ and $H_2$ must only neighbor components in $\{H, H_1, H_2\}$. However, this is impossible; at least one of the $n-3$ hexagons not in $\{H, H_1, H_2\}$ must neighbor some hexagon in $\{H, H_1, H_2\}$. Therefore, our original assumption was false, and we do not have $H_2 = H_4 = H_6$.

Without loss of generality, assume that $H_2 \neq H_6$. Notice also that $H_2 \neq H_5$ and $H_6 \neq H_3$, since alternate sides of $H$ must border different regions (Remark 3.4).

The tiling contains $6n/3$ vertices, which is at least twelve, so we may pick a vertex $v$ not on the boundary of $H$. Starting at $v$, we travel as follows: if we are at one of the vertices marked in Figure 15, travel horizontally as shown in Figure 15; otherwise, travel downward along an edge. It follows that we never travel downward along the edges depicted in Figure 16(a). Furthermore, because $H_6 \notin \{H_2, H_3\}$, we never travel to the right along the edges depicted in Figure 16(b). Similarly, we never travel to the left along the edges depicted in Figure 16(c). It easily follows that we never travel along a horizontal edge only to immediately backtrack along it, and that we never travel along the boundary of $H$ or along any of the edges stemming from $H$.

There are finitely many vertices in the tiling (Proposition 3.1), implying that some portion of our journey begins and ends at the same vertex. Because we never backtrack along a horizontal edge, and because we never travel upward along an edge, this portion of our journey occurs along a closed, noncontractible curve (Figure 17). This curve has length at least 1, and it does not overlap the boundary of $H$, nor does it overlap any of the edges stemming from $H$, which have positive length. Hence, the boundary curves not enclosing $H$ have total length greater than 1, as desired. □

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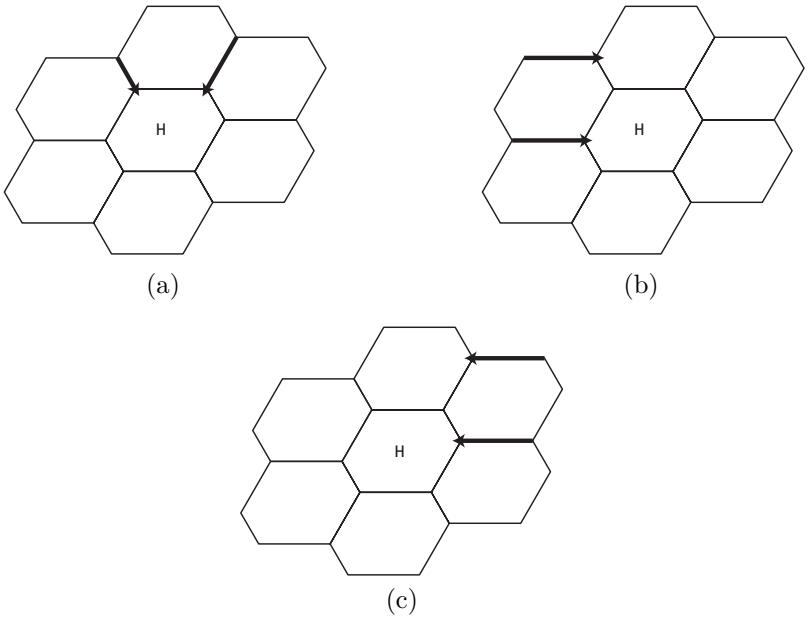



FIGURE 16. In order to bound the perimeter outside any component, we travel along the boundary in an attempt to form a closed noncontractible path. These diagrams show various movements that we do not perform during our journey.

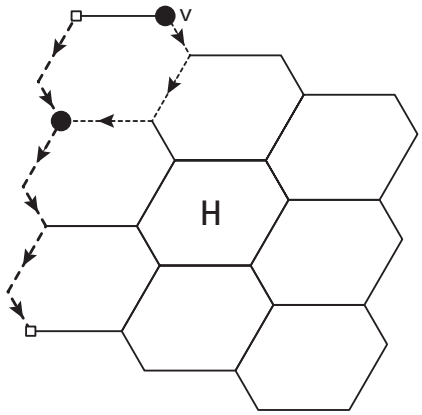



FIGURE 17. Starting at some vertex $v$ and travelling along downward edges — except in those cases where doing so would lead us to a vertex of $H$ — yields a closed, noncontractible curve not passing through the boundary of $H$, for a total perimeter greater than 3 when $H$ is large.

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**Lemma 5.4.6.** *No hexagon tiling with six or more components in the double bubble plus exterior is perimeter minimizing.*

*Proof.* We consider a hexagon tiling with six or more components. We first consider the case when the area $\alpha$ of some component is at least one-third the total area $A$ of the torus, where $A \geq \sqrt{3}/2$ (Remark 2.1.1). By the isoperimetric inequality for hexagons, the perimeter of $H$ is at least $\sqrt{8\sqrt{3}\alpha} \geq \sqrt{8\sqrt{3}(A/3)} \geq 2$. Therefore, by Lemma 5.4.5, the perimeter of the tiling is greater than $2 + 1 = 3$.

Next consider the case in which each component has area less than $A/3$. Because there are more than three components, by Lemma 5.4.3, the tiling has perimeter greater than 3.

In both cases, it follows by our perimeter bound (Proposition 3.8) that the hexagon tiling is not perimeter minimizing. □

*Remark* 5.4.7. We first attempted to prove Lemma 5.4.6 by shrinking and expanding hexagons to preserve perimeter and areas and eventually contradict regularity. Although the perimeter of the tiling remains the same under such variations, the problem — as Wacharin Wichiramala explained to us — is that it is not always possible to preserve the regions' areas in order to reach such a contradiction.

**Lemma 5.4.8.** *In a hexagon with interior angles of $2\pi/3$, the sum of the lengths of any two adjacent sides equals the sum of the lengths of the opposite pair of adjacent sides.*

*Proof.* Orient the hexagon so that the two sides $p_1, p_2$ not being summed are horizontal, with one pair of remaining sides $q_1, q_2$ on the left and the other pair of remaining sides $r_1, r_2$ on the right. Let $d$ be the distance between $p_1$ and $p_2$. The sides $q_1, q_2$ make an angle of $\pi/6$ with the vertical, so the sum of the lengths of $q_1, q_2$ is $d/\sin(\pi/3)$. Similarly, the sum of the lengths of $r_1, r_2$ is $d/\sin(\pi/3)$. Hence, the two sums are equal, as desired. □

**Lemma 5.4.9.** *Given a hexagon tiling with three components in the tiling plus exterior, there exists a second hexagon tiling with the following properties: the edges of the new tiling are parallel to the edges of the original tiling; the perimeters of the two tilings are equal; and any two parallel edges of the new tiling are congruent. In particular, there exists a second hexagon tiling with the same perimeter, which still contains three components (now of the same area) in the tiling plus exterior.*

*Proof.* To get the desired result, we will find a variation that shrinks and expands hexagons as in Figure 18 to make the three areas equal. Perimeter is unchanged, since $dP/dA$ = curvature = 0 through the whole family of double bubbles in the variation. We show that such a variation exists, and we also give an alternative geometric explanation for why perimeter remains unchanged.

Let $a_i, b_i, c_i$ $(1 \leq i \leq 3)$ be the lengths of the edges in the hexagon tiling as in Figure 19; for each $i \in \{1, 2, 3\}$, the sides of some hexagon $H_i$ are assigned the lengths $a_{i-1}, b_{i-1}, c_{i-1}, a_{i+1}, b_{i+1}, c_{i+1}$ (where indices are taken modulo 3). In this proof, as we modify the hexagon tiling, the values of the $a_i, b_i, c_i$ will change accordingly.

Fix some subscript $i_0$. As shown in Figure 20, for some range of real numbers $\epsilon$ (which we will describe shortly), we may transform our hexagon tiling into another hexagon tiling, so that the edge lengths change as follows: $a_{i_0}, b_{i_0}, c_{i_0}$ change by

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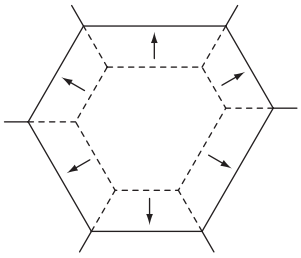


FIGURE 18. Expanding any component in a hexagon tiling or its exterior generates a double bubble enclosing different areas, but does not change perimeter. Hexagons can also be shrunk in a similar fashion.

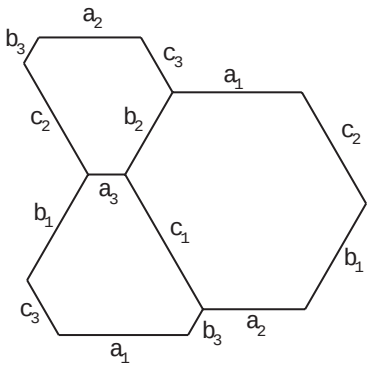



FIGURE 19. A hexagon tiling with three components in the tiling plus exterior.

$-2\epsilon$, and the other $a_i, b_i, c_i$ change by $\epsilon$. More specifically, the range of such $\epsilon$ are those $\epsilon$ for which the side lengths of the resulting tiling are positive; namely, such that $a_i+\epsilon$, $b_i+\epsilon$, and $c_i+\epsilon$ are positive for $i \neq i_0$, and such that $a_{i_0}-2\epsilon$, $b_{i_0}-2\epsilon$, and $c_{i_0}-2\epsilon$ are positive. Adding up the increases and decreases in perimeter shows that under this transformation, the perimeter of the hexagon tiling remains unchanged.

Without loss of generality, assume that $a_1 \geq a_2 \geq a_3$ initially. By Lemma 5.4.8, we have $(a_2 - a_3)/3 = (b_2 - b_3)/3 = (c_2 - c_3)/3$; set $\epsilon$ equal to this common value. As described in the last paragraph, we may shrink $H_3$ so that its edge lengths ($a_i, b_i, c_i$ for $i \neq 3$) decrease by $\epsilon$ and so that $a_3, b_3, c_3$ increase by $2\epsilon$. It is easy to check that this $\epsilon$ is in the allowed range. Furthermore, after this transformation, we have $(a_2, b_2, c_2) = (a_3, b_3, c_3)$.

At this point, we now have $a_1 \geq a_2 = a_3$. We may now set $\epsilon$ equal to $\frac{a_1-a_2}{3}$ and expand $H_1$ so that its edge's lengths increase by $\epsilon$. In the resulting hexagon tiling, we have $(a_1, b_1, c_1) = (a_2, b_2, c_2) = (a_3, b_3, c_3)$, as desired. Furthermore, because the two transformations we used did not change perimeter, the resulting tiling and the initial tiling have the same perimeter. This completes the proof. □

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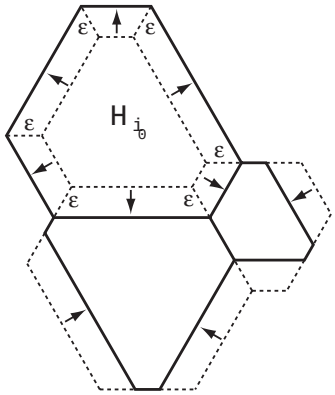



FIGURE 20. In a hexagon tiling with three components in the double bubble plus exterior, we may expand (or shrink) one hexagon so that six edges increase by $\epsilon$ while three edges decrease by $2\epsilon$.

**Lemma 5.4.10.** *Suppose that the hexagonal torus is partitioned into three hexagons with interior angles of $2\pi/3$, where the hexagons are congruent translations of each other. Suppose that the perimeter of the partition is at most 3, or that the sides of the hexagon are parallel to the three short directions. Then the hexagons are regular hexagons of side length $\frac{1}{3}$, with sides parallel to the three short directions of the torus. The total perimeter of the partition is equal to three.*

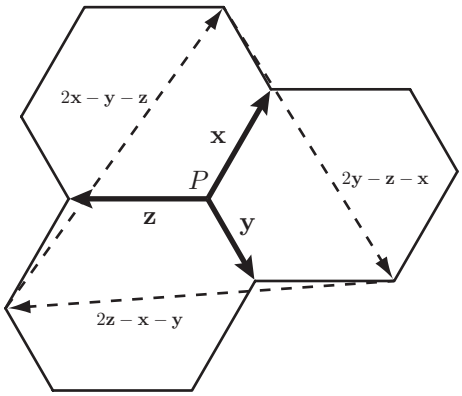



FIGURE 21. In partitions of a torus into three hexagons that are congruent translations of each other, examining three closed geodesics (represented by the dotted vectors) shows that if the hexagons are not regular, then the torus is not hexagonal.

*Proof.* Based on the partition, we draw a fundamental domain of the torus as in Figure 21. Let $P$ be the vertex in the center of the figure where the three hexagons meet; let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be the vectors from $P$ outward along the edges of the hexagon tiling, with magnitudes $x, y, z$.

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The three vectors $3\mathbf{x}-3\mathbf{y}$, $3\mathbf{y}-3\mathbf{z}$, $3\mathbf{z}-3\mathbf{x}$ that start at $P$ on the torus also end at $P$. If in addition $\mathbf{x},\mathbf{y},\mathbf{z}$ are parallel to the short directions, it follows easily that $x,y,z$ are multiples of $1/3$. In fact, $x,y,z$ must then equal $1/3$: if any side length were longer, then the area of the three hexagons would exceed that of the torus, which is impossible.

Otherwise, suppose that the perimeter of the partition is at most 3. The three dotted vectors in Figure 21 all begin and end at the same point on the torus. On the hexagonal torus, any such vector has magnitude equal to 1, equal to $\sqrt{3}$, or greater than or equal to 2.

By assumption, the perimeter $3(x+y+z)$ of the partition satisfies

$$\begin{aligned} 3 &\geq 3(x+y+z) \\ &= \frac{3}{4}((2x+y+z)+(2y+z+x)+(2z+x+y)) \\ &\geq \frac{3}{4}(\|2\mathbf{x}-\mathbf{y}-\mathbf{z}\|+\|2\mathbf{y}-\mathbf{z}-\mathbf{x}\|+\|2\mathbf{z}-\mathbf{x}-\mathbf{y}\|). \end{aligned}$$

It follows that the three magnitudes of the three dotted vectors equal one of the following triples: $1,1,1$; $1,1,\sqrt{3}$; or $1,1,2$. The third case is impossible by the triangle inequality.

Without loss of generality, $2\mathbf{z}-\mathbf{x}-\mathbf{y}$ has maximal magnitude among the three vectors (i.e., either 1 or $\sqrt{3}$). Observe that

$$\begin{aligned} 1 &= (2\mathbf{x}-\mathbf{y}-\mathbf{z})\cdot(2\mathbf{x}-\mathbf{y}-\mathbf{z}) \\ &= 4\mathbf{x}\cdot\mathbf{x}+\mathbf{y}\cdot\mathbf{y}+\mathbf{z}\cdot\mathbf{z}-4\mathbf{x}\cdot\mathbf{y}-4\mathbf{x}\cdot\mathbf{z}+2\mathbf{y}\cdot\mathbf{z} \\ &= 4x^2+y^2+z^2-4xy\cos(2\pi/3)-4xz\cos(2\pi/3)+2yz\cos(2\pi/3) \\ &= (x+y+z)^2+3x^2-3yz, \end{aligned}$$

or

$$1-(x+y+z)^2=3(x^2-yz).$$

Solving this and the two analogous equations gives $(x,y,z)=(1/3,1/3,1/3)$ or $(x,y,z)=(\sqrt{3}/9,\sqrt{3}/9,4\sqrt{3}/9)$, depending on whether $2\mathbf{z}-\mathbf{x}-\mathbf{y}$ has magnitude 1 or $\sqrt{3}$. In the latter case, the partition has perimeter $3(x+y+z)=2\sqrt{3}>3$, a contradiction. In the former case, the partition consists of three regular hexagons of side length $\frac{1}{3}$. It easily follows that the sides of the hexagons are parallel to the three short directions of the torus, and that the perimeter of the partition is three. □

**Corollary 5.4.11.** *If a perimeter-minimizing hexagon tiling on the hexagonal torus contains three components in the tiling plus exterior, then the tiling is a standard hexagon tiling.*

*Proof.* By the perimeter bound (Proposition 3.8), the perimeter of the tiling is at most 3. By this result and Lemma 5.4.9, the edges of the hexagon tiling are parallel to the edges of a hexagon tiling satisfying the hypotheses of Lemma 5.4.10. By Lemma 5.4.10, the edges of the new tiling are parallel to the three short directions of the torus. Therefore, so are the edges of the original tiling. □

**Corollary 5.4.12.** *Any standard hexagon tiling has perimeter three.*

*Proof.* By Lemma 5.4.9, a standard hexagon tiling has the same perimeter as a partition of the torus satisfying the conditions of Lemma 5.4.10. By Lemma 5.4.10, this

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partition has perimeter three, so the original standard hexagon tiling has perimeter three as well. □

**Proposition 5.4.13.** *If a hexagon tiling is perimeter minimizing, then it is the standard hexagon tiling.*

*Proof.* By Lemma 5.4.1, the number of components in a minimizing hexagon tiling plus exterior is divisible by three. By Lemma 5.4.6, a hexagon tiling with six or more components is not perimeter minimizing. Thus, a perimeter-minimizing hexagon tiling must have exactly three components.

By Lemma 5.4.9, a perimeter-minimizing three-component hexagon tiling has the same perimeter as a hexagon tiling in which the area of each component is one-third the area of the torus. By Lemma 5.4.3, this perimeter is greater than three except on the hexagonal torus. Hence, by Proposition 3.8, any perimeter-minimizing three-component hexagon tiling must lie on the hexagonal torus. By Corollary 5.4.11, any such tiling is the standard hexagon tiling. □

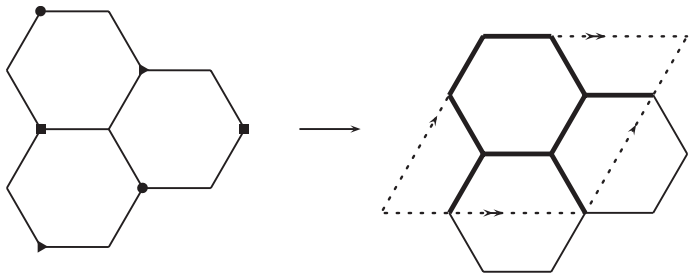


FIGURE 22. This tiling of three congruent hexagons fits on the hexagonal torus and is one of many perimeter-minimizing standard hexagon tilings.

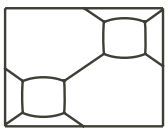


FIGURE 23. An octagon-square tiling always loses to some other double bubble.

**Octagon-square tilings**. Recall that by Definition 7, an octagon-square tiling is a tiling in which one of the three regions consists of two curvilinear quadrilaterals, and each of the other two regions consists of one curvilinear octagon (see Figure 23). The proof that an octagon-square tiling is not perimeter minimizing generalizes an argument described to us by Gary Lawlor eliminating most octagon-square tilings on rectangular tori. Lemma 5.4.14 and Corollary 5.4.15 compare a purported minimizing octagon-square tiling to the band lens in order to establish an upper bound on the length of the interface between the two eight-sided regions. Then, Proposition 5.4.19 and Lemma 5.4.21 establish a lower bound on the length of the same interface by examining projections of the boundary curves of the tiling

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onto four different directions. Proposition 5.4.22 shows that these upper and lower bounds cannot be satisfied at the same time, implying that no octagon-square tiling is perimeter minimizing.

**Lemma 5.4.14.** *Let $A_1 \leq A_2 \leq A_0$ be three areas whose sum is equal to the area of the torus. Consider a lens enclosing area $A_1$ and with perimeter $P$ and diameter $D$. For any pair of areas from $A_0, A_1, A_2$, a minimizing double bubble enclosing those areas has perimeter less than or equal to*

$$2 + P - D = 2 + \sqrt{A_1 \left( \frac{8\pi}{3} - 2\sqrt{3} \right)},$$

*with equality only if there is a band lens that encloses area $A_1$ in the lens and $A_2$ in the band.*

*Proof.* The expression for $P - D$ is easily derived, and we do not include the proof here (see, for instance, the derivations in Proposition 7.3.1).

Since all of the double bubbles that enclose two of the areas from $\{A_0, A_1, A_2\}$ have the same perimeter, it suffices to prove the claim for a minimizing double bubble enclosing the areas $A_1, A_2$.

If $D \geq 1$, then $2 + P - D > 2 + 2D - D \geq 3$. Thus, by the perimeter bound (Proposition 3.8), $2 + P - D$ exceeds the perimeter of a minimizing double bubble enclosing areas $A_1, A_2$.

Suppose instead that $D < 1$. Then the lens can be embedded on the torus so that its axis lies along a shortest closed geodesic $\gamma_1$ (Remark 2.3.2). There exists another shortest closed geodesic $\gamma_2$ such that $\gamma_1$ and $\gamma_2$ divide the complement of the lens into one part with area $A_0$ and another part with area $A_2$. The portions of these geodesics that lie outside the lens, along with the boundary of the lens, thus form the boundary of a double bubble enclosing areas $A_1, A_2$. The perimeter of the double bubble is at most the total perimeter of $\gamma_2$, the lens, and the portion of $\gamma_1$ lying outside the lens: $1 + P + (1 - D) = 2 + P - D$. Equality holds only if $\gamma_2$ does not intersect the interior of the lens — i.e., only if the double bubble is a band lens. Hence we have the desired result. □

**Corollary 5.4.15.** *Let $A_1 \leq A_2 \leq A_0$ be three areas whose sum is equal to the area of the torus. A minimizing double bubble for any pair of areas from $A_0, A_1, A_2$ has perimeter less than*

$$2 + 2\sqrt{\pi A_1}.$$

*Remark* 5.4.16. Rather than examining partitions of the torus into a lens and two bands as in Lemma 5.4.14, one could prove the corollary directly by examining double bubbles for which the region of area $A_1$ is a circle and the other two regions are bands.

**Lemma 5.4.17.** *If two closed geodesics with lengths $L_1, L_2$ meet at an angle $\phi \in (0, \pi/2]$, then $L_1 + L_2 \geq 1 + 2\cos\phi$. If $L_1, L_2 \in [1, 3/2]$, then $\phi \geq \cos^{-1}(7/9)$.*

*Proof.* The torus can be naturally viewed as the plane with various points identified. In the plane, choose a point $P$ that represents the intersection of the two closed geodesics. Draw a segment $\overline{PQ}$ of length $L_1$ representing the corresponding closed geodesic (on the torus). Similarly, draw a segment $\overline{PR}$ of length $L_2$ representing the other closed geodesic, such that segments $\overline{PQ}$ and $\overline{PR}$ meet at an angle $\phi$.

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Because $Q$ and $R$ represent the same point on the torus, $\overline{QR}$ represents a closed geodesic with length at least 1. Thus, applying the Law of Cosines to triangle $PQR$,

$$1 \leq QR^2 = L_1^2 + L_2^2 - 2L_1L_2\cos\phi = (L_1+L_2)^2 - (2+2\cos\phi)L_1L_2.$$

From $0 \leq (L_1-1)(L_2-1)$, we derive $L_1+L_2-1 \leq L_1L_2$, so that the above inequality becomes

$$1 \leq (L_1+L_2)^2 - (2+2\cos\phi)(L_1+L_2-1)$$

or

$$0 \leq (L_1+L_2-1-2\cos\phi)(L_1+L_2-1).$$

Because $L_1+L_2$ is greater than 1, it must be greater than or equal to $1+2\cos\phi$, as desired.

Suppose now that $L_1, L_2 \in [1, 3/2]$. For a fixed $\phi$, $L_1^2+L_2^2-2L_1L_2\cos\phi$ is convex in each variable $L_1, L_2$, so it is maximized for some $(L_1, L_2)$ with $L_1, L_2 \in \{1, 3/2\}$. For the four pairs $(L_1, L_2)$ satisfying this condition, $L_1^2+L_2^2-2L_1L_2\cos\phi < 1$ when $\cos\phi > 7/9$. Therefore, $\phi \geq \cos^{-1}(7/9)$. □

*Remark* 5.4.18. The next result, Proposition 5.4.19, generalizes a result about "taxicab distances" in the plane. The taxicab distance between any two points, defined with respect to two given orthogonal directions, is the minimum length of a piecewise linear path between them that travels only along the given orthogonal directions (analogous to the streets a taxicab travels on). The length of a segment between two points is at least $\frac{1}{\sqrt{2}}$ times the taxicab distance between them. (Setting $\phi = \pi/2$ in Proposition 5.4.19 gives this estimate.) Given two lines that are *not* orthogonal (to take an extreme example, lines parallel to two short directions on the hexagonal torus), it is useful to have an analogous estimate. We derive such an estimate relating length to the sum of two taxicab distances, one corresponding to each of the two given lines.

**Proposition 5.4.19.** *Given two pairs of orthogonal lines in the plane, suppose that each line in the first pair meets each line in the second pair at an angle $\phi$ or $\pi/2-\phi$, with $\phi \in [\pi/4, \pi/2]$. For any segment $\gamma$, let $T(\gamma)$ be the sum of the lengths of the projections of $\gamma$ onto the four lines. Then*

$$T(\gamma) \leq 2(\sin(\phi/2)+\cos(\phi/2))\,\mathrm{Length}(\gamma).$$

*Proof.* Without loss of generality, assume that one pair of lines are the coordinate axes, and that the other lines lie clockwise angles $\phi$ from the positive $x$- and $y$-axes. Without loss of generality, further assume that $\gamma$ has one endpoint at the origin and points into the first quadrant of the plane.

Let $\mathbf{u}$ be a vector parallel to $\gamma$, whose magnitude equals the length of $\gamma$. Let $\mathbf{v}_1$, $\mathbf{w}_1$, $\mathbf{v}_2$, $\mathbf{w}_2$ be unit vectors pointing along the four given lines, chosen as in Figure 24. Then

$$T(\gamma) = \mathbf{u}\cdot(\mathbf{v}_1+\mathbf{w}_1+\mathbf{v}_2) + |\mathbf{u}\cdot\mathbf{w}_2|.$$

If $\mathbf{u}\cdot\mathbf{w}_2$ is negative, then

$$T(\gamma) = \mathbf{u}\cdot(\mathbf{v}_1+\mathbf{w}_1+\mathbf{v}_2-\mathbf{w}_2) \leq \|\mathbf{u}\|\|\mathbf{v}_1+\mathbf{w}_1+\mathbf{v}_2-\mathbf{w}_2\|,$$

with equality when $\mathbf{u}$ is parallel to $\mathbf{v}_1+\mathbf{w}_1+\mathbf{v}_2-\mathbf{w}_2$. When this is true, $\mathbf{u}$ bisects the angle formed by $\mathbf{v}_1$ and $\mathbf{v}_2$, and it is easy to compute $\mathbf{u}\cdot\mathbf{v}_1 = \mathbf{u}\cdot\mathbf{v}_2 = \|\mathbf{u}\|\cos(\phi/2)$ and $\mathbf{u}\cdot\mathbf{w}_1 = -\mathbf{u}\cdot\mathbf{w}_2 = \|\mathbf{u}\|\sin(\phi/2)$. Hence, when $\mathbf{u}\cdot\mathbf{w}_2$ is negative,

$$T(\gamma) \leq \|\mathbf{u}\|(2\cos(\phi/2)+2\sin(\phi/2)) = 2(\cos(\phi/2)+\sin(\phi/2))\mathrm{Length}(\gamma).$$

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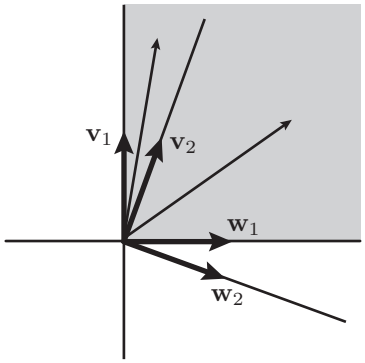



FIGURE 24. If a vector in the first quadrant has fixed magnitude, then the sum of its projections onto the four directions $\mathbf{v}_i$, $\mathbf{w}_i$ drawn above attains a local maximum in two places: when it bisects the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, and when it bisects the angle between $\mathbf{w}_1$ and $\mathbf{v}_2$.

Similarly, if $\mathbf{u} \cdot \mathbf{w}_2$ is nonnegative, then $T(\gamma)/\|\mathbf{u}\|$ is maximized when $\mathbf{u}$ bisects the angle formed by $\mathbf{w}_1$ and $\mathbf{v}_2$. We then find that

$$\begin{aligned} T(\gamma) &\leq 2(\cos(\pi/4 - \phi/2) + \sin(\pi/4 - \phi/2))\text{Length}(\gamma) \\ &\leq 2(\cos(\phi/2) + \sin(\phi/2))\text{Length}(\gamma), \end{aligned}$$

where the second inequality is true because $\phi \in [\pi/4, \pi/2]$. □

**Lemma 5.4.20.** *If* $\cos^{-1}(7/9) \leq \phi \leq \pi/2$, *then*

$$\frac{\max\{2, 1 + 2\cos\phi\}(1 + \sin\phi) - 3}{\sin\left(\frac{\max\{\phi, \pi/2-\phi\}}{2}\right) + \cos\left(\frac{\max\{\phi, \pi/2-\phi\}}{2}\right) - 1} \geq 2.$$

*Proof.* The denominator of the left-hand side is at most $\sqrt{2} - 1$. For $\cos^{-1}(7/9) \leq \phi \leq \pi/4$, we have $\cos\phi \geq \sqrt{2}/2$ and $\sin\phi > 2 - \sqrt{2}$. Hence, the numerator of the left-hand side is greater than $2\sqrt{2} - 2$. Therefore, the left-hand side is greater than 2 when $\phi \leq \pi/4$.

For $\pi/4 \leq \phi \leq \pi/3$, $\max\{2, 1+2\cos\phi\} = 1+2\cos\phi$ and $\max\{\phi, \pi/2-\phi\} = \phi$. It is easy to check that on the interval $\phi \in [\pi/4, \pi/3]$, the numerator of the left-hand side is decreasing while the denominator is increasing. Because the inequality holds when $\phi = \pi/3$ (with equality), it holds along the entire interval.

For $\pi/3 \leq \phi \leq \pi/2$, $\max\{2, 1 + 2\cos\phi\} = 2$ and $\max\{\phi, \pi/2 - \phi\} = \phi$. Clearing the denominator of the desired inequality, then simplifying, yields the following inequalities (each equivalent to the desired inequality):

$$\begin{aligned} 2\sin\phi - 1 &\geq 2\sin(\phi/2) + 2\cos(\phi/2) - 2, \\ 4\sin(\phi/2)\cos(\phi/2) - 1 &\geq 2\sin(\phi/2) + 2\cos(\phi/2) - 2, \\ (2\sin(\phi/2) - 1)(2\cos(\phi/2) - 1) &\geq 0. \end{aligned}$$

The last inequality holds for $\pi/3 \leq \phi \leq \pi/2$. □

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**Lemma 5.4.21.** *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the boundaries of two bands with distinct homologies, where $\mathcal{B}_1$ and $\mathcal{B}_2$ may overlap each other nontrivially (see Figure 25). If the length of $\mathcal{B}_1 \cup \mathcal{B}_2$ does not exceed 3, then the length of $\mathcal{B}_1 \cap \mathcal{B}_2$ is greater than or equal to 2.*

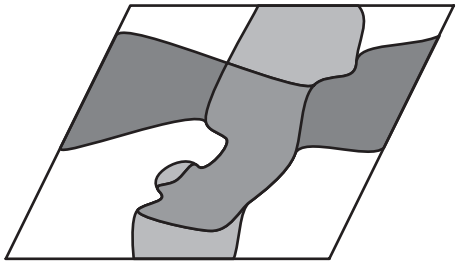
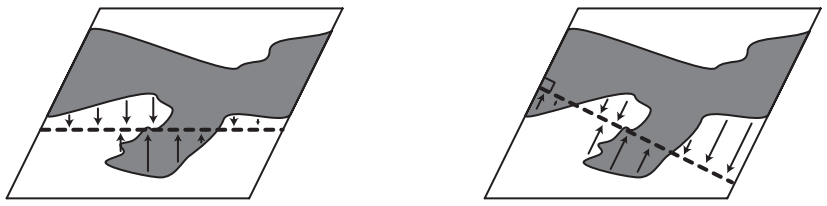


FIGURE 25. Given two bands with overlapping boundary, we can project the boundaries onto various directions to show that the length of the overlap is large. (Above, the bands are filled with dark and light shades, with a medium shade along their intersection.)

*Proof.* Assume that the length of $\mathcal{B}_1 \cup \mathcal{B}_2$ is as at most 3, and let $\gamma_1, \ldots, \gamma_k$ be the components of $\mathcal{B}_1 \cap \mathcal{B}_2$. For $i = 1, 2$, let $L_i$ be the length of the closed geodesic $g_i$ with the same homology as the components of $\mathcal{B}_i$. Observe that $L_i \leq 3/2$, because otherwise $\mathcal{B}_i$ would have length greater than 3. Let $g_1$ and $g_2$ meet each other at an angle $\phi \in (0, \pi/2]$.

Below, we will take each closed curve in $\mathcal{B}_1$ and project it onto two directions — onto the direction of $g_1$ (Figure 26(a)) and onto the direction orthogonal to $g_2$ (Figure 26(b)). We will also take similar projections of the components of $\mathcal{B}_2$. By analyzing these projections' effects on various pieces of $\mathcal{B}_1 \cup \mathcal{B}_2$, we will obtain our desired inequality.

For each $\gamma_i$, let $T(\gamma_i)$ be the sum of the projection of $\gamma_i$ onto four directions: the two directions of the closed geodesics $g_1$ and $g_2$, and the two directions orthogonal to these closed geodesics.



(a) (b)

FIGURE 26. By projecting each component of the boundary of the bands in two directions, we bound the length of the intersection of the two boundaries.

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Take a component of $\mathcal{B}_1$, and project it onto the direction of $g_1$ and onto the direction orthogonal to $g_2$. The first projection has length at least $L_1$, and the second projection has length at least $L_1 \sin\phi$. Project the other component of $\mathcal{B}_1$ in a similar fashion.

Likewise, project each component of $\mathcal{B}_2$ onto the direction of $g_2$ and onto the direction orthogonal to $g_1$. In all, the projections of the components of $\mathcal{B}_1$ and $\mathcal{B}_2$ have total length greater than or equal to

$$2(L_1 + L_2)(1 + \sin\phi).$$

Each component of $\mathcal{B}_1 \cup \mathcal{B}_2 - \mathcal{B}_1 \cap \mathcal{B}_2$ is projected exactly twice, so they contribute at most

$$2 \cdot \text{Length}(\mathcal{B}_1 \cup \mathcal{B}_2 - \mathcal{B}_1 \cap \mathcal{B}_2)$$

to the total length of the projections. Each $\gamma_i$, on the other hand, contributes at most $T(\gamma_i)$. Therefore,

$$\begin{aligned} 2(L_1 + L_2)(1 + \sin\phi) &\leq 2 \cdot \text{Length}(\mathcal{B}_1 \cup \mathcal{B}_2 - \mathcal{B}_1 \cap \mathcal{B}_2) + \sum_{i=1}^{k} T(\gamma_i) \\ &\leq 2\left(3 - \sum_{i=1}^{k} \text{Length}(\gamma_i)\right) + \sum_{i=1}^{k} T(\gamma_i). \end{aligned}$$

We estimate the two sides of this inequality as follows. First, $L_1 + L_2 \geq 2$, and by Lemma 5.4.17, $L_1 + L_2 \geq 1 + 2\cos\phi$. Hence,

$$2\max\{2, 1 + 2\cos\phi\}(1 + \sin\phi) \leq 2(L_1 + L_2)(1 + \sin\phi).$$

Second, from the two pairs of orthogonal directions we project onto, we can choose one direction from each pair so that they meet at an angle of $\max\{\phi, \pi/2 - \phi\} \in [\pi/4, \pi/2]$. Applying Proposition 5.4.19 then shows that

$$2\left(3 - \sum_{i=1}^{k} \text{Length}(\gamma_i)\right) + \sum_{i=1}^{k} T(\gamma_i)$$

is at most

$$6 + \left(2\sin\left(\frac{\max\{\phi, \pi/2 - \phi\}}{2}\right) + 2\cos\left(\frac{\max\{\phi, \pi/2 - \phi\}}{2}\right) - 2\right) \sum_{i=1}^{k} \text{Length}(\gamma_i).$$

Therefore, to prove that $\sum_{i=1}^{k} \text{Length}(\gamma_i) \geq 2$, it suffices to prove that

$$\frac{\max\{2, 1 + 2\cos\phi\}(1 + \sin\phi) - 3}{\sin\left(\frac{\max\{\phi, \pi/2 - \phi\}}{2}\right) + \cos\left(\frac{\max\{\phi, \pi/2 - \phi\}}{2}\right) - 1} \geq 2.$$

By Lemma 5.4.17, $\cos^{-1}(7/9) \leq \phi$; also, by definition, $\phi \leq \pi/2$. By Lemma 5.4.20, the above inequality holds for such $\phi$. □

**Proposition 5.4.22.** *No octagon-square tiling is perimeter minimizing.*

*Proof.* Let $\mathcal{B}_1$ be the boundary of the band enclosing one eight-sided region and one four-sided component (Figure 27(a)). Let $\mathcal{B}_2$ be the boundary of the band enclosing the same eight-sided region and the other four-sided component (Figure 27(b)). If the length of $\mathcal{B}_1 \cup \mathcal{B}_2$ is greater than 3, then by the perimeter bound (Proposition 3.8), the octagon square is not perimeter minimizing.

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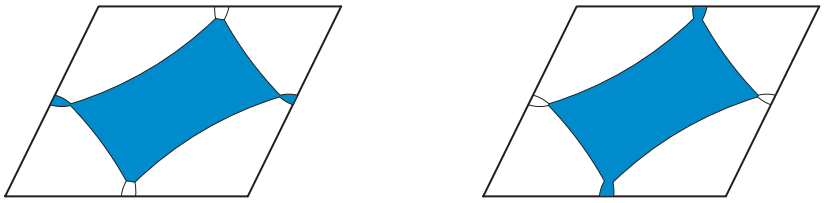



FIGURE 27. Given the above octagon-square tiling, the boundaries of the two colored bands either have a union with length greater than three, or an intersection with length greater than or equal to two. In both cases, it can be shown that the tiling is not perimeter minimizing.

Otherwise, by Lemma 5.4.21, the length of $\mathcal{B}_1 \cap \mathcal{B}_2$ is at least 2. Let $A_1$ be the area of the region containing the four-sided components. By the isoperimetric inequality for the plane, the perimeter of this region is at least $2\sqrt{\pi A_1}$. Therefore, the perimeter of the octagon-square tiling is at least $2 + 2\sqrt{\pi A_1}$. By Corollary 5.4.15, the tiling is not a minimizer. □

**Proposition 5.4.23.** *The only tiling that can be perimeter minimizing is the standard hexagon tiling.*

*Proof.* By Corollary 5.2.9, the tiling must be a hexagon tiling or an octagon-square tiling. By Proposition 5.4.22, no octagon-square tiling is perimeter minimizing. By Proposition 5.4.13, the only minimizing hexagon tiling is the standard hexagon tiling. □

## 6. Perimeter-minimizing double bubbles on the flat two-torus

The Main Theorem characterizes the perimeter-minimizing double bubbles on all flat two-tori, as depicted in Figure 1. Corollaries 6.3 and 6.5 characterize the minimizing double bubbles on the flat infinite cylinder and the flat infinite strip with free boundary. Remark 6.7 makes a conjecture about the double bubble problem on the closely related Klein bottle.

**Main Theorem 6.1.** *A perimeter-minimizing double bubble on a flat two-torus is of one of the following five types (perhaps after relabelling the three regions), depending on the areas to be enclosed:*

(i) *the standard double bubble,*
(ii) *a standard chain,*
(iii) *the band lens,*
(iv) *the double band, or*
(v) *the standard hexagon tiling (only for the hexagonal torus).*

*Proof.* Proposition 4.6 shows that a minimizer must be a contractible double bubble, a tiling, a swath, a band adjacent to a set of components whose union is contractible, or the double band (after perhaps relabelling the two regions and the exterior).

By Proposition 5.1.2, the standard double bubble is the only possible contractible minimizer. By Proposition 5.2.15, a standard chain is the only possible minimizer

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among swaths. By Proposition 5.3.5, the band lens is the only possible minimizer among bands adjacent to a set of components whose union is contractible. By Proposition 5.4.23, the standard hexagon tiling (on the hexagonal torus) is the only possible minimizing tiling. □

*Remark* 6.2. Using the phase portrait computations of Section 8, it can be confirmed that all five types of minimizers do occur.

**Corollary 6.3.** *A perimeter-minimizing double bubble on the flat infinite cylinder is of one of the following four types, depending on the areas to be enclosed:*

(i) *the standard double bubble,*
(ii) *a standard chain,*
(iii) *the band lens, or*
(iv) *the double band.*

*Proof.* Without loss of generality, assume that the cylinder has circumference 1. Let $\Sigma$ be a perimeter-minimizing double bubble on the cylinder that is not of one of the four types listed above. Because $\Sigma$ has finite perimeter, it can be placed on some long rectangular torus (for example, one with length greater than 3). Then by Theorem 6.1, there exists a double bubble $\Sigma'$ on the torus enclosing the same areas as $\Sigma$ but with less perimeter. Because the torus has length greater than 3, and because $\Sigma'$ has perimeter less than or equal to 3, it is contractible or wraps around a short direction of the torus. Therefore, $\Sigma'$ can be placed on the cylinder, and $\Sigma$ cannot be perimeter minimizing. □

**Corollary 6.4.** *Every minimizing double bubble on the infinite cylinder has a line of reflective symmetry that is perpendicular to a shortest closed geodesic, or is congruent to a double bubble with this property.*

*Proof.* Given a minimizing standard double bubble, it can be rotated so that its axis of symmetry is perpendicular to a shortest closed geodesic. Each of the other minimizers in Corollary 6.3 has a line of reflective symmetry with the required property. (A standard chain on the infinite cylinder has a line of reflective symmetry that is parallel to a shortest closed geodesic, but it also has two lines of reflective symmetry perpendicular to a shortest closed geodesic.) □

**Corollary 6.5.** *Any minimizing double bubble on the flat infinite strip with free boundary is the quotient of a minimizing double bubble on an infinite cylinder (with twice the width of the strip) by a reflection across one of its lines of symmetry, using the free boundary of the strip to complete the double bubble, as in Figure 28.*

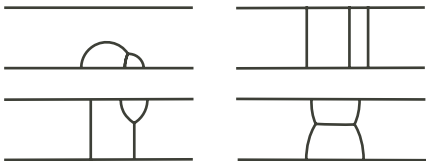


FIGURE 28. The four perimeter-minimizing double bubbles on the flat infinite strip with free boundary are each half of some perimeter-minimizing double bubble on the infinite cylinder.

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*Proof.* The infinite strip is the quotient space of the infinite cylinder by some reflection. Hence we can think of the infinite cylinder as a double cover of the infinite strip. Lifting a minimizer $\Sigma$ on the strip yields a double bubble $\Sigma'$ on the cylinder with twice the perimeter and enclosed areas as $\Sigma$. If $\Sigma'$ is a minimizing double bubble, then we are done. Otherwise, by Corollary 6.4, there exists a minimizer $\Gamma'$ that encloses the same areas as $\Sigma'$ and that has at least one line of reflective symmetry perpendicular to a shortest closed geodesic. Quotienting by the corresponding reflection yields the original infinite strip with free boundary, along with a new double bubble $\Gamma$ enclosing the same areas as $\Sigma$ with less perimeter, a contradiction. □

*Remark* 6.6. For finite rectangular cylinders with free boundary (i.e., the quotient space of a rectangular torus by some reflection), a result analogous to Corollary 6.5 holds if the distance between the two components of the boundary is sufficiently small with respect to the length of each component of the boundary.

*Remark* 6.7 (Klein bottles). We conjecture that perimeter-minimizing double bubbles on the flat Klein bottle are of virtually the same form as on the flat torus. Like the flat torus, we can represent the flat Klein bottle as a planar parallelogram with opposite sides identified and no angle less than 60 degrees, except that we identify the top and bottom sides with a flip. Because of this flip, on a Klein bottle with reasonably long top and bottom, any minimizing double band is bounded by two closed geodesics each of length 2, and has total perimeter 4 (instead of 3). This is an important difference, and for proofs on the torus where we required a perimeter bound of 3, it will be necessary to modify our approach to accommodate the higher bound for the Klein bottle. Note also that the placement of perimeter-minimizing double bubbles on a wide Klein bottle is less flexible than on a torus. The standard chain, band lens, and double band must all be oriented symmetrically about one of the two unique shortest closed geodesics of a wide Klein bottle (which are always vertical). However, all of the conjectured minimizers (including the standard hexagon tiling) do remain well defined, so long as they are placed carefully.

We also conjecture that the perimeter-minimizing double bubbles on the infinite cylinder (which can be thought of as an infinitely long rectangular torus) carry over in the analogous space corresponding to the Klein bottle, that is, a Möbius strip of infinite length — the surface between two horizontal lines in the plane, where the lines are identified with a flip. Here the same comments apply, but the absence of tilings may make the full solution more accessible.

## 7. Formulas for perimeter and area

This section gives formulas for the areas of the enclosed regions and the perimeters of the four potential minimizers different from the double band: the standard double bubble (Proposition 7.1.1), the standard chain (Propositions 7.2.1, 7.2.2), the band lens (Proposition 7.3.1), and the standard hexagon tiling (Proposition 7.4.1). In addition, Lemmas 7.2.4 and 7.4.5 prove that when a standard chain with axis-length one or a standard hexagon tiling *seems* to be minimizing, based on formal perimeter calculations that we can perform even if the double bubble actually does not exist on a given torus, then the double bubble actually exists and is minimizing. As described in Section 8, we use these formulas and results to produce the phase portraits in Figure 2.

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*Remark* 7.1. We begin by stating a few simple geometric formulas that describe circular arcs meeting chords. Given a circular arc subtended by a chord of length $C$ that it meets at an angle $\theta$, the area between the arc and the chord, the length of the arc, and the radius of the circle are given by

$$A(\theta, C) = \frac{C^2(\theta - \sin\theta\cos\theta)}{4\sin^2\theta}, \tag{7.1}$$

$$L(\theta, C) = \frac{C\theta}{\sin\theta}, \tag{7.2}$$

$$R(\theta, C) = \frac{C}{2\sin\theta}. \tag{7.3}$$

7.1. **Formulas for the standard double bubble.** Proposition 7.1.1 gives formulas for perimeter and area of the standard double bubble. These formulas first appeared in [F, Section 2].

**Proposition 7.1.1.** *Let $R_1$ and $R_2$ be the regions of higher and lower pressure enclosed in a standard double bubble. The area and perimeter formulas of a standard double bubble in terms of the separating chord length $C$ and the angle $\theta$ at which it meets the interior arc are*

$$\begin{aligned} A_{R_1}(\theta, C) &= A(2\pi/3 - \theta, C) + A(\theta, C), \\ A_{R_2}(\theta, C) &= A(2\pi/3 + \theta, C) - A(\theta, C), \\ P(\theta, C) &= L(2\pi/3 + \theta, C) + L(2\pi/3 - \theta, C) + L(\theta, C). \end{aligned}$$

*Proof.* The arc between $R_1$ and the exterior meets the separating chord $C$ at $\frac{2\pi}{3} - \theta$, and the arc between $R_2$ and the exterior meets the separating chord $C$ at $\frac{2\pi}{3} + \theta$. The given formulas follow immediately. □

7.2. **Formulas for the standard chain.** To parameterize the standard chain, we distinguish between those in which the enclosed regions have unequal pressure (Proposition 7.2.1) and those in which the enclosed regions have equal pressure (Proposition 7.2.2).

**Proposition 7.2.1.** *Consider a standard chain with axis-length $L_0$ and with two components of unequal pressure. Let $R_1$ and $R_2$ be the enclosed regions, and suppose that $R_1$ has greater pressure than $R_2$. Let $C_1$ be the length of either chord subtending an arc separating $R_1$ from the exterior, where the chord and arc meet at (interior) angles $\theta_1$. (See Figure 29.) The area and perimeter formulas are*

$$\begin{aligned} A_{R_1}(\theta_1, C_1) &= 2A(\theta_1, C_1) + 2A(\pi/6 - \theta_1, C_3) + C_1C_3, \\ A_{R_2}(\theta_1, C_1) &= 2A(\pi/3 - \theta_1, C_2) - 2A(\pi/6 - \theta_1, C_3) + C_2C_3, \\ P(\theta_1, C_1) &= 2L(\theta_1, C_1) + 2L(\pi/3 - \theta_1, C_2) + 2L(\pi/6 - \theta_1, C_3), \end{aligned}$$

*where $C_2$ and $C_3$ are given by*

$$\begin{aligned} C_2 &= L_0 - C_1, \\ C_3 &= \frac{C_1(L_0 - C_1)\sin(\frac{\pi}{6} - \theta_1)}{(L_0 - C_1)\sin\theta_1 - C_1\sin(\frac{\pi}{3} - \theta_1)}. \end{aligned}$$

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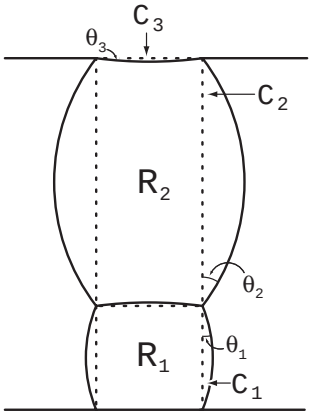



FIGURE 29. This picture illustrates the basic parameters involved in computing perimeter and area for a standard chain.

*A standard chain with these parameters exists on an infinite cylinder of circumference $L_0$ if and only if $C_1 \in (0, \frac{1}{2}L_0)$ and $\theta_1 \in (\sin^{-1}(C_1/L_0), \pi/6)$.*

*Proof.* Let $R_0$ be the low-pressure exterior region. By the results of Section 5.2, the vertices of $R_1$ and $R_2$ form a rectangle. For $i = 1, 2$, let either arc separating $R_i$ from $R_0$ have curvature $\kappa_i$ and radius of curvature $r_i = 1/\kappa_i$, let $C_i$ denote the length of the chord with the same endpoints as the arc, and let $\theta_i$ be the (interior) angle the chord and the arc make. Define $\kappa_3$, $r_3$, $C_3$, and $\theta_3$ similarly with respect to the interface between $R_1$ and $R_2$.

Because the arcs in the boundary of the chain meet at $2\pi/3$, we have

$$\theta_2 = \pi/3 - \theta_1 \qquad \text{and} \qquad \theta_3 = \pi/6 - \theta_1.$$

Of course,

$$C_2 = L_0 - C_1.$$

Also, by the Extended Law of Sines,

$$C_i = 2r_i \sin\theta_i \qquad \text{for } i = 1, 2, 3,$$

allowing us to solve for $r_1$ and $r_2$ in terms of $\theta_1$ and $C_1$. By the cocycle condition (as in Proposition 3.1),

$$-\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = -\kappa_1 + \kappa_2 + \kappa_3 = 0,$$

and we may solve for $r_3$ in terms of $r_1$ and $r_2$. These equations allow us to find $C_3$ in terms of $\theta_1$ and $C_1$:

$$\begin{aligned} C_3 &= 2\frac{1}{1/r_1 - 1/r_2}\sin\theta_3 = \frac{2r_1r_2}{r_2 - r_1}\sin\theta_3 \\ &= \frac{C_1C_2}{C_2\sin\theta_1 - C_1\sin\theta_2}\sin\theta_3 = \frac{C_1(L_0 - C_1)\sin(\pi/6 - \theta_1)}{(L_0 - C_1)\sin\theta_1 - C_1\sin(\pi/3 - \theta_1)}. \end{aligned}$$

It remains to show that $C_1 \in (0, L_0/2)$, $\theta_1 \in (\sin^{-1}(C_1/L_0), \frac{\pi}{6})$, and that a corresponding standard chain exists (on some infinite cylinder) if these bounds hold.

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Because $r_2 > r_1$ and $\theta_2 > \theta_1$, we have $L_0 - C_1 = C_2 = 2r_2 \sin\theta_2 > 2r_1 \sin\theta_1 = C_1$, proving the bound on $C_1$. The upper bound on $\theta_1$ holds because $\pi/6 - \theta_1 = \theta_3 > 0$.

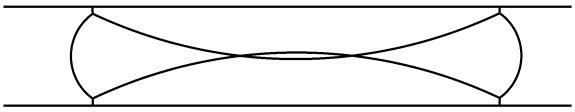


FIGURE 30. If the angles and lengths in the parameterization of a standard chain do not meet certain bounds, then the boundary arcs of the chain intersect each other.

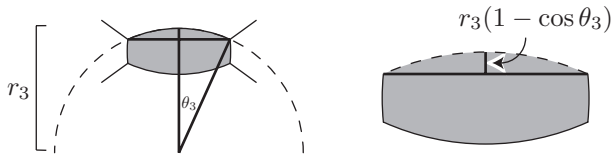



FIGURE 31. The arcs separating the components enclosed in a standard chain cannot intersect each other, yielding bounds on the parameters used to describe standard chains.

Given $C_1 \in (0, L_0/2)$ and $0 < \theta_1 < \pi/6$, a corresponding standard chain exists (on some infinite cylinder) if and only if it can be drawn so that the arcs separating $R_1$ and $R_2$ do not intersect. Consider one of these arcs. As shown in Figure 31, the distance between the midpoint of the arc and the chord subtending the arc is $r_3 - r_3\cos\theta_3$. The two arcs separating $R_1$ and $R_2$ do not intersect each other if and only if this distance is less than $\frac{1}{2}C_2$ — that is, if and only if

$$r_3(1-\cos\theta_3) < \frac{1}{2}C_2.$$

(This inequality fails if and only if the arcs intersect each other as in Figure 30.) Writing $r_3$, $\theta_3$, and $C_2$ in terms of $C_1$ and $\theta_1$, this inequality becomes (after much simplification)

$$C_1 < L_0 \sin\theta_1$$

or $\theta_1 > \sin^{-1}(C_1/L_0)$. □

**Proposition 7.2.2.** *Consider a standard chain with axis-length $L_0$ and with two regions $R_1$ and $R_2$ of equal pressure. The area and perimeter formulas are*

$$\begin{aligned} A_{R_1}(C_3) = A_{R_2}(C_3) &= \frac{2\pi - 3\sqrt{3}}{24}L_0^2 + \frac{1}{2}L_0 C_3 \\ P(C_3) &= \frac{2\pi}{3}L_0 + 2C_3, \end{aligned}$$

*where $C_3$ is the length of the chord separating $R_1$ from $R_2$.*

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*Proof.* For $i = 1, 2, 3$, define $\kappa_i$, $r_i$, $C_i$, and $\theta_i$ as in the first paragraph of the proof of Proposition 7.2.1. (Figure 29 shows many of these variables, although the standard chain in the figure does not contain two regions of equal pressure.)

Observe that $\kappa_1 = \kappa_2$ and hence $r_1 = r_2$. Also, $\theta_3 = 0$ and $\theta_1 = \theta_2 = \pi/6 - \theta_3 = \pi/6$. Therefore, $C_1 = 2r_1 \sin\theta_1 = 2r_2 \sin\theta_2 = C_2$, implying that $C_1 = C_2 = \frac{1}{2}L_0$ and $r_1 = r_2 = \frac{1}{2}L_0/(2\sin(\pi/6)) = \frac{1}{2}L_0$.

Although $C_1$ and $C_2$ are uniquely determined, $C_3$ is not. Still, because all other arc lengths and angles in the chain are fixed, we have simple formulas expressing the areas and perimeters of such chains in terms of $C_3$. Given the fixed angles and chord lengths found earlier, we have $A_{R_1}(C_3) = A_{R_2}(C_3) = 2A(\pi/6, \frac{L_0}{2}) + (\frac{1}{2}L_0)C_3$ and $P(C_3) = 4L(\pi/6, \frac{L_0}{2}) + 2C_3$. Using (7.1)–(7.2) to evaluate these expressions yields the desired formulas. □

*Remark* 7.2.3. The plots based on this parameterization prove the existence of a family of standard chains with fixed axis-length. In particular, we know that there exists a family of standard chains that wrap around a short direction of the torus.

**Lemma 7.2.4.** *Consider any standard chain on the unit-circumference cylinder enclosing areas $A_1, A_2$, and any torus with area greater than $A_1 + A_2$. The perimeter of the chain is greater than or equal to that of the minimizing double bubble(s) enclosing areas $A_1, A_2$ on the torus. If equality holds, the standard chain fits on the torus and is itself a minimizing double bubble on the torus.*

*Proof.* Consider a fundamental domain of the torus, a parallelogram with side lengths 1 and $L$ and with interior angle $\alpha \in [\pi/3, \pi/2]$. We orient the parallelogram so that the side with length 1 is vertical. Also, let $R_1$ be a component of higher (or equal) pressure enclosed in the chain, and let $R_2$ be the other component enclosed in the chain. For $i = 1, 2, 3$ define $C_i$ and $\theta_i$ as in the first paragraph of the proof of Proposition 7.2.1.

We immerse the chain in the torus, such that the chain and the side of length 1 have the same homology. It suffices to prove that if the chain is not embedded, then it has more perimeter than some (embedded) minimizer enclosing areas $A_1, A_2$ on the torus. Because the chain is not embedded in the torus, it does not only wrap around the torus in the direction of the side with length 1; it also wraps around the torus in the direction of the side with length $L$. We consider two cases: first, where the chain wraps around to intersect itself because the two components intersect each other; second, where instead at least one component intersects itself but not the other component.

First suppose that both components of the chain intersect each other. For $i = 1, 2$, consider either boundary arc of $R_i$ that lies along the boundary of the chain. The horizontal distance between the midpoint of this arc and the chord subtending it is $d_i = (C_i/2)\tan(\theta_i/2)$. Also, the vertices of the standard chain lie on two vertical closed geodesics, and the horizontal distance between these closed geodesics is $C_3$. Now, because the chain wraps around the torus in a nonvertical direction, the sum of $d_1$, $d_2$, and $C_3$ must exceed the horizontal distance $L\sin\alpha$ between the two vertical sides of the parallelogram. Hence,

$$\frac{C_1}{2}\tan\left(\frac{\theta_1}{2}\right) + \frac{C_2}{2}\tan\left(\frac{\theta_2}{2}\right) + C_3 > L\sin\alpha \geq \sin(\pi/3) = \frac{\sqrt{3}}{2}.$$

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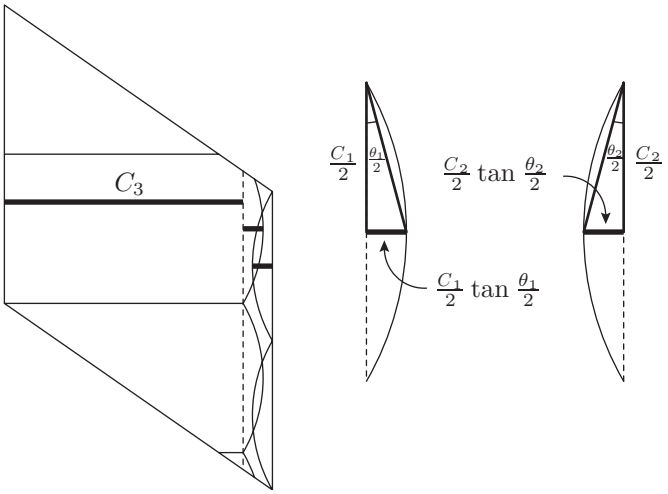



FIGURE 32. One way a standard chain might be nonembedded is if the two components intersect each other. In this case, by considering three segments perpendicular to the short direction with total length at least $\sqrt{3}/2$, it can be shown that the arcs separating the two components in the chain are long enough (greater than .5) to prove that the chain does not have minimizing perimeter.

From our parameterization, $C_1 + C_2 = 1$ and $\theta_2 < \theta_1 < \pi/3$. Therefore,

$$\frac{C_1}{2}\tan\left(\frac{\theta_1}{2}\right) + \frac{C_2}{2}\tan\left(\frac{\theta_2}{2}\right) < \frac{C_1}{2}\tan\left(\frac{\pi}{6}\right) + \frac{C_2}{2}\tan\left(\frac{\pi}{6}\right) = \frac{1}{2}\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{6}.$$

Combining these two inequalities, we find that

$$C_3 > \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{3} > \frac{1}{2}.$$

Hence, each of the arcs separating the two components enclosed in the chain has length greater than $\frac{1}{2}$. It easily follows that the perimeter of the chain is greater than 3. Therefore, by the perimeter bound (Proposition 3.8), the standard chain has perimeter greater than that of the (embedded) minimizer(s) enclosing the same areas.

Next suppose that one of the components — say, $R_i$ — intersects itself but not the other component. The area enclosed by $R_i$ on the torus thus forms a band wrapping around the torus in a nonvertical direction; let $\mathcal{B}_1$ be the boundary of this band. Also, let $\mathcal{B}_2$ be the boundary of a second band containing the two components of the chain (and excluding any area that lies in the overlap of the immersed chain). By Lemma 5.4.21, either the length of $\mathcal{B}_1 \cup \mathcal{B}_2$ is greater than 3, or the length of $\mathcal{B}_1 \cap \mathcal{B}_2$ is greater than or equal to 2.

If the length of $\mathcal{B}_1 \cup \mathcal{B}_2$ is greater than 3, then by the perimeter bound (Proposition 3.8), the perimeter of the standard chain has perimeter greater than that of the (embedded) minimizer(s) enclosing the same areas.

If instead the length of $\mathcal{B}_1 \cap \mathcal{B}_2$ is greater than or equal to 2, then let $R_j$ be the region in the chain different from $R_i$, and let $A_j$ be the area of this region. Then

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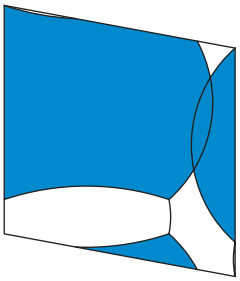


FIGURE 33. A second way a standard chain might be nonembedded is if at least one component intersects itself. The component then forms a band wrapping around the torus, implying (as was the case for octagon-square tilings) that the chain has a large perimeter.

$\mathcal{B}_1 \cap \mathcal{B}_2$ does not intersect the boundary of $R_j$. By the isoperimetric inequality for the plane, the perimeter of the region is at least $2\sqrt{\pi A_j}$. Therefore, the perimeter of the standard chain is at least $2 + 2\sqrt{\pi A_j}$. By Corollary 5.4.15, this exceeds the perimeter of the minimizer(s) enclosing the same areas. □

7.3. **Formulas for the band lens.** By Lemma 5.3.6, any minimizing band lens has the same homology as a shortest closed geodesic. Proposition 7.3.1 gives area and perimeter formulas for these band lenses.

**Proposition 7.3.1.** *Consider a band lens with the same homology as a shortest closed geodesic. Let* $r = (1/\text{curvature})$ *be the radius of curvature of the lens, and let* $d$ *be the width of the band. The area and perimeter formulas for the band lens in terms of* $r$ *and* $d$ *are*

$$\begin{aligned} A_{\text{Lens}}(r) &= 2r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right), \\ A_{\text{Band}}(r,d) &= d - r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right), \\ P(r) &= \left(\frac{4\pi}{3} - \sqrt{3}\right) r + 2. \end{aligned}$$

*Furthermore,* $r < \frac{1}{\sqrt{3}}$ *and* $d > \frac{r}{2}$.

*Proof.* The chord between the vertices of the lens, with length $C = 2r\sin(\pi/3) = \sqrt{3}r$, must fit along a short direction of the torus. Hence, $r < \frac{1}{\sqrt{3}}$. Also, because the band and lens do not overlap, the width of the band is greater than half the width of the lens. Hence, $d > r/2$.

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Using (7.1)–(7.2), we can compute the area and perimeter formulas:

$$\begin{aligned} A_{\text{Lens}}(r) &= 2A\left(\frac{\pi}{3}, \sqrt{3}r\right) = 2r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right), \\ A_{\text{Band}}(r,d) &= d - \frac{1}{2}A_{\text{Lens}}(r) = d - r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right), \\ P(r) &= 2L\left(\frac{\pi}{3}, \sqrt{3}r\right) + (2 - \sqrt{3}r) = \left(\frac{4\pi}{3} - \sqrt{3}\right)r + 2. \end{aligned}$$

□

7.4. **Formulas for the standard hexagon tiling.** Corollary 5.4.12 showed that the perimeter of every standard hexagon tiling is equal to three. This section gives area formulas for the standard hexagon tiling (Proposition 7.4.1), proves that at most one standard hexagon tiling encloses two given areas (Lemma 7.4.2), and describes exactly which pairs of areas can be enclosed by a standard hexagon tiling (Lemma 7.4.4).

**Proposition 7.4.1.** *In any standard hexagon tiling, there exist $a, b, c$ with sum 1 such that one region $R_1$ is a hexagon with side lengths alternating between $a$ and $b$, another region $R_2$ is a hexagon with side lengths alternating between $b$ and $c$, and the exterior region $R_0$ is a hexagon with side lengths alternating between $c$ and $a$. Conversely, given positive $a, b, c$ with sum 1, there exists a corresponding standard hexagon tiling.*

*The areas of the three regions are*

$$\begin{aligned} A_{R_1} &= \frac{\sqrt{3}}{4}(a^2 + 4ab + b^2), \\ A_{R_2} &= \frac{\sqrt{3}}{4}(b^2 + 4bc + c^2), \\ A_{R_0} &= \frac{\sqrt{3}}{4}(c^2 + 4ca + a^2). \end{aligned}$$

*Proof.* Consider the closed curve formed by the four segments $p, q, r, s$ depicted in Figure 34(a). Segments $p$ and $r$ are parallel to a short direction of the torus, and the segment $\sigma$ connecting the endpoints of the curve is parallel to the same short direction of the torus. Therefore, the segments $q$ and $s$ must veer the same distance away from $\sigma$. Because they both meet $\sigma$ at $2\pi/3$, they have equal length.

Just as $q$ and $s$ must be congruent, *any* two sides of a hexagon must be congruent if they are separated by one other side. It follows that the sides of the hexagons have one of three lengths $a, b, c$, as depicted in Figure 34(b). By Corollary 5.4.12, the tiling has perimeter three, implying that $a + b + c = 1$. Conversely, given any positive $a, b, c$ such that $a+b+c = 1$, one can construct the corresponding standard hexagon tiling as in Figure 34(b).

To find the area of $R_1$, extend the sides of $R_1$ with length $a$, as in Figure 35, in order to form an equilateral triangle with sides of length $a+2b$. The new equilateral triangle consists of $R_1$ and three small equilateral triangles with sides of length $b$. Therefore, the area of $R_1$ equals

$$\frac{\sqrt{3}}{4}(a+2b)^2 - 3 \cdot \frac{\sqrt{3}}{4}b^2 = \frac{\sqrt{3}}{4}(a^2 + 4ab + b^2).$$

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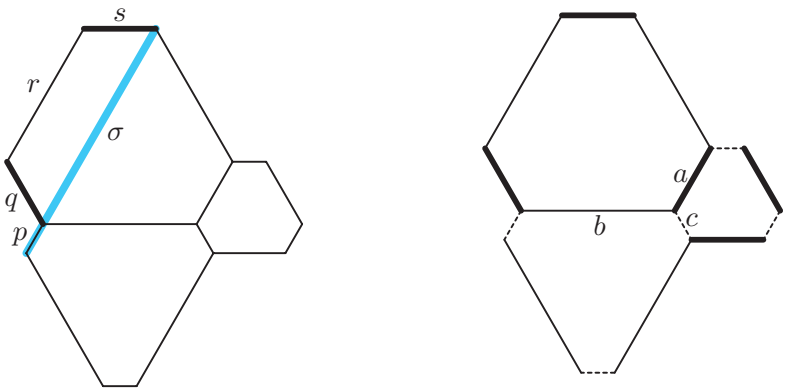



FIGURE 34. By examining how far various edges of a standard hexagon tiling veer from shortest closed geodesics, one can show that the sides of each hexagon alternate between two different lengths.

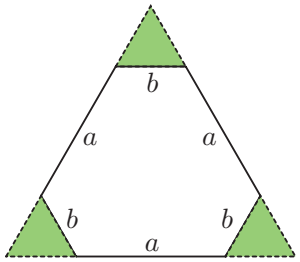



FIGURE 35. If the side lengths of a hexagon with interior angles of $2\pi/3$ alternate between two lengths, one can calculate the area of the hexagon by extending three sides to form an equilateral triangle.

We find the analogous formulas for the areas of $R_2$ and $R_0$ in a similar manner. □

**Lemma 7.4.2.** *Given positive real numbers $A_0, A_1, A_2$, there exists at most one triple $(a, b, c)$ of nonnegative numbers with sum 1 such that*

$$\begin{aligned} A_1 &= \frac{\sqrt{3}}{4}(a^2 + 4ab + b^2), \\ A_2 &= \frac{\sqrt{3}}{4}(b^2 + 4bc + c^2), \\ A_0 &= \frac{\sqrt{3}}{4}(c^2 + 4ca + a^2). \end{aligned}$$

*Proof.* Suppose, for sake of contradiction, that there existed two such triples $(a_1, b_1, c_1)$ and $(a_2, b_2, c_2)$. Without loss of generality, assume that $a_1 < a_2$. Because $a_1^2 + 4a_1b_1 + b_1^2 = a_2^2 + 4a_2b_2 + b_2^2$, it follows that $b_1 > b_2$. Similarly, because

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$b_1^2 + 4b_1c_1 + c_1^2 = b_2^2 + 4b_2c_2 + c_2^2$, it follows that $c_1 < c_2$. Likewise, $c_1 < c_2$ implies that $a_1 > a_2$, a contradiction. □

**Corollary 7.4.3.** *Given prescribed areas $A_1, A_2$, there exists at most one standard hexagon tiling enclosing those areas.*

*Proof.* The result follows from our parameterization (Proposition 7.4.1) and Lemma 7.4.2. □

**Lemma 7.4.4.** *Given positive areas $A_0, A_1, A_2$ with sum $\sqrt{3}/2$ (the area of the hexagonal torus), there exists a standard hexagon tiling enclosing any two of those areas if and only if $\frac{2}{\sqrt[4]{3}}\left(\sqrt{A_j} + \sqrt{A_k}\right) > 1$ for $0 \le j < k \le 2$. That is, if and only if the side length of an equilateral triangle with area $A_j$ and the side length of an equilateral triangle with area $A_k$ have a sum greater than 1. (When equality holds for some pair, say $A_1, A_2$, a degenerate hexagon tiling exists, consisting of two equilateral triangles enclosing areas $A_1, A_2$ and with an exterior hexagonal region of area $A_0$.)*

*Proof.* Consider the following map from $\{(a, b) \mid 0 \le a, b \text{ and } a + b \le 1\}$ to $\mathbb{R}^2$:

$$(a, b) \mapsto \left( \frac{\sqrt{3}}{4}(a^2 + 4ab + b^2), \frac{\sqrt{3}}{4}(b^2 + 4b(1 - a - b) + (1 - a - b)^2) \right).$$

By our parameterization (Proposition 7.4.1), this map sends a pair of side lengths of a hexagon tiling to the areas of the region with side lengths $a, b$ and of the region with side lengths $b, 1 - a - b$. Thus, as we vary $a$ and $b$, we obtain all ordered pairs of areas that can be enclosed by a standard hexagon tiling.

By Lemma 7.4.2, the above map is injective. Also, it maps the boundary of its domain continuously to a simple closed curve $\mathcal{C}$ in $\mathbb{R}^2$, as depicted in Figure 36(a). (The figure is drawn with barycentric coordinates, not Cartesian coordinates.) Because the map is injective and continuous, it follows that its range is precisely the closed region bounded by $\mathcal{C}$. This boundary $\mathcal{C}$ consists of three pieces, corresponding to $a = 0, b = 0, c = 0$. When $a = 0$, we have $1 = b + c = \frac{2}{\sqrt[4]{3}}(\sqrt{A_1} + \sqrt{A_0})$; similarly, along the other two pieces of the boundary, we have $1 = \frac{2}{\sqrt[4]{3}}(\sqrt{A_j} + \sqrt{A_k})$ for $(j, k) = (1, 2)$ and $(0, 2)$. Hence, the interior of the region bounded by these three pieces consists of those area triples $(A_0, A_1, A_2)$ such that $\frac{2}{\sqrt[4]{3}}(\sqrt{A_j} + \sqrt{A_k}) > 1$ for $0 \le j < k \le 2$. Therefore, for each of these triples and no others, some standard hexagon tiling encloses some two of the three areas. □

**Lemma 7.4.5.** *If a minimizing double bubble enclosing areas $A_1, A_2$ on the hexagonal torus has perimeter equal to three, then there exists a (perimeter-minimizing) standard hexagon tiling enclosing those areas.*

*Proof.* Let the exterior region of the given double bubble have area $A_0$, and without loss of generality assume that $A_1 \le A_2 \le A_0$. By Lemma 5.4.14,

$$2 + \sqrt{A_1\left(\frac{8\pi}{3} - 2\sqrt{3}\right)}$$

is greater than or equal to the given minimal perimeter, three. The (nonsharp) bound $A_1 \ge \frac{4}{25}$ follows easily. Hence, $A_0 \ge A_2 \ge A_1 \ge \frac{4}{25}$. For $0 \le j < k \le 2$, we

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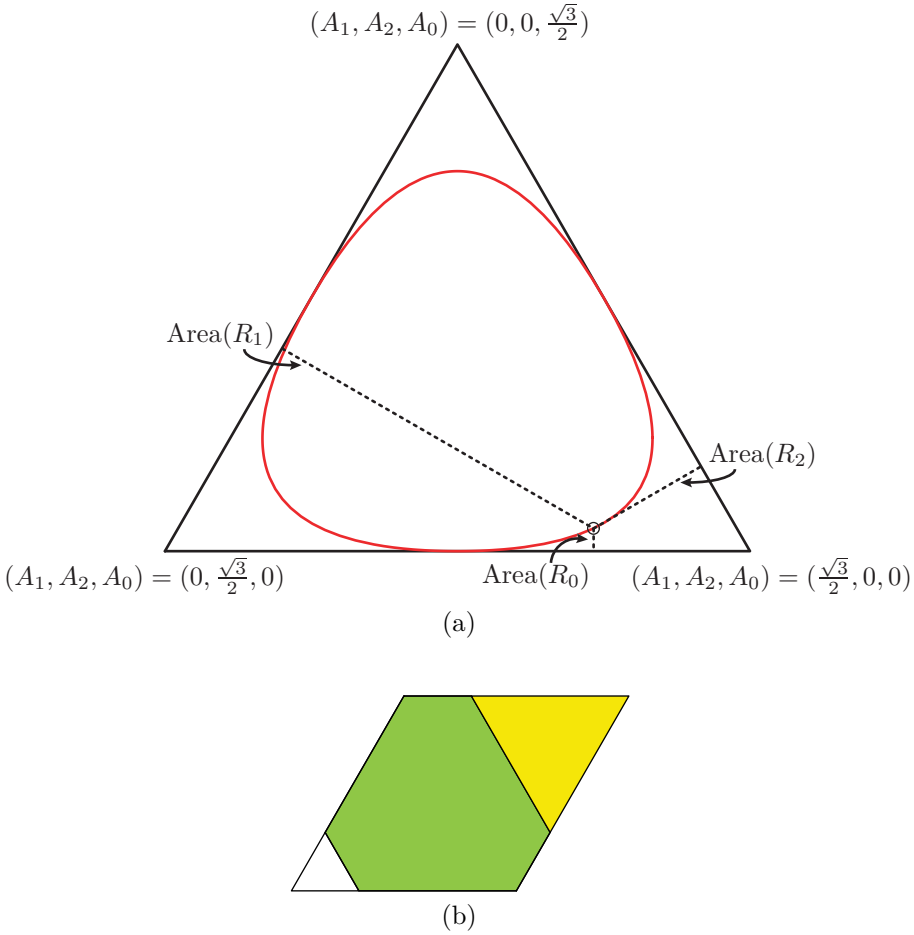



FIGURE 36. As in the phase portraits in Figure 2, the triangle in (a) represents the possible area pairs $(A_1, A_2)$ on the hexagonal torus, where the areas of the three regions are given by the distances to the three sides. The region bounded by the curve represents exactly those area pairs enclosed by some standard hexagon tiling. The curve itself represents area pairs enclosed by degenerate hexagon tilings consisting of a hexagon and two triangles, such as the one depicted in (b).

thus have

$$\sqrt{A_j} + \sqrt{A_k} \geq \frac{4}{5} \geq \frac{\sqrt[4]{3}}{2}.$$

By Lemma 7.4.4, there exists a standard hexagon tiling enclosing areas $A_1, A_2$. By Corollary 5.4.12, the tiling has perimeter three, so it is perimeter minimizing. □

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## 8. Phase portraits

The phase portraits of Figure 2 show which of the minimizers of Theorem 6.1 win for which pairs of prescribed areas on four different tori — where we represent each torus as a parallelogram with side lengths $1, L$ and interior angle $\theta$. To create these phase portraits, we plot perimeter as a function of area pairs for each candidate in Theorem 6.1 using the parameterizations from Section 7. For each candidate in Theorem 6.1 besides the standard chain, we know that at most one double bubble exists enclosing any given pair of areas, so we can solve for its perimeter numerically. Because we do not know that at most one standard chain (of a fixed homology) exists enclosing two given areas, we instead create the perimeter-areas plot for the standard chain parametrically. Also, using Lemma 4.2, it is possible to show that any minimizing standard chain has axis-length 1, $L$, or $\sqrt{1 + L^2 \pm 2\angle \cos\theta}$, so we do not plot perimeter as a function of area pairs for other types of standard chains.

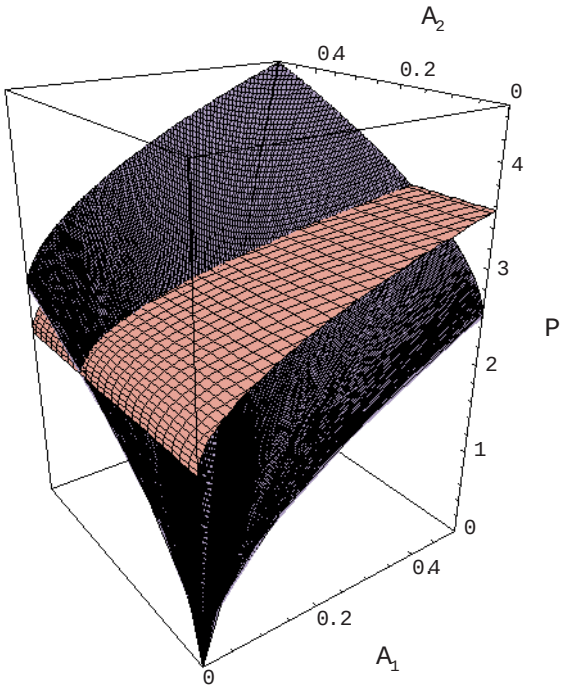



FIGURE 37. This *Mathematica* plot of the intersection of the perimeter plots for the standard double bubble and band lens illustrates the process used to create the phase portraits in Figure 2. The plots suggest that perimeters do not fluctuate wildly along certain sets of unusual area pairs, increasing our confidence that the phase portraits are accurate.

After plotting all these perimeter-areas plots simultaneously, we then view the combined plot from below so that at each area pair the plot(s) of the minimal calculated perimeter is visible. Figure 37 illustrates a simplified version of this

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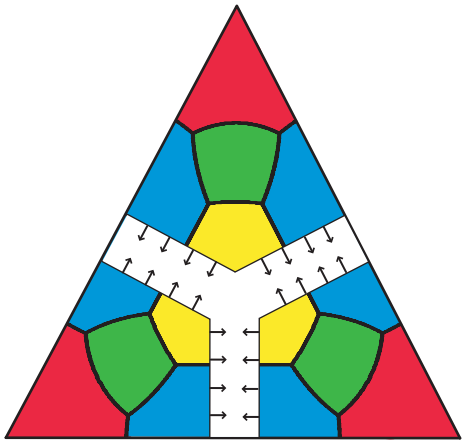


FIGURE 38. Removing pieces of this phase portrait for a rectangular torus with length 1.2 yields the phase portrait plot for some torus with less area.

process, comparing the perimeters of only two candidates, the standard double bubble and the band lens.

This process works if whenever the perimeter calculated for a given double bubble is minimizing for two given areas, the double bubble actually exists on the torus for those areas. Indeed, this is the case. When the standard double bubble does not fit on the torus, it has diameter greater than 1, so by Propositions 5.1.1 and 3.8 it cannot have minimizing perimeter. Lemma 5.4.14 shows that when the calculated perimeter for the band lens is minimizing, that band lens exists. By Lemma 7.2.4, when a standard chain with axis-length one has minimizing perimeter, it can be embedded on the torus; and according to the plots, no standard chain with axis-length greater than one has minimizing perimeter. As the simplest case, we know that the double band exists for all area pairs. Finally, by Lemma 7.4.5, when the minimal perimeter on the hexagonal torus is 3 for two given areas, there exists a standard hexagon tiling enclosing those two areas.

The following conjecture describes a simple connection *among* the various phase portraits. As usual, we assume that the shortest geodesics on the torus and the cylinder have unit length.

**Conjecture 8.1.** *Given areas $A_0 \geq A_2 \geq A_1$ and a torus with area $A_0 + A_1 + A_2$, the nontiling minimizing double bubble(s) enclosing $A_1, A_2$ on this torus are the same as the minimizing double bubble(s) enclosing $A_1, A_2$ on the infinite cylinder. Consequently the phase portrait of any flat two-torus can be obtained from the phase portrait of any flat two-torus of larger area (or from the phase portrait of the infinite cylinder) as in Figure 38. In particular, tori of equal areas have identical phase portraits.*

To prove Conjecture 8.1 it would suffice to show that every minimizing standard chain has axis-length one. The proof would also use the fact (which we can show using Theorem 6.1) that in a minimizer, a region of greater pressure has no more

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area than a region of less pressure, as well as Proposition 5.1.1, Lemma 5.3.6, and Lemma 7.2.4.

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# Double Bubbles in the Three-Torus

Miguel Carrión Álvarez, Joseph Corneli, Genevieve Walsh, and Shabnam Beheshti

CONTENTS

1. Introduction
2. Preliminaries
3. Some Examples and Construction Techniques
4. Simple Groups
5. Conclusions and Further Research
Acknowledgments
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We present a conjecture, based on computational results, on the area-minimizing method of enclosing and separating two arbitrary volumes in the flat cubic three-torus, $\mathbf{T}^3$. For comparable small volumes, we prove that the standard double bubble from $\mathbf{R}^3$ is area-minimizing.

## 1. INTRODUCTION

The classical isoperimetric problem seeks the least-area way to enclose a single region of prescribed volume. About 200 BC, Zenodorus argued that a circle is the least-perimeter enclosure of prescribed area in the plane (see [Heath 60]). In 1884, Schwarz [Schwarz 1884] proved by symmetrization that a sphere minimizes surface area for a given volume in $\mathbf{R}^3$. Isoperimetric problems arise naturally in many parts of modern mathematics; Ros provides a nice survey [Ros 01].

The double bubble problem is a two-volume generalization of the classical isoperimetric problem. In 2000, Hutchings, Morgan, Ritoré, and Ros proved that the standard double bubble was the unique least-area way to enclose and separate two volumes in $\mathbf{R}^3$ ([Hutchings et al. 00], [Hutchings et al. 02], [Morgan 00, Chapter 14]). In this paper, we present a conjecture, based on substantial numerical evidence, on the least-area way to enclose and separate two volumes in the three-torus.

**Conjecture 1.1. (Central Conjecture 2.1.)** *The ten different types of two-volume enclosures pictured in Figure 1 comprise the complete set of surface area-minimizing double bubbles for the flat cubic three-torus* $\mathbf{T}^3$.

The numerical results summarized in Figure 2 indicate the volumes for which we conjecture each type of double bubble minimizes surface area.

In Section 4, we show that for two small comparable volumes, the standard double bubble from $\mathbf{R}^3$ is optimal in $\mathbf{T}^3$. Specifically, we prove the following theorem:

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Keywords: Area minimization, three-torus, double bubble, Surface Evolver

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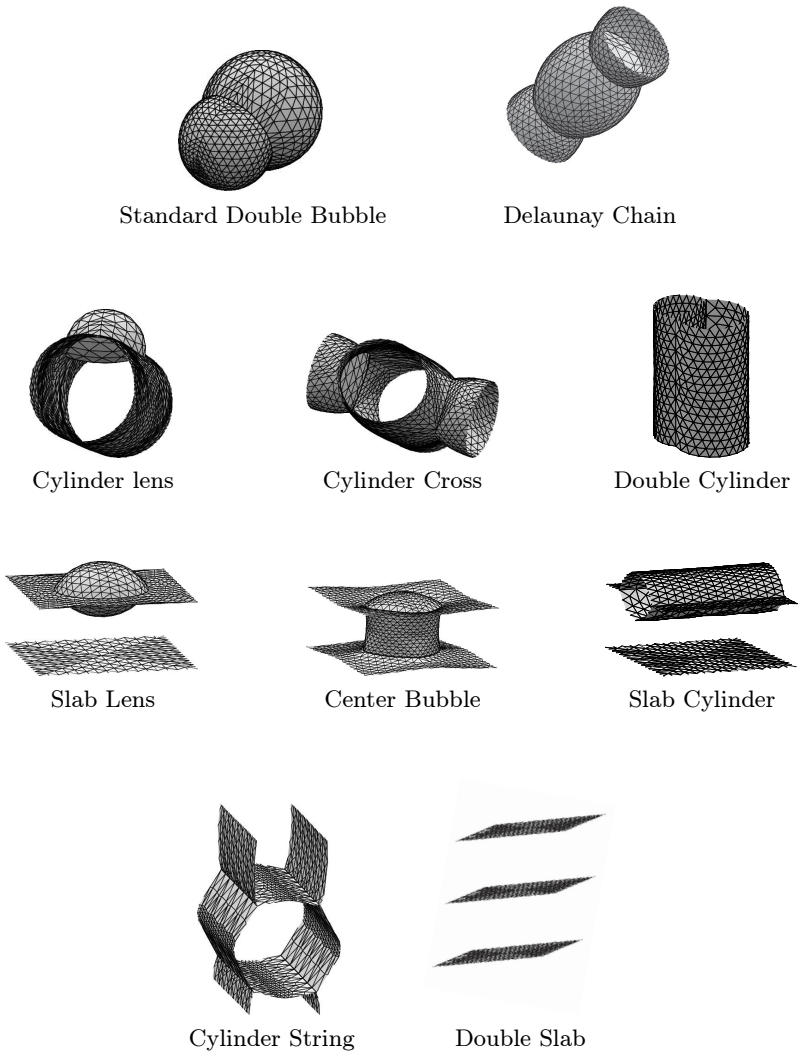



**FIGURE 1**. Catalog of conjectured minimizers.

**Theorem 1.2.** *Let $M$ be a flat Riemannian manifold of dimension three or four such that $M$ has compact quotient by its isometry group. Fix $\lambda \in (0, 1]$. Then there is an $\epsilon > 0$ such that, if $0 < v < \epsilon$, an area-minimizing double bubble in $M$ of volumes $v, \lambda v$ is standard.*

We conclude the paper with some extensions to the primary conjecture that address noncubic tori and additional volume constraints.

### 1.1 Existence and Regularity

Using the language of geometric measure theory, bubble clusters can be expressed as rectifiable currents, varifolds, or $(\mathbf{M}, \epsilon, \delta)$-minimal sets [Morgan 00]. In three dimensions, area-minimizing bubble clusters exist and consist of constant-mean-curvature surfaces meeting smoothly in threes at 120° along smooth curves, which meet in fours at a fixed angle of approximately 109° ([Taylor 76, Theorems II.4, IV.5, IV.8], or [Morgan 00, Section 13.9]).

### 1.2 Recent Results

In 2000, Hutchings, Morgan, Ritoré, and Ros announced a proof that the standard double bubble, the familiar shape consisting of three spherical caps meeting one another at 120-degree angles, provides the area-minimizing method of enclosing and separating two volumes in $\mathbf{R}^3$ ([Hutchings et al. 02], [Hutchings et al. 00]). The key features of the proof are a component bound developed by Hutchings [Hutchings 97, Theorem 4.2] and an insta-

bility argument. Reichardt et al. [Reichardt et al. 03] extended this result to $\mathbf{R}^4$.

In 2002, Corneli et al. [Corneli et al. 03] solved the double bubble problem for the flat two-torus, showing that there are five types of minimizers (one of which only occurs on the hexagonal torus). The proof relies on regularity, a variational component bound due to Wichiramala [Morgan and Wichiramala 02], and combinatorial and geometric classification.

### 1.3 Single Bubbles in $\mathbf{T}^3$

Even the single bubble for the three-torus is not yet completely understood. There are, however, partial results. The smallest enclosure of half of the volume of the torus was shown by Barthe and Maurey [Barthe and Maurey 00, Section 3] to be given by two parallel two-tori. Morgan and Johnson [Morgan and Johnson 00, Theorem 4.4] showed that the least-area enclosure of a small volume is a sphere. Spheres, tubes around geodesics, and pairs of parallel two-tori are shown to be the only types of area-minimizing enclosures for most three-tori by work of Ritoré and Ros ([Ritoré 97, Theorem 4.2], [Ritoré and Ros 96]).

### 1.4 Small Comparable Volumes

We prove Theorem 4.1 by showing that every sequence of area-minimizing double bubbles with decreasing volume and fixed volume ratio contains standard double bubbles. The main difficulty lies in bounding the curvature. Once this is accomplished, it is possible to show that the bubble lies inside a small ball that lifts to $\mathbf{R}^3$, where by [Hutchings et al. 02], a minimizer is known to be standard.

The main idea of the argument is as follows: From a given sequence of double bubbles with shrinking volumes and fixed volume ratio, we generate a new sequence by rescaling the ambient manifold at each stage so that one of the volumes is always equal to one. We can then apply compactness arguments and area estimates to show that certain subsequences of sequences obtained by rigid motions of the ambient manifold have nontrivial limits. These limits are used to obtain a curvature bound on a subsequence. With such a bound, we apply a monotonicity result to conclude that for small volumes, the double bubble is contained in a small ball. We conclude that it must be the same as the minimizer in $\mathbf{R}^3$ or $\mathbf{R}^4$, i.e., it must be the standard double bubble by [Hutchings et al. 02] or [Reichardt et al. 03]. The result applies to any flat three- or four-manifold with compact quotient by the isometry group.

### 1.5 Plan of the Paper

Section 2 reviews the methods that led to our Central Conjecture 2.1 and to Figures 1 and 2. Section 3 surveys some sub-conjectures. Section 4 is the statement and proof of the main theorem on small volumes. Section 5 shifts from the cubic torus to other tori and discusses other conjectures and candidates, including a "Hexagonal Honeycomb."

## 2. THE CONJECTURE

### 2.1 Generating Candidates

Participants in the 2001 CMI/MSRI Summer School proposed many possibilities for double bubbles in the three-torus $\mathbf{T}^3$ in brainstorming sessions.

Standard Double Bubble. The least-area double bubble in $\mathbf{R}^3$.

Delaunay Chain. Two stacked "beads" or "drums" wrapping around one of the periods of the torus. The lateral surfaces are Delaunay surfaces (constant-mean-curvature surfaces of rotation).

Cylinder Lens. A cylinder wrapping around a period of the torus, with a small bubble attached.

Double Cylinder. Two cylinders wrapping around a period of the torus. Transverse sections by an orthogonal plane (i.e., a two-torus) are the standard double bubble in two dimensions. This is the first of a family of solutions we could describe as "cylinders over minimizing double bubbles in the two-torus."

Cylinder String. A cylinder over a "Symmetric Chain" in $\mathbf{T}^2$.

Slab Cylinder. Two parallel flat two-tori with a cylindrical bubble attached to one of them. This is a cylinder over a "Band Lens" in $\mathbf{T}^2$.

Double Slab. Three parallel flat two-tori (cylinder over a "Double Band" in $\mathbf{T}^2$).

Slab Lens. Two parallel flat two-tori, one of which has a small lens-shaped bubble stuck in it.

Center Bubble. Two close-to-parallel two-tori with a close-to-cylindrical bubble between them. The planes buckle slightly (as can be shown by elementary stability considerations) and the bubble stuck between them is not quite round (it bulges slightly in the direction of each of the four corners of the fundamental domain pictured in Figure 1).

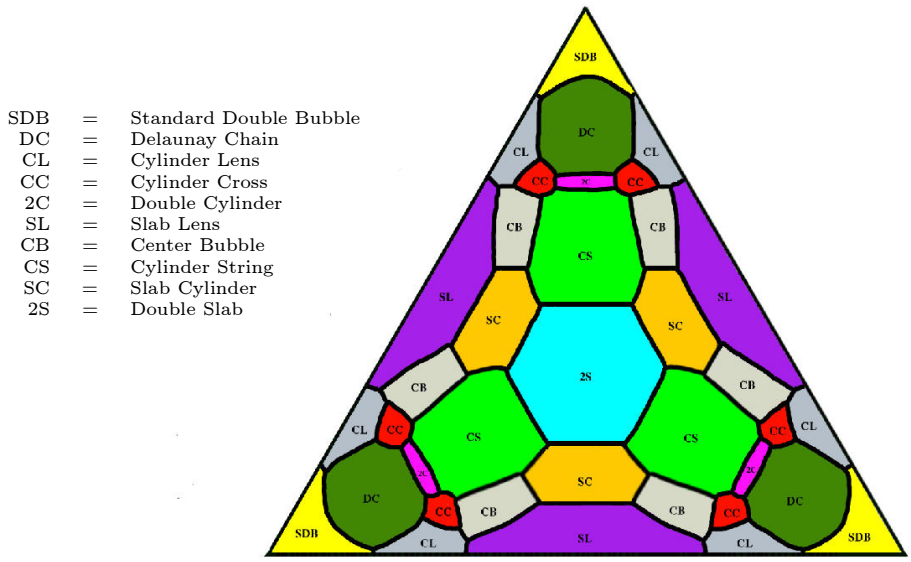



**FIGURE 2.** Phase portrait: volumes and corresponding double bubble. In the center both regions and the complement have one third of the total volume; along the edges, one volume is small; in the corners, two volumes are small.

Transverse Cylinders. Two cylinders wrapping around two different periods of the three-torus that touch each other to reduce surface area.

Hydrant Lens. The Schwarz $P$ surface with a small bubble attached.

Double Hydrant. Another fanciful proposal based on the Schwarz $P$ surface, known as "Scary Gary" after its inventor, Gary Lawlor.

Inner Tube. A cylinder that wraps around one of the periods of the three-torus with a toroidal bubble wrapped around it.

When preparing for the calculations that resulted in the phase diagram of Figure 2, we anticipated the possibility that a minimizing bubble topology might have been missed. In fact, two additional candidate topologies were found by systematic application of this heuristic:

> When the two volumes are very small, the standard double bubble should be optimal. As one or both of the two volumes grow, a bubble that collides with itself should open up to reduce perimeter, whereas if two different bubbles collide, they should stick.

This procedure incorporates the assumption that all three regions associated with a minimizer will be connected. The new candidates we found were as follows:

Cylinder Cross. A cylinder wrapping around one period of the torus with an attached bubble that wraps around one of the perpendicular directions. (Obtained from the Cylinder Lens when the small region grows.)

Center Cylinder. Two parallel flat two-tori with a cylinder going through both of them and wrapping around a period of the three-torus. (Obtained from the Center Bubble when the small region grows or from the Transverse Cylinders as one of the cylinders grows.)

### 2.2 Producing the Phase Diagram

Brakke's *Surface Evolver* [Brakke 99] was used to closely approximate the minimal area that a double bubble of each type needs to enclose specified volumes. Our initial simulations were done with volume increments of 0.05 (from a total volume of 1) for the initial set of brainstormed candidates. The results of these calculations were sufficient to convince us that the Hydrant Lens and the Double Hydrant were inefficient and that the Inner Tube was in fact unstable (which is why we could not draw a picture of it with *Surface Evolver* for inclusion in

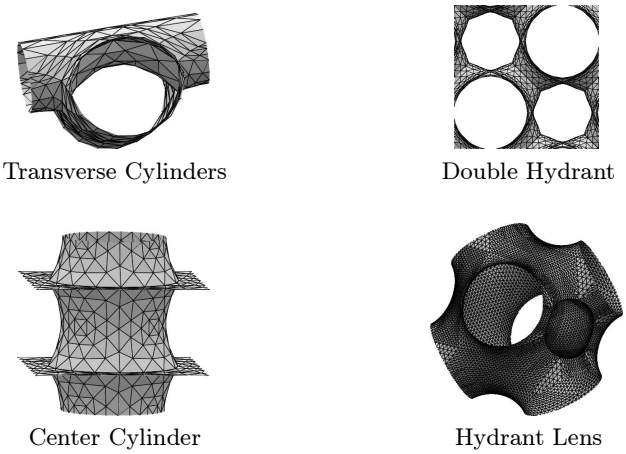



**FIGURE 3**. Inefficient double bubbles.

this paper). A rough version of Figure 2 was then obtained using a volume increment of 0.01. From the data obtained in these computations, we found that each of the double bubbles pictured in Figure 1 was the least-area competitor for some volume triple $v_1 : v_2 : v_3$. The phase diagram appearing in Figure 2 is the result of refining our second series of computations along the boundaries between the regions using a volume increment of 0.005.

**Conjecture 2.1. (Central Conjecture.)** *The ten double bubbles pictured in Figure 1 represent each type of surface area-minimizing two-volume enclosure in a flat, cubic three-torus, and these types are minimizing for the volumes illustrated in Figure 2.*

### 2.3 Comments

One might guess that minimizers would be formed from topological spheres and tori that go around at least one period of $\mathbf{T}^3$. This was borne out in our computations. It is interesting to note that although the Transverse Cylinders pictured in Figure 3 match this description, they were never seen to be minimizing.

A challenging problem related to the problem of area minimization is that of finding all of the stable double bubbles in $\mathbf{T}^3$; there are probably quite a few that we have not looked at. Note that a given type from Figure 1 might be stable for a much wider range of volumes than those for which it actually minimizes surface area.

It is worth observing that all of the conjectured minimizers for the double bubble problem on $\mathbf{T}^2$ are echoed here in at least two ways: several conjectured minimizers are $\mathbf{T}^2$ minimizers $\times$ $\mathbf{S}^1$, and others are more direct analogues, for example, the Delaunay and Symmetric Chains. See Corneli et al. [Corneli et al. 03] for more on the $\mathbf{T}^2$ minimizers.

A few words about the accuracy of the simulations are in order. The commands used in our final stage of numerics were as follows:

```
// five cycles of refining the mesh and
// decreasing area
{{u; g; u} 10; {u; r; u}} 5;
// decrease area ten more times at the end
{u; g; u} 10;
```

These commands produced pictorial output that perfectly matches our idea of what each conjectured minimizer should look like.

## 3. SUBCONJECTURES

We now present a list of natural subconjectures about bubbles in the flat cubic three-torus $\mathbf{T}^3$ that are suggested either by the phase diagram or by examination of the pictures made by *Surface Evolver*.

One immediate observation is that the edges of our phase diagram appear to characterize single bubbles in $\mathbf{T}^3$.

**Conjecture 3.1.** (Ritoré and Ros [Ritoré 97], [Ritoré and Ros 96],[Ros 01].) *The optima for the isoperimetric problem in* $\mathbf{T}^3$ *are the sphere, cylinder, and slab.*

The following two conjectures are perhaps the most intuitive things we would expect to be true about bubbles in $\mathbf{T}^3$:

**Conjecture 3.2.** *An area-minimizing double bubble in* $\mathbf{T}^3$ *has connected regions and complement.*

**Conjecture 3.3.** *There is an* $\epsilon > 0$ *such that if* $v_1, v_2$ *are both less than* $\epsilon$*, the standard double bubble of volumes* $v_1$ *and* $v_2$ *is optimal in* $\mathbf{T}^3$*.*

In Theorem 4.1, we prove that there is such an $\epsilon$ for every fixed ratio of volumes.

If it were known for some reason that the number of components of a minimizer was relatively small, then we think the next two conjectures might admit fairly easy proofs:

**Conjecture 3.4.** *For one very small volume and two moderate volumes, the Slab Lens is optimal.*

**Conjecture 3.5.** *The first phase transition as small equal or close to equal volumes grow is from the standard double bubble to a chain of two bubbles bounded by Delaunay surfaces.*

Delaunay surfaces are constant-mean-curvature surfaces of rotation, and as such contain an $\mathbf{S}^1$ in their symmetry group. From the *Surface Evolver* pictures, it appears that the conjectured surface area minimizers always have the maximal symmetry, given the constraints, a fact that leads to the following natural conjecture.

**Conjecture 3.6.** *All isotopies that are not isometries decrease a minimizer's symmetry group.*

We will conclude the section with a proposition that establishes a very limited symmetry property for all double bubbles in the two torus (including nonminimizing bubbles). Specifically, we prove that for any double bubble, there is a pair of parallel two-tori that cut both regions in half. Hutchings [Hutchings 97, Theorem 2.6] was able to show that in $\mathbf{R}^n$ a minimizing double bubble has two perpendicular planes that divide both regions in half. He used this result in the proof that the function that gives the least-area to enclose two given volumes in $\mathbf{R}^n$ is concave. We conjecture that a similar concavity result also holds for the three-torus:

**Conjecture 3.7.** *The least area to enclose and separate two given volumes in the three-torus is a concave function of the volumes.*

If one could prove Conjecture 3.7, one would then be able to apply other ideas in Hutchings' paper [Hutchings 97, Section 4] to obtain a functional bound on the number of components of a minimizing bubble.

**Proposition 3.8. (Deluxe Ham Sandwich Theorem.)*** *If a double bubble lies inside a solid two-torus inside a rectangular three-torus,* $D^2 \times \mathbf{S}^1 \subset \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$*, then there is a pair of parallel two-tori that divide both volumes in half.*

*Proof:* This is a generalization of the standard argument for the ham sandwich theorem in Euclidean space. We will regard the three-torus as the result of identifying the faces of a rectangular solid $[-n, n] \times [-m, m] \times [0, p]$ in $\mathbf{R}^3$. We may assume that the solid torus is a vertical cylinder with base centered at the origin in the $x - y$ plane. Now we rotate this solid cylinder, and index the rotation by $\alpha \in [0, 2\pi]$. Then for each $\alpha \in [0, \pi]$, rotate the cylinder (along with the surface that lies inside) by $\alpha$ in the $x - y$ plane and consider the family of pairs of planes $\{y = c, y = c + m\}$, where $c \in [-m, 0]$ For each $\alpha \in [0, \pi]$, there is at least one such pair of planes that cuts the volume $V_1$ in half, in the sense that half of $V_1$ is in the inside of the planes, and half of $V_1$ is on the outside of the two planes. There may be an interval of such pairs for a given $\alpha$. However, some one-parameter family of these planes may be chosen which varies continuously as a function of $\alpha$, and coincides at 0 and $\pi$. Since by the time $\alpha$ reaches $\pi$ the two portions of $V_2$ have switched sides, there must be some pair of planes that divides $V_2$ in half. The restriction of these planes to the domain $[-n, n] \times [-m, m] \times [0, p]$ glue up to a pair of two-tori in the three-torus that divides both regions in half. □

## 4. SMALL COMPARABLE VOLUMES

Conjecture 3.3 asserts that two small volumes in $\mathbf{T}^3$ are best enclosed by a standard double bubble. In this section, we will prove Theorem 4.1, which says that for any fixed volume ratio, the standard double becomes optimal when the volumes are sufficiently small. This result holds for a relatively broad class of three- and four-dimensional manifolds (and see Remark 4.4).

The standard double bubble consists of three spherical caps meeting at 120 degrees (see Figure 2). If the volumes are equal, the middle surface is planar. The standard double bubble is known to be minimizing for all volume pairs in $\mathbf{R}^3$ [Hutchings et al. 02] and $\mathbf{R}^4$ [Reichardt et al. 03].

*Name suggested by Eric Schoenfeld, who gave an independent proof for $\mathbf{T}^2$.

**Theorem 4.1.** *Let $M$ be a flat Riemannian manifold of dimension three or four such that $M$ has compact quotient by its isometry group. Fix $\lambda \in (0, 1]$. Then there is an $\epsilon > 0$ such that, if $0 < v < \epsilon$, an area-minimizing double bubble in $M$ of volumes $v, \lambda v$ is standard.*

**Remark 4.2.** Our assumption on the isometry group guarantees solutions to the double bubble problem for all volume pairs (the proof is the same as in Morgan [Morgan 00, Section 13.7]). Not every flat 3-manifold has compact quotient by its isometry group. Euclidean 3-space modulo a glide reflection or a glide rotation is an example.

It is helpful to have the overall strategy of the proof in mind before we go on. Our goal is to prove that a minimizing double bubble of volumes $v$, $\lambda v$ lies inside a trivial ball in $M$ when $v$ is small enough. The results for Euclidean space then prove our claim.

For the proof, we will regard a double bubble as a pair of three- or four-dimensional rectifiable currents, $R_1$ and $R_2$, each of multiplicity one, of volumes $V_1 = \mathbf{M}(R_1)$ and $V_2 = \mathbf{M}(R_2)$. The total area of such a double bubble is $\frac{1}{2}(\mathbf{M}(\partial R_1) + \mathbf{M}(\partial R_2) + \mathbf{M}(\partial(R_1 + R_2)))$. Here $\mathbf{M}$ denotes the mass of the current, which can be thought of as the Hausdorff measure of the associated rectifiable set (counting multiplicities). For a review of the pertinent definitions, see the notes from Morgan's course at the CMI/MSRI Summer School [Morgan and Ritoré 01] or the texts by Morgan [Morgan 00] or Federer [Federer 69]. By the Nash and subsequent isometric embedding theorems, we may assume $M$ is a submanifold of some fixed $\mathbf{R}^N$.

We will consider a sequence of area-minimizing double bubbles in $M$ enclosing the volumes $v$ and $\lambda v$, as $v \to 0$. If the theorem were false, there would be some such sequence with no standard double bubbles in it. We prove that every sequence of fixed volume ratio double bubbles with shrinking volumes contains standard double bubbles.

For each $v$, $M_v$ will denote $s_v(M)$ in $\mathbf{R}^N$, where $s_v$ is the scaling map that takes regions with volume $v$ to similar regions with volume 1. In particular, $s_v$ maps our area-minimizing double bubble enclosing volumes $v$ and $\lambda v$ to a double bubble which we call $S_v$ that encloses volumes 1 and $\lambda$, and is of course area-minimizing for these volumes.

The first step is to show that once $M_v$ has been suitably translated and rotated, the sequence $S_v$ has a subsequence that converges as $v \to 0$ with $V_1 \neq 0$ in the limit. Our argument also shows that there is a differently modified subsequence that converges to a limit with $V_2 \neq 0$, but does not show that there is a subsequence where both volumes are nonzero in the limit. This is because while we are exerting ourselves trapping the first volume in a ball, the second one may wander off to infinity.

The second step in the proof is to use these limits to obtain a weak bound on the curvature of a subsequence of $S_v$. We need this in order to apply the monotonicity theorem for mass ratio [Allard 72, Section 5.1(1)], which says that a small ball around any point on a surface with weakly bounded mean curvature contains some substantial amount of area.

The third step is to show that all of the surface area of each element of our subsequence is contained in some ball in $M_v$ that has fixed radius for all $v$. Eventually, as $v$ shrinks and $M_v$ grows, this ball will have to be trivial in $M_v$.

We then use the result that the optimal double bubble in $\mathbf{R}^3$ is standard to show that our subsequence has a tail comprised of standard double bubbles. The desired conclusion then follows from the fact that $S_v$ is simply a scaled version of the original double bubble containing volumes $v$ and $\lambda v$. Thus there is no sequence of shrinking fixed-ratio area-minimizing double bubbles in $M$ that does not contain standard double bubbles.

We focus on the case of dimension three; the proof for dimension four is essentially identical.

The following lemma is needed in the first step of the proof.

**Lemma 4.3.** *There is a $\gamma > 0$ such that if $R$ is a region in an open Euclidean 3-cube $K$ and $\mathrm{vol}(R) \leq \mathrm{vol}(K)/2$, then*

$$\mathrm{area}(\partial R) \geq \gamma(\mathrm{vol}(R))^{2/3}.$$

*Proof:* Let $\gamma_0$ be such an isoperimetric constant for a cubic three-torus, so that $\mathrm{area}(\partial P) \geq \gamma_0(\mathrm{vol}(P))^{2/3}$ for all regions $P \subseteq \mathbf{T}^3$. Such a $\gamma_0$ exists by the isoperimetric inequality for compact manifolds [Morgan 00, Section 12.3]. Make the necessary reflections and identifications of the cube $K$ to obtain a torus containing a region $R'$ with eight times the volume and eight times the surface area of $R$. The claim follows, with $\gamma = \gamma_0/2$. □

*Proof of Theorem 4.1:*

*Step 1. The sequence $S_v \subset M_v$, where each $M_v$ has been suitably translated and rotated, has a subsequence that converges with $V_1 \neq 0$ in the limit.*

We first show the existence of a covering $\mathcal{K}_v$ of $M_v$ with bounded multiplicity, consisting of 3-cubes contained in $M_v$, each of side-length $L$, for any $L > 0$. Lemma 4.3 will give us a positive lower bound on the volume of the part of $R_1$ that is inside one of these cubes for each $S_v$. We will then apply a standard compactness theorem to show that a subsequence of the sequence of $S_v$ converges.

Take a maximal packing of $M_v$ by balls of radius $\frac{1}{4}L$. Enlargements of radius $\frac{1}{2}L$ cover $M_v$. Circumscribed 3-cubes in $M_v$ of edge-length $L$ provide the desired covering $\mathcal{K}_v$. To see that the multiplicity of this covering is bounded, consider a point $p \in M_v$. The ball centered at $p$ with radius $2L$ contains all the cubes that might cover it, and the number of balls of radius $\frac{1}{4}L$ that can pack into this ball is bounded, implying that the multiplicity of $\mathcal{K}_v$ is also bounded by some $m > 0$.

Now let $\mathcal{K}_v$ be a covering as above, with $L = 2$. By Lemma 4.3, there is an isoperimetric constant $\gamma$ such that

$$\text{area}(\partial(R_1 \cap K_i)) \geq \gamma(\text{vol}(R_1 \cap K_i))^{2/3},$$

and therefore, since $\max_k \text{vol}(R_1 \cap K_k) \geq \text{vol}(R_1 \cap K_i)$ for any $i$,

$$\text{area}(\partial(R_1 \cap K_i)) \geq \gamma \frac{\text{vol}(R_1 \cap K_i)}{(\max_k \text{vol}(R_1 \cap K_k))^{1/3}}. \quad (4\text{–}1)$$

Note that the total area of the surface is greater than $1/m$ times the sum of the areas in each cube, and the total volume enclosed is less than the sum of the volumes, so summing Equation 4–1 over all the cubes $K_i$ in the covering $\mathcal{K}_v$ yields

$$\text{area}(S_v) \geq \text{area}(\partial R_1) \geq m\gamma \frac{V_1}{(\max_k \text{vol}(R_1 \cap K_k))^{1/3}}$$

and

$$(\max_k(\text{vol}(R_1 \cap K_k)))^{1/3} \geq m\gamma \frac{V_1}{\text{area}(S_v)} \geq \delta,$$

for some $\delta > 0$, because $V_1 = 1$ and $\text{area}(S_v)$ is bounded (there is a bounded way of enclosing the volumes, and $S_v$ has the same amount of area or less).

Translate each $M_v$ so that a cube $K_k$ that maximizes $\text{vol}(R_1 \cap K_k)$ is centered at the origin of $\mathbf{R}^N$, and rotate so that the tangent space of each $M_v$ at the origin is equal to a fixed $\mathbf{R}^3$ in $\mathbf{R}^N$. The limit of the $M_v$ will be equal to this $\mathbf{R}^3$. Since a cube with edge-length $L$ centered at the origin fits inside a ball of radius $2L$ centered at the origin, we have

$$\text{vol}(R_1 \cap B(0, 2L)) \geq \delta^3$$

for every $S_v$. By the compactness theorem for locally integral currents ([Morgan 00, pp. 64, 88], [Simon 84, Section 27.3, 31.2, 31.3]), we know that a subsequence of the $S_v$ has a limit, which we will call $D$, with the property that $\text{vol}(R_1) \geq \delta^3$. This completes the first step.

Since $D$ is contained in the limit of the $M_v$, namely, the copy of $\mathbf{R}^3$ chosen above, and each $S_v$ is minimizing for its volumes, a standard argument shows that the limit $D$ is the area-minimizing way to enclose and separate the given volumes $\text{vol}(R_1) \geq \delta^3$ and $\text{vol}(R_2)$ in $\mathbf{R}^3$ (see [Morgan 00, 13.7]). In the limit, $\text{vol}(R_2)$ could be zero, in which case $D$ is a round sphere. If both volumes are nonzero, $D$ is the standard double bubble ([Hutchings et al. 02] and [Reichardt et al. 03], or see [Morgan 00, Chapter 14]).

*Step 2. By taking a subsequence if necessary, we may assume there is a weak bound on the curvature of the $S_v$.*

To show that the mean curvature is weakly bounded is to show that there is a $C$ such that for any $S_v$, and any direction in the two-dimensional space of possible volume changes, $|\frac{dA}{dV}| \leq C$ for smooth variations. Thus it suffices to produce smooth variations such that changes in the volume of $S_v$ and in the area of $\partial S_v$ are controlled, that is, that the magnitude of the change in volume is bounded from below and the magnitude of the change in area is bounded from above.

We first show that the magnitude of the change in volume is bounded below. Take a smooth variation vector field $F$ in $\mathbf{R}^n$ such that for $D$,

$$dV_1/dt = \int_{\partial R1} (F \cdot n)\, dA = c \neq 0$$

and

$$dV_2/dt = 0.$$

(Note that we need the first volume to be nonzero, or its variation could be zero.) For $v$ small enough, the subsequence of $S_v$ associated with the limit $D$ has the property that $dV_1/dt$ is approximately equal to the constant $c$ and $dV_2/dt$ is approximately equal to 0.

By the argument in the first step, we can translate and rotate each $M_v$ so that, after taking a subsequence again if necessary, $S_v$ converges to a minimizer $D'$ in $\mathbf{R}^3$ where the second volume is nontrivial. This time we take a smooth variation vector field $F'$, such that the images of $S_v$ under the appropriate rigid motions have the property that $dV_1/dt$ is approximately equal to 0 and $dV_2/dt$ is approximately equal to some constant $c' \neq 0$, for $v$ small enough.

Note that the change in volume is independent of rigid motions; in particular, we can translate both $F$ and $F'$

back along the rigid motions to act on $S_v$. This proves that for this sequence, the magnitude of the change in volume is bounded below.

Now we need to show that the change in area is bounded above. This follows from the fact that every rectifiable set can be thought of as a varifold [Morgan 00, Section 11.2]. By compactness for varifolds [Allard 72, Section 6], the $S_v$, situated so that the first volume does not disappear, converge as varifolds to some varifold, $J$. The first variations of the varifolds also converge, i.e., $\delta S_v \to \delta J$; see [Allard 72]. The first variation of a varifold is a function representing the change in area. Therefore, far enough out in the sequence the change in area of the $S_v$ under $F$ is bounded close to the change in area of $J$ under $F$, which is finite. Similarly, the change in area of the $S_v$ under $F'$ is bounded above.

We conclude that, after restricting to a subsequence, $S_v$ has the property that $|\frac{dA}{dV}|$ is bounded for two independent directions in the two-dimensional space of volume changes, and hence for the entire space. This completes the second step.

*Step 3. When $v$ is small, all of the elements of a subsequence of $S_v \subset M_v$ are contained in a ball with a fixed radius that can be lifted to* $\mathbf{R}^3$.

As $v$ shrinks, the area of each element of the subsequence $S_v$ found in Step 2 is bounded above by $A > 0$, since our double bubbles are area-minimizing, and a nearly Euclidean double bubble is one candidate. By monotonicity of mass ratio [Allard 72, Section 5.1(1)], every unit-radius ball centered at a point of $S_v$ contains at least area $\epsilon$, for some $\epsilon > 0$. Therefore, there are at most $A/\epsilon$ such disjoint balls.

We claim that the $S_v$ are eventually connected, which will imply that the diameter $S_v$ is bounded above by $2A/\epsilon$. Indeed, unless the $S_v$ are eventually connected, one can arrange to get in the limit a disconnected minimizer in $\mathbf{R}^3$ by translating and centering nontrivial volume on two different points. This is false, so we conclude that $S_v$ is contained in a ball of radius $2A/\epsilon$ for all small $v$.

Since our original manifold has compact quotient by its isometry group, we may quotient by isometries to get a compact 3-manifold. This implies that balls of a certain radius or smaller are topologically trivial. Hence, as we expand the manifold, balls of radius $2A/\epsilon$ can be lifted to $\mathbf{R}^3$, which means that they are Euclidean. This completes the third step.

Since $S_v$ is eventually contained in a Euclidean ball, the tail of this sequence is made up entirely of standard double bubbles ([Hutchings et al. 02], [Morgan 00, Chapter 14]). Recall that our sequence $S_v$ is simply a scaled version of the original sequence of minimizing double bubbles in $M$ enclosing volumes $v$ and $\lambda v$, where we have restricted to subsequences as necessary. Therefore, any sequence of area-minimizing double bubbles with shrinking volumes and fixed volume ratio contains standard double bubbles. We conclude that the minimizing double bubble with volume ratio $\lambda$ is standard when $v$ is small enough. □

**Remark 4.4.** Given $n$ and $m$, essentially the same argument shows that for any smooth $n$-dimensional Riemannian manifold with compact quotient by the isometry group, given $0 < \lambda \leq 1$, there are $C$, $\epsilon > 0$, such that for any $0 < v < \epsilon$, a minimizing cluster with $m$ prescribed volumes between $\lambda v$ and $v$ lies inside a trivial ball. For the case of flat three- or four-dimensional manifolds with compact quotient by the isometry group, this means that small clusters of bubbles with comparable volumes are the same as the Euclidean minimizer containing the same volumes. To deduce the corresponding result in the nonflat case would require knowing that convergence weakly and in measure, under bounded mean curvature, implies $C^1$ convergence, as is known for hypersurfaces without singularities ([Allard 72, Section 8], see [Morgan 01, Section 1.2]).

## 5. SPECIAL TORI

Changing the shape of the torus, by stretching it or by skewing some or all of its angles, would certainly change the phase diagram of Figure 2.

**Conjecture 5.1.** *In the special case of a very long three-torus, i.e., with side lengths $\{\frac{1}{\epsilon^2}, \epsilon, \epsilon\}$ $(\epsilon \to 0)$, the Double Slab is optimal for most volumes.*

Special tori may have special minimizers:

**Conjecture 5.2.** *For the special case of a torus based on a short hexagonal prism, the Hexagonal Honeycomb prism of Figure 4 is an area-minimizing double bubble when both regions and the exterior all have equal or close to equal volumes.*

Indeed, for such volumes, the Hexagonal Honeycomb ties the Double Slab, just as in the hexagonal two-torus, a Hexagonal Tiling ties the Double Band [Corneli et al. 03].

For multiple enclosed regions, we would expect to find minimizers that lift to $\mathbf{R}^3$ as periodic foams with cells of

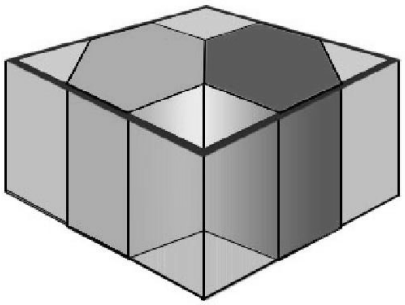


**FIGURE 4**. Hexagonal honeycomb

finite volume, such as Kelvin's foam or the Weaire-Phelan foam [Kusner and Sullivan 96].

However, there is no minimizing *double* bubble in any torus that lifts to a division of $\mathbf{R}^3$ into bounded regions. Indeed, as pointed out by Adams, Morgan, and Sullivan [Adams et al. 2003], by Lebesgue's Covering Theorem [Hurewicz and Wallman 41, Theorem IV2], such regions must sometimes meet in fours or more.

This suggests that it is likely that there are no other special minimizers for the double bubble problem.

**Conjecture 5.3.** *The double bubbles of Figure 1 together with the Hexagonal Honeycomb of Figure 4 comprise the complete set of area-minimizing double bubbles for all three-tori.*

In light of the fact that the triple bubble problem in the torus seems likely to produce many interesting candidates, we conclude with one final conjecture.

**Conjecture 5.4.** *For the triple bubble problem in a flat cubic three-torus in the case where one of the volumes is small, the minimizers will look like the double bubbles of Figure 2 with a small ball attached. The phase diagram will look just like our Figure 3.*

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# DOUBLE BUBBLES IN SPACES OF CONSTANT CURVATURE

BY
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A Thesis

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# Dedication

*To Bea Canchester and Helen Corneli, who encouraged me to pursue my education and have made significant material contributions that have made it possible.*

# Acknowledgements

My father's mother, to whom, together with my mother's mother, this thesis is jointly dedicated, told me at an early age that "it takes a village to raise a child". If this is true of my personal life (and it is) it is even more true of my academic life. This leaves me with many people to thank.

First of all there is my academic sponsor and thesis advisor, Pat McDonald. Pat taught me lots of math and has been very generous with both time and energy spent talking to me about my thesis, and about many other things. It is no coincidence that while I was his student I began to think seriously about how my mind works, to have solid hope for the future, to give respect where it is due and to value intellectual freedom. In short, Pat showed me what a mathematician is and opened the way for me to try it out for myself. In addition to all of this, he has a number of remarkable personal qualities, including his own special blend of modesty and confidence, that I have learned an immense amount by trying (and usually failing) to emmulate.

Then there is Frank Morgan of Williams College. Frank was my devoted advisor during a full year of research that lead to three and a half papers, that, in revised form, constitute the body of this thesis. While I was in his Geometry Group at Williams, Frank made daily comments on daily revisions of our papers every weekday morning (starting on our second day). Afterwards, he continued to advise me and my collaborators, mostly through an extensive email correspondence, in which he made hundreds of thoughtful suggestions about both research and writing. Working for Frank was always fun for me, in part because I was learning so much, but also because of his pleasant enthusiasm and congeniality. In addition to this, he arranged a number of special opportunities for the Geometry Group, including the unprecidented invitation of a group of undergraduates to the Mathematical Sciences Research Institute in Berkeley, California, and full participation in his special session on soap bubble geometry at MathFest'01 in Madison, Wisconsin. By observing Frank in action, I learned an immense amount about how to interact positively with people.

Without the hard work of these two men, both exemplary scholars and teachers, this thesis would not exist. In addition to making that acknowledgement, I want to take this moment to thank each of them for taking a friendly interest in my personal and moral development as well as in my academics.

In addition to being jointly responsible for the majority of the content of this thesis, my collaborators during this past year were both my peers and my teachers, and they formed an invaluable asset to my general education in mathematics.

My collaborators in the 2001 Geometry Group at the Williams REU were Nick Leger of the University of Texas at Austin, and Paul Holt and Eric Schoenfeld of Williams College. Ben Steinhurst from Williams and George Lee from Harvard University will be in the 2002 Geometry Group and they worked with my group during 2001-2002 academic year. Revised versions of our results appear in Chapters 4 and 6.

For three weeks of the summer, the 2001 Geometry Group attended the Clay Mathematics Institute's Summer School on the Global Theory of Minimal Surfaces at the Mathematical Sciences Research Institute in Berkeley, California. Collaborators I met at the summer school were Miguel Carriòn Àlvarez of the University of California at Riverside, Genevieve Walsh of the University of California at Davis, and Shabnam Beheshti of Texas Tech University. A revised version of our joint paper appears in Chapter 7.

Aside from these people who I worked intimately with, there are many others in the world of mathematics at large who made important contributions. In Berkeley, I had a number of helpful conversations with Michael Hutchings, whose theory of double bubble structure underlies the current understanding of the double bubble problem in three and more dimensions, and is treated here in Chapter 3. There I also met David Futer of Stanford University, with whom I had a lengthy correspondence with about the Euclidean version of the Curvature Conjecture appearing in Chapter 4. David's careful work and thoughtful letters contributed greatly to my understanding of the subject

of double bubbles and mathematics in general. During the Summer School, conversations with Matt Kudzin of Indiana State were also very illuminating. I also had several helpful and fun conversations with Manuel Ritoré of the University of Grenada about bubbles in the three-torus and John Sullivan of the University of Illinois at Champaign/Urbana about tilings (and later, by email, about foams).

Andy Cotton and David Freeman, both graduating from Harvard this year, were members of the 2000 Geometry Group, and my group continued their work on double bubbles in spherical and hyperbolic space. I had the pleasure of meeting Andy at MathFest'01 in Madison, and then met again with Andy and David in Cambridge, Massachusetts this past fall, where we talked about double bubbles. At MathFest I also met Wacharin Wichiramala of the University of Illinois Champagne-Urbana, whose results on planar clusters include a lemma that was very important for developing the theory of double bubbles in the two-torus. Wacharin made a number of very helpful suggestions by email during the academic year. At MathFest I also met Joel Foisy, whose Geometry Group in 1992 solved the double bubble problem in the plane, and who proved in his Williams honors thesis a result that is important in Hutchings' structure theory, and whose talk on double bubbles in corners contributed to the Geometry Group's understanding of bubbles on the cylinder and torus.

Notable contributions to the three torus project were made in Berkeley by three of the faculty participants, Michael Dorff, Denise Halverson, and Gary Lawlor, of Brigham Young University, who got us started with *Surface Evolver* computations, and also by David Futer and Matt Kudzin (who I already mentioned), Baris Coskunuzer, Brian Dean, Tom Fleming, Jim Hoffman, Jon Hofmann, Jesse Ratzkin, and Jean Steiner, as well as the entire Geometry Group.

Back at New College, where lots of the writing and tons of the editing took place, Mary Whelan and was a big overall help. In particular, going well beyond her call of duty as an employee of the New College Writing Resource Center, made detailed comments on the Geometry appendix. Conversations with her helped me set the tone for the thesis, and I think enormously contributed to its overall comprehesibility. Yair Kagan produced the illustration of the Hexagonal Honeycomb

appearing at the end of Chapter 7. Holly Barone is the person in charge of interlibrary loans and article copies at the New College library. She worked hard to obtain copies of the books and articles for me. References that were too hard for her to find I gave up on. Tim Teräväinen of New College has had lots of helpful conversations with me about everything; we've lived around each other in three cities and I appreciate his steadfastness. Rob Meyers and Caroline Arruda had the difficult task of being friends with me during my first two years at New College. They helped make things interesting for me here. In particular, Rob and I had lots of fun talking about math together, and it was out of conversations related to his thesis [31] that I began to have some conception of the direction I wanted to go with mine. I am most happy that my dreams about that have come true. Friends who were especially helpful in one way or another during the writing process are Kartina Amin, Justin Clarke-Doane, Selena Lee, Siggy Meek, Ya'el Morowati, Jake Silverstein, Tik Sun, and Stella Tinnirello.

Long-distance from Minneapolis, my sister Zoë Corneli has been a support to me throughout my time at New College.

My dad, Steve Corneli, has had many many wonderful conversations with me over the years. He has also generously sustained all of my living expenses this past semester while I was in the process of writing this thesis, and has covered various other expenses. I confess it here, that I first began to study math at his prompting. I also first began to love it under his tutelage (in geometry and basic calculus) when he asked me point blank whether I really wanted to understand the stuff or not (I did). My dad was also the first person to teach me anything memorable about writing. Judy Knapp is a close friend of my dad's. She has listened to me and encouraged me about many personal issues, and has also moderated my dad's influence on me.

During both the research and writing phases of this project my mother, Lynne Corneli, has made a number of suggestions at key times that improved the quality of my work or writing and that re-focussed my mind. Most of this thesis was composed on a Hewlett-Packard computer she gave

me.

My research at Williams College was supported by the NSF. The Mathematics and Statistics Department at Williams College generously gave me travel support. Expenses incurred in connection with the Summer School were covered by the Clay Mathematics Institute and the Mathematical Sciences Research Institute. I should like to personally thank the organizers of the Summer School: Joel Hass, Arthur Jaffe, Antonio Ros, Harold Rosenberg, Richard Schoen, and Michael Wolf, and especially David Hoffman, who bravely agreed to support the participation of me and three other undergraduates. The Mathematical Association of America provided travel support for my trip to MathFest'01 in Madison, Wisconsin.

A semester of advanced study at the Massachusetts Institute of Technology contributed greatly to this thesis and to my general education in mathematics. This endeavor was underwritten by my paternal grandparents, Kip and Helen Corneli. My professors at MIT were Victor Guillemin, Richard Melrose, and Andras Vašy.

My time at New College was underwritten by my maternal grandparents, Jack and Bea Canchester, and by what used to be called the Dean and Warden's Leadership Scholarship, as well as various other grants from the state of Florida. At New College my professors were Pat McDonald, David Mullins, and Erini Poimenidou (in mathematics), George Rupeiner and Don Colladay (in physics), Catherine Elliot (in economics), Sandra Gilchrist (in biology), and Mac Miller and John McDiarmid (in English).

In addition to all of these people, my thesis is heavily indebted to writings by Almgren [25], Chavel [8], Hutchings [22], Nomizu and Kobayashi [39], Morgan [34], Rudin [45], Spivak [49], and is somewhat better than metaphysically linked to writings by Vinberg [12], Wolf [59], and numerous others.

Mathematics is owned by no one, but writing most certainly is, and I endeavor to point out in the text of the thesis everywhere I borrowed material and to give credit to all of the authors I read

while I wrote. At places where a specific citation is not given, that does not necessarily imply that what I am writing about at that moment is in any way an original idea of my own. In general, I am fairly sure that the distinction between new and old ideas will be clear enough to those who care to think about such things.

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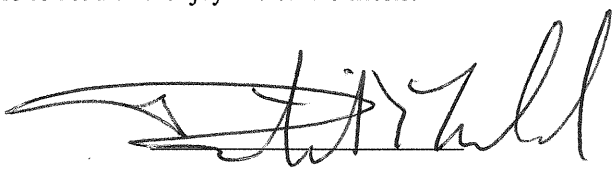
DOUBLE BUBBLES IN SPACES OF CONSTANT CURVATURE

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New College of Florida, 2002

ABSTRACT

The aim of this thesis is to be a comprehensive and self-contained treatment of the known results on area minimizing double bubbles in constant curvature spaces. Difficult proofs of existence and regularity theorems are omitted, as are several more readible proofs that are available (or soon to be made available) in the literature. Included in the thesis are a number of new results, including the reduction of the double bubble conjecture in spherical and hyperbolic space to a problem of asymptotic analysis, a proof of that the least-area way to divide Gauss space ($\mathbf{R}^n$ with Gaussian measure) into three equal pieces consists of three half-hyperplanes meeting at the origin at 120°, and results due to the author and various collaborators on double bubbles it two- and three-dimensional tori. The theoretical background of these developments is surveyed in some detail. A proof of the isoperimetric property of spheres in the classical constant curvature space forms is presented. Various detailed appendices are provided with the hope that persons with little or no background in differential geometry or variational methods will be able to read and enjoy the entire thesis.



Patrick T. McDonald

Division of Natural Sciences

'Perchè l' animo tuo tanto s' impiglia,'

Disse il Maestro, 'che l' andare allenti?'

Che ti fa ciò che quivi si pispiglia?

Vien rentro a me, e lascia dir le genti;

Sta come torre ferma, che non crolla

Giammai la cima per soffiar de' venti.

Chè sempre l' uomo in cui pensier rampolla

Sopra pensier, da sè dilunga il segno,

Perchè la foga l' un dell' altro insolla.'

Dante Alighieri, *Purgartorio* V, 10-18

# Chapter 1

# themes

- 'Double bubbles'?

- 'Spaces of constant curvature'?

Bubbles are regions in a geometric space (like Euclidean space $\mathbf{R}^n$) that minimize surface area given a volume constraint.

The single bubble or isoperimetric problem seeks the least area way to enclose one volume in a given geometric space.

The double bubble problem seeks the least area way to enclose and separate two given volumes in given geometric space – and so on.

Greek mathematicians knew the solution to the isoperimetric problem in Euclidean space well before the start of the common era. The answer is that the least-area enclosure of a given volume is a sphere, or in two dimensions, a circle. Euler is credited with the first rigorous proof of the isopermetry of circles in the plane. In the mid-eighteen-hundreds, analysts Schwarz (see [47]) and Steiner (see [50]) developed the methods that finally made possible a rigorous proof of the

isoperimetry of spheres in three dimensions. The key idea in their arguments is that symmetrization always reduces perimeter.

The idea of minimizing the surface area for given volume constraint(s) is a classic example of a variational problem. (We have some functional, namely area, defined on the space of all enclosures; we want to know which enclosures with a fixed volume are extrema for this functional.) This type of problem readily generalizes to spaces with non-Euclidean geometry. (The isoperimetry of spheres in the classical non-Euclidean spaces was first shown by Schmidt (see [46]) in the mid-nineteen-hundreds.)

The idea that bubbles or clusters of bubbles solve the least-area enclosure problem comes from our experience in the real world. Empiricism tells us that we should be able to restrict ourselves from the set of all enclosures, on which we impose very few constraints (say only that their boundaries be 'not too bad'), to a subset of enclosures that about which we can say something geometrically substantial. Thus, one of the primary tasks in this problem area was to characterize the geometry of minimizing idealized bubble clusters – an even higher-priority task was to say when a minimizer for the least area problem exists. The basic geometrical properties of the smooth (regular) and non-smooth (singular) sets of minimizing bubble clusters also characterize a class of 'potential minimizers', which are said to satisfy *regularity*. This term is common in variational calculus and partial differential equations, mainly because it turns out that the solutions (minimizers) are usually smooth, that is, regular, almost everywhere. This was proved for the least-area enclosure problem by Almgren [25]; a more specific characterization of the geometry of minimizers was given by Taylor ([53], [54]) and subsequently generalized by Brian White (unpublished).

Once we understand regularity, the problem becomes mostly geometric (and/or computational). Even with regularity, we do not know whether there are a finite number of components in a minimizing bubble cluster, which regular clusters are stable in the sense that they stay regular under volume preserving variationsm, and at the bottom of it all, what shapes are actually minimizing. In non-

standard geometries, the intuition gathered from every-day empiricism wears somewhat thin, and we may have to do some serious thinking to even come up with a reasonable conjecture. Computer simulations can be an aid to intuition.

Spaces with constant curvature are among the simplest things that a geometer can study. This class includes the plane (see Figure 1.1), the sphere (see Figure 1.2), the hyperbolic plane (see Figure1.3), the cylinder, and the flat torus (rectangle or box with identified sides, as in Figures 1.4 and 1.5), as well as the Euclidean, Spherical, and Hyperbolic spaces, and many other examples. Each of these geometrical spaces has the property of being 'pretty much the same' everywhere. This property gives us a relatively large set of tools. In particular, area minimizing bubble clusters in these spaces are described by regularity theorems similar to those proved for Euclidean space. However, a wide range of geometrical and topological complications unknown in Euclidean space make themselves felt in the more general category. These lead to new computational and theoretical challenges.



Figure 1.1: The Euclidean plane, or rather a segment of it, shown here tiled with regular hexagons.

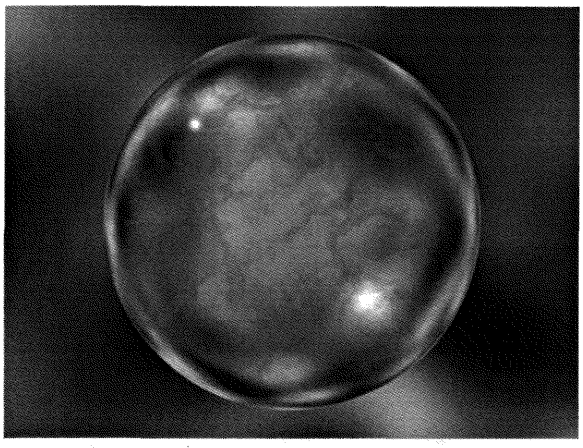


Figure 1.2: The sphere, filled with various media. This computer-generated image looks remarkably like a soap bubble.

**Plan of the Work** Chapter 2 is a review of the results relevant to the subsequent chapters and in other ways important to this thesis. A proof of the minimizing property of spheres for the single bubble problem in the classical constant curvature spaces (spheres, Euclidean space, and hyperbolic space) is presented, and the known results on the regularity properties of minimizing bubble clusters are described. Chapter 3 reviews the known results on the structure of area-minimizing double bubbles. In Chapter 4 we present a partial solution to the double bubble for spherical space $\mathbf{S}^3$ and hyperbolic space $\mathbf{H}^3$. In Chapter 5 we present a solution to a special case of the double bubble problem for $n$-dimensional Gauss space ($\mathbf{R}^n$ with gaussian measure), namely the case in which the two given volumes each take up exactly one third of the volume of the whole space. (This is sort of an 'interlude' since even though this is a perfectly fine space of constant curvature, we won't be directly concerned with curvature properties of the space in this chapter. We do however rely on the known theory for spheres.) In Chapter 6 we present a (more or less complete) solution to the double bubble problem for the flat two torus $\mathbf{T}^2$. In Chapter 7 we present a conjectured solution to the double bubble problem for the flat three torus $\mathbf{T}^3$. And at the end we include a short chapter (really an epilog) that puts together all these various examples and talks in a general way about

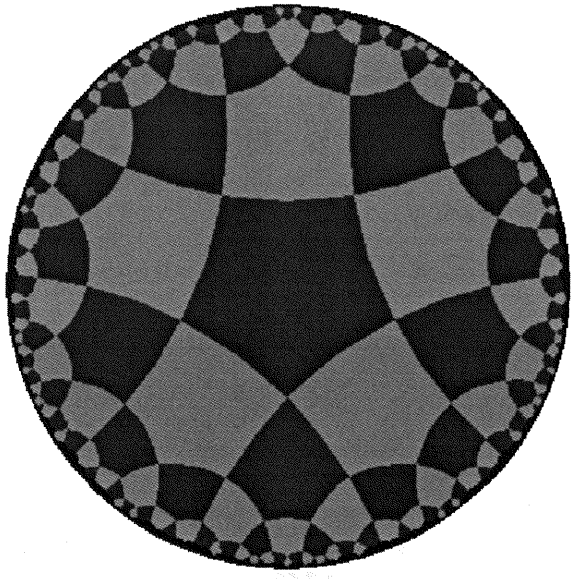


Figure 1.3: The Hyperbolic plane, shown here tiled with regular pentagons. It is not possible to tile the Euclidean plane with regular pentagons, but we can use them to tile the hyperbolic plane because of the way it curves. In particular, the curvature causes areas to expand quickly as you move away from any point – the pentagons look like they are different shapes because of hyperbolic perspective; they are actually exact isometric copies of one another.

what might be true about other related spaces.

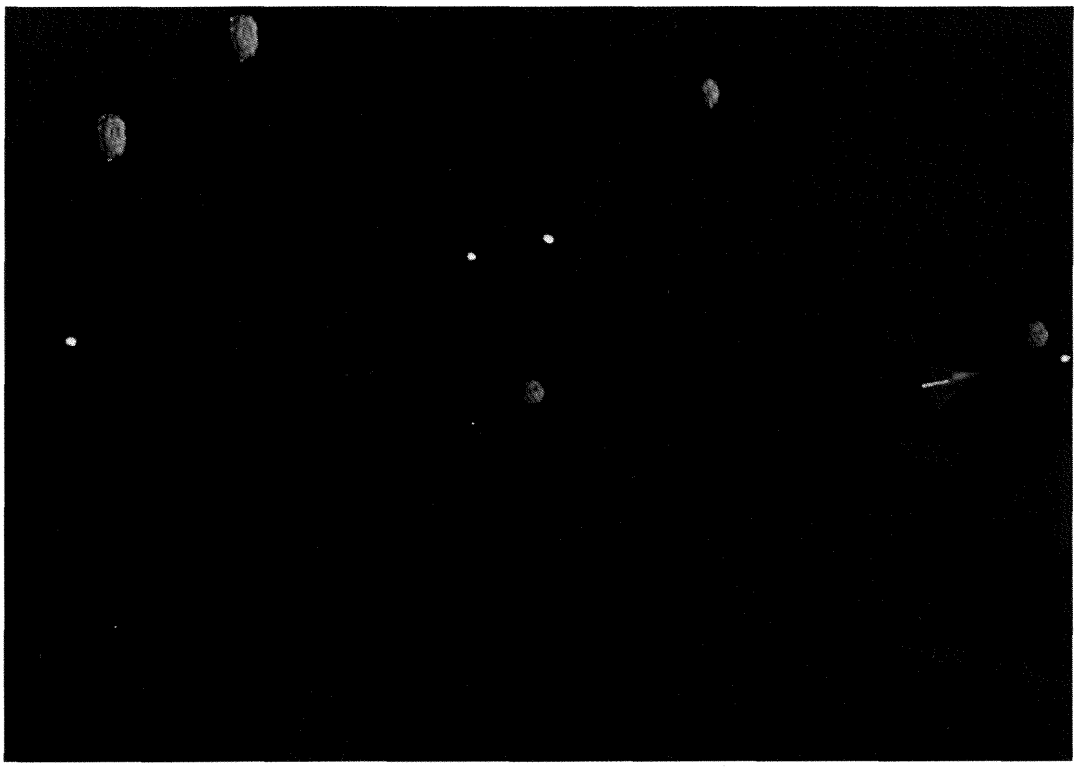


Figure 1.4: The two-torus, viewed from above, with a motif similar to the one in which it it made its debut in popular culture back in 1979. Shots, ships, and asteroids that fly off of the screen on one side instantly reappear on the opposite side (similarly with top and bottom).

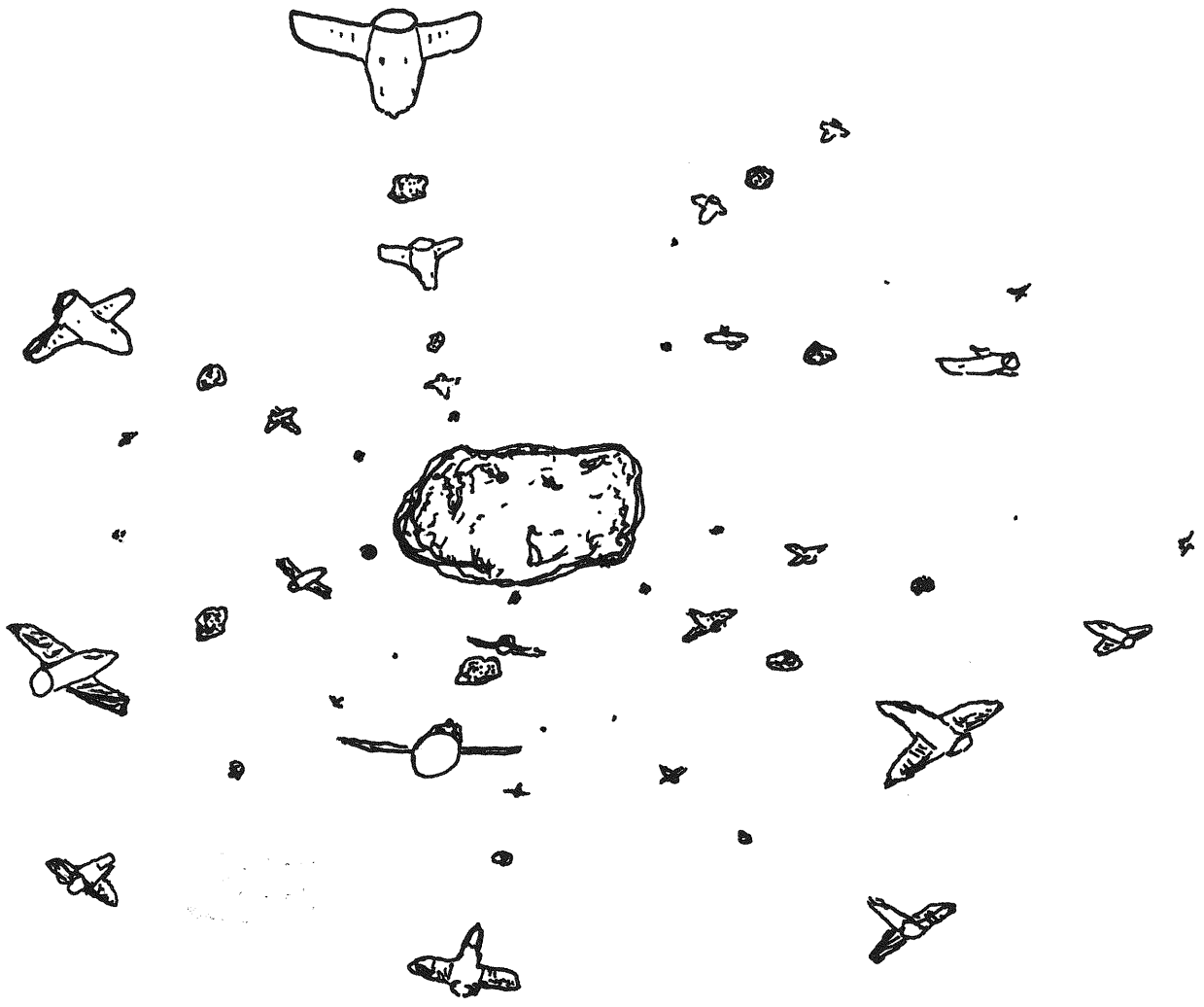


Figure 1.5: The three-torus, viewed from inside. What appears to be multiple ships bearing down on multiple asteroids are actually all just one ship (ours) bearing down on one asteroid, viewed in 'toroidal perspective'. Drawing based loosely on a screenshot from the Jeff Weeks/Geometry Center video, *The Shape of Space* [58]

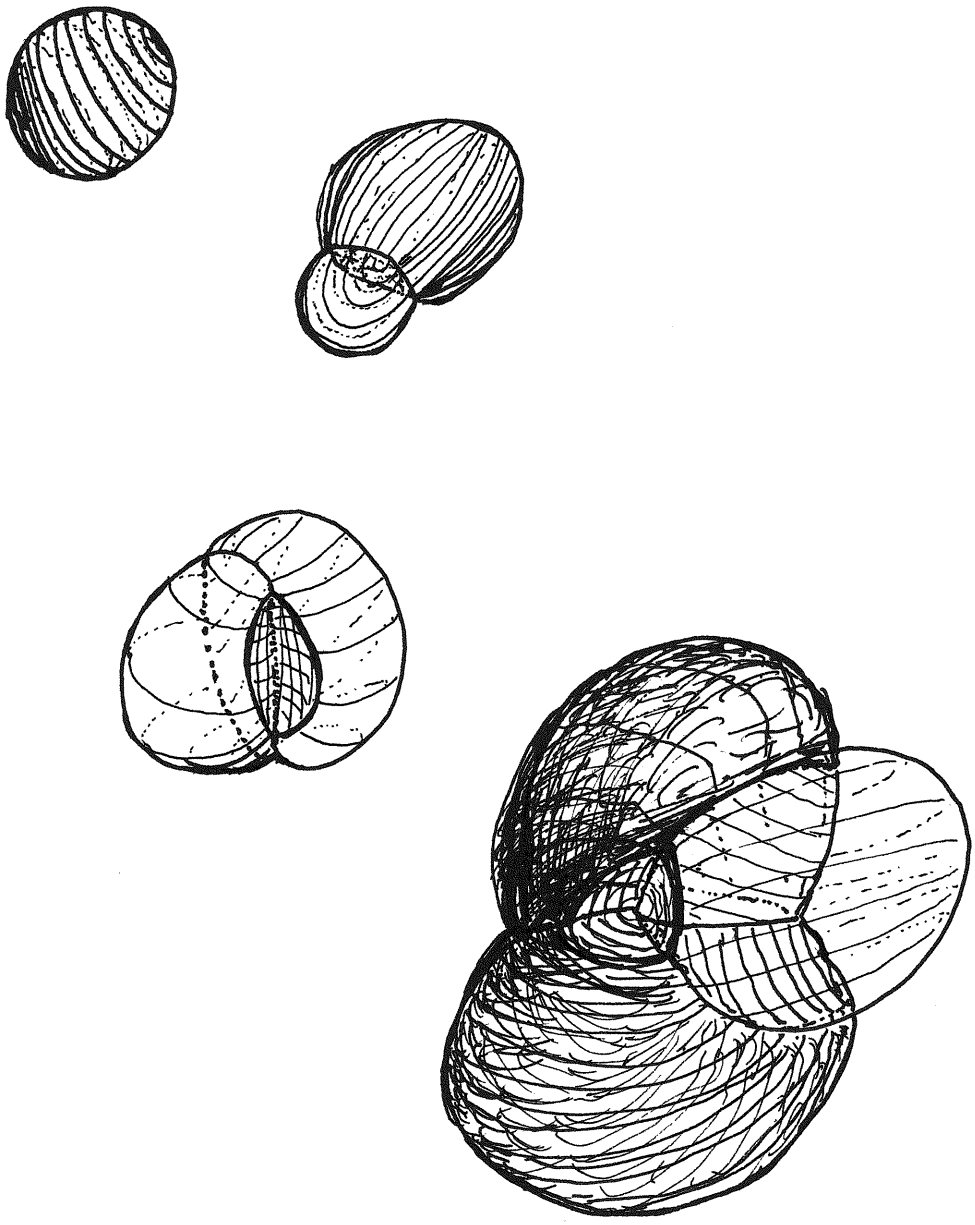


Figure 1.6: Bubble Clusters

# Chapter 2

# general background

## Introduction

**The beginning.** We want to get some feeling for the known results. Other than Euclidean space, the best-understood geometric spaces are the 'classical' non-Euclidean geometries. Understanding the geometry of these spaces and the basic theory of area-minimizing bubbles in them is what is emphasized in this chapter.

**The basics.** We should know what the the simply connected spaces with constant curvature are and some of their basic properties, including their symmetries. These are the classical Euclidean and non-Euclidean geometries; they are collectively called *constant curvature space forms*. The basic properties of constant curvature space forms are reviewed in section 2.1. We should understand the important, well-known, result on the isoperimetric or 'single bubble' problem, which states that of regions with a fixed volume in constant curvature space forms, the ones with the least surface area are round balls. We will present a geometric proof of this fact in section 2.2. We should understand

the basic results of geometric measure theory on the existence and regularity properties of solutions to surface area minimization problems when there are multiple volumes to enclose and separate. These results are reviewed in section 2.3.

In this thesis we are primarily interested in understanding the case of two volumes, or the 'double bubble' problem, and the historical development of this problem up to the current date. It is known that a 'standard double bubble', like that pictured in Figure 2.1, is the least surface area way to enclose and separate two given volumes in $\mathbf{R}^3$ [23], that the equivalent of up one dimension minimizes the analogue of surface area in $\mathbf{R}^4$ [41], and that higher dimensional versions minimize higher dimensional equivalents of surface area when the two volumes to be enclosed are disparate from one another in size. The popularly accepted conjecture is that the standard double bubble is the least surface area way to enclose and separate two arbitrary volumes in Euclidean spaces of arbitrary dimension. We conjecture that standard double bubbles (after being subjected to a subtle redefiniton) give the least surface area way to enclose and separate two given volumes in constant curvature space forms as well. (In Chapter 4 we give a partial proof that this is indeed the case in three dimensions.) We will present the necessary background specific to double bubbles in Chapter 3.

There are some basic features of the double bubble problem that are worth keeping in mind when reading the current chapter however. For constant curvature spaces, one of the key differences between the double bubble problem (and all problems with a higher number of volume constraits) and the single bubble problem is that when there are multi-volume constraints each region in a perimeter minimizing bubble cluster might be disconnected into a number of fragments. In an arbitrary geometric space, a minimizing single bubble might disconnect as well (see Figure 2.2). One way understand why single bubbles in constant curvature spaces do not disconnect is the regularity theorem of Taylor that we discuss in subsection 1.3. This theorem gives explicit conditions on the singular sets present in perimeter minimizing bubble clusters. In particular, in three dimensions, the

boundary surfaces for an area-minimizing bubble cluster always meet at one hundred and twenty degrees along analytic curves. Given a disconnected candidate solution to the single bubble problem, we can use a series of local isometries to move one of the components so that it touches another at a point. *Since this transformation preserves area and perimeter and the final configuration violates regularity, neither the final configuration nor its preimage can be actually be minimizing*. The geometry of the double bubble problem is considerably more complicated than the geometry of the single bubble problem. It is not true, for example, that we can simply translate components of a non-standard double bubble to bring about a violation of regularity (see Figure 2.3 for a simple counter example).

Each of the sections of this chapter and the next requires a good-sized amount of 'modern' geometry and analysis. Many of the ideas (symmetrization, instability, etc.) are straightforward and the reader should be able to get a fairly clear understanding of the theorems without undo difficulty. However, in the hopes that readers will also be able to familarize themselves with the *proofs* of these known results (which I am not squeamish about presenting more or less in full here, to what I hope will be the greater glory of their originators), the standard definitions and results of geometry and analysis needed for the full appreciation of the results of this chapter and its sequel are presented in the appendix.

## 2.1 Constant Curvature Space Forms

**Charter** We should understand what *constant curvature space forms* are and some of their basic properties, including their symmetries. We should understand why there are in some sense only three simply connected spaces with constant curvature.

Three of the most useful geometric spaces in existence are Euclidean space, spherical space, and hyperbolic space (also sometimes called 'Lobachevski' space). These spaces are characterized by,

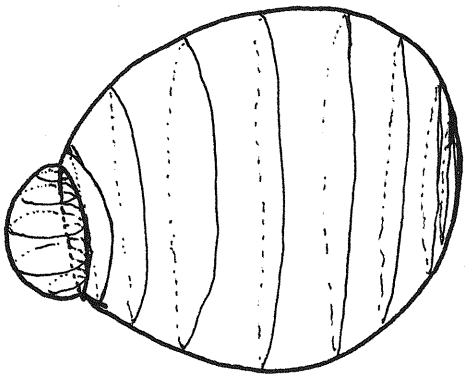


Figure 2.1: Standard Double Bubble

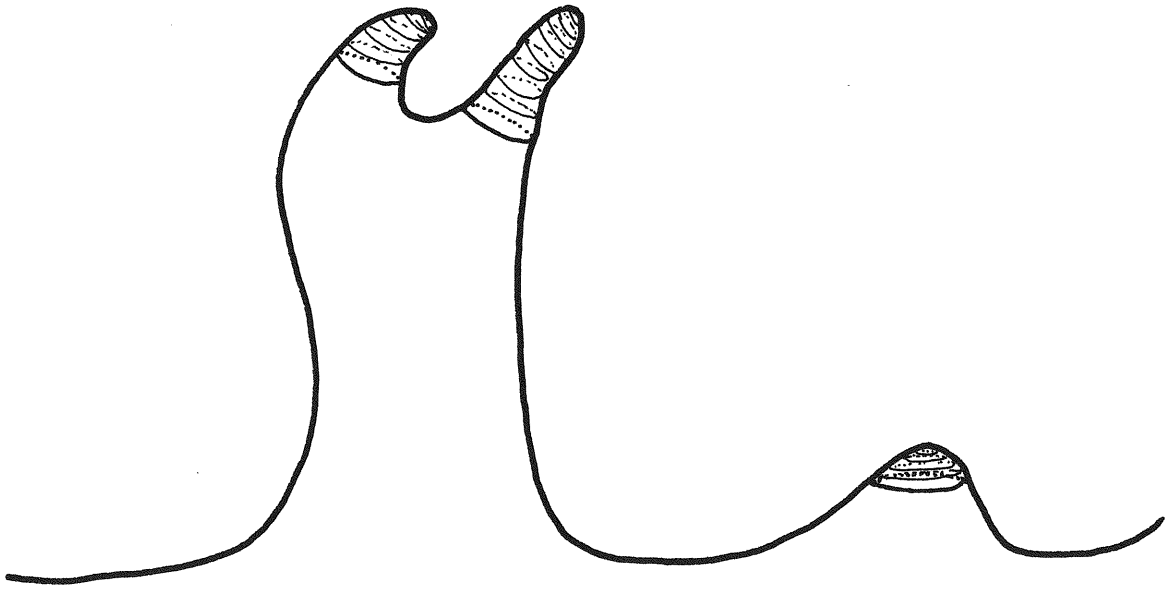


Figure 2.2: Disconnected Single Bubble Minimizer

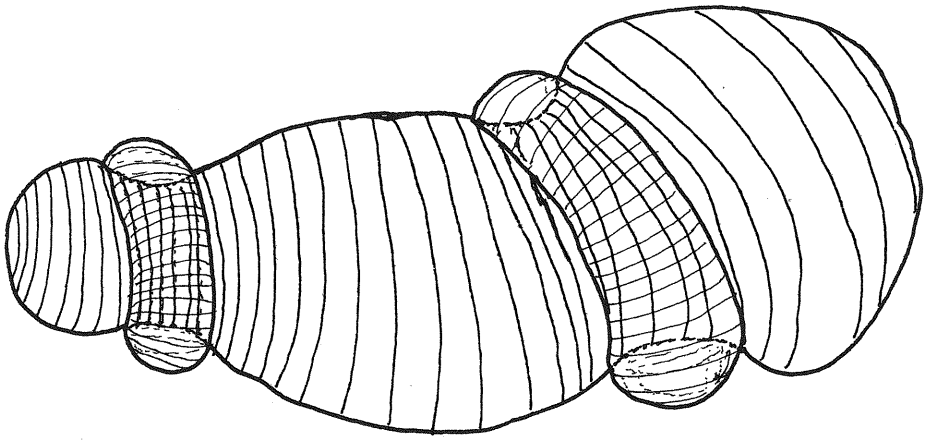


Figure 2.3: Non Standard Double Bubble

among other things, having *constant curvature* of zero, one, and negative one, respectively. By curvature we mean here the sectional curvature, which for the time being we will think of quite loosely as being 'a metric property of two-dimensional subspaces of a given space'. (To speak precisely, sectional curvature is a property of two dimensional subspaces of a tangent space, that is, a local property; we will come to this below. We won't reason in a circle like this in proofs, but in making a first approach to the subject of sectional curvature for constant curvature space forms, it wouldn't be too bad to think first about the curvature properties of subspaces, or, more accurately, submanifolds – such as lines, planes, meridians, and great circles for example – and then make the switch to the correct local formulation when we need it.)

We might mention at this point that there are very classical ways of characterizing these spaces that do not mention curvature. Non-Euclidean geometries were first developed at about the same time that modern geometry had its origins, and were studied using an amalgam of techniques. There are many books on non-Euclidean geometry, of which [3] seems to be very good for both mathematical and historical context.

Here we will work from the basic modern definitions, and we start with Euclidean space $\mathbf{R}^n$.

**Definition 2.1.** We define *n-dimensional Euclidean space* $\mathbf{R}^n$ to be the real vector space made up of

$n$-tuples of real numbers, such as $x = (x_1, x_2, ..., x_n)$, with addition and scalar multiplication defined element-wise, and with distance between points defined by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}.$$

We understand $\mathbf{R}^n$ to carry the metric topology and standard smooth structure (cf. the appendices).

### 2.1.1 A detailed look at Euclidean space $\mathbf{R}^n$

In order to get anywhere in geometry we need to understand Euclidean space very well. The following paragraphs specialize the general treatment of topology and geometry found in the appendix to review the standard facts about Euclidean space.

**1.** We have *curves*, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$, where $t$ lies in some interval of real numbers. Curves are frequently taken to be *smooth* in the sense that each of the component functions $x_k$ is smooth, *i.e.* infinitely differentiable.

**2.** From this we get an extended (beyond the calculus 1 version) notion of differentiation, namely that of the *tangent vector to a curve*, written as $x'(t)$, and obtained by differentiating a curve component-wise, so that

$$x'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t)).$$

**3.** We also have maps between sets in $\mathbf{R}^n$ and $\mathbf{R}^m$, such as

$$x(t_1, \ldots, t_n) = (x_1(t_1, \ldots, t_n), x_2(t_1, \ldots, t_n), \ldots, x_m(t_1, \ldots, t_n))$$

and these maps also have derivatives, this time requiring matrices to represent (so we have for example, the Jacobian, the Hessian, etc.). The maps are said to be of class $C^k$ if the first $k$ of these matrices are non-singular.

**4.** We look at the set of curves through a point $x$ and say that $\mathcal{C}_1 \sim_x \mathcal{C}_2$ if their tangent vectors are the same at $x$. If we identify curves that $\sim_x$ one another, the space of curves through the point $x$ takes on a new identity, namely that of *the tangent space to* $\mathbf{R}^n$ *at* $x$, denoted $T_x\mathbf{R}^n$. So we have tangent spaces, and lots of linear algebra. (This definition of the tangent space varies slightly from the one given in the appendix, but as discussed in Remark C.3, the two definitions are really the same. There we show that the tangent space to $\mathbf{R}^n$ at each point is an isomorphic copy of $\mathbf{R}^n$.) The collection of all the tangent spaces is the *tangent bundle*, which we denote by $T\mathbf{R}^n$.

**5.** We have the Euclidean metric

$$dx_1^2 + dx_2^2 + \ldots + dx_n^2$$

defined on tangent spaces. The meaning of this is as follows: if $\langle \cdot, \cdot \rangle$ denotes the standard inner product of vectors in $\mathbf{R}^n$ (that is, $\langle \cdot, \cdot \rangle : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}, \langle v, w \rangle = \sum_{j=1}^n v_j w_j$) and if $\iota_p : T_p\mathbf{R}^n \to \mathbf{R}^n$ denotes the isomorphism from $T_p\mathbf{R}^n$ to $\mathbf{R}^n$, then *standard Riemannian metric on* $\mathbf{R}^n$ (the Euclidean metric) is defined by

$$T_p\mathbf{R}^n \times T_p\mathbf{R}^n \ni \xi, \eta \mapsto \langle \iota_p(\xi), \iota_p(\eta) \rangle.$$

We often omit the '$\iota_p$'s. At each point, the metric can be written in local coordinates, which fix a basis for the tangent space. In terms of this basis, the metric becomes

$$\xi, \eta \mapsto \xi^T (g_{ij}) \eta,$$

where $g_{ij} = \delta_{ij}$. That is, at each point, the Euclidean metric can be represented by the identity matrix.

**And, the point of all this?** Broadly speaking, we understand curvature to be a smooth function (actually a quadratic form) on the tangent spaces of a manifold. To understand what this means,

we have to understand how to reduce questions about smoothness in a given geometric space (*i.e.* a manifold) to questions about smoothness in $\mathbf{R}^n$, and so we have to be comfortable doing the computations in $\mathbf{R}^n$, where the idea is that everything makes sense and is easy (or, as easy as its going to get).

### 2.1.2 Spherical and hyperbolic spaces

We define spherical and hyperbolic spaces as submanifolds of Euclidean space.

**Definition 2.2.** We define $(n-1)$-*dimensional spherical space* $\mathbf{S}^{n-1}$ to be the set of points a distance one from the origin in $\mathbf{R}^n$, carrying the (metric) topology induced by the metric topology on $\mathbf{R}^n$ and the smooth structure induced by the standard smooth structure on $\mathbf{R}^n$.

**Definition 2.3.** We define $(n-1)$-*dimensional hyperbolic space* $\mathbf{H}^{n-1}$ to be one sheet of a hyperboloid of two sheets

$$x_1^2 + x_2^2 + \ldots + x_{n-1}^2 - x_n^2 = -1 \tag{2.1}$$

(so, this, wherever $x_n > 0$) in the pseudo-Euclidean space $\mathbf{R}^{n-1,1}$ which has the same point-set as Euclidean space but is given the pseudo-Euclidean metric

$$dx_1^2 + dx_2^2 + \ldots + dx_{n-1}^2 - dx_n^2. \tag{2.2}$$

(The pseudo-Euclidean metric here is defined along exactly the same lines as the Euclidean metric above.) We understand $\mathbf{H}^{n-1}$ to carry the same topology and smooth structure as $\mathbf{R}^n$.

*Remark 2.1.* We can change the geometrical properties of spherical and hyperbolic space slightly by purturbing the above definitions; keeping the rest of the definitions the same, we put $\mathbf{S}^{n-1}(\rho)$ equal to the sphere of radius $\rho$, and $\mathbf{H}^{n-1}(\rho)$ equal to one sheet of the hyperboloid defined by the equation

$$x_1^2 + x_2^2 + \ldots + x_{n-1}^2 - x_n^2 = -\rho^2.$$

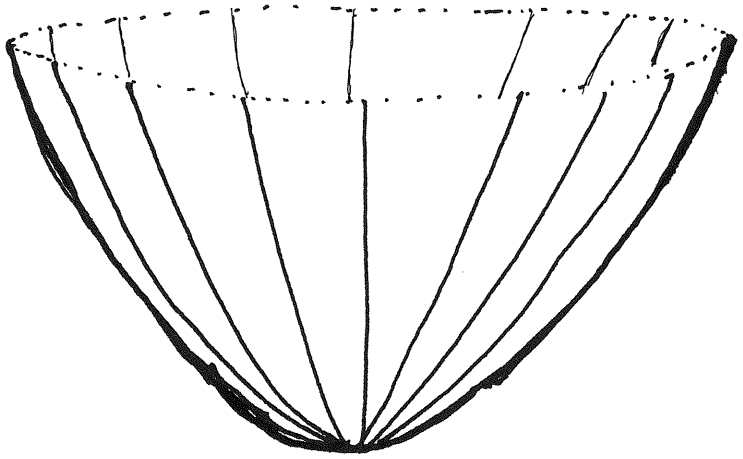


Figure 2.4: One sheet of the hyperboloid of two sheets

**Visualizing higher dimensions and weird spaces.** With all of this tricky abstraction at such an early stage in the game, it seems desirable to take a little time for reflection. The definitions above are, after all, supposed to be intuitive, in one way or another. The key is to figure out a way to actually think about everything geometrically. The basis for geometrical thinking is visualization and that is always lots of fun (one can do this lying on the couch with closed eyes). So for starters, how do we make sense of $\mathbf{R}^4$?

I like to think of this is as space-time – not anything tricky, just nice Cartesian space going on forever in all directions, and nice Galilean time (for lack of a better word) attached to each point in space. So, each point has a clock attached to it, say, and these clocks are syncronized. I like to use the fact that $\mathbf{R}$ is homeomorphic to an open interval to make everything happen in one infinitely long day. Right now its noon, everywhere, but in a few minutes, it will be 12:03. Time's running out! but don't worry, it will take forever to get to 6pm (the scheduled time for our infinite day comes to an end). On the other hand, 1:00 is coming up fairly soon, so we should be getting on with business.

More dimensions are easy: add more hands to the clock – say a hand indicating temperature. Lets say that temperature ranges from an extremely chilly $-90$ degrees to an unendurably hot 120000. Right now, its about 80 degrees in Sarasota, which is nicely described by – not to be self-centered

or anything – $(0, 0, 0, 1200, 80)$. Williamstown Massachusetts is located in $\mathbf{R}^5$ at $(1, 0, 0, 1200, 60)$. On the far side of the moon it is $(0, 15, 0, 1200, -60)$. Perpendicular in space to the plane determined by Williamstown, the moon, and Sarasota we find a point in the upper atmosphere of earth at $(0, 0, 1, 1200, -35)$.

Lets make things a little easier notationally and relabel the time and temperature axes so that 12:00 is 0, and 80 degrees is zero. Then Sarasota is at $(0, 0, 0, 0, 0)$, Williamstown is at $(1, 0, 0, 0, -.12)$, the far side of the moon is at $(0, 15, 0, 0, -3)$ and that special place in the upper atmosphere is $(0, 0, 1, 0, -1.5)$.

Ok, there. Now lets think about things geometrically. The most obvious thing is that Williamstown is pretty close to the north pole $(1, 0, 0, 0, 0)$ of the unit sphere in $\mathbf{R}^5$. The moon is not doing so well in this regard. The upper atmosphere is close to the sphere in most dimensions, but it would have to warm up a fair amount to make it. You might think about this as being sort of like the point $(0, 1, -1.5)$ in three dimensions – which lies due west of the equator of the unit sphere. If the unit sphere in three dimensions was being modeled by the surface of a pool ball, that special place way up in the sky might be stuck to the point on the edge of an 8-ball sitting between your and the corner pocket. The far side of the moon might show up on the tip of your nose at $(15, 0, -3)$, disturbing your concentration as you think about the shot. (If you are wondering how I chose the points in three space, the answer is that I projected onto the second, third, and fifth coordinates. I could have chosen a different model (a very loose sort of 'analogy') by projecting onto three different coordinates. On the other hand what you're probably wondering is how I chose the setting, and in that case you only get a hint – its not really noon in Sarasota.)

Anyway, fun and games aside, we have more exploring to do. We note that there are lots of different lower dimensional spheres. Everywhere the same distance from Sarasota as Sarasota is from Williamstown in the three-dimensional space given by the first three dimensions of $\mathbf{R}^5$ is a *two-sphere*, per usual. Everywhere that is this same distance from that special point in the upper

atmosphere is another two-sphere (and Sarasota is a point on its surface). If I worked on it a little, I could build a standard (2-)double bubble. Now, things get slightly more complicated as we note that the set of points every*when* a distance 1 from now (so, in terms of our original coordinate system, *everywhere* at 11:00 am together with *everywhere* at 1:00 pm) contains within it a set of points on the unit *three-sphere* – only two, actually – the point *here an hour ago* and the point *here an hour into the future*. These points are antipodal, being the poles of the sphere along the time axis. To interpolate between these points (*i.e.* to travel from one to the other and stay inside the sphere) we have to move just right. Let's say we were here an hour ago (not an unsafe assumption) and that we want to be here an hour from now (again, for most of us, this is a fair thing to say). What should we do? Should we sit around and wait? (Some people nod). Not if we want to traverse the three-sphere! Sitting here would take us through the center of the three-sphere, which would be fairly boring*. We have to travel to someplace a unit distance away in space in one hour and then get back in another hour. This means we are going to have to go fast. How can we get to Williamstown in an hour? Not much chance of that happening! Where can we go? Up, I suppose. But, thinking about this from a long-term perspective, that could be risky. The exercise would be better carried out starting in space, where we could designating some point clear of debris as (0,0,0,-1) and then come up on the point fast, and pass it, then gradually slow, turn, go back, and finally pass quickly through the point again at the designated time with no risk of bodily injury. Space also has the benefit of our being able to move in all directions, so we could visit any point on the two-sphere of directions in three-space that makes up the equator of the three-sphere.

So that is the three-sphere. It only a little trickier to build a standard 3-double bubble (which is made out of pieces of three-spheres meeting one another at 120 degrees), but it might take a long time to sort out the details if one wanted to go from scratch – so I'll leave that as a visualization exercise for the reader, should he or she wish to wrack his or her brain. There will be a hint at the

*Funny, am I not? – but in truth, the joke lacks content.

end of this paragraph for those who want the exercise to go more easily.

By a similar construction, everywhere-when-temperature that interpolates at the right rate between points that are one Sarasota-to-Williamstown-length off, one hour off, and one degree off makes up the unit 4-sphere we were talking about a few minutes ago. Materially speaking, this is much easier to traverse, since we can start at one pole in the time dimension, cool things down quickly and then more slowly so that we are one degree colder than we were an hour ago, and then reverse the operation over the course of the following hour. One could also easily combine cooling with driving, and move along a different trajectory from pole to pole. Again, we would be able to see much more of the 4-sphere if we did everything in outer space. However, it is nice to know that there is some higher-dimensional object that we can navigate in our lifetime (with little risk). *Indeed, such things actually show up all over the place when one is looking for them* – so the moral of the story *should* be that higher dimensions can be useful as well as entertaining.

As for hyperbolic space, I should say a few words about that here too. To understand this space in terms of our definition, we should first understand something about $\mathbf{R}^{n-1,1}$. To make things easy on ourselves at first we will restrict to the case that $n = 4$. We will think in a parallel way about the space $\mathbf{R}^{3,1}$ and the space $\mathbf{R}^4$. Indeed, $\mathbf{R}^{3,1}$ is often interpreted as another model for space-time – in this case, we have Cartesian space again, but Einsteinian or relativistic time [29]. Although we think of it as another model for space-time, we also need to keep the two different notions straight, so henceforth we will call $\mathbf{R}^{3,1}$ by its official name *Minkowski space*, The way Minkowski space models space-time is as follows: we have assume the speed of light $c$ is constant (usually we chose this constant to be 1) and we think of vectors in the space as $(x_1, x_2, x_3, ct)$. The pseudo-norm (Minkowski length) $|x|^2 = \sum_{i=1}^{3} x_i^2 - (ct)^2$ induced by the pseudo-Euclidean metric 2.2 gives us an immediate way to define the distance between points $x = (x_1, x_2, x_3, ct_1)$ and $y = (y_1, y_2, y_3, ct_2)$ in $\mathbf{R}^{3,1}$, namely $d(x, y) = |x - y| = \left(\sum_{i=1}^{3}(x_i - y_i)^2\right) - (ct_1 - ct_2)^2$; this is a global definition for distance, *i.e.* it works in much the same way as the Euclidean norm. Here however, distances

are allowed to be negative. The vectors with Minkowski length zero form a cone, since $|\cdot|^2$ is homogeneous. This cone is called the *light cone*, because points that move in straight lines away from the origin with the speed of light stay in the cone. It has two halves, the *forward light cone*, consisting of vectors with $t > 0$ and the *backward light cone*, consisting of vectors with $t < 0$ (similarly, there are forward light cones and backward light cones centered at each point in Minkowski space). All vectors in the light cone are called *light-like*. Vectors with positive Minkowski length are called *space-like* and vectors with negative Minkowski length are called *time-like*, where the names designate the coordinates that are dominant in the vector. The time-like vectors lie in region of $\mathbf{R}^{3,1}$ 'inside' the light cone. The physical interpretation is that particles can not travel faster than light, so tangent vectors must satisfy the inequality $|\frac{\partial x}{\partial t}|^2 = \sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial t}\right)^2 - c \leq 0$. Continuing in the physical vein, we assume that only particles with zero mass have light-like tangent vectors, and particles with positive mass have time-like tangent vectors. The arclength $s = \int_a^b |\frac{\partial x}{\partial t}|^2 d\tau$ in the case of a particle moving with constant velocity becomes $\int_{t_1}^{t_2} \sqrt{c^2 - \sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial t}\right)^2} d\tau = \int_{t_1}^{t_2} c\sqrt{1 - \frac{v^2}{c^2}} d\tau = ct\sqrt{1 - \frac{v^2}{c^2}}\Big|_{t_1}^{t_2}$ and $s/c$ is interpreted as the amount of time the particle takes to travel between two points in space as seen by a fixed observer. If $v \neq 0$, this measurement is the amount of time a clock traveling with the particle takes, scaled by a factor of $\sqrt{1 - \frac{v^2}{c^2}}$. This is a basic property of special relativity; others follow similarly. Three-dimensional hyperbolic space consists of (half of) the vectors in $\mathbf{R}^{3,1}$ with Minkowski length $-1$. This set is asymptotic to the light cone as $t$ increases. One interpretation might be that hyperbolic space consists of all of the positions of a set of particles that explode out of the origin in $\mathbf{R}^3$ at time $t = 1$, and their positions are then plotted with respect to time. The key idea is that the particles speed up as they fly outwards in space at a certain rate, and over time they approach the speed of light. They also spread out as they fly, so that the total mass along each direction goes to zero (assuming the particles were infinitesimal to begin with).

The basic idea is that hyperbolic space spreads out from any given point very quickly, and that it essentially the same as $\mathbf{R}^n$ in all other respects.

Figure 2.5 shows some of the important features of the hyperbolic plane in the Poincaré disk model (which does not look so terribly different from the hyperboloid - but it is different enough to warrent further investigation. Unfortunately we do not have time to show how to map between these representations in this thesis; we direct the interested reader to [16].

The first important feature of this model (which was also used to draw Figure 1.3) is that the boundary of the disk is at infinity, so we are looking at a truly infinite space, although it is captured in finite area. The other important feature of the model is that angles are correct. The circles in the interior of the space *are* circles (*i.e.* boundaries of metric balls in $\mathbf{H}^2$) although their centers are not always in the center of the circles as they are drawn here. The straight lines and pieces of circles that meeting the boundary of the disk at 90 degrees are geodesics. Geodesics that look like pieces of circles have an additional special name – *hypocircles*; we extend this term to all geodesics, and in the future will use these two terms more or less synonymously. If we changed coordinates, so that we were looking down straight onto what is currently the apex of a curved hypocircle, it would look straight – so the only time a distiction can be made is when we work with a given representation of the plane. Circles tangent to the boundary of the disk in the figure are not actually circles in the hyperbolic plane but *horocircles*; they do not have well-defined radii. These can not be unstuck from infinity by a coordinate change. A related fact that sheds some light on the situation is that two geodesics that appear to meet at infinity never actually intersect, but become arbitrarily close to one another.

The picture is similar in three dimensions: things that look like pieces circles in this picture will become a pieces of spheres, the straight lines will become a flat disks, and the Poincaré model itself will become a solid ball. The new vocabulary that goes along with the construction just replaces 'sphere' everwhere 'circle' was found previously. Geodesics are again straight lines and pieces of circles that meet the boundary of the ball at 90 degrees. If we add more dimensions, the definitions and terminology develops along the same lines as it did here.

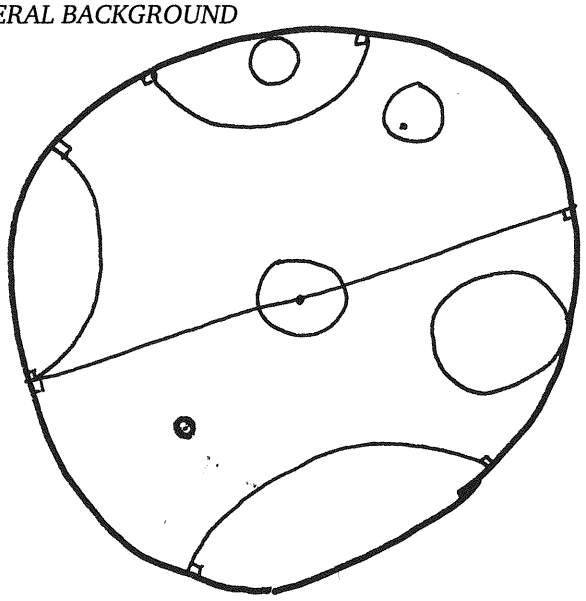


Figure 2.5: Horocircles, hypocircles, circles.

One definition that really needs to be understood clearly in a number of places later on in the thesis is that of a *hyperplane*. In $\mathbf{R}^n$, this is just an embedded isometric copy of $\mathbf{R}^{n-1}$. In Euclidean space, hyperplanes cut the space in two pieces: the line divides the plane, the plane divides space, space divides space-time (for example, everywhere-now is a hyperplane) and so on. Embedded isometric copies of Euclidean spaces with dimension less than $n$ do not separate $\mathbf{R}^n$ – a line does not separate space, a plane does not separate space-time. This is seen to be true because a line will *never* seperate space and a *line-always* is a two-plane in space-time, whereas *space-always* is space-time. In other words, after rotating, at any given point in time a two-plane in space-time is just a line in space, so it doesn't ever space. Furthermore, by definition, a line-always goes on for all time, so it doesn't separate time either, and hence does not separate space-time. And similarly for higher dimensions. In spheres and in hyperbolic space, similar ideas apply - so, in the $n$-sphere, an equatorial $(n-1)$-sphere is a *hypersphere*; in hyperbolic space, a (hyperbolic) $(n-1)$-plane is a *hyper-hyperbolic plane*.). With tori, and other more tricky geometries, it is a little different, since here hyperplanes do not always cut the space in two. Hyperplanes are just one example of a broader category, namely *hypersurfaces*, which are embedded submanifolds of one dimension lower than the ambient manifold.

We will say one last thing about spheres. This is that $\mathbf{S}^n$ is in general not only a hypersurface in $\mathbf{R}^{n+1}$, but a *hypersurface of revolution*. That $\mathbf{S}^2$ is a surface of revolution probably does not come as a major revelation. The fact that $\mathbf{S}^3$ is a hypersurface of revolution may take a little explaining. First, lets walk through an illustrative example of rotation *in* $\mathbf{S}^3$. From this example we will get the ideas we need to define hypersurfaces of revolution.

There are two steps to the illustration. The first step should be familiar. Start with a half circle $C$ with radius $r$ in the upper half plane $\mathbf{R}^2_+$, sitting on the $x$-axis. We revolve $C$ around the $x$-axis and we call the set of points it hits '$\mathbf{S}^2$'. Now the second step: we will take the upper half of $\mathbf{S}^2$ (let's call it $C'$) and perform similar operations to generate $\mathbf{S}^3$. We will regard $C'$ as sitting on $\mathbf{R}^2$ inside of $\mathbf{R}^3_+$ which is itself sitting inside of $\mathbf{R}^4$. But we also keep in mind the idea of $C'$ as sitting inside $\mathbf{S}^3$ (specificall, we will think of $C'$ as half of an equatorial two-sphere). We now rotate $C'$ by interpolating from each of its points in the manner described previously. What we see in $\mathbf{R}^4$ are images of $C'$, getting smaller and earlier, all centered on the same point $p$ in $\mathbf{R}^2$ (*i.e.* the same point in space). Eventually these images shrink all the way down to $p$ (the time then is exactly $-r$), reemerge *reflected across* $\mathbf{R}^2$, and begin to grow and get later again. When the the image in $\mathbf{R}^3$ is equal to the reflection of $C'$ across $\mathbf{R}^2$, the time is zero again. Now the images shrink down to $p$ again, getting later all the while, and when they hit $p$ this time, the clock reads $r$. To make the final turn, the images grow again, and when they get back to $C'$, the clock is back to zero; this completes the illustration.

At this point it would be natural to ask what we have actually done. This is a question perhaps as much due to the fact that spheres are so fearfully symmetric as it is to the fact that we are working in four dimensions. But we also note is that there are some big differences between the lower-dimensional case (the first step of the example) and the higher-dimensional case (the second step). The main difference that concerns us here is that none of the points on $C'$ were fixed in the second step (whereas the two endpoints of $C$ were fixed in the first step). Something that *is* fixed (in space-

time) is the location of the point $p$, since its distance from itself is zero (and distance-from-$p$ was the parameter we used to determine how far out in the time direction we had to go). If we now examine a surface with less symmetry – the top half of a cylinder for example – we can define a rotation about the (spatial) axis of symmetry of the full cylinder by treating each half-circle segment of the half-cylinder in the same way we treated the meridinal segments of $C'$. Taking what we learn from this example, we can define the *rotation into the fourth dimension* for a rotationally symmetric surface $S$ by treating each of the circles that make up $S$ as if it was a meridian in the two-sphere sitting inside $\mathbf{S}^3$. In particular, we can take $\mathbf{S}^2$ and viewing it as a subset of $\mathbf{R}^3$, think of it as being a surface of revolution about some fixed axis $\ell$. If we now rotate into the fourth dimension as described, we trace out a copy of the three sphere. As in the first step of the illustration, two points are fixed for the entire rotation, namely the points where $\ell$ intersects $\mathbf{S}^2$. Since the time at these points is always zero, it might be fitting to call them East and West poles of the three sphere, since they lie on line that is perpendicular to the time axis (which we usually say runs north and south).

Now, the hint for how to construct a standard 3-double bubble should make it very easy: start with a standard 2-double bubble and revolve it around its spatial axis of symmetry.

### 2.1.3 Three constant curvature space forms.

Given the intuitive definitions for Euclidean, spherical, and hyperbolic space we saw above, it may seem reasonable that they should have constant sectional curvature. (We're still thinking of this in very loose terms; the point being that we know intuitively which two dimensional subspaces of $\mathbf{R}^n$, $\mathbf{S}^n$, and $\mathbf{H}^n$ we will be looking at. They are the hyperplanes, equatorial spheres, or hyper-hyperplanes.)

A quick proof that the curvatures are constant and 'signed', so that Euclidean space has zero curvature, $\mathbf{S}^n$ has constant positive curvature, and $\mathbf{H}^n$ has constant negative curvature, will be given in this subsection. A simple corollary shows that by scaling the ambient $\mathbf{R}^n$ we can normalize the

curvature of $\mathbf{S}^{n-1}$ to be 1 and by scaling the ambient $\mathbf{R}^{n-1,1}$ we can normalize the curvature of $\mathbf{H}^{n-1}$ to be $-1$.

This suggests that there are in some sense only three constant curvature space forms. The exact sense in which this is true is that *up to scaling these are the only simply connected spaces with constant curvature, i.e.* that $\mathbf{R}^n$, $\mathbf{S}^n$, and $\mathbf{H}^n$ are the only constant curvature spaces in which all loops are contractible to a point, *i.e.* they are the only constant curvature space forms. (The fact that these spaces are all simply connected is easy to see, if one is good at visualizing things in many dimensions.) The *proof* that these are the only simply connected spaces with constant curvature is not particularly easy, and we omit it (see [12] or [59] for the details).

We henceforth give the constant curvature space forms a *new* collective name, $\mathcal{R}^n$, where the $\mathcal{R}$ refers to the (Riemann) curvature (tensor), which is all that we need to specify in order to fix a particular constant curvature space form.

**The Manifold Curvatures.** There is not much left to do but to plunge in and recall the official definitions. The various curvatures of Riemannian geometry are as incestuous and as plentiful could possibly be desired, so there could be a lot for us to recall if we liked; we will however try to be moderate. *It may be helpful to refer to the appendix on Geometry Preliminaries when reading this section.*

**Definition 2.4.** We define the *Riemann curvature operator* $R$ of a Riemannian manifold $M = (X, \langle\cdot,\cdot\rangle)$ with Levi-Civita connection $\nabla^{\mathrm{LC}}$ and commutator $[\cdot,\cdot]$ to be the map from $\mathrm{vect}(M) \times \mathrm{vect}(M) \times \mathrm{vect}(M)$ to $\mathrm{vect}(M)$ determined by the following action on vector fields:

$$R(\mathcal{U},\mathcal{V})\mathcal{W} = \nabla^{\mathrm{LC}}(\mathcal{V}, \nabla^{\mathrm{LC}}(\mathcal{U},\mathcal{W})) - \nabla^{\mathrm{LC}}(\mathcal{U}, \nabla^{\mathrm{LC}}(\mathcal{V},\mathcal{W})) - \nabla^{\mathrm{LC}}([\mathcal{V},\mathcal{U}],\mathcal{W}).$$

*Remark 2.2.* Note that $R$ is local in the sense that $(R(\mathcal{U},\mathcal{V})\mathcal{W})|_p$ depends only on $\mathcal{U}|_p, \mathcal{V}|_p$ and $\mathcal{W}|_p$.

To be precise, in terms of local coordinates $\{u_i\}_{i=1}^n$, we have

$$R\left(\frac{d}{du_j},\frac{d}{du_k}\right)\frac{d}{du_i}=\sum_{l=1}^{n}R^l_{ijk}\frac{d}{du_l} \tag{2.3}$$

where the '$R^l_{ijk}$'s are functions from the manifold to the reals defined by

$$R^l_{ijk}=\frac{d}{du_k}\Gamma^l_{ik}-\frac{d}{du_j}\Gamma^l_{ik}+\sum_{r=1}^{n}\Gamma^r_{ij}\Gamma^l_{rk}-\Gamma^r_{ik}\Gamma^l_{rj}. \tag{2.4}$$

The proof is just by expanding $R\left(\frac{d}{du_j},\frac{d}{du_k}\right)\frac{d}{du_i}$ in terms of its definition and the definition of a connection (DefinitionC.11). From the definition of the Riemann curvature operator, this is

$$\nabla^{\mathrm{LC}}\left(\frac{d}{du_k},\nabla^{\mathrm{LC}}\left(\frac{d}{du_j},\frac{d}{du_i}\right)\right)-\nabla^{\mathrm{LC}}\left(\frac{d}{du_j},\nabla^{\mathrm{LC}}\left(\frac{d}{du_k},\frac{d}{du_i}\right)\right)-\nabla^{\mathrm{LC}}\left(\left[\frac{d}{du_k},\frac{d}{du_j}\right],\frac{d}{du_i}\right)$$

(but $\left[\frac{d}{du_k},\frac{d}{du_j}\right]$ is zero; we apply the definiton of the connection coefficients (Equation C.5) to the first two terms)

$$=\nabla^{\mathrm{LC}}\left(\frac{d}{du_k},\sum_{m=1}^{n}\Gamma^m_{ij}\frac{d}{du_m}\right)-\nabla^{\mathrm{LC}}\left(\frac{d}{du_j},\sum_{m=1}^{n}\Gamma^m_{ij}\frac{d}{du_m}\right)$$

(by additivity in the second component for derivations (Equation C.2) this is)

$$=\sum_{m=1}^{n}\left(\nabla^{\mathrm{LC}}\left(\frac{d}{du_k},\Gamma^m_{ij}\frac{d}{du_m}\right)\right)-\sum_{m=1}^{n}\left(\nabla^{\mathrm{LC}}\left(\frac{d}{du_j},\Gamma^m_{ij}\frac{d}{du_m}\right)\right)$$

(combining the summations)

$$=\sum_{m=1}^{n}\left(\nabla^{\mathrm{LC}}\left(\frac{d}{du_k},\Gamma^m_{ij}\frac{d}{du_m}\right)-\nabla^{\mathrm{LC}}\left(\frac{d}{du_j},\Gamma^m_{ij}\frac{d}{du_m}\right)\right)$$

(and by the Leibnitz rule for connections (Equation C.3) and the definition of the connection coefficients again this is)

$$=\sum_{m=1}^{n}\left(\left(\left(\frac{d}{du_k}\Gamma^m_{ij}\right)\frac{d}{du_m}+\Gamma^m_{ij}\sum_{r=1}^{n}\Gamma^r_{mk}\frac{d}{du_r}\right)-\left(\frac{d}{du_k}\Gamma^m_{ik}\right)\frac{d}{du_m}+\Gamma^m_{ik}\sum_{r=1}^{n}\Gamma^r_{mj}\frac{d}{du_r}\right)$$

(which, upon collecting like terms, shows that the scalar coefficients are the '$R^l_{ijk}$'s defined by Equation 2.4.)

**Definition 2.5.** With $M$ as above, we define the *Riemann curvature tensor* to be the map

$$(\mathrm{vect}(M))^4\times M\ni(\mathcal{U},\mathcal{V},\mathcal{W},\mathcal{X},p)\mapsto\langle R(\mathcal{U},\mathcal{V})\mathcal{W},\mathcal{X}\rangle_p\in\mathbf{R}. \tag{2.5}$$

*Remark 2.3.* If we have local coordinates, the Riemann curvature tensor is locally computable in terms of the connection coefficients (and so, in terms of the metric). Using the local definition the Riemann curvature operator (Equations 2.3 and 2.4), we have

$$\left\langle R\left(\frac{d}{du_j}, \frac{d}{du_k}\right)\frac{d}{du_i}, \frac{d}{du_m}\right\rangle = R^l_{ijk} = \frac{d}{du_k}\Gamma^l_{ik} - \frac{d}{du_j}\Gamma^l_{ik} + \sum_{r=1}^{n}\Gamma^r_{ij}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rj}. \tag{2.6}$$

In particular, a computation in the appendix (Corollary C.1) shows that the connection coefficients all vanish in the case of $\mathbf{R}^n$, which implies that the Riemann curvature tensor associated with $\mathbf{R}^n$ is identically zero.

**Definition 2.6.** With $M$ as above, let $p \in M$ and let $\xi, \eta$ be vectors in $T_pM$. Define

$$\kappa(\xi, \eta) = \langle R(\xi, \eta)\xi, \eta\rangle,$$

and define

$$\kappa_1(\xi, \eta) = \langle \xi, \xi\rangle\langle \eta, \eta\rangle - \langle \xi, \eta\rangle^2$$

Finally, let

$$K(\xi, \eta) = \frac{\kappa(\xi, \eta)}{\kappa_1(\xi, \eta)}.$$

$K$ is the *sectional curvature at p of the plane determined by $\xi$ and $\eta$.*

*Remark 2.4.* The idea is that in $\mathcal{R}^n$, $K$ depends trivially on $\xi$ and $\eta$, and $p$. For example, in Remark 2.6 we saw that the Riemann curvature tensor on $\mathbf{R}^n$ is identically zero; this implies that the sectional curvature of $\mathbf{R}^n$ is identically zero.

**$\mathbf{R}^n$, $\mathbf{S}^n$, and $\mathbf{H}^n$ have the right curvatures.** We have already done the case of Euclidean space; as summarized in Remark 2.4, the sectional curvature of $\mathbf{R}^n$ is identically zero. We say that Euclidean space is *flat*.

Now, spheres. Using stereographic projection from the north pole,

$$\mathbf{S}^n(\rho) \setminus \text{south pole} \ni y = (y_1, \ldots, y_{n+1}) \mapsto u = (u_1, \ldots, u_n) \in \mathbf{R}^n \subset \mathbf{R}^{n+1}.$$

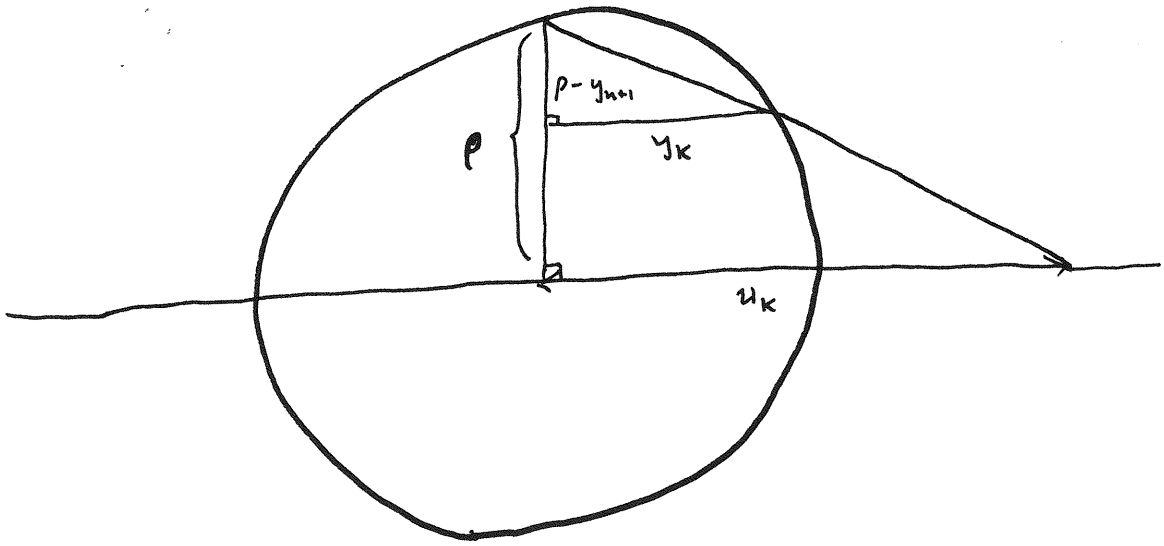



Figure 2.6: Stereographic projection

By similar triangles in Figure 2.6, we have

$$\frac{y_k}{\rho - y_{n+1}} = \frac{u_k}{\rho},$$

*i.e.*

$$y_k = \frac{u_k(\rho - y_{n+1})}{\rho},$$

hence

$$y_{n+1} = \frac{|u|^2 - \rho^2}{|u|^2 + \rho^2}\rho,$$

and

$$(y_1, \ldots, y_n) = \frac{2\rho^2}{\rho^2 + |u|^2}u.$$

We put $\bar{y} = (y_1, \ldots, y_n)$ and compute:

$$\begin{aligned} d\bar{y} &= 2\rho^2 \frac{(\rho^2 + |u|^2)du - 2u(u \cdot du)}{(\rho^2 + |u|^2)^2} \\ dy_{n+1} &= 4\rho^3 \frac{u \cdot du}{(\rho^2 + |u|^2)^2} \end{aligned}$$

and so

$$\begin{aligned} |d\bar{y}|^2 &= 4\rho^4 \frac{(\rho^2 + |u|^2)^2|du|^2 - 4\rho^2(u \cdot du)^2}{(\rho^2 + |u|^2)^4} \\ dy_{n+1}^2 &= 16\rho^6 \frac{(u \cdot du)^2}{(\rho^2 + |u|^2)^4}. \end{aligned}$$

Summing, we have

$$ds = \sum_{k=1}^{n+1} dy_k^2 = 4\rho^4 \frac{|du|^2}{(\rho^2 + |u^2|)^2},$$

*i.e.*

$$g_{jk} = \left\langle \frac{d}{du_j}, \frac{d}{du_k} \right\rangle = \frac{4\delta_{jk}}{(1 + |u|^2/\rho^2)^2}.$$

Using the formula for connection coefficients found in the appendix (Equation C.15), this implies that

$$\Gamma^i_{jk}(u) = \frac{-2}{\rho^2 + |u|^2}(\delta_{ik}u_j + \delta_{ji}u_k - \delta_{jk}u_i).$$

We now invite the reader to use the local formula for the Riemann curvature tensor (Equation 2.6) to show (by a long and nasty computation) that the sectional curvature of $\mathbf{S}^n(\rho)$ is $\rho$.

Finally, hyperbolic space. There are a couple of nice models for hyperbolic space: the disk model, and the upper half space model. It is sometimes helpful to be able to switch between the models.

If we start with a disk with the metric

$$ds = \frac{4|dz|^2}{(1 - |z|^2/\rho^2)^2},$$

then the sectional curvature will turn out to be $-1/\rho$. We show that in the case $\rho = 1$, the sectional curvature is $-1$. It is preferential to work in $\mathbf{R}^n_+$ with the metric

$$ds = \frac{|du|^2}{(u_n)^2},$$

*i.e.*,

$$g_{ij} = \frac{\delta_{ij}}{(u_n)^2}$$

which implies that

$$\Gamma^i_{ij} = -(u_n)^{-1}(\delta_{jn}\delta_{ik} + \delta_{kn}\delta_{ji} - \delta_{in}\delta_{jk}),$$

and so for $j, k \in \{1, \ldots, n-1\}$ we have

$$\nabla^{\mathrm{LC}}_{\mathbf{H}^n}\left(\frac{d}{du_j}, \frac{d}{du_k}\right) = -(u_n)^{-1}\delta_{jk}\frac{d}{du_n}$$

$$\begin{aligned}
\nabla^{\mathrm{LC}}_{\mathbf{H}^n}\left(\frac{d}{du_n},\frac{d}{du_k}\right) &= -(u_n)^{-1}\frac{d}{du_k}\\
\nabla^{\mathrm{LC}}_{\mathbf{H}^n}\left(\frac{d}{du_n},\frac{d}{du_n}\right) &= -(u_n)^{-1}\frac{d}{du_n}\\
\nabla^{\mathrm{LC}}_{\mathbf{H}^n}\left(\frac{d}{du_k},\nabla^{\mathrm{LC}}_{\mathbf{H}^n}\left(\frac{d}{du_j},\frac{d}{du_n}\right)\right) &= -(u_n)^{-1}\delta_{jk}\frac{d}{du_k}\\
\nabla^{\mathrm{LC}}_{\mathbf{H}^n}\left(\frac{d}{du_j},\nabla^{\mathrm{LC}}_{\mathbf{H}^n}\left(\frac{d}{du_k},\frac{d}{du_k}\right)\right) &= -(u_n)^{-1}\frac{d}{du_j}.
\end{aligned}$$

Therefore, when $j \neq k$ we have

$$R\left(\frac{d}{du_k},\frac{d}{du_j}\right)\frac{d}{du_k} = -(u_n)^{-2}\frac{d}{du_j}$$

which implies by the definition of sectional curvature that $K(\frac{d}{du_j},\frac{d}{du_j}) = -1$ and similarly

$$R\left(\frac{d}{du_n},\frac{d}{du_j}\right)\frac{d}{du_n} = -(u_n)^{-2}\frac{d}{du_j}$$

so that $K(\frac{d}{du_j},\frac{d}{du_n}) = -1$.

**Geodesics and Metric Balls.** This paragraph mainly reviews and specializes material found in the appendix. We also define spherical coordinates for constant curvature space forms.

The shortest paths between points are called geodesics. One can go through the derivations in some detail, but to summarize: in Euclidean space $\mathbf{R}^n$, geodesics are straight lines; in spherical space $\mathbf{S}^n$, geodesics are '$\mathbf{S}^1$'s with the same curvature as $\mathbf{S}^n$; and in hyperbolic space, geodesics are '$\mathbf{H}^1$'s with the same curvature as $\mathbf{H}^n$.

More generally, we have the following:

**Definition 2.7.** A submanifold $T \subset M$ is *totally geodesic* if every geodesic in $M$ that has an open segment in $T$ never leaves $T$.

Because the geodesics in $\mathcal{R}^n$ are easy, it is easy to see what the totally geodesic submanifolds are; they are the $k$-dimensional planes or spheres with the same curvature as $\mathcal{R}^n$.

For the sake of understanding the next several section(s) we need to know what metric balls (also interchangably called *disks*) in $\mathcal{R}^n$ look like. They are for the most part like balls or disks in Euclidean space. Disks on spheres are sometimes also called *spherical caps*, and they do not always look a lot like a disk in Euclidean space. Every round circle in $\mathbf{S}^2$ cuts off two spherical caps, so hemispheres as well as things both bigger and smaller are examples.

Round balls are associated with a very convenient coordinate system, namely, spherical coordinates. These are very easy to write down in $\mathcal{R}^n$ [7, page 39]. If $K > 0$, we take $(t, \xi) \in [0, \pi/\sqrt{K}) \times \mathbf{S}^{n-1}$, and if $K \leq 0$ we take $(t, \xi) \in [0, \infty) \times \mathbf{S}^{n-1}$; then there is some coordinate system with respect to which the metric can be written as

$$ds^2 = (dt)^2 + S_K^2(t)|d\xi|^2,$$

where $S_K^2(t)$ is the solution to the differential equation

$$y'' + Ky = 0,$$

with the initial conditions

$$S_K(0) = 0,\ S_K'(0) = 1.$$

If $K > 0$, the solution is $S_K(t) = (1/\sqrt{K}) \sin \sqrt{K}t$ , if $K < 0$, the solution is $S_K(t) = (1/\sqrt{-K}) \sinh \sqrt{-K}t$ and if $K = 0$, the solution is $t$.

This is a nice way to summarize facts that can be determined by direct calculation, using the coordinates $x = p + t\xi$ in $\mathbf{R}^n$, $x = (\cos t/\rho)p + (\sin t/\rho)\xi$ in $\mathbf{S}^n(\rho)$, or (here we center on 0) $x = \rho \tanh(t/2\rho)\xi$ in $\mathbf{H}^n(\rho)$ and then finding the differentials.

## 2.2 Isoperimetric problem in constant curvature space forms

**Charter** We should understand the know results on the 'single bubble' or isoperimetric problem, which states that of regions with a fixed volume in constant curvature space forms, the ones with the least surface area are round balls.

In order to eliminate any potential confusion at such an early stage, and because it is at least somewhat relevant in the arguments which follow, the first note we'll make is that 'isoperimetric' (same-perimeter) is a fine, if somewhat obscure, name for the problem of finding the *least* perimeter needed to enclose a *fixed* area. The key idea to the simple argument that proves this is that maximal area increases monotonically with perimeter. This is obvious for Euclidean and hyperbolic space; for spheres this is only true for volumes that take up up to half of the volume of the space, but after that, we switch to think of the other smaller volume as the one being enclosed, so the same argument applies.

Here is the argument. Each perimeter $P$ has a set of maximal-area objects $S(P)$ associated to it. Obviously all the elements of $S(P)$ must have the same area, which we'll call $A$. Change perspectives, and each area $A$ has a set of perimeter-minimizing obects associated to it. We claim that this set is exactly $S(P)$. The proof is that if $p < P$ and there is some enclosure $E$ with perimeter $p$ and area $A$, so perimeter does not increase monotonically; this is a contradiction.

There are in fact a number of other equivalent formulations of the isoperimetric problem, but the one we want to keep in the forefront of our minds is the cannonical 'least perimeter for a given area' version. More generally, we will talk about the domains with the least surface area for a given volume, where we think in terms of the following formal definitions.

**Definition 2.8.** A *domain* in a manifold $M$ is an open set whose boundary fits the needs at hand;

usually our proofs only require that the boundaries be embedded $C^1$ or $C^2$ submanifolds.

In defining domains, we need to pick something that works as a model for bubbles. In the later sections, when we work with more than one enclosed region, we will be allowing a lot of singular behavior on the boundaries, but at this stage we want regularity up front. This means, for example, that unlike Zenodorus and Pappus, we won't be thinking about how competative the Euclidean solids are with the sphere. In the much more general category we will be working with later, it is possible to show that the boundaries of competative single bubbles really are smooth, so our restrictive-seeming differentiability assumption will be vindicated.

**Definition 2.9.** The *volume* of a domain $\Omega$ in $\mathbf{R}^n$ is the Lebesgue measure of $\Omega$. We write $\mu_n(\Omega)$ for the volume of a domain $\Omega$. In $\mathcal{R}^n$ volume is defined using the analogue of Lebesgue measure (see the appendix on Analysis).

*Remark 2.5.* This is one of the reasons we require differentiability; we want to make the set of domains to be in the $\sigma$-algebra of Lebesgue measurable subsets of $\mathbf{R}^n$. (Or, for $(R)^n$, in the analogous $\sigma$-algebra.)

**Definition 2.10.** We define the *surface area* of $\Omega$ to be the derivative at zero with respect to $\epsilon$ of the volume of an $\epsilon$-tube around $\Omega$ (an *$\epsilon$-tube* $\Omega_\epsilon$ is the set of points $p \in \mathcal{R}^n$ such that $d(p, \Omega) \leq \epsilon$). We write this in symbols as

$$\mu_{n-1}(\partial\Omega) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mu_n(\Omega_\epsilon).$$

Note that $\mu_{n-1}$ as surface area is the same as $\mu_{n-1}$ as $(n-1)$-dimensional measure; Definition 2.10 gives a way to restrict $n$-dimensional measure to $(n-1)$-dimensional sets. However, as with the definition of volume, the definition of surface area relies on the assumption that our domains are bounded by differentiable curves. (We could have defined surface area with a slightly relaxed hypothesis; for example, *piecewise* differentiable would be ok.) Definition 2.10 is only the beginning of a sequence of very interesting things that can be done with tubes. See Gray [17] for some of the

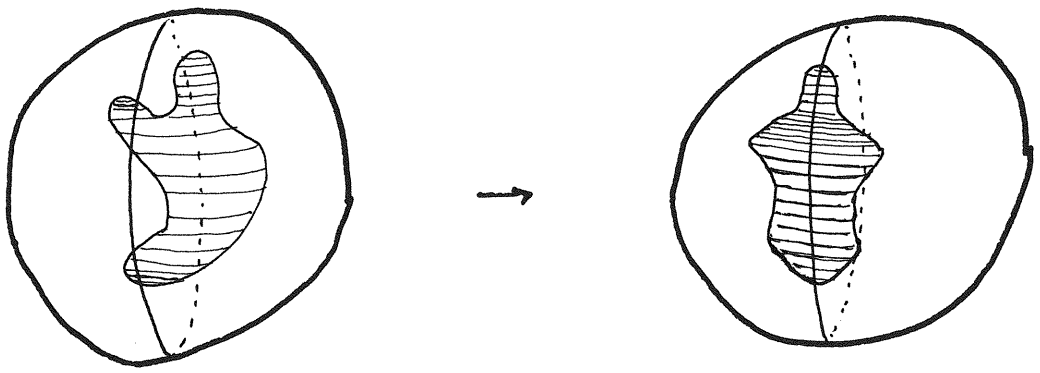


Figure 2.7: The Steiner symmetrization of a domain on the sphere.

most important details.

**Theorem 2.1 (Isoperimetric property of metric balls).** In constant curvature space forms, the domains with least surface area that enclose a given volume are the metric balls.

### 2.2.1 Proof of the isoperimetric property of round balls

We present a proof with a fairly geometric flavor based on an argument due to Figiel, Lindenstrauss, and Milman [52] which covers the spherical case. This proof was first encountered by the author in the text of Chavel [8] where it is modified to treat the Euclidean and hyperbolic cases. This restatement of Theorem 2.1 is based on that found in Chavel (*op. cit.*, page 283, Theorem 6.7) although I prefer in the notation of Figiel *et al.* and I reproduce their argument for the spherical case.

**Restatement of Theorem 2.1.** Let $A$ be the closure of a domain in $\mathcal{R}^n$. For a given $A$, let $B$ be a closed metric ball in $\mathcal{R}^n$ with the same volume as $A$. For every $\epsilon > 0$ let $A_\epsilon$ be the $\epsilon$-tube in $\mathcal{R}^n$ about $A$. Then

$$\mu_n(B_\epsilon) \leq \mu_n(A_\epsilon) \tag{2.7}$$

and consequentially,

$$\mu_{n-1}(\partial B) \leq \mu_{n-1}(\partial A). \tag{2.8}$$

*Proof.* The fact that Inequality 2.7 is sufficient to prove Inequality 2.8 is an immediate consequence of our definition of surface area. We need some addititional definititions. For any bounded subset $C \subset \mathcal{R}^n$ we let

$$r(C) = \min\{\rho; \exists x \in \mathcal{R}^n : C \subset \overline{B(x,\rho)}\}. \tag{2.9}$$

The function $r$ is continuous on the set of compact non-empty subsets of $\mathcal{R}^n$ with respect to the metric topology induced by the metric

$$\delta(C,D) = \min\{\epsilon; C \subset D_\epsilon, D \subset C_\epsilon\}. \tag{2.10}$$

Each bounded subset $C$ of $\mathcal{R}^n$ then has a circumdisk

$$\text{circ}(C) = \text{the disk with radius } r(C) \textit{ containing } C. \tag{2.11}$$

We define

$$\text{cent}(C) = \text{center of circ}(C). \tag{2.12}$$

Note that with respect to $\delta$, the volume functional $\mu_n$ is upper semi-continuous, ie.

$$\delta(C^k, C) \to 0 \Rightarrow \limsup_{k\to\infty} \mu_n(C_k) \leq \mu_n(C). \tag{2.13}$$

Let $\Gamma$ be a geodesic in $\mathcal{R}^n$. For each $x \in \Gamma$ let

$$H^x = \text{the totally geodesic submanifold perpendicular to } \Gamma \text{ at } x. \tag{2.14}$$

Note that the map $y \mapsto \mu_{n-1}(H^y)$ is upper semi-continuous.

For a compact subset $C$ of $\mathcal{R}^n$, for $x \in \Gamma$ we let

$$C^x = C \cap H^x \tag{2.15}$$

and let

$$\mathbf{D}^x(C) = \text{the disk in } H^x \text{centered at } x \text{ with } \mu_{n-1}(\mathbf{D}^x(C)) = \mu_{n-1}(C^x). \tag{2.16}$$

The set

$$\sigma_\Gamma C = \cup_{x\in\Gamma}\mathbf{D}^x_C \tag{2.17}$$

is called the (Steiner) symmetrization of $C$ with respect to $\Gamma$ (see Figure 2.7). It follows from the definition of Riemannian measure and the fact that the cut locus of a point is a set of measure zero (see Chavel, pp. 113, 134), or just Fubini's theorem, that $\mu_n(\sigma_\Gamma C) = \mu_n(C)$. A basic but central fact in the proof is that $\text{circ}(C)$ is invariant under symmetrization about geodesics that pass through $\text{cent}(C)$.

For the particular compact set $A$ given in the statement of the theorem, we put

$$M(A) = \{C \in \mathcal{R}^n; C \text{ closed}, \mu_n(C) = \mu_n(A) \text{ and } \forall\epsilon > 0 : \mu_n(C_\epsilon) \leq \mu_n(A_\epsilon)\}. \tag{2.18}$$

The assertion of the theorem is that disks with the same volume as $A$ are in $M(A)$ (see Inequality 2.7). The proof is broken into three main steps.

1. For all compact $A \in \mathcal{R}^n$, there is an element of $M(A)$ that is minimal for $r$.

2. Given any compact $A$ in $\mathcal{R}^n$ and geodesic $\Gamma$, $\sigma_\Gamma A \subset M(A)$.

3. For any compact $A \in \mathcal{R}^n$ which is not a disk, there exists a finite set $\{\Gamma_1, \ldots, \Gamma_k\}$ such that

$$r(\sigma_{\Gamma_k}\sigma_{\Gamma_{k-1}}\ldots\sigma_{\Gamma_1}A) < r(A).$$

By steps 2 and 3, an $r$-minimal element of $M(A)$ must be a disk. By step 1, an $r$-minimal element (which we may now fittingly call '$D$') exists.

These steps are carried out in the following paragraphs.

**1. Existence of $r$-minimal element.** It suffices to consider elements $C \in M(A)$ with $r(C) < r(A)$, and by an isometry we can restrict further to those elements $C$ with $\text{cent}(C) = \text{cent}(A)$; we will call the set of lower-radius, concentric† sets $m(A)$. With respect to the metric $\delta$, the set $m(A)$ is compact in the set of compact sets of $\mathcal{R}^n$ . We consider a sequence of such sets $C^k$ with $r(C^k)$ converging to the infimum of the '$r$'s associated with sets in $m(A)$. There is an associated subsequence which converges with respect to $\delta$ to a limit which we'll call $B$. We want to show that $B \in M(A)$.

For all $\eta > 0$, we trivially have $B \subset B_\eta$ and $B_\epsilon \subset B_{\eta+\epsilon}$, and hence for all $k$ large enough, if $\epsilon > 0$, we have $B_\epsilon \subset B^k_{\eta+\epsilon}$. Therefore since $B^k \in m(A)$ we have that

$$\mu_n(B_\epsilon) \leq \mu_n(B^k_{\eta+\epsilon}) \leq \mu_n(A_{\eta+\epsilon}). \tag{2.19}$$

Hence

$$\begin{aligned} \mu_n(B_\epsilon) &\leq \inf_\eta \mu_n(A_{\eta+\epsilon}) && (2.20)\\ &= \mu_n(\cap_{\eta>0} A_{\eta+\epsilon}) && (2.21)\\ &= \mu_n(A_\epsilon) && (2.22) \end{aligned}$$

where 2.22 is true because $A$ is compact. This shows that $\mu_n(B) \leq \mu_n(A)$. The upper semi-continuity of $\delta$ implies that

$$\mu_n \geq \limsup_{k\to\infty} \mu_n(B^k) = \mu_n(A), \tag{2.23}$$

and so we are done with the first step.

**2. Symmetrization reduces measure of tubular neighborhood (non-compact case).** We want to show that the symmetrization of $A$ is in $M(A)$. We need only show that

$$\mu_n((\sigma_\Gamma A)_\epsilon) \leq \mu_n(A_\epsilon).$$

†*I.e.* concircumcentric.

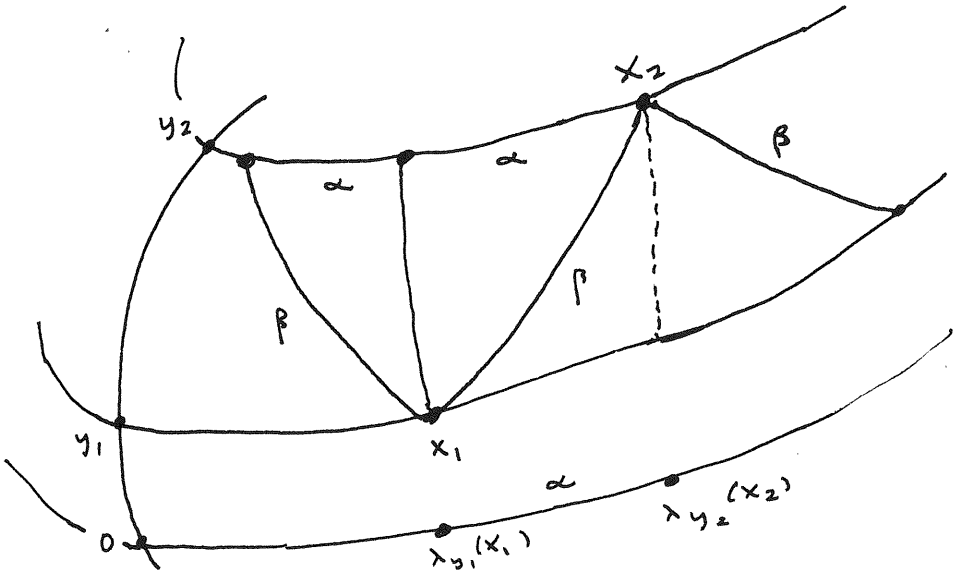



Figure 2.8: $\Psi(y_1, y_2, d(\lambda_{y_1}(x_1), \lambda_{y_2}(x_2)))$ gives the distance between $x_1$ and $x_2$.

Fix a point $o \in \Gamma$, and for any other point $y \in \Gamma$, let $\tau_y$ denote parallel translation along $\Gamma$ from $y$ to $o$. Recall that for every point $x \in \Gamma$ we are letting $H^x$ denote the totally geodesic subspace perpendicular to $\Gamma$ at $x$ (see Equation 2.14). We now define an isometry

$$\lambda_y : H^y \to H^o$$

by

$$\lambda_y = \exp_o t\cdot|_{\left(\frac{d\Gamma}{dt}|_{(\Gamma^{-1}(o))}\right)^\perp} \circ \tau_y \circ \left(\exp_y t\cdot|_{\left(\frac{d\Gamma}{dt}|_{(\Gamma^{-1}(y))}\right)^\perp}\right)^{-1}. \tag{2.24}$$

(This map makes sense because the inverse of $\exp_y$ maps points in the submanifold to tangent vectors; we then move these tangent vectors using $\tau_y$ and project them back to the manifold using $\exp_o$; it is clearly an isometry because all of the $(n-1)$-dimensional totally geodesic subspaces of $\mathcal{R}^n$ have the same metric properties.)

As illustrated in Figure 2.8, there is some function $\Psi : \Gamma \times \Gamma \times [0, \infty) \to \mathbf{R}$ such that for $x_1 \in H^{y_1}$, $x_2 \in H^{y_2}$ we have

$$d(x_1, x_2) = \Psi(y_1, y_2, d(\lambda_{y_1}(x_1), \lambda_{y_2}(x_2))). \tag{2.25}$$

Recall that for a general compact set $C \subset \mathcal{R}^n$ we are letting $C^x$ denote $C \cap H^x$. Equation 2.25 implies that if $y_1, y_2 \in \Gamma$ and $\epsilon > 0$ such that $d(y_1, y_2) \leq \epsilon$ then there is some number $\eta$, depending

on $y_1, y_2$ and $\epsilon$ such that if $Z \subset H^{y_1}$ then $(Z_\epsilon)^{y_2} = (\lambda_{y_2}^{-1} \circ \lambda_{y_1})((Z_\eta)^{y_1})$. (That is, we start with an $(n-1)$-dimensional subset $Z$ of $H^{y_1}$, thicken it in $\mathcal{R}^n$ by $\epsilon$ and then slice by $H^{y_2}$. The result is the same as thickening $Z$ by $\eta$ in $H^{y_1}$ and then projecting to $H^{y_2}$.)

Note that $d(y_1, y_2) > \epsilon$ implies that for any $(n-1)$-dimensional subset $Z$ of $H^{y_1}$ that $(Z_\epsilon)^{y_2} = \emptyset$. Therefore, given $\epsilon > 0$, and $y \in \Gamma$ we have

$$(A_\epsilon)^y = \bigcup_{\substack{z \in \Gamma \\ d(z,y)<\epsilon}} \lambda_y^{-1} \circ \lambda_z((A^z)_{\eta(z,y,\epsilon)}), \tag{2.26}$$

and of course we also have

$$((\sigma_\Gamma A)_\epsilon)^y = \bigcup_{\substack{z \in \Gamma \\ d(z,y)<\epsilon}} \lambda_y^{-1} \circ \lambda_z(((\sigma_\Gamma A)^z)_{\eta(z,y,\epsilon)}). \tag{2.27}$$

We have

$$\mu_{n-1}((A_\epsilon)^y) = \mu_{n-1}\left(\bigcup_{\substack{z \in \Gamma \\ d(z,y)<\epsilon}} \lambda_y^{-1} \circ \lambda_z((A^z)_{\eta(z,y,\epsilon)})\right) \tag{2.28}$$

$$\geq \sup_{\substack{z \in \Gamma \\ d(z,y)<\epsilon}} \mu_{n-1}(\lambda_z(A^z)_{\eta(z,y,\epsilon)}). \tag{2.29}$$

Replacing $A$ by $\sigma_\Gamma A$ in 2.28 and 2.29 yeilds equality, since $\lambda_y^{-1} \circ \lambda_z(((\sigma_\Gamma A)^z)_{\eta(z,y,\epsilon)})$ is an $(n-1)$-dimensional disk centered at $y$. Assuming that Theorem 2.1 is true in dimension $n-1$ means that

$$\mu_{n-1}(((\sigma_\Gamma A)^z)_{\eta(z,y,\epsilon)}) \leq \mu_{n-1}((A^z)_{\eta(z,y,\epsilon)})$$

and so we have

$$\mu_{n-1}(\lambda_z((\sigma_\Gamma A)^z)_{\eta(z,y,\epsilon)}) \leq \mu_{n-1}(\lambda_z((A^z)_{\eta(z,y,\epsilon)})) \tag{2.30}$$

for all $z \in \Gamma$, which implies

$$\mu_{n-1}(((\sigma_\Gamma A)_\epsilon)^y) = \sup_{z \in \Gamma : d(z,y)<\epsilon} \mu_{n-1}(\lambda_z((\sigma_\Gamma A)^z)_{\eta(z,y,\epsilon)}) \tag{2.31}$$

$$\leq \sup_{z \in \Gamma : d(z,y)<\epsilon} \mu_{n-1}(\lambda_z(A^z)_{\eta(z,y,\epsilon)}) \tag{2.32}$$

$$\leq \mu_{n-1}(A_\epsilon^y). \tag{2.33}$$

Therefore, all $y \in \Gamma$ are such that

$$\mu_{n-1}(((\sigma_\Gamma A)_\epsilon)^y) \leq \mu_{n-1}(A_\epsilon^y), \tag{2.34}$$

and so by further properties of the cut-locus (see Chavel [8], Equation (3.51) page 134) we have $\mu_n((\sigma_\Gamma A)_\epsilon) \leq \mu_n(A_\epsilon)$.

**2. Symmetrization reduces size of tubular neighborhood (Sphere case).** Let $A$ be a closed subset of $\mathbf{S}^{n-1}$ and let $\gamma$ be a half-circle on $\mathbf{S}^{n-1}$ joining $x_0$ with $-x_0$. Let $u$ be the midpoint of $\gamma$. Identify the sphere $\mathbf{S}_u^{n-2}$ (the equator) with $\mathbf{S}^{n-2}$ (this preserves measure and distances).

For every $y \in \gamma$ ($y \neq \pm x_0$) we define a mapping $\tau_y$ from $\mathbf{S}_y^{n-2}$ onto $\mathbf{S}_u^{n-2} = \mathbf{S}^{n-2}$ by projecting along the meridians (that is, for $x \in \mathbf{S}_y^{n-2}$, $\tau_y(x)$ is the point on $\mathbf{S}_u^{n-2}$ which belongs to the half circle joining $x_0, x$, and $-x_0$).

There is a function $f$ such that if $x_1 \in \mathbf{S}_{y_1}^{n-2}$ and $x_2 \in \mathbf{S}_{y_2}^{n-2}$ then

$$d(x_1, x_2) = f(y_1, y_2, d(\tau_{y_1}(x_1), \tau_{y_2}(x_2))). \tag{2.35}$$

Hence for every $y_1, y_2 \in \gamma$ and $\epsilon > 0$ such that $d(y_1, y_1) \leq \epsilon$, there is an $\eta = \eta(y_1, y_2, \epsilon) \geq 0$ such that every subset $C \subset \mathbf{S}_{y_1}^{n-2}$ has the property that

$$C_\epsilon \cap \mathbf{S}_{y_2}^{n-2} = \tau_{y_2}^{-1}((\tau_{y_1} C)_{\eta(y_1, y_2, \epsilon)}).$$

(If $d(y_1, y_2) > \epsilon$ then $C_\epsilon \cap \mathbf{S}_{y_2}^{n-2} = \emptyset$.)

Hence for every $\epsilon > 0$ and $y \in \gamma, y \neq \pm x_0$,

$$\tau_y((A_\epsilon^y)) = \cup_{z \in \gamma, d(z,y) \leq \epsilon} (\tau_z A^z)_{\eta(z,y,\epsilon)} \tag{2.36}$$

Now let $B = \sigma_\gamma A$. We have automatically that

$$\tau_y((B_\epsilon^y)) = \cup_{z \in \gamma, d(z,y) \leq \epsilon} (\tau_z B^z)_{\eta(z,y,\epsilon)} \tag{2.37}$$

For every $z \in \gamma$ the set $\tau_z B^z$ is by definition a cap in $\mathbf{S}^{n-1}$ such that $\mu_{n-2}(\tau_z B^z) = \mu_{n-2}(\tau_z A^z)$.

By the induction hypothesis (the theorem for $n-1$) we get that

$$\mu_{n-2}((\tau_z B^z)_{\eta(z,y,\epsilon)}) \leq \mu_{n-2}((\tau_z A^z)_{\eta(z,y,\epsilon)}) \tag{2.38}$$

for all 'admissable' $y, z, \epsilon$. Since all the sets appearing in the union on the right-hand side of Equation 2.37 are caps with the same center (*i.e.* $u$) we get by combining our equations that

$$\begin{aligned} \mu_{n-2}(\tau_y((B_\epsilon)^y)) &= \sup_{z\in\gamma, d(z,y)\leq\epsilon} \mu_{n-2}((\tau_z B^z)_{\eta(z,y,\epsilon)}) && (2.39)\\ &\leq \sup_{z\in\gamma, d(z,y)\leq\epsilon} \mu_{n-2}((\tau_z A^z)_{\eta(z,y,\epsilon)}) && (2.40)\\ &\leq \mu_{n-2}((\tau_y A^y)). && (2.41) \end{aligned}$$

So we have for every $y \neq \pm x_0$ in $\gamma$ that

$$\mu_{n-2,y}((B_\epsilon^y)) \leq \mu_{n-2,y}((A_\epsilon)^y), \tag{2.42}$$

and by Fubini's theorem, this implies that $\mu_{n-1}(B_\epsilon) \leq \mu_{n-1}(A_\epsilon)$.

**3. Repeated symmetrization reduces radius.** We want to show that we can reduce $r$ by repeated symmetrization. So, suppose $A$ is not a disk. We note that symmetrization with respect to a geodesic $\Gamma$ through $\mathrm{cent}(A)$ maps $\mathrm{circ}(A)$ isometrically onto itself. Since $\sigma_\Gamma$ pulls $A$ off of the boundary and towards $\Gamma$ whenever there are gaps in $A \cap H^x$ $\sigma_\Gamma(A)$ (where we recenter[‡] $\sigma_\Gamma(A)$ on $\mathrm{cent}(A)$ if necessary) can not contain all of $\partial(\mathrm{circ}(A))$. We will show that by successive symmetrizations, we can reduce the intersection with the boundary of $\mathrm{circ}(A)$ to nothing, and so reduce $r$.

If $B \subset \mathrm{circ}(A)$, $B$ closed, and $x \in \partial(\mathrm{circ}(A)) \setminus B$ then $x \notin \sigma_\Gamma(B)$.

If $B \subset \mathrm{circ}(A)$, $B$ closed, and $C$ is subset of $\partial(\mathrm{circ}(A))$ open with respect to the metric induced by $d$ such that $C \cap B = \emptyset$, then since $C$ is open it contains a spherical cap subtended by some

[‡]*I.e.* recircumcenter

angle $\alpha > 0$. If $\alpha > \pi$ then $C$ contains a pair of antipodal points. Symmetrizing with respect to the geodesic determined by this pair of points finishes the argument If $\alpha < \pi$ then there is some $\Gamma$ such that there exists some relatively open subset $C'$ of $\partial(\mathrm{circ}(A))$ which is disjoint from $\sigma_\Gamma(A)$ and subtends an angle $3\alpha/2$. This then is enough. QEF

## 2.3 Existence and regularity

**Charter** We should understand the known results on the existence and regularity of solutions to area minimization problems with multiple volume constraints.

### 2.3.1 Generalized manifolds

Geometric measure theory works with classes of geometric sets much more general than the class of manifolds. These generalizations were developed, it seems, for two main reasons: (1) Physics, since many phenomena exhibit behavior that is not smooth eg. bubble clusters, black holes, crystals, and so on; (2) Mathematics, since generalized manifolds arise naturally, in much the same way as generalized functions (*i.e.*, distributions). Insofar as (1) and (2) are related (and it happily turns out that they are) we do a disservice to both if we do not note that the main place these extended ideas of manifolds are likely to be interesting is in mathematical physics, where they appear as solutions to differential equations (eg. the variational problems studied in this thesis).

The idea is to use quite general geometrical objects at first and then show that when further natural constraints are imposed (eg. physically motivated ones), the only objects that satisfy the constraints are actually the nice (eg. smooth almost everywhere) things that we were expecting to see. This allows us to do away with potentially very dissatisfying hypotheses, such as 'if $A$ has smooth boundary' (cf. Theorem 2.1 and Definition 2.8). This does not come cheaply. *The sequence of definitions that follows assumes a basic familiarity with vector space duals, differential forms, and*

*measures, as might be gained by reading the appendices.*

**Definition 2.11.** We will denote by $D^m$ the space of $C^\infty$ differential $m$-forms on $\mathbf{R}^n$ with compact support.

**Definition 2.12.** We let $D_m$ denote the space of linear functionals on $D^m$ endowed with the weak topology, *i.e.* open sets are defined so that if $\{T_j\}$ is a sequence in $D_m$, it converges to $T \in D_m$ if and only if $T_j\varphi \to T\varphi$ for all $\varphi \in D^m$. An element of $D_m$ is called an *$m$-dimensional current*.

**Definition 2.13.** We define the boundary of current by duality, that is, if $T$ is an $m$-dimensional current, $\partial T$ is an $(m-1)$-dimensional current defined by $(\partial T)(\omega) = T(d\omega)$ for all $(m-1)$-forms $\omega$.

**Definition 2.14.** We also define the pushforward of a rectifiable current using duality, so that that if $T$ is an $m$-dimensional current and $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ then $\varphi_* T$ is an $m$-dimensional current defined by $\varphi_* T(\omega) = T(\varphi^* \omega)$ for all $m$-forms $\omega$.

*Remark 2.6.* We might have suspectated that $\varphi_*$ and $\varphi^*$ would turn out to be dual operations from the beginning. Definition 2.14 makes this precise.

**Definition 2.15.** For $m \leq n$, we define the *$m$-dimensional Hausdorff measure of $A \subset \mathbf{R}^n$*, denoted $\mathcal{H}^m(A)$, in terms of covering classes $\{\mathcal{C}_\delta\}_{\delta>0}$ with the property that $S_\delta \in \mathcal{C}_\delta$ implies $S_\delta$ is countable and if $S_\delta = \{S_j \subset \mathbf{R}^n\}_{j=1}^n$ then $A \subset \cup_{j=1}^\infty S_j$ and every $S_j$ has $r(S_j) < \delta$, where $r$ is the circumradius of $S_j$, defined by Equation 2.9. Letting $\alpha_m$ denote the Lebesgue measure of the closed unit ball in $\mathbf{R}^m$, the Hausdorff measure of $A$ is then given by

$$\mathcal{H}^m(A) = \lim_{\delta \to 0^+} \inf_{S_\delta \in \mathcal{C}_\delta} \sum_{j=1}^{\infty} \alpha_m (r(S_j))^m \tag{2.43}$$

where the limit is seen to exist (in $\mathbf{R} \cup \infty$) because the infima do not decrease as $\delta$ decreases.

**Definition 2.16.** We say that a set $A \subset \mathbf{R}^n$ is an *$m$-dimensional rectifiable set* if $A$ is $\mathcal{H}^m$-measurable, $\mathcal{H}^m(A) < \infty$, and $\mathcal{H}^m$-almost all of $A$ is contained in the image of some countable collection of Lipschitz maps from $\mathbf{R}^m$ to $\mathbf{R}^n$.

*Remark 2.7.* The set of smooth $m$-manifolds is clearly inside the set of rectifiable sets since by paracompactness (and the rest of the definition) a smooth manifold is contained it the image of some countable collection of smooth maps from $\mathbf{R}^m$.

Tangent planes to a rectifiable set are defined $\mathcal{H}^m$-almost everywhere. We can choose an orientation for each tangent plane, and we then write its (now, ordered) basis of tangent vectors as the $m$-vector $\Xi = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$.

**Definition 2.17.** An *$m$-dimensional rectifiable current* is an element $\mathcal{S}$ of $D_m$ that has compact support in $D^m$ and that is associated with a rectifiable set $S$ with ordered basis of tangent vectors $\Xi$ in the sense that the action of $\mathcal{S}$ on $\omega \in D^m$ is given by

$$\mathcal{S}\omega = \int_{\{x \in S\}} \langle \omega(x), \Xi(x) \rangle \, d\mathcal{H}^m.$$

If $\mu : S \to \mathbb{Z}$, we may define

$$\mathcal{S}\omega = \int_{\{x \in S\}} \langle \omega(x), \Xi(x) \rangle \mu(x) \, d\mathcal{H}^m;$$

we then say that $\mathcal{S}$ has *multiplicity* $\mu$.

*Remark 2.8.* The general idea is to integrate some $m$-section of the $m$-tangent bundle over the underlying set. It almost goes without saying that these ideas work very naturally for the set of smooth orientable manifolds, so that we find that set sitting inside of the set of rectifiable currents. In general, if the underlying set we are working with does not admit some kind of global orientation, there could be trouble. Note that by Stokes' Theorem, the definition of boundary given above corresponds to the usual definition of boundary in the case of a rectifiable current that is associated to a smooth oriented manifold with boundary.

It does not necessarily have to be the case that the boundary of a rectifiable current (Definition 2.13) is again a rectifiable current. When this is the case give the current a special name.

**Definition 2.18.** An *integral current* $T$ is a rectifiable current whose boundary $\partial T$ is also a rectifiable current.

**Definition 2.19.** A *locally integral current* $T$ is an element of $D_m$ such that for all $x \in \mathbf{R}^n$ there is some integral current $S$ such that $x \notin \text{supp}(T - S)$.

**Definition 2.20.** We define the *mass norm* (actually semi-norm) on $D_m$ in terms of the comass norm on $m$-forms (Definition C.24) by

$$\mathbf{M}(T) = \sup_{\omega \in D^m} \{T\omega; \sup_{x \in \mathbf{R}^n} \{\text{comass}(\omega(x))\} \leq 1\}.$$

We can compute the mass norm, for example, of the boundaries of bubble cluster components.

**Definition 2.21.** A subset $S$ of $\mathbf{R}^n$ is an $(\mathbf{M}, 0, \delta)$-*minimal set (with respect to the null set)* if it is bounded and any Lipschitz deformation $\Phi$ of $\mathbf{R}^n$ that is equal to the identity outside of some $\delta$-ball has the property that $\mathcal{H}^m(S) \leq \mathcal{H}^m(\Phi(S))$.

**Definition 2.22.** We may relax the previous definition and talk about $(\mathbf{M}, \epsilon, \delta)$-*minimal sets (with respect to the null set)*, where $\epsilon$ is some function of the form $\epsilon(r) = Cr^\alpha$ and the condition of the previous definition becomes $\mathcal{H}^m(S) \leq (1 + \epsilon(r))\mathcal{H}^m(\Phi(S))$ for all $r < \delta$.

*Remark 2.9.* $(\mathbf{M}, \epsilon, \delta)$-minimal sets are *rectifiable* [25] and have other nice properties.

### 2.3.2 Soap bubble clusters

The statements of the theorems are as follows:

**Theorem 2.2 ('Existence of compact minimizing bubble clusters').** In $\mathcal{R}^n$, given volumes $v_1, ..., v_m > 0$, there is a boundary mass-minimizing rectifable current comprised of bounded regions $V^i$ with volumes $v_i$.

This theorem was proved in the more general context of $(\mathbf{M}, \epsilon, \delta)$-minimal sets (which are also rectifiable sets, and so rectifiable currents) by Almgren [25]. The theorem motivates the following definition:

**Definition 2.23.** A *bubble cluster* is a disjoint union of bounded $n$-dimensional locally integral currents of multiplicty 1.

If we denote the complement of the cluster $C = \sqcup_{i=1}^{m} V^i$ by $V^0$, there is a convenient expression for the surface area in terms of the mass norm,

$$\text{Area}(C) = \frac{1}{2}\sum_{i=0}^{m} \mathbf{M}(\partial V^i). \tag{2.44}$$

In [25], Almgren also proved a basic regularity result.

**Theorem 2.3 ('Almgren Regularity').** A boundary-mass minimizing bubble cluster is bounded by $C^{1,\alpha}$ surfaces almost everywhere.

Note that where the surfaces are regular, a classical variational argument yeilds that they have constant mean curvature. Almgren's result been improved upon to give a more geometrical description of minimizers, with the first and best results in three dimensions by Taylor.

**Theorem 2.4 ('Taylor Regularity').** A soap bubble cluster in $\mathcal{R}^3$ consists of real analytic constant-mean curvature surfaces meeting smoothly in threes at one hundred and twenty degrees along smooth curves, which in turn meet in fours at angles of $\cos^{-1}(-1/3) \approx 109°$.

The curves described in Taylor's regularity theorem are called *singular curves*. The points at which these curves meet are *singular points*.

Taylors' proof is in two pieces; the Euclidean case in [53] and a generalization in [54]. Both are increadibly difficult to read. The singular curves were only proved to be $C^{1,\alpha}$ by Taylor; the analyticity came later; see Morgan [34] for a more complete overview of the history and proof. Morgan points out that this result has been partially extended to higher dimensions.

**Theorem 2.5 ('White's extension', unpublished).** Soap bubble clusters in $\mathbf{R}^n$ consist of smooth constant mean curvature hypersurfaces meeting in threes at one hundred and twenty degrees along smooth $(n-2)$-dimensional hypersurfaces, which in turn meet in an $(n-4)$-dimensional set.

*Remark 2.10.* Theorem 2.2 holds in any smooth Riemanian manifold with compact quotient by the isometry group. (As far as I know, this statement is a 'folk theorem'; the idea behind it it that we can pull all the pieces of a minimizer into some compact set where the general ideas of the theorems above can be applied.) It need not hold otherwise – for example, if there is a sequence of bumps with ever-increasing curvature headed off to infinity, any candidate minimizer could be improved upon by moving it to a different bump. The exact statement by Taylor is that the same results hold in a manifold with Hölder continuously differentiable metric is; a discussion of the relationship of these two criteria would take us outside of the scope of this thesis.

**Pressure and curvature.** One of the most important tools that we can use to study bubble clusters is a physically motivated condition, namely that pressures have to equilibrate. We use the phrase 'pressures add' as short-hand for the longer phrase: the sum of the pressure differences around a singular curve or vertex is zero. This, combined with the fact that pressure difference across walls is proportional to curvature, gives us a useful geometrical tool. A simple proof that pressure is proportional to curvature follows, based on a letter from Miguel Carrión Álvarez.

The basic principle of statics is that energy is minimized at equilibrium. In our case, surface tension is assumed to be constant throughout the whole system, and we have

$$\text{energy} = (\text{surface tension}) * (\text{total area}).$$

Work is the change in energy produced by motion. For example,

$$\text{work} = \text{torque} * (\text{angle rotated})$$

$$\text{work} = \text{force} * \text{displacement}.$$

By definition,

$$\text{pressure} = \text{force}/\text{area}.$$

Since we are in equilibrium the pressure *difference* across a wall

$$\text{pressure}_1 - \text{pressure}_2 = (\text{force}_1 - \text{force}_2)/\text{area}$$

has to be equal to

$$(\text{surface tension})/\text{area}.$$

Hence, if a bubble cluster changes its shape slightly, we have

$$\begin{aligned} \text{work} &= (\text{surface tension}) * (\text{change in area}) \\ &= (\text{pressure difference}) * (\text{change in volume}) \end{aligned}$$

and so

$$(\text{pressure difference}) = (\text{surface tension}) * (\text{area change})/(\text{volume change}).$$

From the appendix, we have

$$(\text{area change})/(\text{volume change}) = (\text{mean curvature})$$

whence,

$$(\text{pressure difference}) = (\text{surface tension}) * (\text{mean curvature}).$$

The fact that pressure is higher for smaller volumes (and so smaller bubbles will curve into larger neighbors) is an experimental observation – made, for example, by Robert Boyle in the 1600's. The above argument provides geometrical intuition for this (smaller things should have higher curvature) but not quite a proof.

# Chapter 3

# dbs in ccsfs

**Charter** The known results on double bubbles in constant curvature space forms are to be presented here. (This chapter is based on the results of Michael Hutchings [22] and his collaborators [23], and others who have worked subsequently with these results ([10], [41].)

In the previous two chapters, we became familiar with the geometry of $\mathcal{R}^n$. We studied single bubbles and proved that metric balls give the least perimeter enclosure of a given volume. For the double bubble problem in $\mathcal{R}^n$, the conjectured minimizer is more complicated; it is especially complicated in the negative curvature case. Figure 2.5 and the surrounding discussion motivates the following definition:

**Definition 3.1.** A *hypersphere-like submanifold* of $\mathcal{R}^n$ is a sphere in the positive or zero curvature cases, and a sphere, a horosphere, or a hyposphere in the negative curvature case.

**Lemma 3.1.** For prescribed volumes $v$ and $w$, there is a unique double bubble in $\mathcal{R}^n$, up to an isometry, that consists of three pieces of hypersphere-like submanifolds meeting at 120°. The outer caps are always pieces of spheres. This unique double bubble is called a *standard double bubble.*

To prove this lemma, one has to consider hypersphere-like submanifolds with all of the possible

curvatures and show that sticking them together at 120° yields all the possible volumes exactly once.

**Conjecture 3.1 (double bubble conjecture for $\mathcal{R}^n$).** The least-surface area way to enclose and separate two given volumes in $\mathcal{R}^n$ is the standard double bubble.

The results of this chapter grew up mainly in the context of the double bubble problem it $\mathbf{R}^n$, where the double bubble conjecture is known to be true for dimensions two [24], three [23], and four, and for some volumes in high dimensions [41]. However the conjecture has been thought of in the context of $\mathcal{R}^n$ for a long time.

Michael Hutchings [22] produced two key simplications of the problem in this general setting. The first simplification applies ideas of symmetrization to show that a non-standard minimizer must be bounded by a surface of revolution. This parallels the symmetrization techniques used in the proof of the single bubble problem in Chapter 2. Once we have this rotational symmetry the geometry becomes simple enough that we might hope to characterize non-standard competitors. The primary question then is, exactly how many regions can a minimizing bubble have? The second key simplification is an explicit bound on the number of components of each region in terms of the volumes. This makes a complete characterization possible. Indeed if the bound tells us that the largest of the two regions is connected, the a variational argument developed in the $\mathbf{R}^3$ by Hutchings, Morgan, Ritoré and Ros, successfully applied in the $\mathbf{R}^4$ version of the problem by Reichardt *et al.* and revised to work in $\mathbf{S}^n$ and $\mathbf{H}^n$ by Cotton and Freeman [10] can be applied to shows the instability of all remaining non-standard competitors. The two key simplifications of Hutchings and the instability argument for constant curvature space forms will be described in this chapter.

## 3.1 Hutchings structure theory

Hutchings proved many results for $m$ prescribed volumes. We need only two, but will do things in the general language when possible, because usually when it is possible it is also usually no harder.

**Definition 3.2.** We define the *least-area function* or *isoperimetric profile* of a given manifold $M$ to be the map that takes a given volume to the surface area of a least-area enclosure in the space that contains the given volume. We denote this function by $I_M$.

More generally, we define the *m-volume least-area function* to be the map that takes $m$ given volumes to the surface area of a least-area enclosure in the space that contains these volumes in separate chambers. We denote this function by $I_M^m$.

We are especially interested in the *two-volume least-area function*, which we denote by the special symbol $\mathit{\Pi}_M$.

We frequently will drop the subscripts of these maps when there is no chance of this making things confusing.

*Remark 3.1.* The isoperimetric profile is a primative of mean curvature for minimizers, *i.e.*,

$$dI(x)/dx = dA_{\text{minimizer}}(x)/dx = H_{\text{minimizer}}(x).$$

This is in fact a proof if you accept the second equality. The reason for that the equality is a simple variational argument (see [33] p. 15).

### 3.1.1 Symmetry

We saw in Chapter 2 how important symmetry was for solving the isoperimetric problem in constant curvature space forms. Symmetry is just as important for the double bubble problem, even though it is weaker (minimizing double bubbles are not spherically symmetric) and does not finish the job. *For double bubbles, the central symmetry result is that if $n \geq 3$ any area-minimizing double bubble in $\mathbf{R}^n$, $\mathbf{S}^n$ or $\mathbf{H}^n$ is symmetric about some line.* This is a corollary to the broader-reaching

**Theorem 3.1 (Symmetry Theorem ).** Let $B$ be a minimal enclosure of $m$ volumes in $\mathbf{R}^n$. If $m$ is less than or equal to $n-1$ then $B$ is symmetric about some $m-1$ plane.

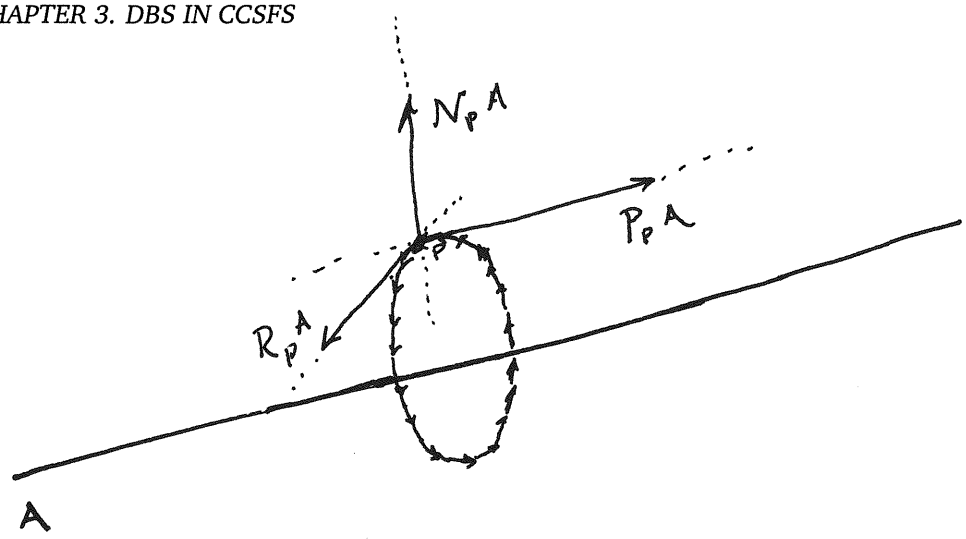



Figure 3.1: Hutchings' decomposition of $T_p(\mathbf{R}^3 \setminus A)$.

Hutchings proves Theorem 3.1 in a series of lemmas. We will do the same, after introducing some notation.

In the statement of the theorem and in all that follows, $B$ is taken to be an $n$-dimensional rectifiable current made up of a disjoint collection of $n$-dimensional regions $V^i$, such that $V^i$ has volume $v_i$. Until we say otherwise, these regions may be broken into many disjoint components.

Since it is area-minimizing its boundary is mainly made up of smooth manifold points. We let $B_{\text{reg}}$ denote the set of smooth manifold points in the boundary of $B$.

**Definition 3.3.** Let $A$ be an affine subspace of $\mathbf{R}^n$ with $\dim A \leq n-2$. We decompose the tangent bundle of $\mathbf{R}^n \setminus A$ as follows:

$$T(\mathbf{R}^n \setminus A) = \mathcal{P}A \oplus \mathcal{N}A \oplus \mathcal{R}A, \tag{3.1}$$

where $\mathcal{P}A$ are vectors parallel to $A$, $\mathcal{N}A$ are vectors normal to $A$, and $\mathcal{R}A$ are vectors that are perpendicular to both $\mathcal{P}A$ and $\mathcal{N}A$. It is important that we exclude $A$, because the decomposition degenerates at points in $A$ (see Figure 3.1). At a given point $p \in \mathbf{R}^n \setminus A$ we write the associated decomposition of the tangent space in the same way, so $T_p(\mathbf{R}^n \setminus A) = \mathcal{P}_pA \oplus \mathcal{N}_pA \oplus \mathcal{R}_pA$.

*Remark 3.2.* The case in which $A$ is a line in $\mathbf{R}^3$ is illustrated in Figure 3.1. Here it is clear that $\mathcal{R}A$

are vectors taken from vectorfields that determine a rotation about $A$. It works the same way in higher dimensions. For example, if $A$ is a $2-$plane in $\mathbf{R}^4$ then $\mathcal{P}_p A$ is two dimensional, $\mathcal{N}_p A$ is one dimensional, and so there is an extra dimension left over for $\mathcal{R}_p A$.

**Lemma 3.2 ('Infinitesimal Symmetry Implies Global Symmetry' ).** Let $B$ be a minimizing enclosure of $m$ volumes in $\mathbf{R}^n$, and let $A$ be an affine subspace of $\mathbf{R}^n$ with $\dim(A) \leq n-2$. Suppose that for almost all $p \in B_{\text{reg}} \setminus A$ that we have $\mathcal{R}_p A \subset T_p B$. Then $B$ is symmetric about $A$.

*Proof.* Since $\dim A \leq n-2$, an object mapped to itself invariantly under rotations about $A$ will also be mapped to itself invariantly under reflections about $A$.

Let $\varphi$ be a rotation about $A$. Our goal is to show that $\varphi_* \partial V^i = \partial V^i$ for all $i$. By the definition of the pushforward of a current (Definition 2.14), this means that the boundary of $V^i$ is fixed under rotation about $A$.

Associated to $\varphi$ is a one parameter group $\{\varphi_t\}$ of rotations about $A$ such that $\varphi = \varphi_1$. These induce a vector field $\mathcal{V}$ on $\mathbf{R}^n$. If $\omega$ is an $(n-1)$-form on $\mathbf{R}^n$,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \langle \partial V^i, \varphi_t^* \omega \rangle &= \langle \partial V^i, (d\iota(\mathcal{V}) + \iota(\mathcal{V})d)\omega \rangle && (3.2) \\ &= \langle \partial V^i, d\iota(\mathcal{V})\omega \rangle + \langle \partial V^i, \iota(\mathcal{V}) d\omega \rangle && (3.3) \\ &= \langle \partial V^i, \iota(\mathcal{V}) d\omega \rangle, && (3.4) \end{aligned}$$

where the equality between 3.3 and 3.4 is due to Stokes' theorem. If the members of this chain of equalities are equal to zero, then the value of the pairing is constant to first order at $t = 0$, and this implies that the pairing is constant for all $t$. If $\mathcal{V}(p) \in T_p B$, Equation 3.4 implies that for almost all $p \in B_{\text{reg}} \cap \partial V^i$ we have

$$\langle \iota_{\mathcal{V}} d\omega(p), \Xi(p) \rangle = 0, \tag{3.5}$$

where $\Xi(p)$ denotes the ordered basis of $T_p(\partial B)$. This shows that the members of 3.4 are equal to zero. Since $\mathcal{V}(p) \in \mathcal{R}_p A$ for all $p \notin A$ and $\mathcal{R}_p(A) \subset T_p B$ for almost all $p \in B_{\text{reg}}$ such that $p \notin A$, we

have

$$\langle \partial V^i, \omega \rangle = \langle \partial V^i, \varphi^* \omega \rangle \tag{3.6}$$

Since $\langle \partial V^i, \varphi^* \omega \rangle = \langle \varphi_* \partial V^i, \omega \rangle$ by definition, this completes the proof. QEF

**Lemma 3.3 (Bisectors Orthogonal ).** Let $B$ be a minimizing enclosure of $m$ volumes in $\mathbf{R}^n$, and let $H$ be a hyperplane that bisects all of the $V^i$. Then $H$ and $B_{\text{reg}}$ are orthogonal wherever they intersect.

*Proof.* Let $B_+$ and $B_-$ be the halves of $B$ on either side of $H$. Since $H$ bisects all of the regions, we can make a new cluster $\widehat{B_+}$, equal to the union of $B_+$ the reflection of $B_+$ across $H$. The regions $\widehat{V^i}$ that make up $\widehat{B_+}$ have the same volume as the $V^i$. Since $B$ is minimizing, $\text{area}(\widehat{B_+}) \geq \text{area}(B)$, so $\text{area}(B_+) \geq \text{area}(B_-)$. Similarly, $\text{area}(B_-) \geq \text{area}(B_+)$, hence they must be equal, and so we must have $\text{area}(\widehat{B_+}) = \text{area}(\widehat{B_-}) = \text{area}(B)$.

Hence, $\widehat{B_+}$ and $\text{area}(\widehat{B_-})$ are minimizing too. By regularity, the tangent spaces to the regular points of $B$ contained in $H$ are either orthogonal to or parallel to $H$.

The final step is to show that if $B$ is tangent to $H$ at a regular point then either $\widehat{B_+}$ or $\widehat{B_-}$ can be modified to decrease area. (This is enough because the tangent spaces will only be parallel to $H$ at points where $\widehat{B_+}$ is tangent to $H$ or at points where $\widehat{B_+}$ and $H$ do not intersect.)

Note that $\mathbf{R}^n$ can be parameterized by $H \times \mathbf{R}$. If $\mathbf{D}_p(r)$ is a closed ball of radius $r$ about $p \in H$ then $B$ is regular in $\mathbf{D}_p(r) \times [-\epsilon, \epsilon] \subset H \times \mathbf{R}$. Since $B$ is tangent to $H$ at $p$, chosing $r$ small enough to have

$$B \cap \partial(\mathbf{D}_p(r) \times [-\epsilon, \epsilon]) \subset (\partial \mathbf{D}_p(r) \times [-\epsilon, \epsilon])$$

for small $\epsilon$, and so that $\mathbf{D}_p(r) \times \{\pm\epsilon\}$ are in the same component of $\widehat{B_+}$, we remove $\widehat{B_+}$ from its intersection with $\mathbf{D}_p(r) \times [-\epsilon, \epsilon]$ and add back $\partial \mathbf{D}_p(r) \times [-\epsilon, \epsilon]$. This decreases area to the order of $r^{n-1}$, and changes the areas of $\widehat{V^i}$ of at most order $\epsilon r^{n-1}$.

By a standard first variation argument, we can restore the volumes by adjusting the cluster

elsewhere, with an area increase of at most order $\epsilon r^{n-1}$. This is smaller than the initial increase for $r$ sufficiently small. QEF

**Lemma 3.4 (Symmetry about intersections).** Let $B$ be a minimizing enclosure of $m$ volumes in $\mathbf{R}^n$. Let if $k > 1$ and $H_1, \ldots, H_k$ be mutually orthogonal hyperplanes. If $B$ is symmetric about each $H_i$ then $B$ is symmetric about $A = \cap_{i=1}^{k} H_i$.

*Proof.* $B$ maps to itself invariantly under reflection across each $H_i$, so also maps to itself invariantly under reflection across $A$. So every hyperplane containing $A$ bisects $V^1, \ldots, V^m$. Since $B_{\text{reg}}$ meets its bisectors orthogonally wherever they intersect, and the directions perpendicular to all of the $H_i$ are the directions of rotation about $A$, $B$ is symmetric about $A$, because infinitesimal symmetry implies global symmetry. QEF

**Lemma 3.5 ( Linear Algebra).** If $A_1$ and $A_2$ are affine subspaces of $\mathbf{R}^n$ with non-empty intersection, and $p \notin \text{span}(A_1, A_2)$, then $\mathcal{R}_p(A_1 \cap A_2) = \mathcal{R}_p A_1 \oplus \mathcal{R}_p A_2$

*Proof.* Any rotation fixing $A_1$ or $A_2$ fixes $A_1 \cap A_2$ so

$$\mathcal{R}_p(A_1 \cap A_2) \supset \mathcal{R}_p A_1 \oplus \mathcal{R}_p A_2.$$

If

$$\mathcal{R}_p(A_1 \cap A_2) \not\subseteq \mathcal{R}_p A_1 \oplus \mathcal{R}_p A_2,$$

then since

$$\dim \mathcal{R}_p A_1 \cap \mathcal{R}_p A_2 > n - \dim \text{span}(A_1, A_2) - 1,$$

and since

$$\mathcal{R}_p A_1 \cap \mathcal{R}_p A_2 = (\mathcal{P}_p A_1 \oplus \mathcal{P}_p A_2 \oplus \mathcal{N}_p A_1 \oplus \mathcal{N}_p A_2)^{\perp}$$

and, lastly, since

$$\dim(\mathcal{P}_p A_1 \oplus \mathcal{P}_p A_2) = \dim \text{span}(A_1, A_2),$$

we must have

$$\mathcal{N}_p A_1 \subset \mathcal{P}_p A_1 \oplus \mathcal{P}_p A_2.$$

Since $\mathcal{N}_p A_1$ is perpendicular to $\mathcal{P}_p A_1$ and $\mathcal{N}_p A_1 \subset \mathcal{P}_p A_2$, we must have $p \in \text{span}(A_1, A_2)$. This contradicts our assumption, so we have the claim. QEF

**Lemma 3.6 (Assembly, Euclidean case ).** Let $B$ be a minimizing enclosure of $m$ volumes in $\mathbf{R}^n$. Let $H$ be a hyperplane, and let $B_+$ and $B_-$ be the two symmetrizations of $B$ across $H$. Suppose that both $B_+$ and $B_-$ minimize area for the volumes they enclose. Let $A_+$ and $A_-$ be non-empty affine subspaces of $H$ of dimension at most $n-2$. If $B_\pm$ is symmetric about $A_\pm$ then $A_+ \cap A_- \neq \emptyset$ and $B$ is symmetric about $A_+ \cap A_-$.

*Proof.* First note thet $B \cap H$ is symmetric about $A_1$ and $A_2$ in $H$. We immediately rule out the possibility that $A_1 \cap A_2 = \emptyset$ since that would imply that $B$ is not compact, which contradicts its minimality. To prove symmetry it is enough to show that for $\mathcal{H}^{n-2}$-almost every $p \in B_{\text{reg}}$ such that $p$ is not in $A_1 \cap A_2$ that

$$\mathcal{R}_p(A_1 \cap A_2) \subset T_p B.$$

We first show that for almost every $p \in B \cap H$ that is not in $A_1 \cap A_2$ that each of $B$, $B_1$ and $B_2$ is regular at $p$ and for these '$p$'s we have $\mathcal{R}_p(A_1 \cap A_2) \subset T_p B$.

Since $B_i$ is regular almost everywhere and is symmetric about $A_i$ we know that almost every point in $(B_i \cap H) \setminus A_i$ is regular. Hence almost every point of $B \cap H$ that is not in either $A_1$ or $A_2$ is a regular point of both $B_1$ and $B_2$. Hence, by the Bisectors Orthogonal lemma (Lemma 3.3), we have that $B_1$ and $B_2$ are orthogonal to $H$ at each $p \in (B \cap H) \setminus (A_1 \cup A_2)$, and so a small ball $\mathbf{D}_\epsilon(p)$ meets only two of the '$V^i$'s that make up $B$, so $B$ is regular in $\mathbf{D}_\epsilon(p)$ except on a set of Hausdorff dimension at most $n-8$. From this it follows that almost every point in $(B \cap H) \setminus (A_1 \cup A_2)$ is a regular point for $B$, $B_1$ and $B_2$, as we wanted.

If $U$ is a regular neighborhood of such a point, then by uniqueness of analytic continuation we

have

$$U \cap B = U \cap B_1 = U \cap B_2.$$

Hence we have

$$\mathcal{R}_p(A_1), \mathcal{R}_p(A_2) \subset T_pB$$

for all $p \in U \setminus (A_1 \cup A_2)$. By the Linear Algebra lemma (Lemma 3.5), for all $p \in U \setminus H$ we have

$$\mathcal{R}_p(A_1 \cap A_2) \subset T_pB.$$

By continuity this holds for all $p \in U$.

Let $W_i$ be the closed half-space in which $B_i$ lies. Let $O_1$ be the union of orbits under rotation about $A_2$ whose intersection with $H$ is contained in $A_1$. Note that $O_1$ is half of a $k$-plane where $k \leq \dim(A_1) + 1$. Define $O_2$ similarly. By our work in the previous paragraph, for $\mathcal{H}^{n-1}$-almost every $p \in (B \cap W_1) \setminus O_2$ there is a rotation $\varphi$ about $A_1$ such that $\varphi(p) \in H$ and $\varphi(p)$ is a regular point for both $B$ and $B_1$, and furthermore, $\mathcal{R}_p(A_1 \cap A_2) \subset T_pB$. Since $B_1$ is invariant under $\varphi$ and $\varphi$ is a rotation about $A_1 \cap A_2$ it must be the case that $B$ is regular at $p$ and that $\mathcal{R}_p(A_1 \cap A_2) \subset T_pB$. Similar statements can be made for almost every $p \in (B \cap W_2) \setminus O_1$.

Now suppose that $p \in B_{\text{reg}} \cap O_i$. Either there is some regular neighborhood of $p$ contained in $O_i$ or $p$ is a limit of regular points of $B$ not in $O_i$ whose tangent spaces contain $\mathcal{R}_p(A_1 \cap A_2)$. In the latter case, continuity implies that $\mathcal{R}_p(A_1 \cap A_2) \subset T_pB$. In the former case, using the rotation invariance of both $B_1$ and $B_2$, we must have that the point in $\mathbf{R}^n$ on either side of $U$ lie in the same $V^i$ so $U$ can be removed to decrease area, contradicting the minimality of $B$. QEF

We can now prove our big theorem.

*Proof of Symmetry Theorem 3.1.* Since $B$ is compact, we can apply the ham sandwich theorem to produce a hyperplane $H_1$ that bisects all of the regions. By Assembly (Lemma 3.6) it is enough to show that each of the symmetrizations of $B$ across $H_1$ are symmetric about an $(m-1)$-plane in $H_1$. Let $B_1$ be one of the two possible symmetrizations. By the Borsuk-Ulam theorem, there is a

hyperplane $H_2$ orthogonal to $H_1$ that bisects all of the regions. By Assembly again, it is enough to show that each of the symmetrizations of $B_1$ across $H_2$ is symmetric about an $(m-1)$-plane in $H_1 \cap H_2$. We let $B_2$ be one of the two possible symmetrizations, and continue in this fashion until we have obtained hyperplanes $H_1, \ldots, H_{n-m+1}$ and a minimal cluster $B_{n-m+1}$. By Lemma 3.4 this cluster is symmetric about the $(m-1)$-plane $A = \cap H_i$. QEF

**Corollary 3.1.** Let $B$ be a minimal double bubble in $\mathbf{R}^n$ for $n \geq 3$ and let $H \subset \mathbf{R}^n$ be a hyperplane. If each symmetrization of $B$ across $H$ is area-minimizing for the volumes it encloses, then $B$ is symmetric about a line in $H$.

*Proof.* Use the symmetry theorem, starting with $H_1 = H$. QEF

### 3.1.2 Concavity of the least-area function

In this subsection, $I^m$ means the $m$-volume least-area function associated with a constant curvature space form of arbitrary dimension, and $\mathit{\Pi}$ means the 2-volume least-area function associated with a constant curvature space form of arbitrary dimension. The proof of the main theorem in this section (Theorem 3.2) relies on the symmetry we have in constant curvature space forms, and does not apply when there are more than two prescribed volumes. We begin with a simple lemma that does hold for any number of prescribed volumes.

**Lemma 3.7 ('Continuity').** For all $m$, $I^m$ is a continuous function.

*Proof.* Let $C$ be a least-area bubble cluster for volumes $v_1, \ldots, v_m$. Add a small round ball of radius $\epsilon$ somewhere away from the cluster, and say that it is now part of $V^1$. This increases the volume of $V^1$ by less than $\epsilon$ and increases surface area by less than $\epsilon$. This is an upper bound on the increase of surface area for a minimizing bubble cluster with volumes $v_1 + \epsilon, v_2, \ldots, v_m$, hence $I^m$ is continuous in the first component, and so by the same argument, in all.

QEF

**Theorem 3.2 ('Concavity').** $\Pi$ is a concave function in the sense that if $\mathbf{v}$ and $\mathbf{w}$ are two pairs of volumes, then for $t \in (0,1)$

$$\Pi(t\mathbf{v} + (1-t)\mathbf{w}) > t\Pi(\mathbf{v}) + (1-t)\Pi(\mathbf{w}).$$

*Proof.* We must prove that

$$\Pi(tv_1 + (1-t)w_1, tv_2 + (1-t)w_2) > t\Pi_(v_1, v_2) + (1-t)\Pi(w_1, w_2) \tag{3.7}$$

for all $t \in (0,1)$. For a contradiction, suppose not.

By continuity, the function $f$ defined by

$$f(t) = \Pi(tv_1 + (1-t)w_1, tv_2 + (1-t)w_2) - t\Pi(v_1, v_2) - (1-t)\Pi(w_1, w_2) \tag{3.8}$$

assumes a minimum on the interval [0,1] at some $t_0$.

Let $B$ be a minimizing cluster of two regions $V$ and $W$, with volumes

$$(v, w) = t_0(v_1, v_2) + (1-t_0)(w_1, w_2) = (t_0v_1 + (1-t_0)w_1, t_0v_2 + (1-t_0)w_2).$$

By the Symmetry Theorem (Theorem 3.1), $B$ is symmetric about some line $\ell$.

We can parameterize the set of oriented hyperplanes in $\mathbf{R}^n$ by $\mathbf{S}^{n-1} \times \mathbf{R}$, since each oriented hyperplane is determined by a normal direction and a distance from the origin in that direction.

We define a map $h_+$ from the set of oriented hyperplanes, so that given $H \in \mathbf{S}^{n-1} \times \mathbf{R}$ we let $H^+$ denote the upper half space determined by $H$, and we put

$$h_+(H) = (\mathrm{vol}(V \cap H^+), \mathrm{vol}(W \cap H^+)) \in \mathbf{R}^2$$

Since regularity implies $B$ is compact, $h_+$ is continuous.

By an isometry, we may assume 0 is in the line of symmetry $\ell$. We then choose $x \in \mathbf{S}^{n-1}$ perpendicular to $\ell$. The hyperplane determined by $(x, r) \in \mathbf{S}^{n-1} \times \mathbf{R}$ contains $\ell$ if and only if $r = 0$, and is parallel to $\ell$ otherwise. For all $r \in \mathbf{R}$ we have that

$$h_+(x, r) + h_+(x, -r) = (v, w), \tag{3.9}$$

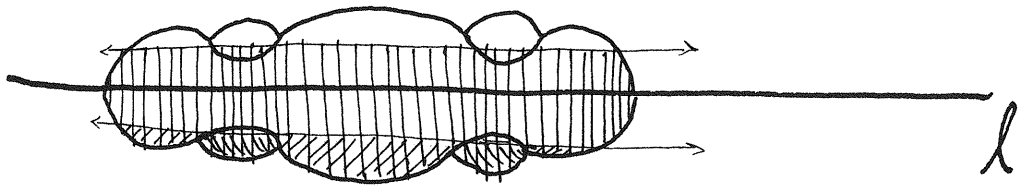



Figure 3.2: Cutting the bubble by oriented planes equidistant from $\ell$ gives all the volume.

since by symmetry the volumes that lie above $(x, r)$ are equal the volumes that lie below $(x, -r)$, hence the total ammount of volume collected by the two maps is equal to the total volume of the bubble (see Figure 3.2).

The line segment

$$\mathcal{K} = \left\{ \frac{t(v_1, v_2) + (1-t)(w_1, w_2)}{2}; t, 2t_0 - t \in (0,1) \right\} \tag{3.10}$$

in the volume plane has the property that either $h_+(x, r) \in \mathcal{K}$ for some $r \neq 0$ or $h_+(y \times [0, \infty)) \in \mathcal{K}$ whenever $y$ is close to $x$.

In both cases, there is a hyperplane $H_{x,t} \not\supseteq \ell$ such that

$$h_+ H_{x,t} = \frac{t(v_1, v_2) + (1-t)(w_1, w_2)}{2} \in \mathcal{K}.$$

Let

$$\begin{aligned} a_0 &= \text{area}(B \cap H_{x,t}) \\ a_+ &= \text{area}(B \cap H^+_{x,t}) \\ a_- &= \text{area}(B \cap H^-_{x,t}). \end{aligned}$$

Replacing $B \cap H^-_{x,t}$ with the reflection of $B \cap H^+_{x,t}$ across $H$, we find that

$$\begin{aligned} a_0 + 2a_+ &\geq \Pi(t(v_1, v_2) + (1-t)(w_1, w_2)) \\ &= f(t) + t\Pi(v_1, v_2) + (1-t)\Pi(w_1, w_2) \\ &\geq f(t_0) + t\Pi(v_1, v_2) + (1-t)\Pi(w_1, w_2) \\ &= \Pi(t_0(v_1, v_2) + (1-t_0)(w_1, w_2)) \\ &\quad + (t-t_0)\Pi(v_1, v_2) + (t_0 - t)\Pi(w_1, w_2). \end{aligned}$$

If we symmetrize in the other direction, we find that

$$\begin{aligned} a_0 + 2a_- &\geq \Pi(t_0(v_1, v_2) + (1-t_0)w) + ((2t_0 - t) - t_0)\Pi(v_1, v_2) \\ &\quad + (t_0 - (2t_0 - t))\Pi(w_1, w_2). \end{aligned}$$

Adding these, we get

$$2(a_0 + a_+ + a_-) \geq 2\Pi(t_0(v_1, v_2) + (1-t_0)(w_1, w_2)).$$

By the definition of the '$a$'s, this must be an equality, so each of the non-strict inequalities in the chain above must be equalities.

We can therefore conclude that each symmetrization of $B$ across $H_{x,t}$ is minimizing for the enclosed volumes.

By Corollary 3.1, $B$ is symmetric about some line $\ell' \subset H_{x,t}$. Since $B$ is compact, $\ell$ and $\ell'$ must intersect. But $\ell \neq \ell'$, so by applying the Assembly Lemma 3.6 to a hyperplane containing $\ell$ and $\ell'$ we get that $B$ is a union of concentric spheres. Then $B$ must contain only one sphere by regularity. *So $B$ encloses only one volume, and $(v_1, w_1)$ both lie on the same coordinate axis in $\mathbf{R}^2$.*

But $\Pi$ is strictly concave along any line through the origin, since (in general) by scaling

$$I^m_n(\lambda v_1, \ldots, \lambda v_m) = \lambda^{(n-1)/n} I^m_n(v_1, \ldots, v_m) \tag{3.11}$$

But this contradicts our assumption that $\Pi$ is not concave along this line.

This completes the proof in the case of $\mathbf{R}^n$. For the other spaces, we just have some comments: the proof for $\mathbf{H}^n$ is essentially the same, the only significant thing that one must check is that $I_{\mathbf{H}^n}$ is concave and this is very easy (see Lemma 3.8 in the next section); the theorem for $\mathbf{S}^n$ is a straightforward rewording of the theorem for $\mathbf{R}^n$– some things to note for the *proof* are (1) oriented hyperplanes in $\mathbf{S}^n$ are parameterized by $\mathbf{S}^n$, (2) instead of looking at paths of the form $h_+(x \times [0, \infty))$, we look at images of $h_+$ under geodesics, and (3) we have strict concavity along the edge of the simplex $\{v_1 + v_2 + v_3 = \text{vol}(\mathbf{S}^n)\}$ because in a minimal sphere dividing $\mathbf{S}^n$ into two equal volumes, the pressure difference between the two regions decreases as two of the volumes become.

QEF

### 3.1.3 Connectivity (component bound)

**Lemma 3.8 ('Ratio of area to volume is decreasing for spheres' ).** In $\mathbf{R}^n$, $\mathbf{S}^n$, and $\mathbf{H}^n$, $\frac{I(x)}{x}$ is decreasing in $x$.

*Proof.* We use the following standard formulas for area and volume as functions of radius ($\alpha_n$ is again the Lebesgue measure of the unit ball in $\mathbf{R}^n$):

| | $\mathbf{R}^n$ | $\mathbf{S}^n$ | $\mathbf{H}^n$ |
|---|---|---|---|
| $A_{\text{sphere}}$ | $n\alpha_n r^n$ | $n\alpha_n \sin^{n-1} r$ | $n\alpha_n \sinh^{n-1} r$ |

Working from the appendix on variational calculus, we have

$$H_{\text{sphere}} = \frac{dA_{\text{sphere}}}{dV_{\text{sphere}}} = \frac{dA_{\text{sphere}}(r)/dr}{dV_{\text{sphere}}(r)/dr} = \frac{A'_{\text{sphere}}(r)}{A_{\text{sphere}}(r)},$$

By a very easy computation we get

| | $\mathbf{R}^n$ | $\mathbf{S}^n$ | $\mathbf{H}^n$ |
|---|---|---|---|
| $H_{\text{sphere}}$ | $n/r$ | $(n-1)\cot(r)$ | $(n-1)\coth(r)$ |

A glance at the graphs of these functions (on the appropriate domains) proves that mean curvature of spheres is a decreasing function of radius in each space. Since $I(x)$ is a primative of mean curvature,

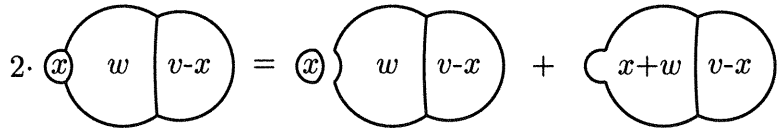



Figure 3.3: Decomposition of a double bubble in which $V$ has a component of volume $x$

this proves that $I$ is concave. From this we see directly that the ratio $I(x)/x$ is decreasing, since if $I$ is is not decreasing, its slope is. QEF

**Lemma 3.9 ('Basic Estimate').** Consider a minimal enclosure of volumes $v$ and $w$ in $\mathcal{R}^n$. Suppose that the region $V$ has a connected component with volume $x$. Then

$$2\Pi(v,w) \geq I(w) + \frac{v}{x}I(x) + I(v+w).$$

*Proof.* From Figure 3.3,

$$2\Pi(v,w) \geq \Pi(v-x,w) + I(x) + \Pi(v-x,w+x). \tag{3.12}$$

From Concavity (Theorem 3.2), with $v_1 = 0, v_2 = v+w, w_1 = v, w_2 = w$ and $t = x/v$ in Inequality 3.7, we get

$$\Pi(v-x,w+x) \geq \frac{v-x}{v}\Pi(v,w) + \frac{x}{v}I(v+w),$$

and again with $v_1 = 0, v_2 = w, w_1 = v, w_2 = w$ and $t = x/v$ we get

$$\Pi(v-x,w) \geq \frac{v-x}{v}\Pi(v,w) + \frac{x}{v}I(w).$$

Substitution of these bounds into Inequality 3.12 gives our result, upon making the following rearrangements:

$$\begin{aligned} 2\Pi(v,w) &\geq \frac{v-x}{v}\Pi(v,w) + \frac{x}{v}I(w) + I(w)\frac{v-x}{v}\Pi(v,w) + \frac{x}{v}I(v+w) \\ &= 2\Pi(v,w) - 2\frac{x}{v}\Pi(v,w) + \frac{x}{v}I(w) + I(x) + \frac{x}{v}I(v+w) \\ \Rightarrow 2\frac{x}{v}\Pi(v,w) &\geq \frac{x}{v}I(w) + I(x) + \frac{x}{v}I(v+w). \end{aligned}$$

Now divide through by $x/v$. QEF

*Remark 3.3.* The theorem we just proved and the following corollaries work in any space where $I\!I$ is concave.

**Corollary 3.2.** In a double bubble enclosing and separating the pair of volumes $(v, w)$ where $(v, w)$ satisfies

$$F(v,w) = 2I\left(\frac{v}{2}\right) + I(w) + I(v+w) - 2I\!I(v,w) > 0, \tag{3.13}$$

$V$ is connected.

*Proof.* Suppose Inequality 3.13 holds. Let the *smallest* component of the region $V$ be $V_1$. By concavity of $I\!I$, (Theorem 3.2) we can apply the Basic Estimate (Lemma 3.9). This yields

$$\frac{v}{v_1} I(v_1) \leq 2I\!I(v,w) - I(w) - I(v+w). \tag{3.14}$$

Since the left hand side of this inequality is decreasing in $v_1 \leq v$ by Lemma 3.8, if $v_1 \leq v/2$, we have

$$\frac{v}{v/2} I(v_1) \leq 2I\!I(v,w) - I(w) - I(v+w) \tag{3.15}$$

*i.e.*

$$2I(v_1) \leq 2I\!I(v,w) - I(w) - I(v+w) \tag{3.16}$$

*i.e.*

$$2I(v_1) - 2I\!I(v,w) + I(w) + I(v+w) \leq 0, \tag{3.17}$$

which implies

$$2I(\frac{v}{2}) - 2I\!I(v,w) + I(w) + I(v+w) \leq 0, \tag{3.18}$$

which is a direct contradiction of Inequality 3.13. Therefore we must have $v_1 > v/2$ and therefore the *smallest* component of $V$ must have volume *greater* than half of the volume of $V$. Any other component would have to have still greater volume, which is impossible, so $V$ must be connected.

QEF

Essentially the same proof works to show

**Corollary 3.3 ('Component Bound').** If

$$F_k(v,w) = kI\left(\frac{v}{k}\right) + I(w) + I(v+w) - 2\Pi(v,w) > 0,$$

$V$ has less than $k$ components.

## 3.2 Instability arguments

Hutchings' result on the rotational symmetry of area minimizing double bubbles (Theorem 3.1 in the previous section) underlies a series of arguments that showed that area minimizing double bubbles are standard in $\mathbf{R}^3$, $\mathbf{R}^4$, and for a range of volumes, in higher dimensions, as well as for equal volumes in $\mathbf{S}^3$ and $\mathbf{H}^3$. The symmetry, plus the component bound that follows from symmetry by way of Concavity (Theorem 3.2), give key simplifications to the geometry of the problem. However, these results do not in and of themselves finish the task.

Hutchings, Morgan, Ritoré, and Ros [23] developed an instability argument that rules out all the non-standard double bubbles left after these simplifications. (One of the most important steps was to precisely describe which non-standard double bubbles were still possible.) Reichardt *et al.* [41] adapted the instability argument to $\mathbf{R}^4$, and Cotton and Freeman [10] generalized the Hutchings-Morgan-Ritoré-Ros-Reichardt instability argument to $\mathbf{S}^3$ and $\mathbf{H}^3$, to use in their proof for the equal volumes cases of the double bubble problem in these spaces. Not a lot has to change, but there are lots of details to check; we review the statements of the main theorems from Cotton and Freeman's REU report.

### 3.2.1 Structure and Stability

**Lemma 3.10 ('Fundamental Instability Argument').** Let $\Sigma$ be a stable double bubble in $\mathbf{R}^n$, $\mathbf{S}^n$, or $\mathbf{H}^n$. Let $\mathcal{V}$ be a vectorfield corresponding to some isometric motion of $\mathbf{R}^n$, $\mathbf{S}^n$, or $\mathbf{H}^n$. Suppose that

the points where $\mathcal{V}$ is tangent to $\Sigma$ separate the bubble into at least four pieces. Then the normal component of $\mathcal{V}$ vanishes on any smooth component of $\Sigma$.

**Lemma 3.11.** Consider a stable double bubble of revolution $\Sigma \subset \mathbf{S}^n, n \geq 3$ with a piecewise smooth generating curve $\Gamma = \cup\bar{\Gamma}_i$ such that $\Gamma_i$ end either on the axis or meet in threes at vertices $v_{ijk}$. Let the axis of symmetry be denoted by $\ell$. At each $p \in \cup\Gamma_i$ let $N(p)$ denote the geodesic normal to $\Gamma$ at $p$. Consider the map

$$f : \cup\Gamma_i \ni p \mapsto N(p) \cap \ell \subset \ell \tag{3.19}$$

(that is, $f$ takes $p$ to the pair of antipodal points on $\ell$ that are determined by intersecting $\ell$ with the geodesic normal to $\Gamma$ at $p$). Let $\{p_i\}_{i=1,\ldots,k}$ be the smallest set of points in $\cup\Gamma_i$ that separates $\Gamma$ with the additional property that $f$ maps all of the '$p_i$'s to the same pair of antipodal points, which we call $(x, x')$. Then every connected component of $\Sigma$ which contains one of the points $p_i$ is part of a sphere centered at $x$ (and $x'$).

**Definition 3.4.** Two geodesics in $\mathbf{H}^n$ are *parallel* if they do not intersect but for any positive $\epsilon$ there is a point on each line such that the distance between the two points is less than $\epsilon$.

**Definition 3.5.** Two geodesics in $\mathbf{H}^n$ are *disjoint* if they do not intersect and are not parallel.

**Lemma 3.12.** Consider a stable double bubble of revolution $\Sigma \subset \mathbf{H}^n, n \geq 3$ with generating curve $\Gamma$ and axis of symmetry $\ell$ as in the previous lemma. We let $\bar{\ell}$ denote $\ell \cup \{-\infty, \infty\}$. At each $p \in \cup\Gamma_i$ we again let $N(p)$ denote the geodesic normal to $\Gamma$ at $p$, and we consider the map

$$f : \cup\Gamma_i \to \{0, 1\} \times \bar{\ell},$$

defined so that

- if $N(p)$ intersects $\ell$ then $f$ maps $p$ to $(0, N(p) \cap \ell)$,
- if $N(p)$ is parallel $\ell$ then $f$ maps $p$ to $\pm\infty$, depending on which end of $\ell$ comes arbitrarily close to $N(p)$,

- if $N(p)$ and $\ell$ are disjoint, there is a unique line $\ell'$ which intersects both $\ell$ and $N(p)$ orthogonally, and $f$ maps $p$ to $(1, \ell' \cap \ell)$.

Let $\{p_i\}_{i=1,\ldots,k}$ be the smallest set of points in $\cup\Gamma_i$ that separates $\Gamma$ with the additional property that $f$ maps all of the '$p_i$'s to the same point, which we call $x$. Then every connected component of $\Sigma$ which contains one of the points $p_i$ is part of

- a sphere centered at the second coordinate of $x$ when $x$ is of the form $(0, q)$ with $q \in \ell$,
- part of a horosphere centered at one end of $\ell$ if $x = \pm\infty$
- part of a hyposphere intersecting $\ell$ orthogonally if $x$ is of the form $(1, q)$ with $q \in \ell$.

In the hypospherical case, the hyposphere in question is generated by revolving a hypocircle that is a constant distance from the perpendicular to $\ell$ through $q$ about $\ell$.

### 3.2.2 Criteria for standardness

Given enough connectedness, the characterization of stable bubbles appearing above provides the geometrical engine that proves the double bubble theorem in constant curvature space forms. The idea is that the propositions above limit the possibilities for a non-standard double bubble with enough connectivity to be among some set of double bubbles that we can apply some volume and perimeter preserving variation to that brings about a violation of regularity. (It is the role of the Fundamental Instability Argument to carrying out this final critical step.)

**Corollary 3.4 (Structure Summary).** In $\mathbf{R}^n$, $\mathbf{S}^n$, or $\mathbf{H}^n$, any $\Gamma_i$ (as above) which can be disconnected by removing one point in its interior is a constant curvature arc which, if completed, would hit the axis of rotation $\ell$ orthogonally. In $\mathbf{R}^n$ it is a piece of a circle; similarly in $\mathbf{S}^n$; and in $\mathbf{H}^n$ it is a piece of a circle, horocycle, or hypercircle.

Adapting an argument from the William's honors thesis of Joel Foisy [**?**] about on double bubbles in $\mathbf{R}^3$, Cotton and Freeman showed that an area minimizing double bubble in $\mathbf{H}^n$ intersects its axis

of symmetry. With the addition of this final piece, Cotton and Freeman then had the right tools to prove:

**Criterion for standardness in $\mathbf{H}^n$.** An area minimizing double bubble in $\mathbf{H}^n$ in which both regions are connected must be the standard double bubble.

**Criterion for standardness in $\mathbf{S}^n$.** An area minimizing double bubble in $\mathbf{S}^n$ in which both regions and the exterior are connected must be the standard double bubble.

# Chapter 4

# s3h3

The culmination of the previous chapter was that

- If both enclosed regions and the exterior are connected, an area minimizing double bubble in $\mathbf{S}^n$ is standard.

- If both enclosed regions are connected, an area minimizing double bubble in $\mathbf{H}^n$ must be the standard double bubble.

The *other* culmination of the previous chapter (which took place in an earlier section) was that we have a criterion for connectivity, Proposition 3.2. That proposition said that if

$$2I(\frac{v}{2}) + I(w) + I(v+w) > 2\mathit{\Pi}(v,w) \tag{4.1}$$

then the region containing volume $v$ is connected. We can check this criterion by plotting

$$2I\left(\frac{v}{2}\right) + I(w) + I(v+w) \tag{4.2}$$

and

$$2\tilde{\mathit{\Pi}}(v,w), \tag{4.3}$$

(where $\tilde{\Pi}(v, w)$ gives the surface area of a standard double bubble containing volumes $v$ and $w$, and so an upper bound on $\Pi(v, w)$) and comparing the plots to see which function is bigger. In fact, both 4.2 and 4.3 are increasing functions in both variables (in $\mathbf{H}^3$; in $\mathbf{S}^3$, they are only increasing on a certain range). Where the functions are increasing, there won't be 'jumps' between plotted points. Hence to plot these functions is to prove the double bubble theorem in $\mathbf{S}^3$ and $\mathbf{H}^3$, modulo asymptotic analysis to show that the graphs of 4.2 and 4.3 continue to bear the correct relationship to each other outside of the range of the plot.

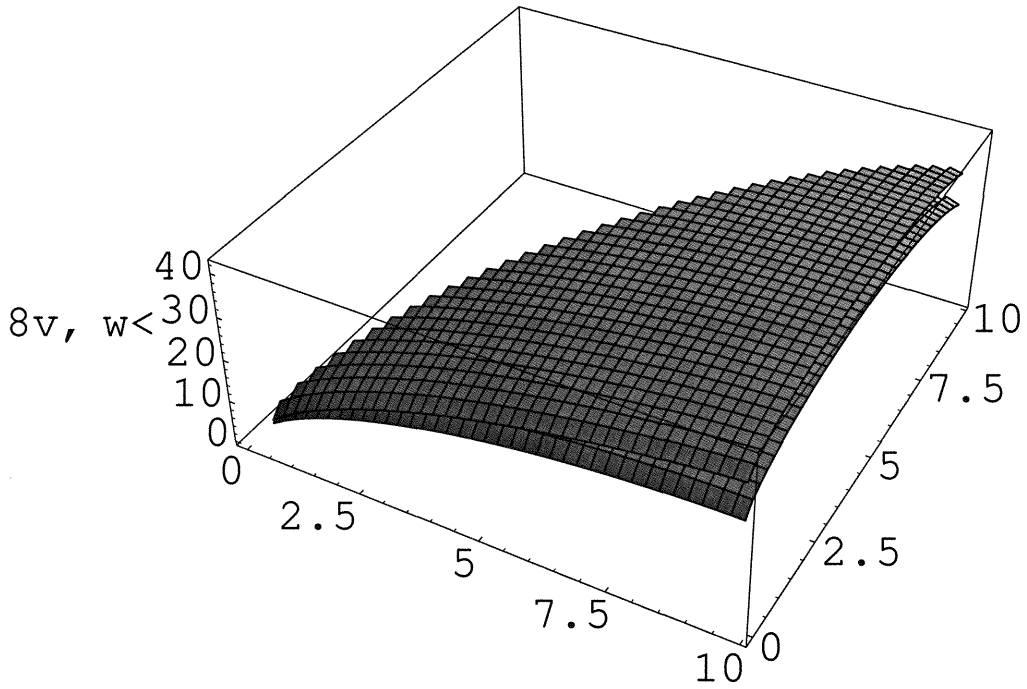



Figure 4.1: This plot proves connectivity in $\mathbf{S}^3$.

**Theorem 4.1.** The standard double bubble is the least-area way to partition $\mathbf{S}^3$ into three regions.

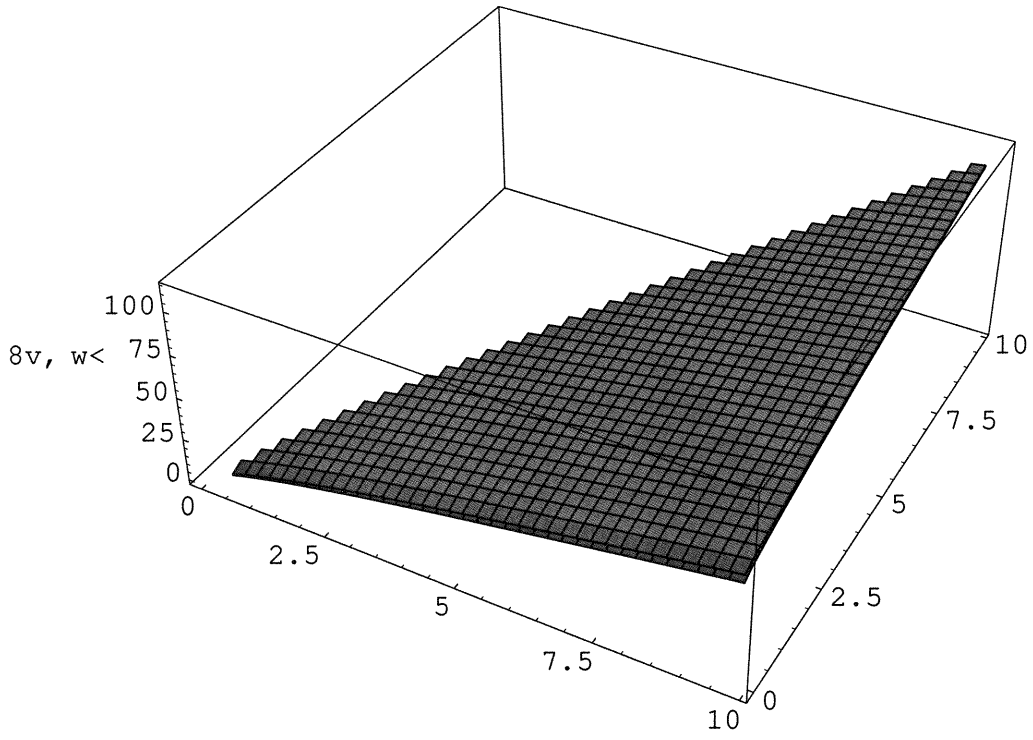



Figure 4.2: This plot is a little harder to read, but nonetheless, exhibits the bound we were looking for, proving connectivity in $\mathbf{H}^3$ for a finite range of volumes.

*"Proof".* Pending asymptotic analysis at zero, where the functions plotted are equal, the fact that the plots show that $2I(\frac{v}{2}) + I(w) + I(v+w) > 2\tilde{I}(v,w)$ on the domain where $v < |\mathbf{S}^3|/2$ and $w \leq v$ proves the claim, because the functions are increasing an this domain so the top function can not drop below the bottom function in between the mesh of the plot, and this domain is a general one in the sense that all we require is that one of the volumes takes up less than half of the total volume of the space and be larger than one of the other volumes; hence, by switching labels the properties of the given domain are representative of the properties of all volume pairs. QEF

**Theorem 4.2.** The standard double bubble is the least-area way to enclose and separate two volumes $v$ and $w$ in $\mathbf{H}^3$.

*"Proof".* The logic of this proof is the same as the proof of Theorem 4.1. Note that here, to finish the proof, not only must we study asymptotics at zero, but also through an entire quarter circle of directions heading out towards infinity. QEF

We don't have time to do the asymptotic analysis in this thesis. We satisfy ourselves with showing that the plotted formulas are correct.

## 4.1 Area and Volume Formulas

We can get a simple upper bound on $\mathit{\Pi}$ by using $\tilde{\mathit{\Pi}}$, the area of a standard double bubble, which will of course be greater than or equal to the area of the least-area double bubble. In this section we find parameterizations of the area and volumes of the standard double bubble and spheres in terms of the relevant radii (or curvatures, in the hyperbolic case). Using these parameterizations, we found $I$ and $\tilde{\mathit{\Pi}}$ numerically. In order to create Figures 4.1 and 4.2 it was necessary to take this roundabout path, because no simple closed form for the relevant functions in terms of volumes is known (or likely to be found).

The formulas is this section are derived using spherical and hyperbolic trigonometry as found without explaination in [40]; for the explaination see Thurston [55, Section 2.4].

### 4.1.1 Spheres

Here we present the standard results on volume and surface area of spheres in $\mathbf{S}^3$ and $\mathbf{H}^3$.

*Remark 4.1.* The formulas for surface area of a sphere of radius $r$ in $\mathbf{S}^3$ and $\mathbf{H}^3$ are

$$A_{\mathbf{S}^3} = 4\pi \sin^2 r$$

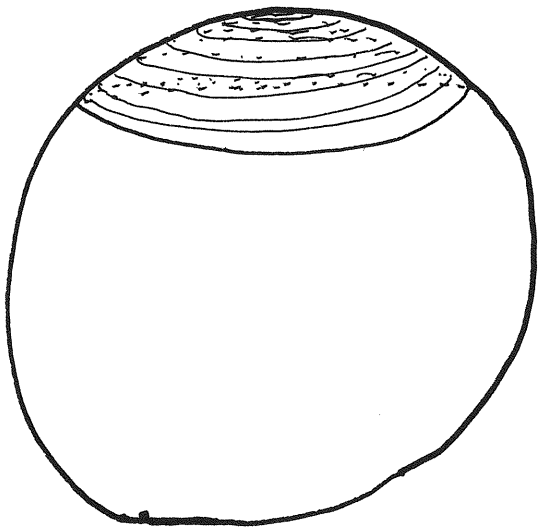


Figure 4.3: Spherical cap on sphere

$$A_{\mathbf{H}^3} = 4\pi \sinh^2 r$$

The volume formulas for a ball of radius r in $\mathbf{S}^3$ and $\mathbf{H}^3$ are

$$V_{\mathbf{S}^3} = \pi(2r - \sin 2r)$$

$$V_{\mathbf{H}^3} = \pi(\sinh 2r - 2r)$$

**Lemma 4.1.** The mean curvatures, $\frac{dA}{dV}$, of spheres of radius $r$ in $\mathbf{S}^3$ and $\mathbf{H}^3$ are

$$(\mathbf{S}^3) \qquad 2\cot r \tag{4.4}$$

$$(\mathbf{H}^3) \qquad 2\coth r. \tag{4.5}$$

*Proof.* This is a special case of the computation appearing in Lemma 3.8. QEF

### 4.1.2 Spherical Caps

Even though it seems out of line with the name of the chapter, in this section we will prove the results we want in the Euclidean setting and then state the $\mathbf{S}^3$ and $\mathbf{H}^3$ versions as corollaries.

**Proposition 4.1.** In Euclidian space the volume of a spherical cap belonging to a sphere of radius $r$

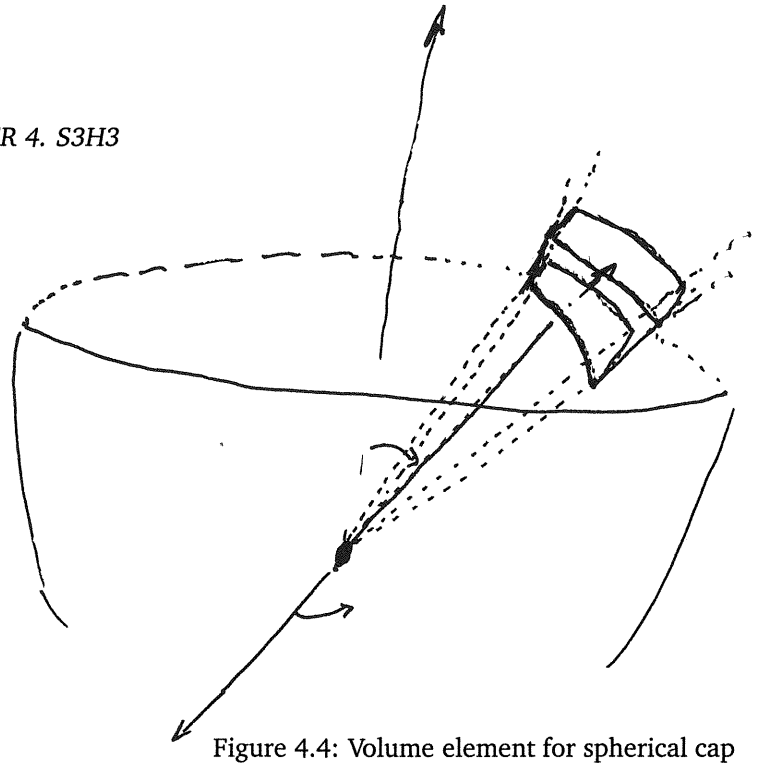


Figure 4.4: Volume element for spherical cap

and subtended by an angle of $2\varphi_0$ is given by

$$\int_0^{2\pi} d\theta \int_0^{\varphi_0} d\varphi \int_{\frac{r\cos\varphi_0}{\cos(\varphi)}}^{r} d\rho\, \rho^2 \sin\varphi$$

*Proof.* Spherical coordinates in $\mathbf{R}^3$

$$\begin{aligned} x &= \rho\cos\theta\sin\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\sin\varphi \end{aligned} \tag{4.6}$$

have an associated Jacobian of $\rho^2\sin\varphi$. We do the integral in the standard way, *around* the sphere, *through* all the azimuthal angles, and *from* the disk that bounds the cap to the surface of the sphere (Figure 4.4). For this parameterization, we choose to have $\varphi$ run *down* from the north pole of the sphere, since this makes the formulas come out simpler.

A little bit of trig gives the correct bounds for the integral. Actually, given the strategy above, the only question is what the lower bound is on the '$\rho$' integral. (See Figure 4.5.)

We want to solve for $\rho_0$. The distance from the diameter $d$ to the point $p$ is $\rho_0\sin(\varphi)$. The distance from the center of the sphere to the base of the cap is $r\cos\varphi$. By the pythagorean theorem, we have

$$\rho_0^2\sin^2(\varphi) + r^2\cos^2\varphi_0 = \varphi_0^2.$$

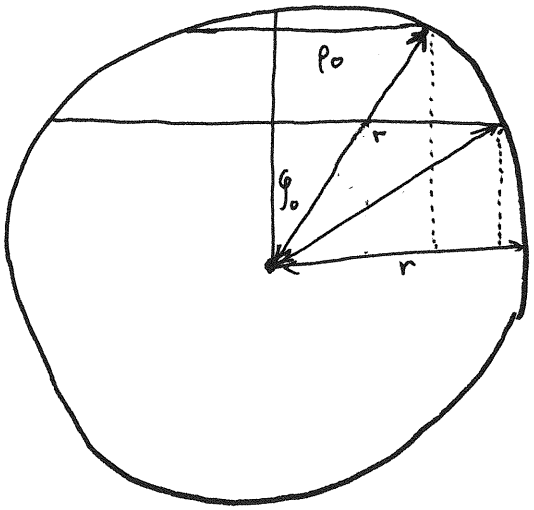



Figure 4.5: Find Lower Bound For $\rho$ Using Trig

Hence,

$$\rho_0^2(1-\sin^2(\varphi)) = r^2\cos^2\varphi_0,$$

so

$$\rho_0 = \sqrt{\frac{r^2\cos^2\varphi_0}{1-\sin^2(\varphi)}} = \frac{r\cos\varphi_0}{\cos(\varphi)}$$

QEF

**Corollary 4.1.** The corresponding formulas in $\mathbf{S}^3$ and $\mathbf{H}^3$ are

$$(\mathbf{S}^3) \qquad \int_0^{2\pi} d\theta \int_0^{\varphi_0} d\varphi \int_{\tan^{-1}(\frac{\tan r\cos\varphi_0}{\cos(\varphi)})}^{r} d\rho\ \sin^2(\rho)\sin\varphi \tag{4.7}$$

$$(\mathbf{H}^3) \qquad \int_0^{2\pi} d\theta \int_0^{\varphi_0} d\varphi \int_{\tanh^{-1}(\frac{\tanh r\cos\varphi_0}{\cos(\varphi)})}^{r} d\rho\ \sinh^2(\rho)\sin\varphi. \tag{4.8}$$

*Proof.* For spherical coordinates in these spaces replace $\rho$ in the parameterization of Equations 4.6 by $\sin\rho$ or $\sinh\rho$ respectively, and use spherical or hyperbolic trigonometry in Figure 4.5. QEF

**Corollary 4.2.** (Formulas for Volume of Spherical Caps) The volume of a spherical cap of radius $r$ that is subtended by an angle $2\varphi_0$ is

$$(\mathbf{S}^3) \qquad -\pi(\tan^{-1}(\cos\varphi_0\tan r) - r\cos\varphi_0 + (-1+\cos\varphi_0)(r-\cos r\sin r)) \tag{4.9}$$

$$(\mathbf{H}^3) \qquad \pi(-r+\tanh^{-1}(\cos\varphi_0\tanh r) - \cos\varphi_0\cosh r\sinh r + \cosh r\sinh r))). \tag{4.10}$$

*Proof.* We did the integrals symbolically in *Mathematica*. QEF

**Proposition 4.2.** In Euclidian space the surface area of a spherical cap belonging to a sphere of radius $r$ and subtended by an angle of $2\varphi_0$ is given by

$$\int_0^{2\pi} d\theta \int_0^{\varphi_0} d\varphi \, r^2 \sin\varphi$$

*Proof.* The equation for a hemisphere of radius $r$ is $f(x,y) = \sqrt{r^2 - x^2 - y^2}$. The area integral is

$$\int_{\text{projection}} \sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dx}\right)^2}\, dxdy.$$

Calculating the derivatives and subsitituting in polar coordinates

$$\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta, \end{aligned} \tag{4.11}$$

this becomes

$$\int_0^{2\pi} d\theta \int_0^{\rho_0} d\rho \, \rho \sqrt{1 + \frac{\rho^2}{r^2 - \rho^2}},$$

From Figure 4.5, $\rho_0$ is seen to be $r\sin\varphi_0$. The following sequence of transformations then gives the stated form:

$$\begin{aligned} \int_0^{2\pi} d\theta \int_0^{r\sin\varphi_0} d\rho \, \rho \sqrt{1 + \frac{\rho^2}{r^2 - \rho^2}} &= \int_0^{2\pi} d\theta \int_0^{r\sin\varphi_0} d\rho \, \rho \sqrt{\frac{r^2}{r^2 - \rho^2}} \\ &= \int_0^{2\pi} d\theta \int_0^{r\sin\varphi_0} d\rho \, \rho \frac{1}{\sqrt{1 - (\frac{\rho}{r})^2}} \end{aligned} \tag{4.12}$$

(putting $\bar\rho = \rho/r$, we note that $d\bar\rho = \frac{1}{r}d\rho$, and that $\rho = r\sin\varphi_0$ implies $\bar\rho = \sin\varphi_0$, so 4.12 can be written:)

$$= \int_0^{2\pi} d\theta \int_0^{\sin\varphi_0} d\bar\rho \, r^2 \bar\rho \frac{1}{1 - \bar\rho^2} \tag{4.13}$$

(4.14)

(in general, $\bar\rho = \sin\varphi$, so $d\varphi = d(\sin^{-1}\bar\rho) = \frac{1}{\sqrt{1-\bar\rho^2}}d\bar\rho$, and we note that if $\rho = \rho_0 = r\sin\varphi_0$, then $\varphi = \varphi_0$, so we end with this change of variables:)

$$= \int_0^{2\pi} d\theta \int_0^{\varphi_0} d\varphi \, r^2 \sin\varphi. \tag{4.15}$$

QEF

**Corollary 4.3.** The corresponding formulas in $\mathbf{S}^3$ and $\mathbf{H}^3$ are

$$(\mathbf{S}^3) \qquad \int_0^{2\pi} d\theta \int_0^{\varphi_0} d\varphi \ \sin^2 r \sin\varphi \tag{4.16}$$

$$(\mathbf{H}^3) \qquad \int_0^{2\pi} d\theta \int_0^{\varphi_0} d\varphi \ \sinh^2 r \sin\varphi \tag{4.17}$$

*Proof.* For spherical coordinates in these spaces replace $\rho$ in the parameterization of Equations 4.11 by $\sin\rho$ or $\sinh\rho$ respectively, and use spherical or hyperbolic trigonometry in Figure 2. QEF

**Corollary 4.4.** (Formulas for Area of Spherical Caps) The area of a spherical cap of radius $r$ subtended by an angle $2\varphi_0$ is

$$(\mathbf{S}^3) \qquad 2\pi \sin^2 r(1-\cos\varphi_0) \tag{4.18}$$

$$(\mathbf{H}^3) \qquad 2\pi \sinh^2 r(1-\cos\varphi_0). \tag{4.19}$$

*Proof.* These integrals can be done in ones head. QEF

### 4.1.3 Standard Double Bubbles

Since each standard double bubble in $\mathbf{S}^3$ and $\mathbf{H}^3$ is symmetric about a geodesic line, it is convenient to compute area and volume of a standard double bubble by considering the revolution of some generating curve consisting of 3 circular arcs meeting at 120°. Figure 4.6 represents an arbitrary generating curve (and its reflection about the line of symmetry) labelled with relevant measurements.

**Spherical Space** From Figure 4.6 and repeated application of an identity of spherical trigonometry (see Figure 4.7) we have

$$\sin\varphi_1 \quad = \quad \frac{\sin x}{\sin r_1};$$

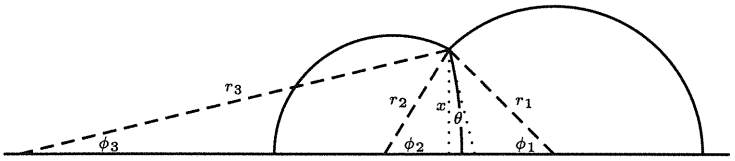



Figure 4.6: Some important geometrical features of the generating curve for a standard double bubble.

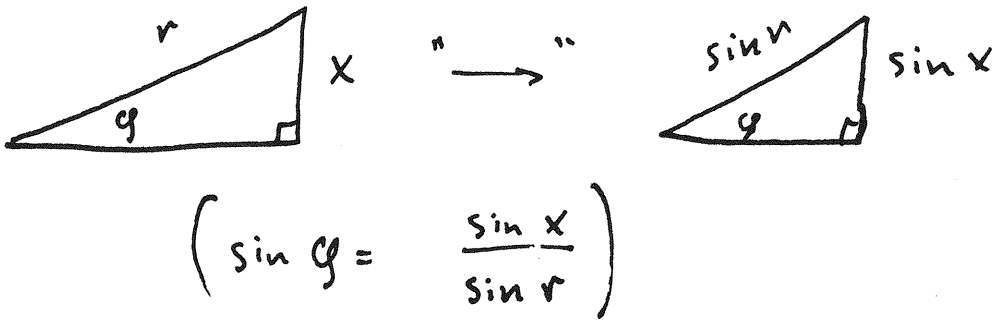



Figure 4.7: Sphere sine relation

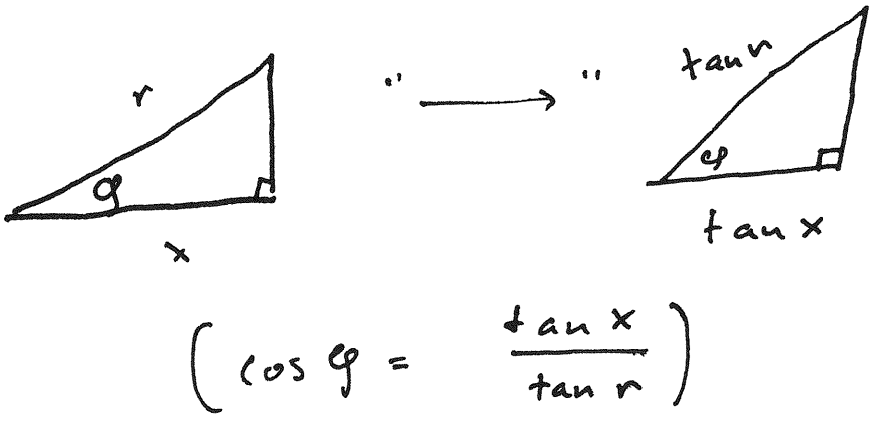



Figure 4.8: Sphere tantancos

$$\begin{aligned} \sin\varphi_2 &= \frac{\sin x}{\sin r_2}; \\ \sin\varphi_3 &= \frac{\sin x}{\sin r_3}. \end{aligned}$$

Another identity (see Figure 4.8) yields

$$\begin{aligned} \tan x &= \tan r_1 \cos(\frac{\pi}{6} - \theta); \\ \tan x &= \tan r_2 \cos(\frac{\pi}{6} + \theta); \end{aligned}$$

or

$$\begin{aligned} \tan x &= \tan r_1 (\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta); \\ \tan x &= \tan r_2 (\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta). \end{aligned}$$

Solving for $\theta$, we get

$$\theta = \tan^{-1}(\frac{\sqrt{3}(\tan r_2 - \tan r_1)}{\tan r_1 + \tan r_2}).$$

We can plug this into the first of the two preceding equations above and rearrange terms to get

$$x = \tan^{-1}(\tan r_1 \cos(\frac{\pi}{6} - \tan^{-1}(\frac{\sqrt{3}(\tan r_2 - \tan r_1)}{\tan r_1 + \tan r_2}))),$$

which we can put into the first triple of equations to obtain a (complicated) expression for the '$\varphi$'s in terms of $r_1$ and $r_2$:

$$\varphi_1 = \sin^{-1}\left(\frac{\sin(\tan^{-1}(\tan r_1 \cos(\frac{\pi}{6} - \tan^{-1}(\frac{\sqrt{3}(\tan r_2 - \tan r_1)}{\tan r_1 + \tan r_2}))))}{\sin r_1}\right);$$

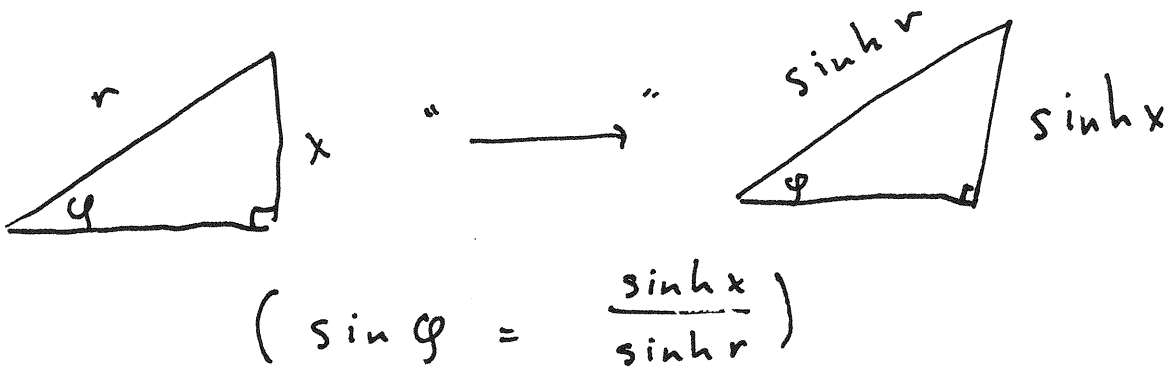



Figure 4.9: Hyperbolic sine relation

$$\varphi_2 = \sin^{-1}\left(\frac{\sin(\tan^{-1}(\tan r_1 \cos(\frac{\pi}{6} - \tan^{-1}(\frac{\sqrt{3}(\tan r_2 - \tan r_1)}{\tan r_1 + \tan r_2}))))}{\sin r_2}\right);$$

$$\varphi_3 = \sin^{-1}\left(\frac{\sin(\tan^{-1}(\tan r_1 \cos(\frac{\pi}{6} - \tan^{-1}(\frac{\sqrt{3}(\tan r_2 - \tan r_1)}{\tan r_1 + \tan r_2}))))}{\sin r_3}\right).$$

Note that there are only two relevant parameters here. Since pressures add, we have $r_3$ in terms of $r_1$ and $r_2$:

$$2\cot r_3 = 2\cot r_1 - 2\cot r_2,$$

or

$$r_3 = \cot^{-1}(\cot r_1 - \cot r_2).$$

**Hyperbolic Space** We derive similar formulas for the angles $\varphi_i$, but this time using trigonometric identities in hyperbolic space (see Figures 4.9 and 4.10):

$$\begin{aligned}
\sin\varphi_i &= \frac{\sinh x}{\sinh r_i}\ (i = 1, 2, 3); \\
\tanh x &= \tanh r_1 \cos(\frac{\pi}{6} - \theta); \\
\tanh x &= \tanh r_2 \cos(\frac{\pi}{6} + \theta).
\end{aligned}$$

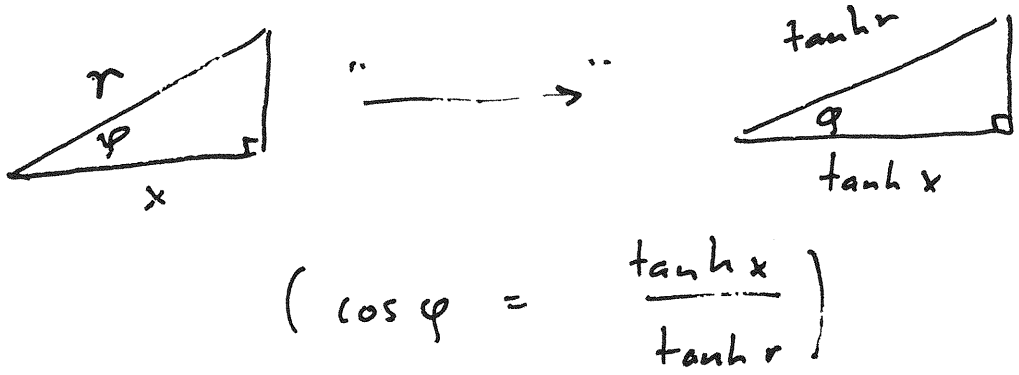



Figure 4.10: Hyperbolic tantancos

Solving for $\varphi_i$ (i = 1, 2, 3) yields

$$\varphi_i = \sin^{-1}\left(\frac{\sinh(\tanh^{-1}(\tanh r_1 \cos(\frac{\pi}{6} - \tan^{-1}(\frac{\sqrt{3}(\tanh r_2 - \tanh r_1)}{\tanh r_1 + \tanh r_2}))))}{\sinh r_i}\right)$$

So far this derivation has been analogous to the work we did in $\mathbf{S}^3$. But in hyperbolic space we have already mentioned the fact that not all hypersphere-like submanifolds in hyperbolic space have well-defined radii. We work around this by computing area and volume of double bubbles in $\mathbf{H}^3$ in terms of curvature. To do this, we derive the area and volume formulas as before, and convert formulas using the relation:

$$k = \coth r \tag{4.20}$$

The wonderful fact is that analytic continuation implies that our new formulas yield the desired areas and volumes.

We can now compute $\varphi_i$ in terms of $k_1$ and $k_2$ (the curvatures of the two outer caps). Here, the curvature constraint gives, $k_3 = k_1 - k_2$.

$$\varphi_1 = \sin^{-1}\left(\frac{\sqrt{1-\frac{1}{k_1^2}}\cos(\frac{\pi}{6}-\tan^{-1}(\frac{\sqrt{3}(k_1-k_2)}{k_1+k_2}))}{\sqrt{1-\frac{\cos(\frac{\pi}{6}-\tan^{-1}(\frac{\sqrt{3}(k_1-k_2)}{k_1+k_2}))}{k_1^2}}}\right)$$

$$\varphi_2 = \sin^{-1}\left(\frac{k_2\sqrt{1-\frac{1}{k_2^2}}\cos(\frac{\pi}{6}-\tan^{-1}(\frac{\sqrt{3}(k_1-k_2)}{k_1+k_2}))}{k_1\sqrt{1-\frac{\cos(\frac{\pi}{6}-\tan^{-1}(\frac{\sqrt{3}(k_1-k_2)}{k_1+k_2}))}{k_1^2}}}\right)$$

$$\varphi_3 = \sin^{-1}\left(\frac{(k_1-k_2)\sqrt{1-\frac{1}{(k_1-k_2)^2}}\cos(\frac{\pi}{6}-\tan^{-1}(\frac{\sqrt{3}(k_1-k_2)}{k_1+k_2}))}{k_1\sqrt{1-\frac{\cos(\frac{\pi}{6}-\tan^{-1}(\frac{\sqrt{3}(k_1-k_2)}{k_1+k_2}))}{k_1^2}}}\right)$$

*Remark 4.2.* Now we can compute the volumes and surface area of standard double bubbles, in terms of the radii or curvatures of the outer caps, by substituting the formulas we have just derived in Corollary 4.2 and Corollary 4.4. Refering to Figure 4.6, we observe that the angle $\varphi_1$ is either acute or obtuse depending on the relative size of the two bubbles; the transition between regimes is when $\varphi_1$ is 90°. Tracing back through the equations, we notice that this happens when

$$\frac{\tan(r_2)-\tan(r_1)}{\tan(r_2)+\tan(r_1)} = 1/3 \quad (\text{in } \mathbf{S}^3)$$

and when

$$\frac{k_1-k_2}{k_1+k_2} = 1/3 \quad (\text{in } \mathbf{H}^3)$$

We must make adjustments to $\varphi_1$ in our computations accordingly. Specifically, if $\varphi_1 < \frac{\pi}{2}$, the formulas return values corresponding to a cap that is larger than a hemisphere, which is what we desired, otherwise they return values corresponding to a cap smaller than a hemisphere, so we have to use $\pi - \varphi_1$ in place of $\varphi_1$.

**Definition 4.1.** In light of the last remark, we define $\widehat{\varphi}_1$ to be equal to $\varphi_1$ if $\varphi_1 < \frac{\pi}{2}$ and to be $\pi - \varphi_1$ otherwise.

Now, the following lemma summarizes how we compute area and volumes for the standard double bubble.

**Lemma 4.2.** The volume of the first bubble is the volume of the spherical cap in the sphere of radius $r_1$ subtended by an angle $2\widehat{\varphi}_1$, plus the volume of the spherical cap in the sphere of radius $r_3$ subtended by an angle $2\varphi_3$. The volume of the second bubble is the volume of the spherical cap in the sphere of radius $r_2$ subtended by an angle $2(\pi - \varphi_2)$, minus the volume of the spherical cap in the sphere of radius $r_3$ subtended by angle $2\varphi_3$.

*Proof.* For each bubble component, we compute the volume and area of the spherical cap which separates it from the exterior. For area of the double bubble, we simply add these areas plus the area of the separating cap. To compute the volume of each bubble component we have to include the volume contained within the separating cap in the smaller region, and exclude it from the larger region. QEF

## 4.2 The Curvature Conjecture

The inequality we have been studying in this chapter is

$$\hat{F}(v,w) = 2I(\frac{v}{2}) + I(w) + I(v+w) - 2\tilde{\Pi}(v,w) > 0 \tag{4.21}$$

$\tilde{\Pi}(v,w)$ denotes the surface area of the standard double bubble enclosing the volumes $v$ and $w$.

We can perturb the definition of $F_k$ of Corollary 6.2 in the same way to deduce that if

$$\hat{F}_k(v,w) = kI(\frac{v}{k}) + I(w) + I(v+w) - 2\tilde{\Pi}(v,w) > 0 \tag{4.22}$$

then $V$ has less than $k$ components.

Lemma 3.8 tells us that $kI(\frac{v}{k})$ is increasing in $k$. Therefore there is some largest $k$ for which the Basic Estimate Lemma 3.9 (with $x = \frac{v}{k}$) applies *i.e.* for this $k$ we have $2\Pi(v,w) = I(w) + \tilde{k}I(\frac{v}{\tilde{k}}) + I(v+w)$. A possibly even greater '$k$', which we will call $\tilde{k}$ satisfies

$$2\tilde{\Pi}(v,w) = I(w) + \tilde{k}I(\frac{v}{\tilde{k}}) + I(v+w). \tag{4.23}$$

This defines $\tilde{k}$ implicitly as a function of $v$ and $w$.

In light of the fact that mean curvature is proportional to the derivative of area with respect to volume, differentiating Equation 4.23 gives

$$\frac{d(\tilde{k}I(\frac{v}{k}))}{dw} = kH_{v/k} - H_w - H_{w+v} \tag{4.24}$$

$$\frac{d(\tilde{k}I(\frac{v}{k}))}{dv} = kH_{v/k} - H_{w+v}. \tag{4.25}$$

(*N. B.* Upon differentiating the surface area of the standard double bubble with respect to one of the volumes, we only pick up the curvature of the outer cap because (a) the standard double bubble is stable under small variations; and (b) therefore all variations must give the same $dA/dv$ or some combination could reduce perimeter, contradicting (a); (c) a variation that raises a small bump on the outer cap picks up its curvature.)

The curvature conjecture says that each of 4.24 and 4.25 is signed. Specifically, in the most important case $k < 2$, we have:

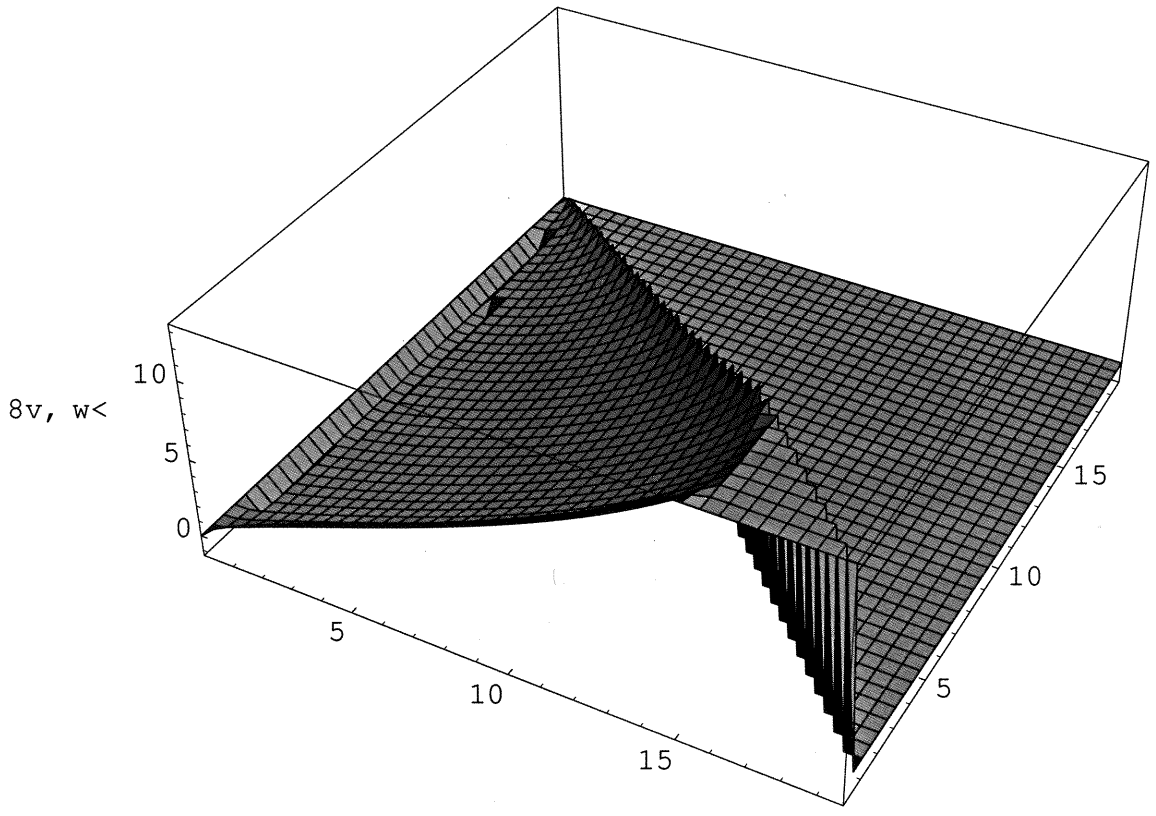
**Conjecture 4.1.** Curvature Conjecture In $\mathbf{R}^n$, $\mathbf{S}^n$, or $\mathbf{H}^n$, the various mean curvatures associated with standard double bubbles satisfy the following inequalities, where we $H_x$ denote the curvature of a sphere of volume $x$ and we let $H_{x,y}$ denote the curvature of the the outer cap of the '$x$ component' of a standard double bubble containing volumes $x$ and $y$

$$H_{v,w} > \frac{H_{v/2} + H_{w+v}}{2}.$$

$$H_{w,v} < \frac{H_{v/2} + H_{w+v}}{2}.$$

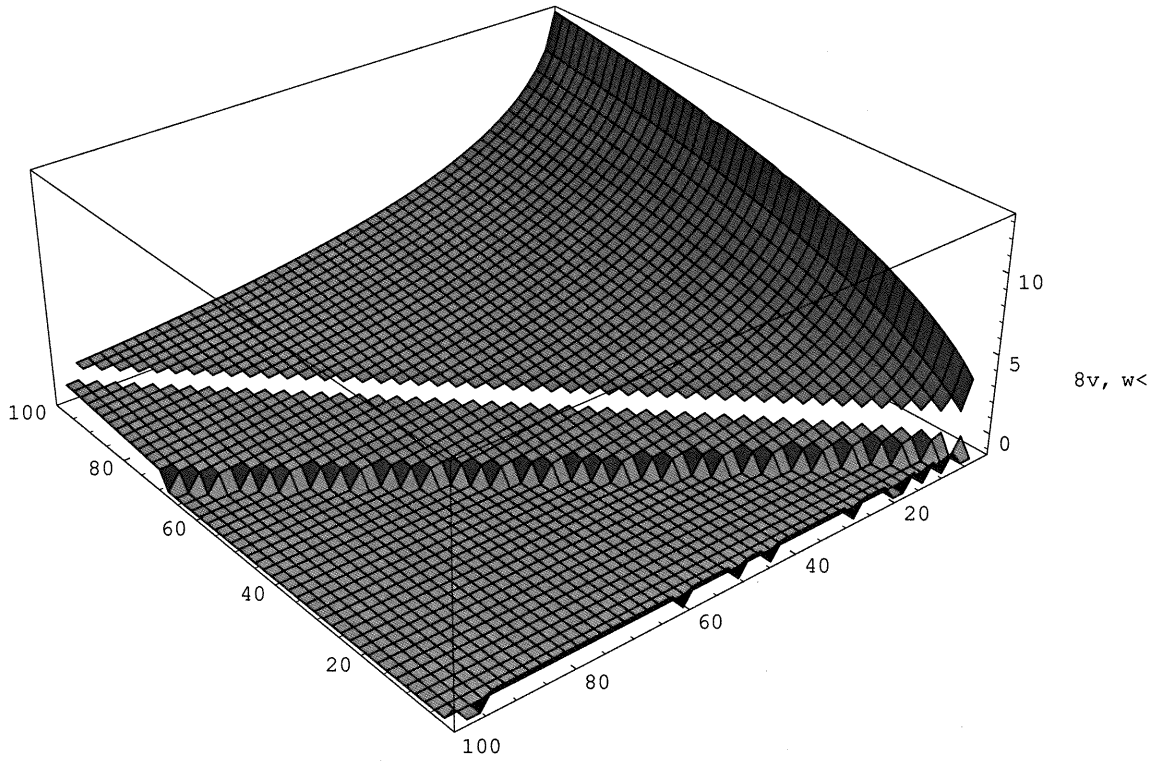
If this conjecture is true, then $\hat{F}$ is a decreasing function one of the variables and an increasing function in the other, and so plots of $\hat{F}$ will not have jumps in between the mesh of the plot. Plots and analysis suggest that the curvature conjecture holds in $\mathbf{R}^n$, although the analysis part is not complete. In $\mathbf{R}^n$, the volumes can be normalized so that one of the volumes is equal to one. David Futer then showed that in the case that the other volume is bigger than one. The proof uses the fact that the standard double bubble is invariant under inversion in the sphere that separating cap is a piece of.

Since we already know that the component functions of $\hat{F}$ are increasing, we are probably better off studying them. At present, plots of the function $\hat{F}$ are useful because they show (non-rigorously) what the gaps between the functions plotted in Figures 4.1 and 4.2 look like.




*The function value is lowered artificially on region where graph is negative for ease of viewing.*

Figure 4.11: The Hutchings function $\hat{F}(v, w)$ in $\mathbf{S}^3$.




*The function value is lowered artificially on regions where graph is negative for ease of viewing; the gap on the diagonal is due to a singularity in our formulas that doesn't show up in the proof-plots for some reason. Cotton and Freeman already treated the diagonal in detail anyway.*

Figure 4.12: The Hutchings function $\hat{F}(v, w)$ in $\mathbf{H}^3$.

# Chapter 5

# gn

**Charter.** We prove that the least-area division of $n$-dimensional Gauss space into three equal pieces is given by the standard configuration of three $(n-1)$-dimensional hyperplanes meeting along an $(n-2)$-dimensional hyperplane at 120°.

## 5.1 Introduction

Let $\mathbf{G}^n$ denote $\mathbf{R}^n$ with the volume form weighted by Gaussian measure and area given by restricting this measure to $(n-1)$-dimensional subsets in the standard fashion (the area of an $(n-1)$-dimensional subset $A$ is given by finding the derivative with respect to $\epsilon$ at zero of the volume of $\epsilon$-tubes about $A$, so, for example, a hyperplane through the origin has total $(n-1)$-dimensional measure of $1/2\pi$). Borell [4], Sudakov-Tsirel'son [51] and others (cf. Barthe and Maurey [2] and the references therein) showed that the solution to the isoperimetric problem in this space (that is, the least-area division of $\mathbf{G}^n$ into two regions of prescribed volume) is given by an $(n-1)$-dimensional hyperplane. Our Theorem 5.1 will show that the least-area division of Gauss space into three pieces of equal size is given by a 'Y', that is, by three half-hyperplanes meeting each other at 120°along an

$(n-2)$-dimensional plane through the origin.

Theorem 5.1 solves a 'double bubble problem', the name given to the problem of enclosing and separating two regions of prescibed volume in a give space with the least area. This problem has recently been solved in $\mathbf{R}^3$ by Hutchings, Morgan, Ritoré and Ros ([23], see [34]).

**The proof.** A classical observation ascribed to Poincaré (Proposition 5.1) says that $\mathbf{G}^n$ can be realized as a limit of projections from $\mathbf{S}^N$ onto $\mathbf{R}^n$ as $N$ goes to infinity and the radius of $\mathbf{S}^N$ grows. This reduces the problem to studying double bubbles in $\mathbf{S}^N$.

Hutchings' theory of the structure of area minimizing double bubbles in spheres [22] shows how to bound the number of components of minimizing bubbles that divide the $\mathbf{S}^N$ into three equal volumes. Using this connectivity result, an argument based on the that used in $\mathbf{R}^3$ [23, Section 6] shows that minimizing double bubbles must be standard. Taking limits of projections onto $\mathbf{S}^n$ then gives the result.

**Acknowledgements** Theorem 5.1 came out of conversations with Frank Morgan about a lecture by Antonio Ros (and the lecture notes [44]). Morgan provided lots of useful ideas about how to *prove* the theorem. Pat McDonald had a number of very helpful conversations with me on this topic.

## 5.2 The Geometry of Gauss Space

Proposition 5.1, attributed to Poincaré shows that $n$-dimensional Gauss space is the limit of projections onto $\mathbf{R}^n$ of uniform measure on spheres whose size and dimension grow in a prescibed way. The proof of Proposition 5.1 is based on that found in McKean [30]. We also point out that it is very easy to compute areas of hyperplanes and pieces of hyperplanes in $\mathbf{G}^n$ (Proposition 5.2), reducing this to computations with a one-dimensional Gaussian.

**Proposition 5.1 (attrib. Poincaré).** Gauss space of $n$ dimensions is realized as the limit as $N \to \infty$ of projections of $\mathbf{S}^N(\sqrt{N})$ onto some fixed $\mathbf{R}^n$.

*Proof.* Because of how we have defined volumes and areas in $\mathbf{G}^n$, it suffices to show that if $\mathbf{x} = (x_1, ...x_N)$ is uniformly distributed on $S^N(\sqrt{N})$, then for fixed $n < \infty$ the probability that $\mathbf{x}$ is in the box $(\mathbf{a}, \mathbf{b}) = (a_1, b_1) \times \ldots \times (a_n, b_n) \subset \mathbf{R}^n$ is equal to the integral of $n$-dimensional Gaussian measure over this box. In symbols, what we wish to prove is:

$$\lim_{N\to\infty} P(\cap_{i=1}^n (a_i \leq x_i \leq b_i)) = \prod_{i=1}^n \int_{a_i}^{b_i} \frac{e^{-x_i^2/2}}{\sqrt{2\pi}} dx_i.$$

The probablity that $\mathbf{x}$ is in $(\mathbf{a}, \mathbf{b})$ is given by the integral of the area form $dA$ induced by the volume form on $\mathbf{R}^{N+1}$ over pieces of the sphere that lie 'above' or 'below' $(\mathbf{a}, \mathbf{b})$, divided by the total measure of $\mathbf{S}^N$, *i.e.* the integral of $dA$ over $\mathbf{S}^N$.

The form of the integral is given in three dimensions by parameterizing by the azimuthal angle $\varphi$, so if $R = \sqrt{N}$, then $x_3 = R\cos\varphi$, and $dx_3 = R\sin\varphi\, d\varphi$, which by trigonometry is $\sqrt{R^2 - x_3^2}$. We also have $dA = 2\pi R \sin\varphi\, d\varphi = 2\pi\sqrt{R^2 - x_3^2}\, d\varphi$, which is therefore just $2\pi\, dx_3$. We rotate so that $x = x_3$, so the area form is proportional to $(1 - \frac{x^2}{N})^0$.

By an analogous procedure in higher dimensions the probability that $\mathbf{x}$ lies above or below an interval $(a, b)$ (that is, the probability that one of the dimesions $x$ of $\mathbf{x}$ is in the range of values for which $\mathbf{x}$ might be in the box $(\mathbf{a}, \mathbf{b})$ that we specified, not taking into account what is going on in the other dimensions) works out to be

$$\frac{\int_a^b (\sqrt{N}^2 - x^2)^{(N-3)/2}\, dx}{\int_{-\sqrt{N}}^{\sqrt{N}} (\sqrt{N}^2 - x^2)^{(N-3)/2}\, dx} = \frac{\int_a^b (1 - x^2/N)^{(N-3)/2}\, dx}{\int_{-\sqrt{N}}^{\sqrt{N}} (1 - x^2/N)^{(N-3)/2}\, dx}. \tag{5.1}$$

Recall that $e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n$, so $e^{-x^2} = \lim_{n\to\infty}(1 - \frac{x^2/2}{n})^n = \lim_{n\to\infty}(1 - \frac{x^2}{n})^{n/2}$. It is easy to check that this is $\lim_{n\to\infty}(1 - \frac{x^2}{n})^{(n-3)/2}$. Therefore as $n \to \infty$, the ratio in Equation 5.1 approximates $\int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$. The linearity of the integral now gives the stated form for the integral over a box. QEF

**Proposition 5.2.** The $(n-1)$-dimensional measure of a hyperplane that divides $\mathbf{G}^n$ into two pieces of any given ratio is the same for all $n$.

*Proof.* To prove this, we need to use the basic fact that an $(n-1)$-hyperplane has an $(n-1)$-dimensional Gaussian on it, whose total measure depends only on its distance from the origin and not on the dimension. This is a consequence of the fact that up to normalization, $\mathbf{G}^n = \mathbf{G}^{n-1} \times \mathbf{G}^1$. For any given volume ratio and any $n > 1$, a hyperplane $P$ whose distance from the origin is the same as the distance from the origin of a point $p$ in $\mathbf{G}^1$ that cuts $\mathbf{G}^1$ into pieces with the given ratio will cut each perpendicular hyperplane $H$ into two pieces of the given ratio, and so it cuts $\mathbf{G}^n$ into two pieces of the given ratio. Since the total measure of this $P$ depends only on its distance from the origin, this proves the claim. QEF

**Corollary 5.1.** The $(n-1)$-dimensional measure of three half-hyperplanes that divides $\mathbf{G}^n$ into three pieces of equal volume is the same for all $n > 2$.

*Proof.* Three half-hyperplanes through the origin have total $(n-1)$-dimensional measure $\frac{1}{\sqrt{2\pi}}\frac{3}{2}$. QEF

## 5.3 Area-minimizing division of $\mathbf{G}^n$

We now introduce some notation. We let $I_{\mathbf{G}^n}$ denote the isoperimetric profile of $\mathbf{G}^n$. By Proposition 5.2, the surface area of a standard single bubble is given by the same function in $\mathbf{G}^n$ and $\mathbf{G}^1$, and can be written down (fairly) explicitly: $I_{\mathbf{G}^n}(v) = I_{\mathbf{G}^1}(v) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, where $x$ is defined by $\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt = v$. We let $I\!I_{\mathbf{G}^n}$ denote the two-volume least-area function of $\mathbf{G}^n$.

We will denote the isoperimetric profile and two-volume least-area function of $\mathbf{S}^m$ by $I_{\mathbf{S}^m}$ and $I\!I_{\mathbf{S}^m}$ respectively. These can be thought of classically or by using a Hausdorff measure construction. Viewed the latter way, we can the realize the functions $I_{\mathbf{G}^n}$ and $I\!I_{\mathbf{G}^n}$ as projections of limits of $I_{\mathbf{S}^m}$ and $I_{\mathbf{S}^m}$.

Hutchings theory [22] says that if

$$F(v,w) = 2I_{\mathbf{S}^m}(v/2) + I_{\mathbf{S}^m}(w) + I_{\mathbf{S}^m}(v+w) - 2\Pi_{\mathbf{S}^m}(v,w) > 0 \tag{5.2}$$

then the region containing the volume $v$ in an area-minimizing double bubble in $\mathbf{S}^m$ that contains volumes $v$ and $w$ is connected. As discussed in the previous two chapters

$$\tilde{F}(v,w) = 2I_{\mathbf{S}^m}(v/2) + I_{\mathbf{S}^m}(w) + I_{\mathbf{S}^m}(v+w) - 2\widetilde{\Pi}_{\mathbf{S}^m}(v,w) > 0, \tag{5.3}$$

where $\widetilde{\Pi}_{\mathbf{S}^m}(v,w)$ denotes the area of a standard double bubble containing volumes $v$ and $w$, is also a condition for connectivity.

Taken together we now have enough tools to prove our theorem.

**Theorem 5.1.** For $n > 1$, the least-area division of $n$-dimensional Gauss Space $\mathbf{G}^n$ into three equal pieces consists of three hyperplanes, meeting at one hundred and twenty degrees along an $(n-2)$-dimensional plane through the origin.

*Proof.* In the case of three equal volumes in $\mathbf{S}^m(\sqrt{m})$, with $v = |\mathbf{S}^m(\sqrt{m})|$, the Hutchings criterion 5.3 becomes

$$2I_{\mathbf{S}^m}\left(\frac{v}{6}\right) + I_{\mathbf{S}^m}\left(\frac{v}{3}\right) + I_{\mathbf{S}^m}\left(\frac{2v}{3}\right) - 2\widetilde{\Pi}_{\mathbf{S}^m}\left(\frac{v}{3},\frac{v}{3}\right) > 0. \tag{5.4}$$

By symmetry, if this inequality holds, all of the regions in are connected.

By the considerations of Proposition 5.1, there is some $N$ such that if $m > N$ then

$$2I_{\mathbf{G}^n}\left(\frac{1}{6}\right) + I_{\mathbf{G}^n}\left(\frac{1}{3}\right) + I_{\mathbf{G}^n}\left(\frac{2}{3}\right) - 2\widetilde{\Pi}_{\mathbf{G}^n}\left(\frac{1}{3},\frac{1}{3}\right) > 0 \tag{5.5}$$

is a criterion for connectivity in $\mathbf{S}^m$.

By the considerations of Proposition 5.2, this last criterion is the same as

$$2I_{\mathbf{G}^1}\left(\frac{1}{6}\right) + I_{\mathbf{G}^1}\left(\frac{1}{3}\right) + I_{\mathbf{G}^1}\left(\frac{2}{3}\right) - 2\left(\frac{1}{\sqrt{2\pi}}\frac{3}{2}\right) > 0, \tag{5.6}$$

which proves that Inequality 5.5 establishes a lower bound that is uniform in $n$. By symmetry, Inequality 5.6 is equivalent to

$$2I_{\mathbf{G}^1}\left(\frac{1}{6}\right) + 2I_{\mathbf{G}^1}\left(\frac{1}{3}\right) - 2\left(\frac{1}{\sqrt{2\pi}}\frac{3}{2}\right) > 0. \tag{5.7}$$

Since

$$\int_{-100}^{-1} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \approx .15866 < .16666\ldots = 1/6, \tag{5.8}$$

we have that $\frac{1}{\sqrt{2\pi}} e^{-(-1)^2/2} = \sqrt{\frac{e}{2\pi}} \approx .657745$ is a lower bound on $I_{\mathbf{G}^1}(1/6)$, and similarly since

$$\int_{-100}^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \approx .30854 < .33333\ldots = 1/3, \tag{5.9}$$

we have that $\frac{1}{\sqrt{2\pi}} e^{-(-1/2)^2/2} \approx .352065$ is a lower bound on $I_{\mathbf{G}^1}(1/3)$.

Combining these approximations we get

$$2I_{\mathbf{G}^1}\left(\frac{1}{6}\right) + 2I_{\mathbf{G}^1}\left(\frac{1}{3}\right) > 2 \cdot .65775 + 2 \cdot .35207 \approx 2.01964 > .7978 \approx 2\left(\frac{1}{\sqrt{2\pi}}\frac{3}{2}\right), \tag{5.10}$$

which proves that the Hutchings connectivity criterion is satisfied for equal volumes for spheres close enough to the Poincaré limit.

An argument like that used in $\mathbf{R}^3$ by Hutchings *et al.* proves that the minimizing enclosure for three equal volumes in these high-dimensional spheres is the standard double bubble; specifically, Cotton and Freeman ([10], or see Chapter 3) prove that in $\mathbf{S}^n$, an area minimizing double bubble that has both regions and the exterior connected must be standard.

By rotating and translating in $\mathbf{R}^{m+1}$ as necessary, we can recenter so that the origin is contained in determined by the $(m-2)$-dimensional singular set in the standard double bubble contains the origin. In addition we require that the first $n$ dimensions of $\mathbf{S}^m$ line up for all $m$. These are the dimensions which become $\mathbf{G}^n$ in the limit of projections. Since the standard double bubble in $\mathbf{S}^m$ is defined as a surface of revolution generated by the standard double bubble in $\mathbf{S}^{m-1}$, it is easy to see in the case $n = 2$ that the limit of the projections of the intersection of the standard double bubble with gthe first two dimensions of $\mathbf{S}^m$ the standard double bubble is the Y-shaped division of $\mathbf{G}^2$. This division is minimizing because it lifts to a minimizer in each of the spheres $\mathbf{S}^N$ as these spheres tend to the Poincaré limit. The same argument is in fact true when $n$ is greater than two, but it is easier to just observe that Y-shaped division of $\mathbf{G}^2$ lifts to the Y-shaped division of $\mathbf{G}^n$ and therefor the Y-shaped division of $\mathbf{G}^n$ must be minimizing. QEF

*Remark 5.1.* The proof shows that for large $N$, the least area division of $\mathbf{S}^N$ into three pieces of equal volume is given by a standard double bubble (in this case, three half-hyperspheres meeting along an $N-2$ sphere). The conjecture is that this is in fact the optimal division for all $N$.

# Chapter 6

# t2

**Charter** Characterize the perimeter minimizing double bubbles in the flat two-torus $\mathbf{T}^2$.

## 6.1 Introduction

Our Main Theorem (Section 6.7) shows that on the flat two-torus $\mathbf{T}^2$, the least-perimeter way to enclose and separate two prescribed areas is a double bubble of one of the six types of Figure 6.1: the standard planar double bubble $\Theta$, the band-lens $\Phi$, the symmetric two-component chain $\Omega$, the double band $\Xi$, the double hexagon $H$ (on the hexagonal torus only), or an octagon-square tiling $X$. We conjecture that the last of these never occurs as a minimizer.

**Conjecture 6.1.** The octagon-square tiling $X$ is never perimeter minimizing.

If Conjecture 6.1 is correct, Figure 6.2 shows which types prevail for which prescribed areas for four different tori.

The proof of the main theorem separates all candidates into five classes, and shows that the special double bubbles of Figure 6.1 are the only ones that are potentially perimeter minimizing: Propo-

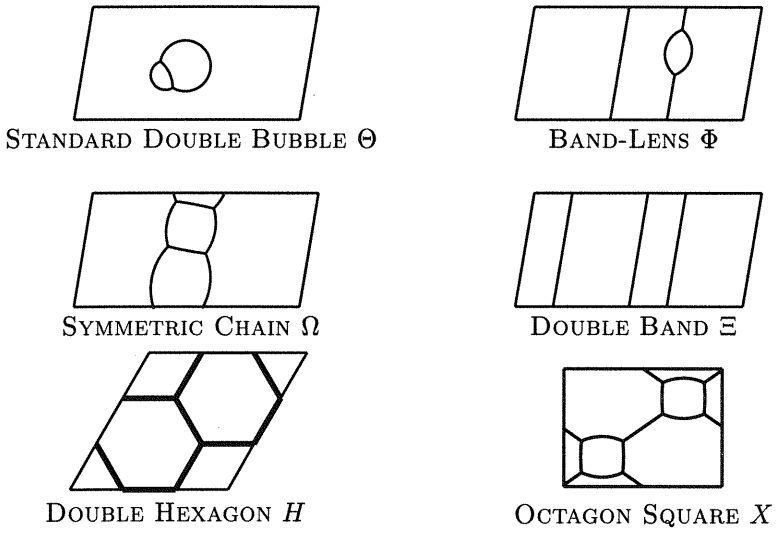



Figure 6.1: The six candidate perimeter- minimizing double bubbles on the flat two-torus.

sition 6.6 says that minimizers must be the double band or belong two one of four other topological classes; section 6.5.1 treats double bubbles in which all of the regions are contractible; section 6.5.2 treats double bubbles whose components taken together wrap around the torus, but whose components are all contractible; and section 6.5.3 treats double bubbles with one non-contractible region. The fourth class includes double bubbles that wrap around the torus in both directions. We call these candidates "tilings." Proposition 6.12 and 6.13 use the Euler characteristic of $\mathbf{T}^2$ and some basic facts about the regularity of minimizing bubbles to show that only two tilings are potential minimizers: the hexagon tiling $H$ on the hexagonal torus, and (perhaps) an octagon-square tiling. When it occurs as a minimizer, $H$ ties the double band $\Xi$ for perimeter. Section 6.6 gives formulas for the perimeter and area of the four finalists. Figure 6.2 gives a computational plot of the minimizing type for given areas.

We have been unable to prove Conjecture 6.1, which would eliminate the octagon-square and thus complete our analysis of tilings. Some tests using *Surface Evolver* suggest that deflating one of the separated bubble and expanding the other will reduce perimeter.

## 6.2 Definitions

Here we rigorously define, among other things, the classes of potential minimizers mentioned above. Thus, at this point it is necessary to introduce a convention that we use throughout the paper. Any double bubble partitions the space into three regions, but in naming and in speaking about these configurations, we will usually only look at two of these regions, $R_1$ and $R_2$ and not the third, which we call the *exterior*. So for example, $\Xi$, which might be otherwise be named the "triple band", is the double band. We should remark that on the torus the exterior need not be the largest of the three regions.

### 6.2.1 Tori and the Cylinder

**Definition 6.1.** A *flat two-dimensional torus* can be represented by a planar parallelogram with opposite sides identified, for which the shortest side is a shortest closed geodesic, and no interior angle is less than sixty degrees. In this paper we normalize so that the shortest closed geodesic has unit length. This also implies that the shortest side of the torus has length 1. We call the length of the other side $L$. If $L = 1$, there are two short directions, and on the hexagonal torus (see below) there are three short directions, one corresponding to the geodesic with homology $(-1, 1)$ .

**Definition 6.2.** The *hexagonal torus* can be represented either by a regular hexagon with opposite sides identified, or by a 60-degree rhombus with opposite sides identified.

**Definition 6.3.** The *flat infinite cylinder* can be represented as the surface contained between two identified infinite parallel lines in the plane. We take the distance between the two lines to be one.

### 6.2.2 Topological Definitions

**Definition 6.4.** A *double bubble* consists of two disjoint regions bounded by piecewise smooth curves, each of which may have multiple components.

**Definition 6.5.** A closed curve has *homology* $(p, q)$ if it wraps around the torus $p$ times in the short direction and $q$ times in the long direction, where $p$ and $q$ are signed to signify the orientation of the curve.

**Definition 6.6.** A *tiling* is a double bubble in which every component is contractible, including components of the exterior.

**Definition 6.7.** A *band* $\Pi$ is a non-contractible annulus. We often refer to the homology of a band, meaning the homology of one of its boundary components.

**Definition 6.8.** A *swath* $\Psi$ is a set of contractible components with non-contractible and connected closure $\overline{\Psi}$ and with non-contractible complement.

**Definition 6.9.** A *chain* is a non-contractible cycle of contractible components. Note that every swath contains at least one chain and that every chain is itself a swath. We often refer to the homology of the closure of the union of a chain, meaning the homology of one of its boundary components.

**Definition 6.10.** A *hexagon tiling* $\Upsilon$ is a tiling in which every component is a curvilinear hexagon.

**Definition 6.11.** An *octagon-square tiling* is a four-component tiling consisting of two curvilinear octagons and two curvilinear quadrilaterals. The two quadrilaterals are components of the same region.

### 6.2.3 Geometric Definitions

**Definition 6.12.** The *standard double bubble* $\Theta$ is comprised of three circular arcs meeting in threes at 120°, such that the curvature of the separating arc is the difference of the curvatures of the outer caps. For prescribed areas, $\Theta$ is unique up to congruence [34, Proposition 14.1].

**Definition 6.13.** The *double band* $\Xi$ is the union of two adjacent bands, each of which is bounded by two short geodesics.

**Definition 6.14.** A *symmetric chain* is a chain that is symmetric about a geodesic circle.

**Definition 6.15.** A *standard 2-component chain* $\Omega$ is comprised of six circular arcs meeting in threes at four common vertices at 120°, such that the curvature of each separating arc is the difference of the curvatures of the outer arcs (see Figure 6.1). Further, by Lemma 6.2 a minimizing standard 2-component symmetric chain has non-zero homology only in a shortest direction of the torus, and by Lemma 6.3 any such standard 2-component symmetric chain is unique up to isometries of the torus.

**Definition 6.16.** A *lens* is a pair of congruent circular arcs meeting at two vertices at 120°.

**Definition 6.17.** The *band lens* $\Phi$ is the union of a band and a lens, such that the boundary curves of the resulting configuration meet in threes at two vertices, at 120°. Further, by Proposition 6.6, a minimizing band-lens has non-zero homology only in a shortest direction of the torus and encloses the smallest area in the lens. By Lemma 6.19 any immersed band-lens is unique up to isometries of the torus.

**Definition 6.18.** The *hexagon tiling H* is a 3-component straight-edge hexagonal tiling on the hexagonal torus.

## 6.3 Existence, Regularity, and Basic Properties

This section provides existence and regularity (Proposition 6.1), a component bound (Proposition 6.2), and some easy consequences essential to all of our analysis.

**Proposition 6.1 (Existence and Regularity Theorem [32, 2.3 and 2.4] ).** In a smooth Riemannian surface S with compact quotient S/$\Gamma$ by the isometry group $\Gamma$, for any two areas $A_1$ and $A_2$ (whose sum is less than area(M)), there exists a least-perimeter enclosure of the two areas. This enclosure consists of smooth constant-curvature curves meeting in threes at 120° angles. All curves separating a specific pair of regions have the same curvature.

*Remark 6.1.* It follows that minimizing bubbles are connected, and that each component of either of the two regions or the exterior is adjacent to both others. Also, note that a double bubble satisfying regularity is in equilibrium, which means that the "first derivative" of its perimeter under any variation fixing areas is zero.

**Proposition 6.2 (Wichiramala Component Bound [38, Proposition 3.1] ).** Unless all three regions have equal pressure, each region of highest pressure has at most two components.

*Proof.* The topology of the torus does not interfere with the Wichiramala proof, as all variations used in his argument are strictly local. QEF

**Proposition 6.3.** The perimeter of a perimeter minimizing double bubble is at most three.

*Proof.* Three geodesic circles can enclose any pair of prescribed areas $A_1$, $A_2$. QEF

**Proposition 6.4.** No perimeter-minimizing tiling can occur on a torus with unit shortest side length and longest side length greater than $\sqrt{3}$.

*Proof.* Consider a torus with a tiling on it. Now, at each coordinate in the long direction of the torus, consider the unique non-contractible shortest closed geodesic associated with that coordinate. This shortest closed geodesic must intersect at least one curve at some point, otherwise the configuration could not be a tiling. Further, it must intersect some other or potentially the same curve in some other point, otherwise two components of the same region would be bordering themselves and the configuration could not be a minimizer. Now, considering the sum over all such shortest closed geodesics, we see that the configuration must span the long direction of the torus at least twice. Using the co-area formula, the configuration must have perimeter at least $2 \cdot L \cdot \sin(\theta) \geq \sqrt{3} \cdot L$. But by Proposition 6.3, $P \leq 3$, so $L \leq \sqrt{3}$. QEF

## 6.4 Topological Classification of Minimizers

In this section we build up several lemmas to prove two important propositions. Proposition 6.5 says that if a perimeter minimizing double bubble is not a tiling, and both regions and the exterior all have equal pressure, then the configuration is the double band. This is important in later sections. Proposition 6.6 is the backbone for the entire proof. It shows that all minimizing double bubbles must either be a tiling, a contractible double bubble, a swath, a band with some contractible blobs, or the double band. The next section and each of its subsections deals with the topological classes developed in this proposition.

**Lemma 6.1.** In a partition of the torus into piecewise smooth non-contractible components, each component is a band.

*Proof.* Every closed curve in the boundary must be non-contractible. Hence the boundary must consist of disjoint closed curves, all in the same non-trivial homology class. It follows that each component is a band. QEF

**Lemma 6.2.** Any non-contractible component, $A$, in a minimizing double bubble on the torus must either be a band or have contractible complement.

*Proof.* If the complement of $A$ is contractible, we are done. Suppose that the complement of $A$ is not contractible. If there are any contractible components of the complement inside of $A$, merge them with $A$. Now we have a partition of the torus into two non-contractible regions, and by Lemma 6.1, $A$ must be a band or a punctured band (since we filled the holes in $A$). But since A is part of a minimizing double bubble, by Remark 6.1 $A$ must be a band. QEF

**Lemma 6.3.** Given a chain or band that is part of a minimizing double bubble, and with homology $(p, q)$, $|p| \leq 1$ and $|q| \leq 1$.

*Proof.* Since a band and the closure of the union of a chain are simply annuli, each chain or band has

at least two disjoint, non-contractible boundary curves. Thus, if $|p| > 1$ or $|q| > 1$ then the perimeter is at least four. By Proposition 6.3 the double bubble cannot be a minimizer. QEF

**Lemma 6.4.** A perimeter minimizing double bubble on the torus, together with its exterior, has at most three disjoint bands or chains. Further, if there are three, then they must all be bands, and the double bubble must be the double band.

*Proof.* Since a band and the closure of the union of a chain are non-contractible annuli, each chain or band has at least two non-contractible boundary curves, each of length at least one. Since each such boundary curve borders at most two such annuli, if there are $n$ annuli, there will be at least $n$ such non-contractible boundary curves. Thus, by Proposition 6.3, $n \leq 3$. Further, if $n = 3$, the perimeter of the double bubble is greater than three unless each of the non-contractible boundary curves is a geodesic in the short direction of the torus and if there is no other boundary, so that the double bubble must be the double band.

QEF

**Proposition 6.5.** A perimeter minimizing double bubble of three equal pressures is either a tiling or the double band.

*Proof.* Assume that the double bubble is not a tiling, so that there is at least one non-contractible component. By Lemma 6.2, this component is either a band, or its complement is contractible. The latter is impossible, for it would be a planar polygon with interior angles of 240 degrees.

Thus, the configuration must contain at least one band, bounded by piecewise geodesic curves with interior angles of 120 degrees. Lifting to the plane shows that the boundary curves must be smooth closed geodesics.

Now, denote the rest of the boundary by $X$. If $X$ is contractible, merely slide $X$ up against the band to contradict regularity. If $X$ is non-contractible, then the double bubble plus its exterior must have at least three bands. By Lemma 6.4, the double bubble must be the double band.

QEF

**Proposition 6.6.** All minimizing double bubbles on the torus must be either a contractible double bubble, a tiling, a swath, a band adjacent to a set of components whose union is contractible, or the double band (after perhaps relabelling the two regions and the exterior).

*Proof.* We may assume that every non-contractible component is a band, since otherwise by Lemma 6.2 it is the complement of a contractible double bubble.

If the double bubble plus its exterior has no bands, then all regions are contractible, and thus by definition the double bubble is a tiling.

If the double bubble plus its exterior has one band, then the complement is a set of contractible components. But the closure of the union of this set of components, being the complement of a band, is itself a band, and thus non-contractible. Thus, by definition the double bubble is a swath.

If the double bubble plus its exterior has two bands, consider the complement, C, of the closure of the union of the two bands. If C is non-contractible, then we have a partition of the torus into non-contractible regions, and thus, by Lemma 6.1 this complement is also a band. But since the two bands are the only non-contractible components, we must have a configuration with two bands and a swath. But the swath contains at least one chain, and so the double bubble plus its exterior contains one chain and two bands, which impossible by Proposition 6.4. So the complement of the closure of the union of the two bands must be contractible. And thus the double bubble must be a band adjacent to a set of components whose union is contractible

If the double bubble plus its exterior has three or more bands, then by Proposition 6.4, the double bubble must be the double band.

QEF

## 6.5 The Four Classes with Contractible Components

By Proposition 6.6, all minimizing double bubbles on the torus must be either a contractible double bubble, a tiling, a swath, a band adjacent to a set of components whose union is contractible, or the double band. While the double band describes a specific geometric configuration, each of the remaining possibilities is merely a topological class of configuration. In this section we identify the minimizer in each of these four remaining topological classes: the standard double bubble among contractible double bubbles (Proposition 6.1), a two-component symmetric chain among swaths (Proposition 6.10), and the band-lens among bubbles with a single band (Proposition 6.11) Proposition 6.12 and 6.13 use the Euler characteristic of $\mathbf{T}^2$ and some basic facts about the regularity of minimizing bubbles to show that only two tilings are potential minimizers: the hexagon tiling $H$ on the hexagonal torus, and (perhaps) an octagon-square tiling.

### 6.5.1 Contractible Double Bubbles

Proposition 6.1 uses the result from $\mathbf{R}^2$ that perimeter-minimizing double bubbles are standard [24], to show that the only potential minimizer among contractible double bubbles is the standard double bubble.

**Lemma 6.1.** The perimeter of a standard double bubble is greater than Pi times its diameter: $P > \pi D$.

*Proof.* We obtain a formula for $P/D$ as a function $f(\theta)$, where $\theta$ is the angle between the interior arc of the standard double bubble and the chord that connects its endpoints. Then we show $f(\theta) > \pi$ for the entire domain $0 < \theta \leq \pi/3$.

Let $r_1, r_2, c_1, c_2$ be the radii and centers of the longer and shorter exterior arcs of the standard double bubble, respectively. Then the diameter is the sum of $r_1$, $r_2$, and the distance between $c_1$ and

$c_2$:

$$D = r_1 + r_2 + r_1 \cos(\pi/3 - \theta) + r_2 \cos(\pi/3 + \theta).$$

Applying the formula for radii in Section 6.6.1, we obtain a expression that can be solved for $C$, the length of the separating chord:

$$D = C((1 + \cos(\pi/3 - \theta))/(2\sin(2\pi/3 + \theta)) + (1 + \cos(\pi/3 + \theta))/(2\sin(2\pi/3 - \theta))).$$

Substituting for $C$ in the equation for perimeter in Section 6.6.1, we obtain the desired function in $\theta$:

$$f(\theta) = P/D = (8\pi \sin\theta\cos\theta + 3\sqrt{3}\theta)/(6\sin\theta\cos\theta + 3\sin\theta)$$

.

To show $f(\theta) \geq \pi$ is equivalent to showing

$$8\pi \sin\theta\cos\theta + 3\sqrt{3}\theta \geq \pi(6\sin\theta\cos\theta + 3\sin\theta).$$

Moving the terms to the left hand side and using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$, we obtain the equivalent inequality

$$\pi\sin(2\theta) + 3\sqrt{3}\theta - 3\pi\sin\theta > 0. \qquad (*)$$

We view the left hand side $L$ as a function of $\theta$; its second derivative is

$$-4\pi\sin(2\theta) + 3\pi\sin\theta = \pi\sin\theta(-8\cos\theta + 3).$$

For $\theta \in (0, \pi/3)$, we have $\sin\theta > 0$ while $\cos\theta > \cos(\pi/3) > 3/8$. Hence, this last expression is strictly negative, and the left hand side $L$ of $(*)$ is strictly concave for $\theta \in [0, \pi/3]$. Because $L$ equals 0 when $\theta$ is at the endpoints of this interval, it must be positive along the interior of this interval, as desired.

QEF

**Proposition 6.1 (Generalization of [28]).** The only possible minimizer among contractible double bubbles is the standard double bubble.

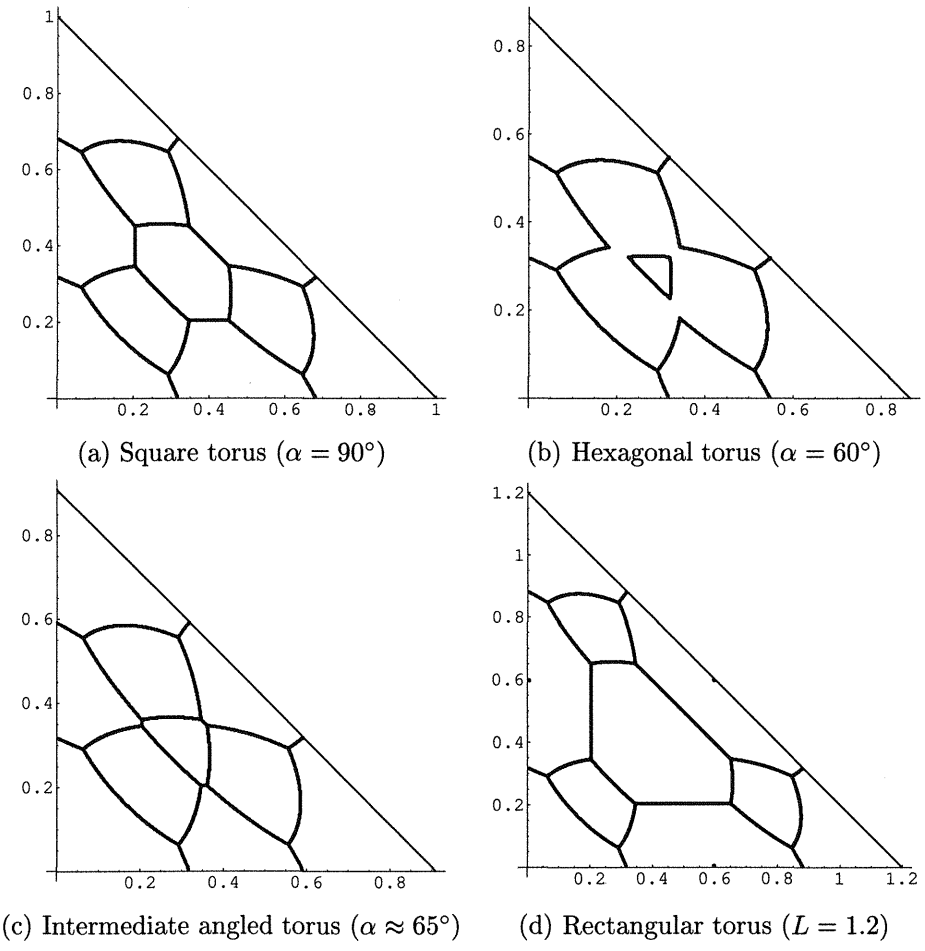
*Proof.* Assume that there is a perimeter minimizing contractible double bubble $\Sigma$ enclosing areas $A_1$ and $A_2$ that is not the standard double bubble. By [24], the unique perimeter-minimizing solution in $\mathbf{R}^2$ for the same prescribed areas is a standard double bubble $\Theta$. If $\Theta$ fits on the torus, it will beat $\Sigma$, since it beats $\Sigma$ in the plane. Therefore $\Theta$ does not fit on the torus.

But since $\Theta$ does not fit on the torus, then its diameter must be greater than the shortest side length of the torus (otherwise, it would fit with the axis of symmetry in any direction). But, by Lemma 6.1, $P > \pi \cdot D > 3$, and thus by Proposition 6.3, $\Theta$ is not a minimizer.

QEF

### 6.5.2 Swaths

By Proposition 6.5, minimizing swaths must be chains consisting of two or four components. By Proposition 6.8, asymmetric chains cannot be minimizers since they have perimeter greater than three. Finally, by Proposition 6.9, minimizing symmetric chains can have only two components.




(a) Square torus ($\alpha = 90°$)

(b) Hexagonal torus ($\alpha = 60°$)

(c) Intermediate angled torus ($\alpha \approx 65°$)

(d) Rectangular torus ($L = 1.2$)

***Square Torus.*** *Twenty-one smooth-looking curves divide the space into ten regions. The approximate locations of the four vertices in the lower left corner are* $\approx (0.34596, 0.20352)$, $\approx (0.20352, 0.34596)$, $\approx (0.06, 0.29)$, *and* $\approx (0.29, 0.06)$. *The two curves parallel to the axes are* $A_1$ *and* $A_2 = (\frac{8\pi}{3} - 2\sqrt{3})^{-1} \approx 0.2035$, *respectively. The curves that touch the axes in the lower left corner touch at* $\frac{1}{\pi} \approx 0.3183$.

***Hexagonal Torus.*** *Eighteen smooth-looking curves divide the space into eight regions.*

Figure 6.2: How minimizing double bubble types depend on the prescribed areas for four flat tori

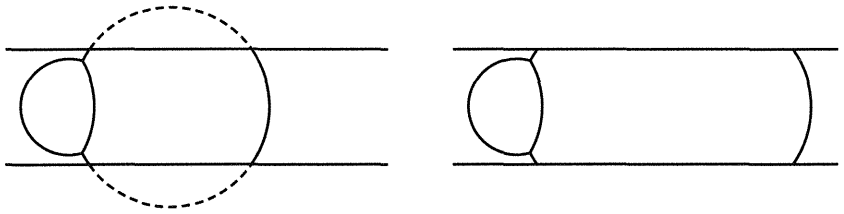


Figure 6.3: On the cylinder, we can make simple adjustments when the standard double bubble does not fit; the same argument does not work on the torus.

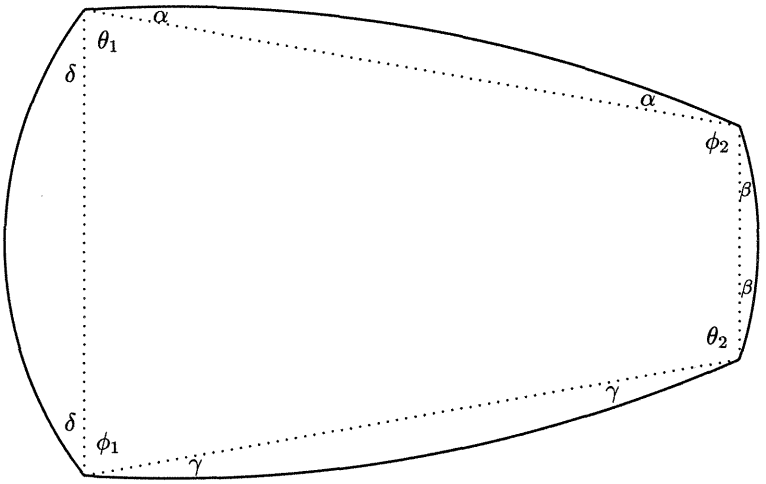



Figure 6.4: Important angles in a regularity satisfying quadrilateral

**Lemma 6.2.** If a component of a region is a curvilinear quadrilateral, then the quadrilateral is cyclic (i.e., the vertices lie on a common circle).

*Proof.* We must show that opposite angles sum to 180°. Consider the chords joining consecutive vertices as in Figure 6.4.

Since each boundary curve must meet its corresponding chord at equal angles $\alpha, \beta, \gamma, \delta$ and these curves meet each other at 120° angles, we have the following set of equations:

$$2(\alpha + \beta + \gamma + \delta) + \theta_1 + \theta_2 + \varphi_1 + \varphi_2 = 480°;$$

$$\theta_1 + \theta_2 + \varphi_1 + \varphi_2 = 360°.$$

From this we easily get $\alpha + \beta + \gamma + \delta = 60°$. Summing opposite angles of the component yields

$$\alpha + \beta + \gamma + \delta + \theta_1 + \theta_2 = 240°;$$

$$\alpha + \beta + \gamma + \delta + \varphi_1 + \varphi_2 = 240°.$$

Whence, we get $\theta_1 + \theta_2 = \varphi_1 + \varphi_2 = 180°$ QEF

*Remark 6.2.* The center of the circle is of course where the two bisectors of any two nonparallel sides intersect. The following Lemma shows that Figure 6.4 describes the general case.

**Lemma 6.3.** The vertices of a curvilinear quadrilateral component of a regularity satisfying planar double bubble form an isosceles trapezoid. Furthermore, if the trapezoid is not a rectangle, then the two boundary components corresponding to the parallel sides of the trapezoid lie on the same circle.

*Proof.* We reflect across the line of symmetry of each arc to see that the component must be an isosceles trapezoid.

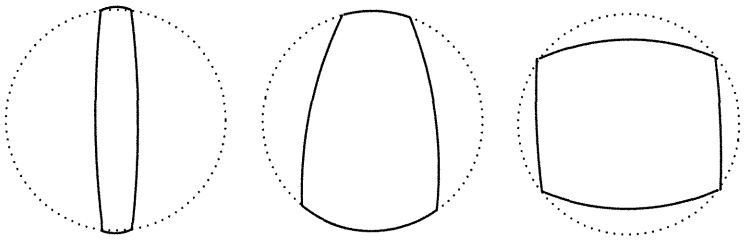


Figure 6.5: Quadrilateral components

Suppose it is not a rectangle. Since the arcs corresponding to the two non-parallel sides make a 120° angle with the longer of the two parallel sides, they will also make a 120° interior angle with the completion of this arc to a circle. Since the component is not rectangular, any arc other than these two with the correct curvature which makes a 120° angle with one of the non-parallel arcs will not make a 120° angle with the other. We therefore have the desired result.

QEF

*Remark 6.3.* In the previous two lemmas it is actually enough to have two parallel sides be circular arcs of the same curvature, the other two boundary curves can then be arbitrary smooth curves. This is relevant for example in Figure 2 of [41].

**Proposition 6.2.** In a regularity satisfying double bubble, then arcs $\lambda_1$ and $\lambda_2$ connecting to one of the two parallel sides of a quadrilateral component that is not a rectangle lie on the same circle.

*Proof.* Consider the digon obtained by replacing $\mu_2$ with the completion of the arc containing $\sigma_1$ and $\sigma_2$. This gives us two boundary components of some standard double bubble. Since pressures add, we can complete this double bubble with the correct circular piece. Since, locally, the original quadrilateral component looks like the standard double bubble, the boundary curves connecting to it lie on the same circle. QEF

**Proposition 6.3.** Unless both regions and the exterior all have equal pressure, every component

of each lowest pressure region of a regularity satisfying double bubble that is not a tiling is non-contractible.

*Proof.* Suppose a component of the lowest pressure region is contractible. Then it is must be a curvilinear $n$-gon, where $n$ is strictly greater than six, since each edge of a lowest pressure region must either be straight or bow into the region, and not all the edges are straight, leaving room for 120°angles only if there are more than six edges. By regularity, consecutive sides must border different regions, so $n$ must be even.

By Proposition 6.2, a region of highest pressure, which we will call $R_1$, has at most two components. Since every component of a highest pressure region can have at most four vertices, $R_1$ can have at most eight vertices.

Therefore, there is exactly one lowest pressure region, and it has exactly eight edges, otherwise $R_1$ would have too many vertices.

Furthermore, we note that none of the components of $R_1$ can be digons, since this would leave six vertices for the other component of $R_1$, which as we have noted is too many. Thus, there are two quadrilateral components of $R_1$ each bounded by a pair of sides on the octagon.

By the results of the previous subsection, the double bubble can not be contractible. Therefore at least one of the quadrilateral components must wrap around the torus. If only one of the quadrilateral components wraps around the torus, we get the absurd configuration in Figure 6 (a), in which two pieces of the quadrilateral that does not wrap are disconnected from one another. If both of the components wrap around the torus, either they wrap around different sides, in which case the configuration is a tiling (the octagon square actually – but at any rate, not addressed in this proposition) or we have a configuration like that of Figure 6 (b).

As illustrated, each of the quadrilaterals must admit the inscription of a rectangle on their vertices, otherwise by Lemma 6.3 the sides which do not border the low pressure octagon would lie on the same circle, and by Proposition 6.2 the two adjacent edges of the low pressure octagon would



A. B.

Figure 6.6: Two absurd low pressure components

have to lie on a circle as well, which is ridiculous.

It now follows from a standard fact about parallel lines and transversals, that the edge of the low pressure octagon that joins the two edges shared with the quadrialteral must meet one of these edges at an acute angle. But this contradicts regularity, proving the proposition.

QEF

**Proposition 6.4.** In a perimeter minimizing swath containing a chain $K$, no component of $K$ is a digon.

*Proof.* Suppose $K$ contains a digon $D$. Since $D$ is adjacent to only two components, if it occurs in $K$, it must be adjacent to components of the same region. This requires either that four arcs meet at a point, or that some region is separated from itself by arcs which could be removed to reduce perimeter. In either case, the original configuraion can not have been minimizing, proving the claim. QEF

**Proposition 6.5.** Minimizing swaths must be chains with a total of two or four components from two regions.

*Proof.* By Proposition 6.5 the three regions in a minimizing swath $\Psi$ do not have the same pressure. Furthermore, by Proposition 6.3 a least pressure region does not appear in the swath. Also, the least pressure region is unique since a swath contains components from two regions. Consider a chain $K$ in $\Psi$. The components in $K$ must alternate between two higher pressure regions $R_1$ and $R_2$. By Proposition 6.2 the region of highest pressure, $R_1$, has at most two components, so $K$ has exactly 2 or 4 total components.

Suppose $K$ has four components. Then any additional components of the $R_2$ in the swath have to be adjacent to components of $R_1$ in the chain (since by definition, the union of swath components is connected). However, this is impossible since components of $R_1$ already have four vertices and a convex non-polygonal component has at most four vertices. Thus, a swath containing a four component chain must be a four-component chain.

Next, suppose $K$ has two components. Further, suppose the swath contains additional components. Then there is some component adjacent to the chain (otherwise the closure of the swath is disconnected). As before, there can be no extra components adjacent to the component of $R_1$ in $K$. So, there is a component of $R_1$ adjacent to the component of $R_2$ in $K$. If this is the only other component in the swath, then it is a digon which can be eliminated by sliding it along $\partial R_2$



Figure 6.7: Extra digons do not attach to the symmetric chain

until it touches another component or a vertex creating an illegal singularity (see Figure 6.7). If, instead, there is another component of $R_2$ attached to the second component of $R_1$, then it is either a digon or it is attached to another component of $R_1$. In the case of the former, we can eliminate this configuration using the same sliding technique. The latter case is impossible since, by Proposition 6.2 it would have to be attached to an existing quadrilateral component of $R_1$. This contradicts our assumption that the swath contains additional components; therefore, a minimizing swath is a chain with two or four components. QEF

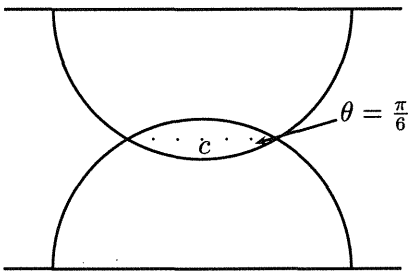



Figure 6.8: Chain violates regularity.

**Proposition 6.6.** In a symmetric 2-component chain where the arcs separating each component from the exterior lie on the same circle, the total perimeter of the chain is strictly greater than 3.

*Proof.* In a cylinder with geodesic circle of length 1, we can parameterize a symmetric chain composed of two components, by two parameters: the length $C$ of a chord connecting vertices shared by both components of the chain, and the angle $\theta$ between this chord and the boundary curve con-

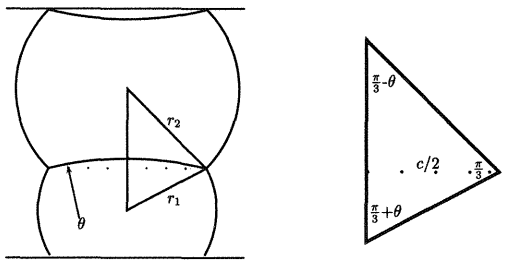



Figure 6.9: Chain constraints.

necting the same vertices. Here $\theta$ must range between 0 and $\frac{\pi}{6}$ radians, since at $\theta = \frac{\pi}{6}$, the chain violates regularity (see Figure 6.8).

At an endpoint of $C$ we draw the radii $r_1$, $r_2$ of the two exterior arcs of the chain that meet at this point (See Figure 6.9). Connecting the opposite endpoints of these radii (the centers of the circular arcs), we have a set of triangles from which we can compute a value of $C$ for a given $\theta$.

Using the law of sines in Figure 6.9, we discover that $\frac{\sin\frac{\pi}{3}}{\frac{1}{2}} = \frac{\sin(\frac{\pi}{3}-\theta)}{r_1}$. Since, $\sin(\frac{\pi}{3}+\theta) = \frac{\frac{C}{2}}{r_1}$, we get

$$\frac{\sin(\frac{\pi}{3}-\theta)}{\sqrt{3}}\sin(\frac{\pi}{3}+\theta) = \frac{C}{2}$$

Further simplification yields:

$$C = \frac{\sqrt{3}}{2} - \frac{2}{\sqrt{3}}\sin^2\theta$$

We can obtain a lower bound on the perimeter of the chain by approximating the length of each arc by its corresponding chord. This length is exactly $2C + 2$. Since $C$ is decreasing in $\theta$ and $C(\frac{\pi}{6}) = \frac{1}{\sqrt{3}} \approx .577$, we conclude that perimeter is greater than 3. By scaling, we get the result for arbitrary cylinders. QEF

**Proposition 6.7.** For any symmetric chain $\Sigma$ consisting of four components, there exists a two-component chain with equal perimeter that is similar to an adjacent pair of components in $\Sigma$.

*Proof.* Since alternating components of a 4 component symmetric chain are congruent (this comes from symmetry about a line perpendicular to the geodesic circle), we construct a two-component chain by taking adjacent components of $\Sigma$ and scaling up by a factor of 2. The perimeter remains constant since we doubled half of the original perimeter of $\Sigma$. Also, by construction, the chain is similar to $\Sigma$. QEF

**Proposition 6.8.** All asymmetric chains have perimeter greater than 3.

*Proof.* By Lemma 6.3, at least one component of an asymmetric chain must have vertices that form a non-rectangular isosceles trapezoid, and the arcs separating this component from the exterior lie on the same circle. By Proposition 6.2, the same is true for the other components in the chain.

We can expand the cylinder continuously, rotating components as needed, until the chain fits symmetrically along a geodesic circle (see Figure 6.10) . By Propositions 6.6 and 6.7, the perimeter of the chain is greater than three times the side length of the expanded cylinder. Hence the perimeter of the chain is greater than 3, and by Proposition 6.3, we have the desired result. QEF

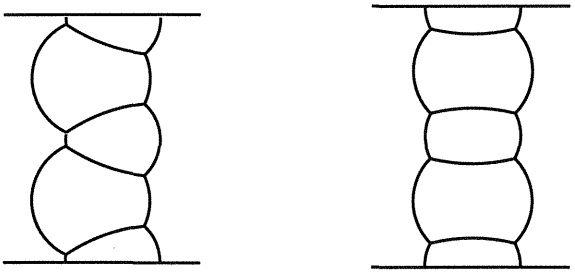


Figure 6.10: Rotate bubbles and stretch cylinder to make asymmetric chain symmetric.

**Proposition 6.9.** A four-component symmetric chain cannot be a minimizer.

*Proof.* Suppose that a four component chain $\Sigma$ is a minimizer for some pair of prescribed volumes. Let $R_1$ be the region of highest pressure. Since each pair of vertices shared by adjacent components is separated by the same horizontal distance, we can rearrange the components so that those belonging to the same region are adjacent (see Figure 6.11). Upon careful relabelling of the regions, area is maintained and we have violated regularity, a contradiction. QEF

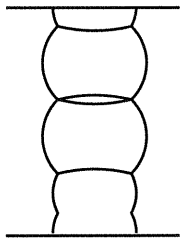


Figure 6.11: A regularity violating rearrangement of the components of a four component symmetric chain.

**Proposition 6.10.** The only potential minimizer among `sws` is a standard two-component symmetric chain.

*Proof.* By Proposition 6.5, minimizing swaths must be chains consisting of two or four components. By Proposition 6.8, asymmetric chains cannot be minimizers since they have perimeter greater than 3. Finally, by Proposition 6.9, minimizing symmetric chains can have only two components. QEF

**Conjecture 6.2.** A minimizing standard two-component symmetric has non-zero homology only in the short direction of the torus.

(This can be checked computationally.)

### 6.5.3 A Single Band Adjacent to a Set of Components with Contractible Union

Proposition 6.11 uses regularity and the results of Proposition 6.2, and Proposition 6.3 to show that the only minimizer in the class of double bubbles with a single band adjacent to a set of components with contractible union is the band-lens, $\Phi$.

We will consider a proposed minimizer $\Sigma$ with regions $R_1$, $R_2$ and the exterior, such that $R_1$ contains a band, $\Pi$.

In several of the proofs, we do some area maintaining deformation which reduces perimeter, or which causes the bubbles to bump up against one other, violating regularity. In either case, the original configuration was not a minimizer.

**Lemma 6.4.** The pressure difference between $R_1$ and the exterior must be zero, i.e. all curves separating $R_1$ from the exterior must be straight lines.

*Proof.* Consider a curve separating the band, $\Pi$, from the exterior. If it is a straight line segment, the proof is complete, since the all curves separating $R_1$ from the exterior have the same curvature.

If the curve to be considered has nonzero curvature, continuously straighten the curve to reduce perimeter, and slide one of the sides of $\partial\Pi$ in order to maintain area. Either we will be able to straighten all such pieces while reducing perimeter, or we will first cause a violation of regularity.

Note that this implies that all curves enclosing components of $R_2$ must have the same curvature.

QEF

**Lemma 6.5.** In the class of double bubbles with a single band, $R_2$ must have a higher pressure than both $R_1$ and the exterior.

*Proof.* Assume not. Then components of $R_2$ are either bounded by straight lines, or are concave with respect to both $R_1$ and the exterior.

Now because the blob is contractible, we can create a closed geodesic $T$ which is not in $\Pi$ and which is not adjacent to any region but the exterior. If we slide $T$ towards one of the blobs, they

must eventually collide. Since all components are either concave or bounded by straight lines, the collision either occurs at a point which must be an illegal singularity (i.e. it cannot be 3 smooth curves meeting a 120 degree angles), or in a line. If it is in a line, the line with which $T$ collided must be a complete closed geodesic, otherwise it would contain an illegal singularity. This configuration either is the meridional band solution $\Xi$, or $\Xi$ beats it. QEF

**Corollary 6.1.** $R_2$ has at most two components.

*Proof.* This follows directly from the previous lemma and Proposition 6.2. QEF

**Corollary 6.2.** All contractible components of a double bubble with a single band consist of components of $R_2$.

*Proof.* By Lemma 6.5 and Lemma 6.4, $R_1$ and the exterior have equal, lowest, pressure. Thus, by Proposition 6.3, their components must be non-contractible. Hence only components of $R_2$ can be contractible. QEF

**Proposition 6.11.** The only potential minimizer among double bubbles with a single band is the band-lens $\Phi$.

*Proof.* By Proposition 6.2, $R_2$ (the region without the band) has either one or two components.

If $R_2$ has a single component, by Corollary 6.2, this is the only contractible component in the double bubble. The side of $\partial\Pi$ which does not intersect $\partial R_2$ is, by Lemma 6.4, a straight line; and, by the definition of a band, closed. Now consider the side which intersects $\partial R_2$. Again by Lemma 6.4, the curve which separates $\Pi$ from the exterior is a straight line. Now, rotate the component until this line lies on a geodesic circle, while sliding the boundary of $\Pi$ to maintain area. This will always save perimeter or cause a violation of regularity. Now the only way in which curvature and regularity conditions can be satisfied is if we have the standard band-lens solution.

If $R_2$ has two components, by Corollary 6.2 these are the only contractible components that share boundary with $\Pi$. Further, either both components are on the same side of $\partial\Pi$ or they are on

opposite sides of $\partial\Pi$.

Assume they are on opposite sides of $\partial\Pi$ and consider one of these sides. By Lemma 6.4, any curve which separates $\Pi$ from the exterior is a straight line. Now, rotate the component until this curve lies on a geodesic circle, while sliding the boundary of $\Pi$ to maintain area. This will always save perimeter or cause a violation of regularity. Repeat for the other side. Move one of the components so that they both lie on the same side. Now we can slide them, while maintaining area, until there is a violation of regularity, showing the configuration to be bad. If the components are too large to fit on the same side, the configuration must have perimeter greater than 3, which by Lemma 6.3 is a contradiction.

If both components of $R_2$ are on the same side of $\partial\Pi$, we must consider their structure in a bit more detail. By Lemma 6.4, any curve which separates $\Pi$ from the exterior is a straight line. Now, each component must meet the line of $\partial\Pi$ in two frames where three curves meet at 120 degree angles. Now rotate and shift the two components until all 4 frames and thus the lines connecting them lie along a shortest closed geodesic. We can maintain area by sliding the boundary of $\Pi$, and we will either have a violation of regularity, or save perimeter by reducing the original configuration to a configuration in which connecting lines are part of a shortest closed geodesic. If we have the latter, slide the two components, while maintaining area, until there is a violation of regularity showing the configuration to be bad.

Thus, the double bubble must be the band-lens, $\Phi$. QEF

**Lemma 6.6.** In a minimizing band-lens, the band must have non-zero homology only in a short direction of the torus and the lens must enclose the smallest of the three areas.

*Proof.* First we show that any pair of areas that can be enclosed by a band-lens along the long direction can be enclosed by a band-lens in the shorter direction. Assume not. Then either the lens is too long or too wide to fit in the short direction. If it is too long to fit in the shorter direction, then the double bubble will have perimeter greater than three, and, by Proposition 6.3, it cannot be a

minimizer. If the lens is too wide, then the geodesic not adjacent to the lens cannot get close enough to make the band thin enough. But if this happens in the short direction, then it will certainly happen in the long direction where there is less leeway.

It is easy to see that the lens must enclose the smallest of the three possible areas. This is because the other two regions can exchange area for free, but the perimeter of the band-lens is strictly increasing with the area of the lens. QEF

### 6.5.4 Tilings

Propositions 6.12 and 6.13 show that the only minimizing tilings are the hexagon tiling $H$ and possibly an octagon-square tiling. The proofs use pressure conditions, the Wichiramala component bound, and a variational argument, as well as elementary graph theory and plane geometry.

**Lemma 6.7.** A component $C$ of a perimeter minimizing tiling $\Sigma$ must be a curvilinear polygon with an even number of edges greater than or equal to four.

*Proof.* Of course, by regularity we know that if $x$ and $y$ are adjacent edges of $C$, then $C$ is adjacent to different regions along each side. This implies that the regions adjacent to $C$ must alternate with the edges of $C$, which is only possible if $C$ has an even number of edges.

To complete the proof, we must show that a digon cannot occur in a tiling solution. To do this, note that by regularity a component that is a digon must be a lens. This lens either lies on a pair of straight lines, or on a pair of curves which lie on the same circle.

Either way, we can slide the lens along the lines or arcs without changing perimeter or area. Then either we reach a violation of regularity, or we can slide the lens around the torus without bumping into anything. If the latter is the case, either the lens lies on a free circle, which can be translated to cause a violation of regularity, or the lens lies on a free, closed geodesic. The latter implies that $\Sigma$ contains a non-contractible component (actually, two), which contradicts our hypothesis. QEF

**Lemma 6.8.** If a tiling of a torus has vertices of degree 3, the average number of edges per component is exactly 6.

*Proof.* The Euler characteristic on the torus is $v - e + c = 0$, where $v$ is the number of vertices, $e$ is the number of edges, and $c$ is the number of components. Now, let $n$ be the average number of edges per component. Then $c = \frac{2}{n}e$, since each edge is adjacent to two components, and each component averages $n$ edges. We also know that $v = \frac{2}{3}e$, since each edge is adjacent to two vertices, each vertex is adjacent to three edges. Substituting into the Euler formula, we obtain $(\frac{2}{3} - 1 + \frac{2}{n})e = 0$, from which we obtain $n = 6$. QEF

**Proposition 6.12.** A perimeter-minimizing tiling $\Sigma$ is either a straight-line hexagon tiling, or an octagon square tiling (a tiling in which the region of highest pressure consists of two curvilinear quadrilaterals, and the other two regions are curvilinear octagons).

*Proof.* If the three regions of $\Sigma$ have equal pressure, then all boundary curves are straight lines. Now, the only straight-edged polygon with 120° interior angles is the hexagon, so by regularity all components must be hexagons.

If the three regions of $\Sigma$ do not have equal pressure, then there is a region $R_1$ with pressure at least as low as all other regions, and a region $R_2$ with pressure at least as high as all other regions. For the following vertex-counting argument, note that by regularity each vertex of $\Sigma$ must be a boundary constituent in components of all three regions.

Now, every boundary curve of an $R_1$ component is either a straight line or bows in, so by regularity every $R_1$ component must have at least eight sides. Then $R_1$ components must each contain at least eight vertices in their boundaries, so $\Sigma$ must contain at least eight vertices.

Similarly, every boundary curve of an $R_2$ component is either a straight line or bows out, so by regularity and by Lemma 6.7, every $R_2$ component must have exactly 4 sides and 4 vertices. Furthermore, we know by the Wichiramala Component Bound that $R_2$ has at most 2 components.

Then $R_2$ components are adjacent to at most $(2\ components)(4\frac{vertices}{component}) = 8\ vertices$, so $\Sigma$ can contain at most 8 vertices.

Thus $\Sigma$ must contain exactly eight vertices, and $R_2$ must consist of exactly two four-sided components. Now the components of the remaining region, $R_3$, contain a total of eight vertices in their all boundaries. are adjacent to eight vertices, so $R_3$ must have either one 8-sided component or two four-sided components. But by Lemma 6.8, the average number of sides per component must be exactly six. The only combination that satisfies this condition is for $R_3$ to have one eight-sided component. This implies by regularity that $R_3$ must have less pressure than $R_2$, since a region of highest pressure cannot have eight-sided components.

Thus if the three regions of $\Sigma$ do not have equal pressure, there is a unique region of highest pressure consisting of two curvilinear quadrilaterals, while the other two regions each consist of a single curvilinear octagon. QEF

**Lemma 6.9.** The number of components in a minimizing hexagon tiling $\Upsilon$ must be a multiple of three, since each region must have the same number of components.

*Proof.* By regularity, every vertex in $\Upsilon$ must be adjacent to exactly 1 component from each of the three different regions. Then every region has exactly the same number of vertices lying in the boundaries of its components. By definition, every component in a hexagon tiling has the same number of vertices (6) in its boundary. Then every region must have the same number of components. Since there are three regions, the total number of components must be a multiple of three. QEF

**Lemma 6.10.** A hexagon tiling with 9 or more components is not a perimeter minimizer.

*Proof.* We first fix some notation. Given a tiling $\tau_0$ with $t = 3n$ hexagons on the 2-torus, the claim is that the total perimeter of the tiling is greater than three.

Extend the tiling $\tau_0$ of the torus to a tiling $\tau$ of the plane, coloring the plane in $t$ colors in the

natural manner. Without loss of generality, we may assume that any segment in the perimeter of the tiling forms an angle of $0$, $\pi/3$, or $2\pi/3$ with a horizontal line in the plane.

We can travel from any fixed hexagon $H$ in the tiling to a neighboring hexagon in any of six *directions*, which we label `N`, `NE`, `SE`, `S`, `SW`, `NW` in clockwise order, where by travelling in the `N` direction, we travel to the hexagon which borders the top edge of $H$. Similarly, we can travel from any vertex in the tiling to a neighboring vertex in several possible directions, in three directions from any fixed vertex but in six directions total. We label these directions `e`, `se`, `sw`, `w`, `nw`, `ne`, where any rays in these directions make clockwise angles of $0$, $\pi/3$, ..., $5\pi/3$ with the horizontal ray pointing to the right.

The first major step is to finding a 'gridlike' fundamental domain

Suppose that the two hexagons $A_1$ and $A_2$ have the same color, and that we can travel from $A_1$ to $A_2$ via some sequence $\mathcal{S}$ of directions $(d_1, \ldots, d_r)$. ($A_1$ and $A_2$ must be associated with different fundamental domains, since by choice each component of $\tau_0$ has a distinct color.)

Let $B_1$ and $B_2$ be any two hexagons separated by the same sequence of directions. *A priori*, $B_1$ and $B_2$ need not have the same color – but the folowing inductive argument shows that, in fact, they do:

We travel from $A_1$ to $B_1$ along some sequence of directions $(e_1, \ldots, e_s)$; the same sequence of directions will take us from $A_2$ to $B_2$. Now, starting from $A_1$ and $A_2$, and travelling along each direction in $(e_1, \ldots, e_s)$ simultaneously from each of the two hexagons, we pass through regions of the same color at each point (either because all the fundamental domains are colored the same, or because they have been identified in a routine way); hence, $B_1$ and $B_2$ are of the same color.

We claim that there exists some $a > 0$ such that any two hexagons separated by a sequence of $a$ 'S's (or equivalently $a$ '`N`'s) are of the same color. Pick any hexagon in $\tau$, and travel in the `S` direction. Among the first $t + 1$ hexagons reached during this journey, some two must have the same color – say, separated by a sequence of $a_0$ `S`s. From our work above, it follows that *any* two hexagons

separated by such a sequence are of the same color. So, we may set $a = a_0$. We can go even further, though, and choose $a > 0$ above to be minimal.

We now construct our desired fundamental domain as follows. Find $a$ as above. Pick any hexagon $H_{1,1}$ in the tiling of the plane, and take the $a$ hexagons $H_{1,1}, \ldots, H_{a,1}$ obtained by starting from $H_{1,1}$ and traveling in the S direction $a-1$ times. By the minimal definition of $a$, these have distinct colors. If there are already $t$ hexagons here, we have our fundamental domain.

Otherwise, travel from $H_{1,1}$ in the SE direction to reach another hexagon $H_{1,2}$, and travel in the S direction $a-1$ times to again yield $a$ hexagons $H_{1,2}, \ldots, H_{a,2}$ of distinct colors (by the minimal definition of $a$). (Note that by traveling from any $H_{r,1}$ in the SE direction, one can reach $H_{r,2}$.) Suppose, for sake of contradiction, that one of $H_{1,2}, \ldots, H_{a,2}$ were the same color as one of $H_{1,1}, \ldots, H_{a,1}$. We would then have that *each* of $H_{1,2}, \ldots, H_{a,2}$ is the same color as one of $H_{1,1}, \ldots, H_{a,1}$; and furthermore, that *all* hexagons in the tiling $\tau$ is the same color as one of $H_{1,1}, \ldots, H_{a,1}$. This contradicts our construction of $\tau$ from $\tau_0$. We now have found an $a \times 2$ cluster of hexagons, each of a different color. If there are already $t$ hexagons here, we have our fundamental domain.

Otherwise, travel from $H_{1,2}$ in the SE direction to reach another hexagon $H_{1,3}$, and define $H_{1,3}, \ldots, H_{n,3}$ in the natural way. These new $a$ hexagons have distinct colors; and as before, these colors must be distinct from those of $H_{1,1}, \ldots, H_{a,1}, H_{1,2}, \ldots, H_{a,2}$. We now have found an $a \times 3$ cluster of hexagons, each of a different color. If there already $t$ hexagons, we have our fundamental domain.

Otherwise, we can continue on, although because $t$ is finite, eventually we're bound to find our fundamental domain. Also, note that the N edges of the N-most hexagons of our fundamental domain our identified with the S edges of the S-most hexagons; similarly with the SE-most and NW-most edges. Indeed, the SE-most and NW-most edges are identified "naturally" with each other.

The next major step is to finding a short noncontractible path along the perimeter of the tiling.

It is sufficient to show that there exists a sequence $\mathcal{S} = (d_1, d_2, \ldots, d_k)$ of directions (along edges)

in $\{\texttt{e}, \texttt{se}, \texttt{sw}, \texttt{w}, \texttt{nw}, \texttt{ne}\}$ with the following properties:

(i) at most $n = \frac{1}{3}t$ of the $d_i$ equal `e` or `w`, at most $t$ of the $d_i$ equal `se` or `nw`, and at most $n$ of the $d_i$ equal `sw` or `ne`;

(ii) from any hexagon's northwest vertex, travelling according to $\mathcal{S}$ gives a closed, noncontractible path around the torus.

(Note that if we have two vertices in the tiling of the plane which correspond to the same points on the torus, then any continuous path between corresponds to a contractible closed path on the torus. With a little work, this fact can be used to ensure that the paths we find below are noncontractible.)

We select a $(3a) \times b$ "grid" of hexagons in the tiling which serve as a fundamental domain, where all four sides of our fundamental domain contain `nw-se` edges and the tom and bottom sides are identified naturally. Note that it suffices to prove the claim starting from the northwest corner $v_0$ of the top edge. Also note that $n = ab$.

If $b \geq 3$, then $(d_1, d_2, d_3, d_4, \ldots, d_{6a}) = (\texttt{sw}, \texttt{se}, \texttt{sw}, \texttt{se}, \ldots, \texttt{sw}, \texttt{se})$ suffices.

If $(a, b) = (1, 2)$, then $(d_1, d_2, d_3, d_4) = (\texttt{e}, \texttt{se}, \texttt{e}, \texttt{ne})$.

Because $t \geq 6$, we cannot have $(a, b) = (1, 1)$. Thus, we may assume that $a \geq 2$ and $b \leq 2$. We begin our journey by travelling from $v_0$ along $2b - 1$ edges, alternating between travelling `e` and `se` so that we travel `e` $b$ times and `se` $b - 1$ times. This brings us to a vertex $v_1$ on the left side of our fundamental domain.

If $v_1$ is the west vertex of one of the middle $a$ hexagons of our grid, then our initial $2b - 1$ movements `e`, `se`, ..., `e` are the "same" as between $2a + 1$ and $4a - 1$ movements `sw`, `se`, ..., `se`. From $v_1$, travel `se`; then repeat our initial $2b - 1$ movements. Our journey is then the "same" as moving between $2(2a + 1) + 1 = 4a + 3$ and $2(4a - 1) + 1 = 8a - 1$ edges `sw`, `se`, ..., `se`. In other words, we are now at some vertex $v_2$ which is the west vertex of either (a) one of the top $a$ hexagons of the first column; or (b) one of the bottom $a$ hexagons of the first column.

In either case, we have travelled e at most $2b$ times and se at most $2b-1$ times. And we are now either (a) on the west vertex of one of the top $a$ hexagons of the first column, or (b) on the west vertex of one of the bottom $a$ hexagons of the first column.

In case (a), we may alternate between travelling ne and nw to reach $v_0$, travelling ne at most $a$ times and nw at most $a-1$ times. Altogether, then, we have travelled: e at most $2b$ times, so total is $2b \leq ab$; se at most $2b-1$ times and nw at most $a-1$ times, so total is $a+2b-2 = ab-(a-2)(b-1) \leq ab$; ne at most $a$ times so total is $a \leq ab$.

Hence, condition (ii) is satisfied.

In case (b), we may alternate between travelling se and sw to reach $v_0$, travelling se at most $a$ times and sw at most $a-1$ times. Altogether, then, we have travelled: e $\leq 2b$ times, so total is $2b \leq ab$; se $\leq a+2b-1$ times, so total is $a+2b-1 = ab-(a-2)(b-1)+1 \leq ab+1$; sw $\leq a-1$ times, so total is $a-1 \leq ab$.

Hence, condition (ii) is satisfied if $a > 2$, because in this case the second inequality above is strict.

If $a = 2$, then it seems that we might have an incredible stroke of bad luck so that all our approximations above are sharp. Actually, though, this doesn't happen: one can verify that with a $6 \times 1$ or $6 \times 2$ grid, we can still find a path of the desired form. (To do so in the $6 \times 2$, we need to use the barbell property.)

We now make the final points to show that $\tau_0$ has perimeter greater than 3.

For each of our $t$ hexagons, start at the northwest vertex of the hexagon and travel along $(d_1, d_2, \ldots, d_k)$ as in the lemma of the previous section. Each of these $t$ overlapping paths uses at most $n = t/3$ edges parallel to the e (resp. se or sw) direction, implying that they use a total of $t^2/3$ such edges. Furthermore, a symmetry argument implies that all of the $t$ edges parallel to the e (resp. se or sw) direction are used the same number of times in these $t$ paths. Therefore, each edge is used at most at most $(t^2/3)/t = t/3$ times in these overlapping paths.

Thus, the total perimeter $P'$ of the $t$ overlapping paths is at most $t/3$ times the perimeter $P$ of $\tau_0$.

Furthermore, each of the $t$ overlapping paths has perimeter greater than 1, implying that altogether their total perimeter $P'$ is greater than $t$.

It follows that

$$t < P' \leq (t/3)P,$$

or $P > 3$. QEF

**Lemma 6.11.** A hexagon tiling $\Upsilon$ with 6 components has perimeter strictly greater than 3.

**Definition 6.1.** Let a *barbell* be a pair of hexagons related as in Figure 6.12. Then the two hexagons each adjacent to both hexagons of the pair must be of different regions (B and C in Figure 6.12), and the two hexagons of the barbell must be of the same region (A in Figure 6.12).

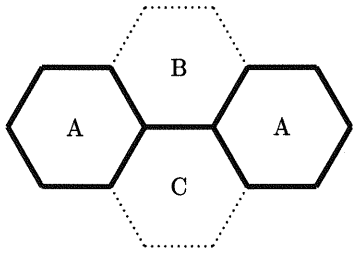



Figure 6.12: Regularity satisfying hexagon tilings exhibit the barbell-lift property.

**Definition 6.2.** Let a *web* $W$ be a set of hexagons $\{x_1, x_2, ..., x_n\}$ in the lift of a tiling such that for any hexagon $x_i \in W$, another hexagon $y$ is also in $W$ if $x_i$ and $y$ are contained in the same barbell.

Then every hexagon in a web must be of the same region. Furthermore, a web contains exactly one third of the hexagons in the tiling, so all components of a region are contained in a single web.

*Proof.* By the perimeter bound, a double bubble whose set of boundary curves contains three non-contractible non-geodesic circles as disjoint subsets is not perimeter-minimizing. We show that $\Upsilon$ must contain three such circles by considering the lift of $\Upsilon$ to the plane.

Let the three regions of the tiling be labelled $A$, $B$, and $C$. Note that in the lift, there are exactly two unique hexagonal components of each region; let these components be labelled $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, and $C_2$.

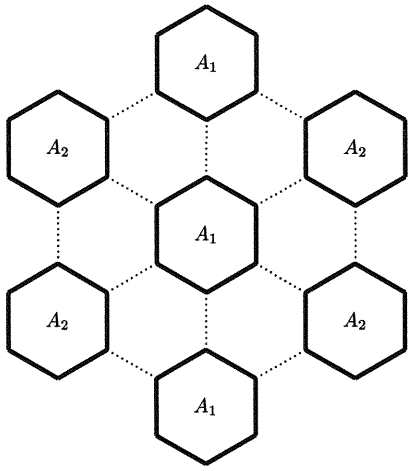



Figure 6.13: Structure of the lift of a six-hexagon tiling.

Now consider a seven-hexagon segment of a web as in Figure 6.13. Let the center hexagon be $A_1$. Then the other six hexagons must all be of region $A$, by the definition of a web. Further, at least one of the other six hexagons must also be $A_1$, since if all of them were $A_2$ then we would have the relation that every hexagon directly above or below $A_2$ in the web was also $A_2$, which would imply that the center hexagon was $A_2$, a contradiction. Let the hexagon immediately below the center hexagon be $A_1$. This immediately implies that the hexagon above the center hexagon is also $A_1$, again by identification. Now if any of the remaining four hexagons is $A_1$, then all of the hexagons in the segment, and therefore all of the hexagons in the web, must also be $A_1$. Thus, the remaining four hexagons must all be $A_2$. Now, considering the web as a whole we see that it consists of alternating vertical stripes of $A_1$ and $A_2$ hexagons. Now consider the hexagon adjacent and to the left of the center hexagon in the original web segment. Let this hexagon be the unique hexagon $B_1$. Then every $A_1$ hexagon has a $B_1$ hexagon adjacent and to its left, and we see that in the $B$ web

we also have alternating vertical stripes of $B_1$ and $B_2$ hexagons. The same pattern holds for the $C$ web. Thus, there is a single fundamental pattern in the arrangement of the six unique hexagons, up to rotation and relabelling. See Figure 6.13.

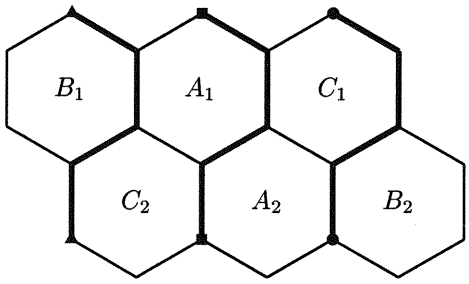



Figure 6.14: A fundamental domain for a six hexagon tiling has perimeter greater than three, since we can exhibit three non-trivial closed curves.

Now the set of six hexagons in Figure 6.14, with boundary curves identified as shown, has precisely the same structure and contains precisely the same perimeter as a six-hexagon tiling of a parallelogram torus. The three paths highlighted in Figure 6.14 are three disjoint non-contractible non-geodesic circles, since for each path the endpoints are in fact the same (identified) point. QEF

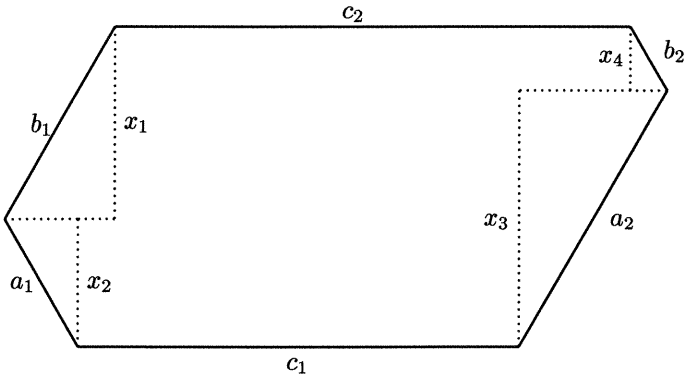



Figure 6.15: In a hexagon with regular angles, a simple geometric construction shows that opposite adjacent side pairs have perimeters which sum to the same total.

**Lemma 6.12.** In a hexagon that satisfies regularity, the sum of the lengths of two sides that are adjacent is the same as the sum of the lengths of the opposite pair of adjacent sides.

*Proof.* Let two adjacent sides of a regular hexagon be labelled $a_1$ and $b_1$, the opposite pair of adjacent sides be labelled $a_2$ and $b_2$, and the remaining two sides be labelled $c_1$ and $c_2$, as in Figure 6.15. Also, let the vertex between $a_1$ and $b_1$ be $V_1$ and the vertex between $a_2$ and $b_2$ be $V_2$. Then since $c_1$ and $c_2$ are parallel, the sum of the perpendicular distance from $V_1$ to each of $c_1$ and $c_2$ is the same as the sum of the perpendicular distance from $V_2$ to each of those sides; labelling these distances as in Figure 6.15, we get $x_1 + x_2 = x_3 + x_4$. Now, by the Law of Sines we have $a_1 = \frac{x_1}{sin(\frac{\pi}{3})}$, $b_1 = \frac{x_2}{sin(\frac{\pi}{3})}$, $a_2 = \frac{x_4}{sin(\frac{\pi}{3})}$, and $b_2 = \frac{x_3}{sin(\frac{\pi}{3})}$. Combining these equations, we get $a_1 + b_1 = a_2 + b_2$, as desired. QEF

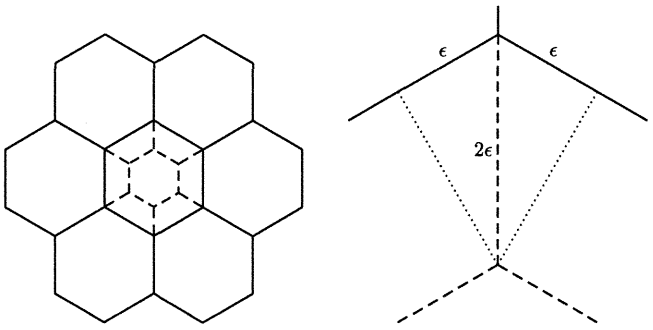



Figure 6.16: Shrinking or expanding components of a tiling of hexagons generates double bubbles enclosing different areas, but does not change perimeter.

**Lemma 6.13.** A 3-component straight-edge hexagon tiling enclosing arbitrary areas has the same perimeter as a 3-component straight-edge hexagon tiling of equal areas.

*Proof.* First, note that "shrinking" or "expanding" a single hexagon in a straight-edged hexagon tiling (as in Figure 6.16 is a smooth variation through equilibrium double bubbles and hence maintains perimeter.

Now consider the three unique hexagons in a 3-component straight-edge hexagon tiling, with boundary curves identified as in Figure 6.17. Let the nine unique sides, in three groups each of three sides parallel to one another, be labelled $a_1$, $a_2$, $a_3$, $b_1$, $b_2$, $b_3$, $c_1$, $c_2$, and $c_3$, such that $a_1 \geq a_2 \geq a_3$, $b_1 \geq b_2 \geq b_3$, and $c_1 \geq c_2 \geq c_3$. Then if $a_1 = a_2 = a_3$, $b_1 = b_2 = b_3$, and $c_1 = c_2 = c_3$, the three hexagons are congruent and therefore have equal area. (In fact, any one of these equality equations implies the other two, by Lemma 6.12). We show that by shrinking or expanding hexagons we can transform any straight-edged hexagon tiling into one with congruent hexagons.

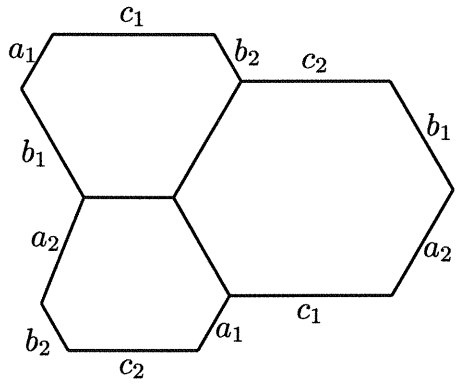



Figure 6.17: A generic fundamental region of a three-hexagon tiling looks something like this.

We start by showing that we can always label the nine sides in such a way that the hexagon $H_1$ with sides labelled $a_1$ and $a_2$ also has sides $b_1$, $b_2$, $c_1$, and $c_2$. Let the remaining sides of $H_1$ besides $a_1$ and $a_2$ be $b_a$, $b_b$, $c_a$, and $c_b$, as in Figure 6.17. Since $a_1 \geq a_2$, by Lemma 6.12 we know that $b_a \geq b_b$ and $c_a \geq c_b$. Similarly, since $a_1 \geq a_3$, we have $b_a \geq b_c$ and $c_a \geq c_c$, and since $a_2 \geq a_3$, we have $b_b \geq b_c$ and $c_b \geq c_c$. Thus we can always set $b_a = b_1$, $b_b = b_2$, $c_a = c_1$, $c_b = c_2$, and $c_c = c_3$ so that $H_1$ has sides with subscripts 1 and 2, while $H_2$ has sides all of subscript 1 or 3, and a hexagon $H_3$ has sides all of subscript 2 or 3.

Now consider a straight-edged three-hexagon tiling with hexagons of unequal area and side labels as described above. Then at least one of the inequalities in $a_1 \geq a_2 \geq a_3$ must be strict, or the tiling would be equal-area.

Suppose $a_1 > a_2$. Then know from Lemma 6.12 that $b_1 > b_2$ and $c_1 > c_2$. Now let us "expand" $H_3$ by lengthening $a_2$, $a_3$, $b_2$, $b_3$, $c_2$ and $c_3$ while shortening $a_1$, $b_1$, and $c_1$, until $a_1 = a_2$. We know we can do this smoothly, since $a_1$ is bounded below by $a_2$. Note also that by Lemma 6.12, $a_1 = a_2$ implies $b_1 = b_2$ and $c_1 = c_2$, so during the transformation we also have $b_1$ bounded from below by $b_2$, and $c_1$ bounded below by $c_2$. Thus none of the sides that we are shortening vanishes to zero length.

Now either $a_1 = a_2 = a_3$ and we are done, or $a_1 = a_2$ and $a_2 > a_3$. If the latter, then let us "shrink" $H_1$ by shortening $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, and $c_2$ while lengthening $a_3, b_3$, and $c_3$, until $a_2 = a_3$. Again, we know we can do this smoothly, since $a_1$ and $a_2$ are bounded from below by $a_3$. As before, $a_2 = a_3$ implies $b_2 = b_3$ and $c_2 = c_3$, so $b_1, b_2, c_1$, and $c_2$ are all bounded from below as well.

Thus, by expanding $H_3$ and/or shrinking $H_1$ as necessary, we can continuously deform any straight-edged hexagon tiling into one with congruent hexagons, and thus equal areas. QEF

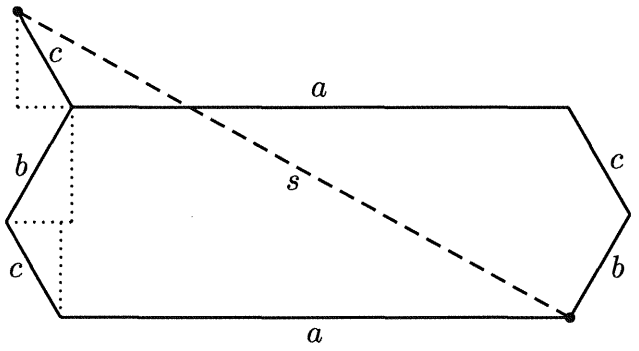



Figure 6.18: A closed geodesic in a three-hexagon tiling. It is a simple geometric argument to show that $a + b + c > s$, except in the case of the hexagonal torus, which implies that the hexagonal tiling in all other cases loses to the double band solution.

**Lemma 6.14.** A 3-component straight-edge hexagon tiling with congruent hexagons has perimeter strictly greater than 3 on all tori except the hexagonal torus.

*Proof.* A 3-hexagon tiling with congruent hexagons has three unique side lengths, $a \geq b \geq c$, and perimeter $P = 3(a + b + c)$. We show that unless $a = b = c$, $a + b + c$ is always greater than the length of a closed geodesic, and therefore the perimeter P is always greater than 3.

Consider some hexagon in such a tiling, and let $V_1$ be a vertex between sides of length $a$ and $b$. Then the line connecting the diametrically opposed vertex $V_2$ (also between sides of length $a$ and $b$) with the copy of $V_2$ that is connected to $V_1$ by a single side of length $c$, is a closed geodesic. See Figure 6.18.

Let $s$ be the length of that geodesic. Then by the Pythagorean Theorem we have $s^2 = (\frac{\sqrt{3}}{2}c + \frac{\sqrt{3}}{2}b + \frac{\sqrt{3}}{2}c)^2 + (a + \frac{c}{2} - \frac{b}{2} + \frac{c}{2})^2$, which reduces to

$$\begin{aligned} s^2 &= a^2 + b^2 + 4c^2 + 2ac + 2bc - ab \\ &= (a^2 + b^2 + c^2 + 2ac + 2bc + 2ab) - 2ab + 3c^2 - ab \\ &= (a + b + c)^2 + 3c^2 - 3ab. \end{aligned}$$

Now suppose at least one of the inequalities in $a \geq b \geq c$ is strict. Then $c^2 < ab$, so $3c^2 - 3ab < 0$ and $(a + b + c)^2 > s^2$. Thus $P = 3(a + b + c) > 3s$, so the perimeter of the tiling is strictly greater than three unless $a = b = c$. Note from Figure 6.19 that the latter condition can only be satisfied on the hexagon torus, and that in this case the geodesic we constructed above is the shortest closed geodesic so the perimeter of the tiling is precisely 3. QEF

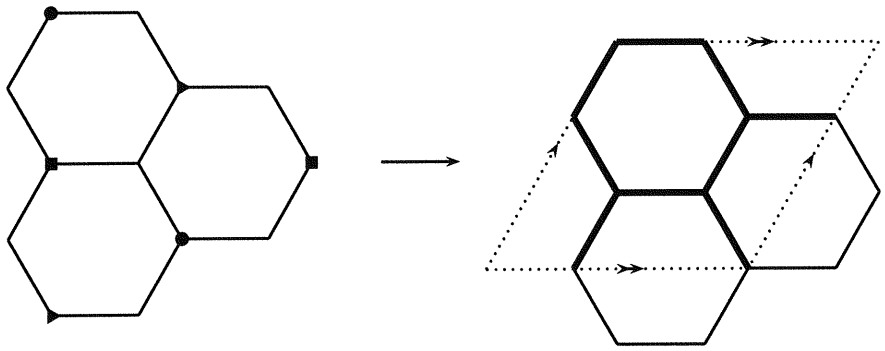


Figure 6.19: A three-hexagon tiling of congruent hexagons fits on the hexagonal torus, and is in fact a minimizer.

**Proposition 6.13.** The only hexagon tiling that is perimeter-minimizing is the 3-component straight-edge hexagon tiling on the hexagonal torus.

*Proof.* By Lemma 6.9, the number of components in a minimizing hexagon tiling must be a multiple of 3. By Lemma 6.10, a hexagon tiling with 9 or more components is not perimeter-minimizing. By Lemma 6.11, a hexagonal tiling with 6 components has perimeter strictly greater than 3, so by the perimeter bound it is not perimeter-minimizing. Thus, a perimeter-minimizing hexagon tiling must have exactly 3 components. By Proposition 6.12, a perimeter-minimizing hexagon tiling must have straight edges. By Proposition s 6.13 and 6.14, a 3-component straight-line hexagon tiling has perimeter strictly greater than 3 on all tori except the hexagonal torus, so by the perimeter bound the only hexagon tiling that can be perimeter-minimizing is the 3-component tiling on the hexagonal torus.

QEF

## 6.6 Formulas for Perimeter and Area

This section gives formulae for area and perimeter of the three potential minimizers from section 6.5: the standard double bubble $\Theta$ (Proposition 6.14), the two-component symmetric chain $\Omega$ (Propositions 6.15, 6.16), and the band-lens $\Phi$ (Proposition 6.18). We use these formulae to obtain the phase portrait of Figure 6.2. Propositions **??** and 6.19 show that the two-component symmetric chain and the band-lens are unique for prescribed areas.

### 6.6.1 Formulas for the Standard Double Bubble

We begin by stating a few simple geometric formulae that describe circular arcs meeting chords.

*Remark 6.4.* Given a circular arc $\lambda$ and a chord of length $C$ meeting the arc at (interior) angles $\theta$, the area between the arc and the chord, the length of the arc, and the radius of the circle are given

by:

$$A(\theta, C) = \frac{C^2(\theta - \sin\theta\cos\theta)}{4\sin^2\theta} \tag{6.1}$$

$$L(\theta, C) = \frac{C(\theta)}{\sin\theta} \tag{6.2}$$

$$R(\theta, C) = \frac{C}{2\sin\theta} \tag{6.3}$$

Here area means the area between an arc and chord.

Formulae for perimeter and area of the standard double bubble as derived by Foisy [24, Section 2] are given in the following proposition.

**Proposition 6.14.** The area and perimeter formulae of a standard double bubble in terms of the separating chord length $C$ and the angle $\theta$ at which it meets the interior arc are (smaller region $R_1$)

$$A_{R_1}(\theta, C) = A(\frac{2\pi}{3} - \theta, C) + A(\theta, C)$$

$$A_{R_2}(\theta, C) = A(\frac{2\pi}{3} + \theta, C) - A(\theta, C)$$

$$P(\theta, C) = L(\frac{2\pi}{3} + \theta, C) + L(\frac{2\pi}{3} - \theta, C) + L(\theta, C)$$

*Proof.* These formulae follow directly from Equations 6.1-6.3. QEF

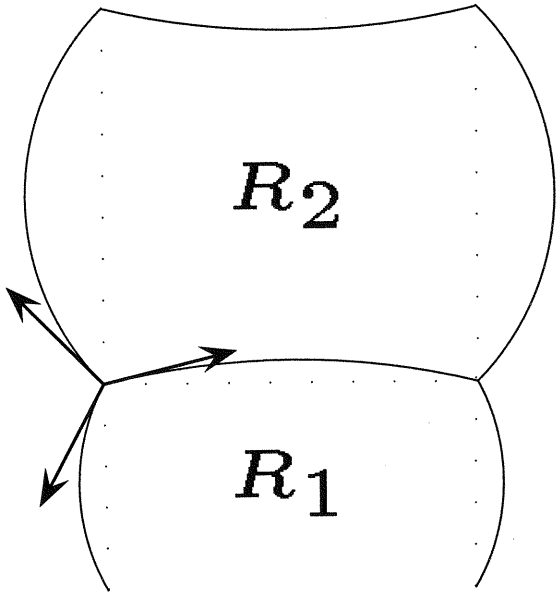



Figure 6.20: Important angles and lengths in a symmetric two component swath.

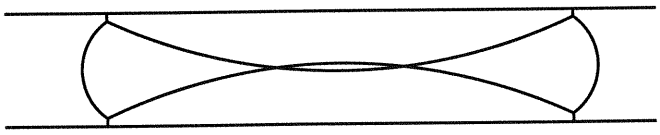


Figure 6.21: Arcs curve the right direction but intersect.

### 6.6.2 Formulas for the Standard 2-Component Symmetric Chain

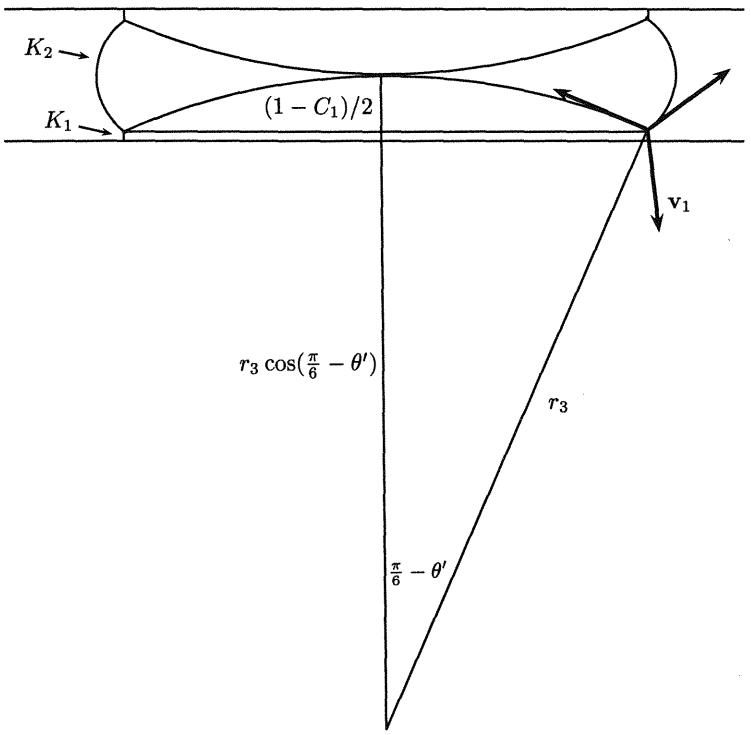



Figure 6.22: Conditions for tangency give the appropriate lower limit of the parameter $\theta_1$.

Representing vertices by rectangles in a 2-component symmetric chain, gives a natural parameterization in terms of an angle $\theta_1$ and a chord length $C_1$ determined by the boundary of the highest pressure region, $R_1$, with the exterior (Figure 6.20). We label this curve $\lambda_1$. Let $\theta_2$ and $C_2$ be the corresponding parameters for each arc $\lambda_2$ between the exterior and $R_2$. Similarly, $\theta_3$ and $C_3$ describe arcs $\lambda_3$ between $R_1$ and $R_2$.

Given $\theta_1$ and $C_1$, we can determine $\kappa_1$. The $\lambda_i$'s meet at 120°, so we get that $\theta_2 = \frac{\pi}{3} - \theta_1$ and $\theta_3 = \frac{\pi}{6} - \theta_1$. Since $C_1 + C_2 = 1$, we recover $C_3$ from the radius formula above using the fact that curvatures add to zero at each vertex. In sum, we have:

$$C_2 = 1 - C_1;$$

$$\kappa_3 = \kappa_1 - \kappa_2;$$

$$\frac{1}{\kappa_1} = \frac{C_1}{2\sin\theta_1};$$

$$\frac{1}{\kappa_2} = \frac{C_2}{2\sin(\frac{\pi}{3} - \theta_1)};$$

$$\frac{1}{\kappa_3} = \frac{C_3}{2\sin(\frac{\pi}{6} - \theta_1)}.$$

In terms of $C_1$ and $\theta_1$, the formula for $C_3$ becomes

$$C_3 = \frac{C_1(1 - C_1)\sin(\frac{\pi}{6} - \theta_1)}{(1 - C_1)\sin\theta_1 - C_1\sin(\frac{\pi}{3} - \theta_1)}.$$

Now we must specify a range of values for $\theta_1$ and $C_1$ where the swath makes sense, i.e. we find a parameter space where swaths do not violate regularity.

By symmetry, we can restrict $C_1$ to the open interval $(0, \frac{1}{2})$ (We exclude the endpoints to ensure uniqueness, and deal with these cases separately.) If we consider a frame of 3 vectors $v_1, v_2, v_3$ meeting at 120° we can model the set of tangent vectors to each arc at a given vertex (see Figure 6.21). In particular, we can place this frame at the intersection of the $\lambda_i$'s and consider the structural change of a bubble under a small rotation of the frame. In the case of $C_1 = \frac{1}{2}$, $v_3$ must coincide with $C_3$ otherwise $\lambda_3$ will bow into the region of higher pressure. For instance, rotating the frame slightly counterclockwise makes $\kappa_2 > \kappa_1$ which implies $R_2$ has higher pressure with respect to the exterior. Since $v_3$ points into $R_2$, this construction cannot work. For an arbitrary $C_1$ (less than $\frac{1}{2}$), we rotate the frame until $\kappa_1 = \kappa_2$, this particular construction cannot work since $v_3$ must point horizontal if the regions have equal pressure. Rotating the frame slightly counterclockwise presents the same problem as before. If we rotate clockwise, we may initially produce a chain which intersects itself (see Figure 6.21)). At a certain value $\theta'$ of $\theta_1$ the $\lambda_3's$ are tangent and beyond that, until $\theta_1 = \frac{\pi}{6}$,

the construction is regular. (Unless $C_1 = \frac{1}{2}$ the angle $\theta_1 = \frac{\pi}{6}$ is not valid. Again, we treat this case separately.)

We see from figure 6.20) that $\theta'$ satisfies

$$r_3 \cos(\frac{\pi}{6} - \theta') + \frac{1 - C_1}{2} = r_3$$

Upon further simplification, this condition reduces to

$$C_1 = \sin\theta'$$

Thus, $\theta_1$ must lie in $(\sin^{-1}(C_1), \frac{\pi}{6})$. We can now write down formulae for the area and perimeter of chains.

**Proposition 6.15.** Using the parameterization of Section 6.6.2, the area and perimeter formulae for symmetric chains with components of unequal pressure are

$$P(\theta_1, C_1) = 2L(\theta_1, C_1) + 2L(\frac{\pi}{3} - \theta_1, C_2) + 2L(\frac{\pi}{6} - \theta_1, C_3)$$

$$A_{R_1}(\theta_1, C_1) = 2A(\theta_1, C_1) + 2A(\frac{\pi}{6} - \theta_1, C_3) + C_1C_3$$

$$A_{R_2}(\theta_1, C_1) = 2A(\frac{\pi}{3} - \theta_1, C_2) - 2A(\frac{\pi}{6} - \theta_1, C_3) + C_2C_3$$

where $C_2$ and $C_3$ are given by

$$C_2 = 1 - C_1$$

$$C_3 = \frac{C_1(1 - C_1)\sin(\frac{\pi}{6} - \theta_1)}{(1 - C_1)\sin\theta_1 - C_1\sin(\frac{\pi}{3} - \theta_1)}.$$

*Proof.* These formulae follow directly from the parameterization and Equations 6.1-6.2. QEF

Now we must treat the case where $C_1 = \frac{1}{2}$. We showed above that $\theta_1 = \frac{\pi}{6}$ at this point, otherwise the construction is irregular. However, this does not determine a unique value for $C_3$, in fact, it can take on any value greater than zero. Since all other arc lengths and angles in the chain are fixed, we have a simple formula relating volume and perimeter of such chains.

**Proposition 6.16.** In symmetric chains comprised of equal pressure components, perimeter as a function of the area of one region is given by

$$P(A_{R_1}) = 4A + \frac{2\pi + 3\sqrt{3}}{6}$$

where $A_{R_1} > 2A(\frac{\pi}{6}, \frac{1}{2}) = \frac{2\pi - 3\sqrt{3}}{24}$

*Proof.* By Equations 6.1-6.2, we know the perimeter and area in terms of $C_3$

$$P = 4L(\frac{\pi}{6}, \frac{1}{2}) + 2C_3 = \frac{2\pi}{3} + 2C_3$$

$$A_{R_1} = A_{R_2} = 2A(\frac{\pi}{6}, \frac{1}{2}) + \frac{C_3}{2} = \frac{2\pi - 3\sqrt{3}}{24} + \frac{C_3}{2}$$

This allows us to get perimeter as a function of area. Combining equations yields

$$P = 4A_{R_1} + \frac{2\pi + 3\sqrt{3}}{6}$$

For, $A_{R_1} > 2A(\frac{\pi}{6}, \frac{1}{2}) = \frac{2\pi - 3\sqrt{3}}{24}$, the chain is regular QEF

**Conjecture 6.3.** Given prescribed areas $a_1$ and $a_2$, and up to isometries of the torus, there is a unique standard two-component symmetric chain with homology $(1, 0)$ enclosing these areas.

Note that even if this conjecture fails, all we really need is the weaker result that the chain is unique *when it is minimizing*. This is probably not at all hard to show.

**Proposition 6.17.** For a given set of area pairs, and based on the parameterization described in this subsection, if the perimeter of an immersed symmetric is less than or equal to the perimeter of the embedded minimizer for those areas, then it is itself embedded.

*Proof.* We must consider the case when the symmetric chain overlaps itself around the torus. Clearly, $C_3$ must be less than .5, or else the double bubble would have perimeter greater than three. $C_3$ is perpendicular to the shortest side of the torus, so $C_3 < L \cdot \sin(\theta)$, where $\theta$ is the smallest interior angle of the torus, since otherwise two vertices which fall on a geodesic, $g_1$, in the short direction,

would wrap around past the other two vertices (which also lie on a geodesic, $g_2$) and there would be no exterior. This case falls outside of our calculation and thus we do not need to consider it.

Thus, there exists a distance $\epsilon = L \cdot \sin(\theta) - C_3 > 1 \cdot \sin(\frac{\pi}{3}) - 1/2 > .365$ that separates $g_1$ and $g_2$. But somewhere in the parallelogram that lies between $g_1$ and $g_2$, the double bubble intersects itself. Now there are three cases. Case 1 is when $R_1$ or $R_2$ intersects one of the vertices of the chain. Case 2 is when $R_1$ intersects $R_2$. Case 3 is when $R_1$ or $R_2$ intersects itself.

Case 1: Assume that $R_1$ intersects one of the vertices of the chain. Thus, given $\lambda_1$ and the height, $h_1$ that it makes with its chord, then $h_1 > \epsilon$.

Now, we'd like to calculate the length of $\lambda_1$,

$$L(\lambda_1) = \frac{C_1 \cdot \theta_1}{\sin(\theta_1)}.$$

But

$$\tan(\theta_1/2) = \frac{h_1}{(C_1/2)}.$$

So

$$C_1 = \frac{2 \cdot h_1}{\tan(\theta_1/2)} > .73/tan(\theta_1/2).$$

Thus,

$$L(\lambda_1) > \frac{2 \cdot \epsilon \cdot \theta_1}{\tan(\theta_1/2) \cdot \sin(\theta_1)}.$$

We put

$$f(x) = \frac{x}{\tan(x/2)\sin(x)} = \frac{x \cdot \cos(x) + x}{\sin(x)^2}.$$

Then

$$f'(x) = \frac{-(x \cdot (\cos(x))^2 - (\sin(x) - 2 \cdot x) \cdot \cos(x) - \sin(x) + x)}{\sin(x)^3}.$$

But since $x > sin(x)$ for $x > 0$, on the interval $[0, \pi/3]$, $f'(x) < 0$ and thus $f$ is decreasing. Thus,

$$L(\lambda_1) > \frac{.73 \cdot \pi/3}{\tan(\pi/6) \cdot \sin(\pi/3)} > 1.528,$$

and so $P > 2 \cdot 1.528 > 3$.

Case 2: Assume that $R_1$ intersects $R_2$ but neither intersects one of the vertices of the chain. Thus, given $\lambda_1$ and $\lambda_2$, and the height, $h_1$ and $h_2$ that each makes with its respective chord, either $h_1 + h_2 > \epsilon$.

Now, we would like to calculate the sum of the lengths of $\lambda_1$ and $\lambda_2$,

$$L(\lambda_1) + L(\lambda_2) = \frac{C_1 \cdot \theta_1}{\sin(\theta_1)} + \frac{C_2 \cdot \theta_2}{\sin(\theta_2)}$$

. But

$$\tan(\theta_i/2) = \frac{h_i}{(C_i/2)}.$$

So

$$C_i = \frac{2 \cdot h_i}{\tan(\theta_i/2)}.$$

Thus,

$$L(\lambda_1) + L(\lambda_2) = \frac{2 \cdot h_1 \cdot \theta_1}{\tan(\theta_1/2) \cdot \sin(\theta_1)} + \frac{2 \cdot h_2 \cdot \theta_2}{\tan(\theta_2/2) \cdot \sin(\theta_2)}.$$

But, from above

$$f(x) = \frac{x}{\tan(x/2)\sin(x)} = \frac{x \cdot \cos(x) + x}{\sin(x)^2}$$

is decreasing. So if we assume, without loss of generality that $\theta_1 \geq \theta_2$, then

$$L(\lambda_1) + L(\lambda_2) \geq \frac{2 \cdot h_1 \cdot \theta_1}{\tan(\theta_1/2) \cdot \sin(\theta_1)} + \frac{2 \cdot h_2 \cdot \theta_1}{\tan(\theta_1/2) \cdot \sin(\theta_1)} > \frac{2 \cdot \epsilon \cdot \theta_1}{\tan(\theta_1/2) \cdot \sin(\theta_1)} > 1.528$$

for all $\theta$ in the interval $[0, \pi/3]$, as above. So $P > 2 \cdot 1.528 > 3$.

Case 3: Now assume that $R_1$ intersects itself. They will clearly intersect in a lens. We can remove the lens, and get the area back with less perimeter in the exterior. (This is where I wanted to have the isoperimetric inequality for free boundary closed surfaces - I employ a different method that needs tuning.) To see this, align the lens perpendicular to the axis of symmetry of the chain and center it over the $\epsilon$-rectangle and where the exterior is widest. If the exterior is not wider at this point than the lens, separate the two circular arcs of the lens and slide them apart until the intersection of these arcs with the exterior encloses the same area that the lens had, and with less perimeter. If this

component of the exterior does not have enough area, take the whole component and repeat with a different component. If there is not enough area in the whole region, then the immersed double bubble lies outside our calculations. We must still consider the case where the exterior is too wide. If the lens fits, we are done. If not, then... I can't explain this without a picture. I will get on that for next time. QEF

This completes the description of symmetric chains. Note that when computing perimeter later, we restrict our parameter space even more by requiring $C_3 < \frac{1}{2}$, since the polygonal arc length of chains not satisfying this inequality is already greater than 3.

### 6.6.3 Formulae for the Band-Lens

The family of Band-Lens bubbles can be parameterized nicely in terms of two parameters, $r$, the radius of curvature ($\frac{1}{curvature}$) of the lens, and $d$, the width of the band. Of course, the lens can only be as tall as the cylinder, so $r$ can be at most $\frac{1}{2\sin\frac{\pi}{3}} = \frac{1}{\sqrt{3}}$. For a given $r$, the width of the band must be at least half the width of the lens. Thus $d$ is at least $\frac{r}{2}$. From this we can compute the area and perimeter of a Band-Lens.

**Proposition 6.18.** The area and perimeter formulae for a band-lens bubble in terms of $r$ and $d$ ($r < \frac{1}{\sqrt{3}}, d > \frac{r}{2}$) are

$$A_{Lens}(r) = 2r^2(\frac{\pi}{3} - \frac{\sqrt{3}}{4})$$

$$A_{Band}(r,d) = d - r^2(\frac{\pi}{3} - \frac{\sqrt{3}}{4})$$

$$P(r,d) = (\frac{4\pi}{3} - \sqrt{3})r + 2$$

*Proof.* Using Equations 6.1-6.2 we can compute the area of both components. The chord length $C$ of the lens is $2r\sin\frac{\pi}{3} = \sqrt{3}r$. This gives us

$$A_{Lens}(r) = 2A(\frac{\pi}{3}, \sqrt{3}r) = 2r^2(\frac{\pi}{3} - \frac{\sqrt{3}}{4})$$

$$A_{Band}(r,d) = d - \frac{1}{2}A_{Lens}(r) = d - r^2(\frac{\pi}{3} - \frac{\sqrt{3}}{4})$$

The perimeter is given by

$$P(r,d) = 2L(\frac{\pi}{3}, \sqrt{3}r) + (2 - \sqrt{3}r) = (\frac{4\pi}{3} - \sqrt{3})r + 2$$

QEF

**Proposition 6.19.** Given prescribed areas $a_1$ and $a_2$, there exists a unique (up to isometry of the torus) minimal band-lens.

*Proof.* The lens is uniquely determined by the length, $C$, of the chord separating its vertices, and the interior angle $\theta$ between the two boundary arcs. By regularity $\theta$ is fixed. Moreover, the area of the lens is strictly increasing in $C$. Thus, given the smallest area $A_1$ of the three regions of the double bubble, there is a unique lens enclosing that area. The chord of the lens must lie parallel to the short side of the torus and thus the lens is positioned uniquely up to translation. Further, the vertices are connected by a uniquely determined line segment also parallel to the short direction of the torus and there exists a unique geodesic in the short direction which will enclose the proper area. QEF

**Proposition 6.20.** For a given set of area pairs, and based on the parameterization described in this subsection, if the perimeter of an immersed band-lens is less than or equal to the perimeter of the embedded minimizer for those areas, then it is itself embedded.

*Proof.* The first case we consider is the case in which the band with smaller area is too narrow, so that the shortest closed geodesic which is a component of its boundary intersects the boundary of the lens. But we can create a new, embedded, double bubble by eliminating the section of the geodesic which intersects the lens and sliding the rest of the geodesic inward to compensate for area. But this double bubble has perimeter less than the band lens, and further, contradicts regularity. So for such "difficult" area pairs, the band-lens is not a minimizer.

The second case to be considered is the case in which the lens does not fit along the short direction of the torus. In this case, the double bubble clearly has perimeter greater than three, and by Proposition 6.3, it cannot be a minimizer. Thus, if the immersed band-lens would minimize perimeter, then it must be embedded. QEF

## 6.7 Perimeter Minimizing Double Bubbles on the Flat 2-Torus and the Infinite Cylinder

**Main Theorem 1.** Each perimeter minimizing double bubble on a flat 2-torus is one of the following six types, depending on the areas to be enclosed:

i) the standard double bubble, $\Theta$,

ii) the band-lens, $\Phi$,

iii) the double band, $\Xi$,

iv) a symmetric two-component chain $\Omega$,

v) the hexagon tiling, $H$, on the hexagonal torus, or

vi) an octagon-square tiling, $X$.

Conjecture 1 says that the octagon-square tiling is never perimeter-minimizing.

*Proof.* Proposition 6.6 shows that a minimizer must be either a a contractible double bubble, a tiling, a swath, a band adjacent to a set of components whose union is contractible, or the double band (after perhaps relabelling the two regions and the exterior).

First we consider all minimizers that do not fall in the class of tilings.

By Proposition 6.1, the only potential contractible minimizer is the standard double bubble $\Theta$. By Proposition 6.10, the only potential swath minimizer is the symmetric two-component chain $\Omega$. By Proposition 6.11, the only potential minimizer among bands adjacent to a set of components whose union is contractible is the band-lens $\Phi$.

We focus here on potential minimizers that are tilings. Propositions 6.12 and 6.13 tell us that such a tiling must be either the special hexagon tiling $H$ or an octagon-square tiling. QEF

**Lemma 6.5.** On the flat 2-torus, there exist area pairs for which the band-lens is perimeter minimizing, and separately for which the standard double bubble is perimeter minimizing.

*Proof.* Consider regions $R_1$ and $R_2$ with area $A_1$ and $A_2$ in a perimeter minimizing double bubble. By [21] , the perimeter minimizing single bubble for area $A_1$ is either a single band, a circle, or the complement of a circle, depending on the area $A_1$. Thus, in the limit through minimizing double bubbles as $A_2$ goes to zero, we must get back the single band, circle, or complement of a circle enclosing $R_1$. This is only possible if for very small $A_2$, the minimizing double bubble is the band-lens or the standard double bubble which will in the limit be the single band or the circle or complement of a circle, respectively. QEF

**Corollary 6.1.** Each perimeter minimizing double bubble on a torus with unit shortest side length and longest side of length greater than $\sqrt{3}$ is of one of the following four types of Figure 6.1, depending on the areas to be enclosed:

i) the standard double bubble $\Theta$,

ii) the band-lens $\Phi$,

iii) the symmetric two-component chain $\Omega$, or

iv) the double band $\Xi$ formed by three geodesic circles.

*Proof.* By Proposition 6.4, no tiling can occur on a torus with unit shortest side length and longest side length greater than $\sqrt{3}$. Thus, by the Main Theorem, each perimeter minimizing double bubble must be of one of the above types.

QEF

**Corollary 6.2.** Each perimeter minimizing double bubble on the infinite cylinder of unit circumference is of one of the following four types of Figure 6.1, depending on the areas to be enclosed:

i) the standard double bubble $\Theta$,

ii) the band-lens $\Phi$,

iii) the symmetric two-component chain $\Omega$, or

iv) the double band $\Xi$ formed by three geodesic circles.

*Proof.* Clearly there cannot be a tiling on the infinite cylinder. As for the non-tiling minimizers, all proofs carry over from the proof of the main theorem. This is because locally, the cylinder looks like a section of the torus. And none of the proof strategies require "wrapping" around on the torus, where something could go on to infinity on the cylinder.

QEF

*Remark 6.5.* Note that on the infinite cylinder and the torus with longest side greater than $\sqrt{3}$, each of the four minimizers occurs. Since we have paramterized each of them, the phase plots in Figure 6.2 provide a complete description of the minimizers for each area pair.

## 6.8 Numerical comparisons between Minimizer Finalists

Now that we have established what the minimizers can be, we will do a numerical comparison to show which double bubbles occur as minimizers given a arbitrary area pairs. To do this, we plot perimeter versus area for each minimizer (using the parameterizations from the previous section) and numerically find the actual minimizer. Figure 6.2 contains 2-d plots showing where each minimizer occurs for the whole parameter space. Further, in Figure 6.23 we have plotted the intersections of the standard double bubble graph and the band-lens graph as evidence that these surfaces do in fact intersect in well-defined curves.

Figure 6.23: This graph of the intersection of the area for the standard double bubble and the band-lens shows that the boundaries between regions in the phase plots are likely to be smooth curves.

It is important to note that because we have not parameterized the octagon square, we can not *a priori* assume that each of these potential minimizers is actually a minimizer since the octagon square *might* beat everything else. Lemma 6.5 proves that the band-lens and the standard double bubble do in fact occur as minimizers.

# Chapter 7

# t3

**Charter.** We present a conjecture, based on computational results, on the area minimizing way to enclose and separate two arbitrary volumes in the flat cubic three-torus $\mathbf{T}^3$. For comparable small volumes, we prove that an area minimizing double bubble in $\mathbf{T}^3$ is the standard double bubble from $\mathbf{R}^n$.

## 7.1 Introduction

Our Central Conjecture 2.1 states that the ten different types of two-volume enclosures pictured in Figure 1 comprise the complete set of surface area minimizing *double bubbles* in the flat cubic three-torus $\mathbf{T}^3$. Our numerical results, summarized in Figure 7.2, indicate the volumes for which we conjecture that each type of double bubble minimizes surface area. Our main theorem, Theorem 4.1, states that given any fixed ratio of volumes, for small volumes, the minimizer is the standard double bubble. This result applies to any smooth flat Riemannian manifold of dimension three or four with compact quotient by its isometry group.

The *double bubble problem* is a two-volume generalization of the famous isoperimetric problem. The isoperimetric problem seeks the least-area way to enclose a single region of prescribed volume. About 200 BC, Zenodorus argued that a circle is the least-perimeter enclosure of prescribed area in the plane (see [19]). In 1884, Schwarz [47] proved by symmetrization that a round sphere minimizes perimeter for a given volume in $\mathbf{R}^3$. Isoperimetric problems arise naturally in many areas of modern mathematics. Ros [44] provides a beautiful survey.

Soap bubble clusters seek the least-area (least-energy) way to enclose and separate several given volumes. Bubble clusters have served as models for engineers, architects, and material scientists (Emmer [13] surveys some architectual applications, and the text by Weaire and Hutzler [57], an introduction to the physics of space-filling bubble clusters or *foams*, discusses numerous other applications).

**Existence and regularity** The surface area minimizing property of bubble clusters can be codified mathematically in various useful ways, using the rectifiable currents, varifolds, or $(\mathbf{M}, \epsilon, \delta)$-minimal sets of geometric measure theory (see [34]). In three dimensions, mathematically idealized bubble clusters consist of constant-mean-curvature surfaces meeting smoothly in threes at 120°along smooth curves, which meet in fours at a fixed angle of approximately 109° ([53, Theorems II.4, IV.5, IV.8], or see [34, Section 13.9]).

**Bubbles in the three-torus** In comparison with $\mathbf{R}^3$ and $\mathbf{T}^2$, the double bubble problem in $\mathbf{T}^3$ appears to be more difficult. In the torus it is not possible to push through a component bound like Hutchings', since a key step in his proof is to show that the double bubble has an axial symmetry. Nor is a variational bound after Wichiramala forthcoming, due to the additional topological complications in three dimensions. Until some new approach provides a component bound, there probably

will be no definitive results. Indeed, the single bubble for the three-torus is not yet completely understood, although there are partial results. The smallest enclosure of half of the volume of the torus was shown by Barthe and Maurey [2, Section 3] to be given by two parallel two-tori. Morgan and Johnson [36, Theorem 4.4] show that the least-area enclosure of a small volume is a sphere. Spheres, tubes around geodesics, and pairs of parallel two-tori are shown to be the only types of area minimizing enclosures for most tori by the work of Ritoré and Ros ([42, Theorem 4.2], [43]).

**Theorem about small volumes** Theorem 4.1 states that any sequence of area minimizing double bubbles of decreasing volume and fixed volume ratio has a tail consisting of standard double bubbles. The central difficulty is to bound the curvature. This accomplished, we show that the bubble lies inside some small ball that lifts to $\mathbf{R}^3$, where a minimizer is known to be standard [23]. The result extends to any flat 3- or 4-manifold with compact quotient by the isometry group.

Our proof goes roughly as follows. From the original sequence of double bubbles we generate a new sequence by rescaling the manifold at each stage so that one of the volumes is always equal to one. We can then apply compactness arguments and area estimates to the rescaled sequence to show that certain subsequences of translates have non-trivial limits. These limits are used to obtain a curvature bound on the original sequence. With such a bound, we can apply monotonicity to conclude that if the volumes are small, the double bubble is contained in a small ball. We conclude that it must be the same as the minimizer in $\mathbf{R}^3$ or $\mathbf{R}^4$, *i.e.* that it must be the standard double bubble by [23] or [41].

**Plan of the Paper.** Section 2 reviews the methods leading to our Central Conjecture 2.1 and to Figures 7.1 and 7.2. Section 3 surveys some subconjectures. Section 4 focuses on the proof of Theorem 1 on small volumes. Section 5 shifts from the cubic to other tori and discusses other conjectures

and candidates, including a "Hexagonal Honeycomb."

## 7.2 The Conjecture

**Generating Candidates.** Many possibilities for double bubbles in the three-torus $\mathbf{T}^3$ were proposed in brainstorming sessions by participants in the Clay/MSRI Summer School. In order to classify the candidates we used the following method. Starting with a standard double bubble, we imagined one of the two volumes growing until the bubble enclosing it wrapped around the torus and encountered an obstruction. Following the principles of regularity for bubble clusters, if a bubble collided with itself we opened the walls up, whereas if two different bubbles collided we allowed them to stick together. We then repeated this procedure for these new double bubbles, sometimes changing the perspective slightly or becoming a bit more fanciful (e.g. the Center Cylinder of Figure 7.1 or Gary Lawlor's Fire Hydrant of Figure 7.3, also known as "Scary Gary").

**Producing the Phase Diagram.** Brakke's *Surface Evolver* [5] was used to closely approximate the minimal area that a double bubble of each type needs to enclose specified volumes. Our initial simulations gave us the approximate surface area for each candidate double bubble, tested on partitions $v_1 : v_2 : v_3$ of a unit volume taken in increments of $0.01$. From the data obtained in these simulations, we found the least-area competitor for each volume triple. Figure 7.1 shows the candidates we found to be minimizing for some set of volumes. The phase diagram appearing in Figure 7.2 is the result of refining our initial computations along the boundaries with a 0.005 increment.

**Central Conjecture 1.** The ten double bubbles pictured in Figure 7.1 represent each type of surface area minimizing two-volume enclosure in a flat, cubic three-torus, and these types are minimizing

for the volumes illustrated in Figure 7.2.

**Comments.** One might expect that minimizers would be found among the various regularity-satisfying conglomerations of topological spheres and homotopically non-trivial tori, other possibilities being excessively complex. This was borne out in our computations. It is interesting to note, however, that not all of the simple possibilities appeared as minimizers (at least one of the enclosures in Figure 7.1 was always more efficient than the Transverse Cylinders pictured in Figure 7.3). The various double bubbles of Figure 7.3, while stable for a certain range of volumes, were never area minimizing. A challenging unsolved problem is to find all of the stable non-minimizing bubbles. Note that a given type from Figure 7.1 might be stable for a much wider range of volumes than those for which it actually minimizes surface area.

It is worth observing that all of the conjectured minimizers for the double bubble problem on $\mathbf{T}^2$ (a Standard Double Bubble, Band with Lens, Symmetric Chain, and Double Band) are echoed here in at least two ways. The Double Cylinder, the Slab Cylinder, the Double Slab, and the Cylinder String are $\mathbf{T}^2$ minimizers $\times$ $\mathbf{T}^1$. There are also more direct analogues, as is seen by comparing, for example, the three- and two-dimensional Standard Double Bubbles, or the Délauney and Symmetric Chains. See Corneli *et al.* [9] for more on the $\mathbf{T}^2$ minimizers.

## 7.3 Subconjectures

We now present a list of natural subconjectures, suggested either by the phase diagram or by examination of the pictures made with *Surface Evolver*.

One immediate observation is that the edges of our phase diagram appear to characterize single bubbles in $\mathbf{T}^3$.

**Conjecture 7.1 (Ritoré and Ros [42], [43], [44]).** The optima for the isoperimetric problem in a cubic $\mathbf{T}^3$ are the sphere, cylinder, and slab.

We still do not know proofs for the following intuitive conjectures about the double bubble problem (although Theorem 4.1 gives partial results on Conjecture 3.3):

**Conjecture 7.2.** An area-minimizing double bubble in $\mathbf{T}^3$ has connected regions and complement.

**Conjecture 7.3.** For two small volumes the standard double bubble is optimal.

**Conjecture 7.4.** For one very small volume and two moderate volumes, the Slab Lens is optimal.

**Conjecture 7.5.** The first phase transition as small equal, or close to equal, volumes grow is from the standard double bubble to a chain of two bubbles bounded by Délaunay surfaces.

Délaunay surfaces are constant-curvature surfaces of rotation, and as such have full rotational symmetry. From the *Surface Evolver* pictures, it appears that the conjectured surface area minimizers always have the maximial symmetry, given the constraints, a fact which leads to the following natural conjecture.

**Conjecture 7.6.** A minimizer is as symmetric as possible, given its topological type.

We will conclude this section with a proposition which represents the first step towards establishing a symmetry property. Specifically, we prove that for any double bubble, there is a pair of parallel planes (actually two-tori) that cut both regions in half. Hutchings [22, Theorem 2.6] used the fact that in $\mathbf{R}^n$ a double bubble has two perpendicular planes that divide both regions in half. This is an easy step in a difficult proof that the function that gives the least-area to enclose two given volumes is concave. Before proving a basic result of a similarly elementary flavor for the torus, we mention that we conjecture that the much deeper concavity result also holds:

**Conjecture 7.7.** The least area to enclose and separate two given volumes in the three-torus is a concave function of the volumes.

If one could prove Conjecture 3.7, one would then be able to apply other ideas in Hutchings' paper [22, Section 4] to obtain a functional bound on the number of components of a minimizing bubble.

**Proposition 7.1 (Deluxe Ham Sandwich Theorem).** In a rectangular torus, if a double bubble lies inside a cylinder $S^1 \times D^2$, then there is a plane that cuts both volumes in half.

*Proof.* This is a generalization of the standard argument for the ham sandwich theorem in Euclidean space. Assume that the identification occurs in the vertical direction. Take a circle in the $xy$ plane indexed by $\alpha \in [0, \pi]$. Then for each $\alpha$, rotate the surface by $\alpha$ in the $xy$ plane and consider the family of pairs of flat tori parallel to the $xz$ plane that are at distance 1/2 from each other. Then there is at least one such pair of planes that cuts the volume $V_1$ in half, for each $\alpha$. There may be an interval of such pairs for a given $\alpha$. However, some one parameter family of these planes may be chosen which varies continuously as a function of $\alpha$, and one such plane cuts $V_2$ in half. QEF

## 7.4 Small Volumes

Conjecture 7.3 stated that small volumes are best enclosed by a standard double bubble. Theorem 4.1 proves that for any fixed volume ratio, the standard double becomes optimal when the volumes are sufficiently small. This result holds for many flat three- and four-dimensional manifolds (and see Remark 7.2).

The standard double bubble consists of three spherical caps meeting at 120 degrees. (If the volumes are equal, the middle surface is planar.) It is known to be minimizing in $\mathbf{R}^3$ [23] and $\mathbf{R}^4$ [41].

**Theorem 7.1.** Let $M$ be a smooth flat Riemannian manifold of dimension three or four, such that $M$ has compact quotient by the isometry group. Fix $\lambda \in (0, 1]$. Then there is an $\epsilon > 0$, such that if $0 < v < \epsilon$, a perimeter-minimizing double bubble of volumes $v, \lambda v$ is standard.

*Remark 7.1.* Solutions to the double bubble problem exist for all volume pairs in manifolds with compact quotient by their isometry group. The proof is the same as in Morgan [34, Section 13.7].

For the proof we will regard a double bubble as a pair of 3- or 4−dimensional rectifiable currents,

$R_1$ and $R_2$, each of multiplicity one, of volumes $V_1 = \mathbf{M}(R_1)$ and $V_2 = \mathbf{M}(R_2)$. The total perimeter of such a double bubble is $\frac{1}{2}(\mathbf{M}(\partial R_1) + \mathbf{M}(\partial R_2) + \mathbf{M}(\partial(R_1 + R_2)))$. Here $\mathbf{M}$ denotes the mass of the current, which can be thought of as the Hausdorff measure of the associated rectifiable set (counting multiplicities). For a review of the pertinent definitions, see notes from Morgan's course at the Summer School [37] or the texts by Morgan [34] or Federer [14].

Before proceeding it is helpful to fix some notation. By the Nash and subsequent embedding theorems, we may assume $M$ is a submanifold of some fixed $\mathbf{R}^N$. We will consider a sequence of perimeter-minimizing double bubbles in $M$, containing the volumes $v$ and $\lambda v$, as $v \to 0$. For each $v$, $M_v$ will denote $s_v(M)$ in $s_v(\mathbf{R}^N)$, where $s_v$ is the scaling map that takes regions with volume $v$ to similar regions with volume 1. In particular, $s_v$ maps our perimeter-minimizing double bubble containing volumes $v$ and $\lambda v$ to a double bubble which we call $S_v$ which contains volumes 1 and $\lambda$, and which is of course perimeter-minimizing for these volumes.

We focus on the case of dimension three; the proof for dimension four is essentially identical.

**Lemma 7.1.** There is a $\gamma > 0$ such that if $R$ is a region in an open Euclidean 3-cube $K$ and $\mathrm{vol}(R) \leq \mathrm{vol}(K)/2$, then

$$\mathrm{area}(\partial R) \geq \gamma(\mathrm{vol}(R))^{2/3}.$$

*Proof.* Let $\gamma_0$ be such an isoperimetric constant for a cubical 3-torus, so that $\mathrm{area}(\partial P) \geq \gamma_0(\mathrm{vol}(P))^{2/3}$ for all regions $P \subseteq \mathbf{T}^3$. Such a $\gamma_0$ exists by the isoperimetric inequality for compact manifolds [34, Section 12.3]. Make the necessary reflections and identifications of the cube $K$ to obtain a torus containing a region $R'$ with eight times the volume and eight times the boundary area of $R$. The claim follows, with $\gamma = \gamma_0/2$. QEF

**Proof of Theorem 4.1.** The first step is to show that the sequence $S_v$, suitably translated and rotated, has a subsequence that converges as $v \to 0$ and has $V_1 \neq 0$ in the limit. Our argument also shows that there is a subsequence that converges to a limit with $V_2 \neq 0$, but does not show that

there is a subsequence where both volumes are non-zero in the limit. This is because while we are exerting ourselves trapping the first volume in a ball, the second one may wander off to infinity.

We first show the existence of a covering $\mathcal{K}_v$ of $M_v$ with bounded multiplicity, consisting of cubes contained in $M_v$, each of side-length $L$. Lemma 7.1 will give us a positive lower bound on the volume of the part of $R_1$ that is inside one of these cubes for each $S_v$. We will then apply a standard compactness theorem to show that a subsequence of the $S_v$'s converges.

Take a maximal packing by balls of radius $\frac{1}{4}L$. Enlargements of radius $\frac{1}{2}L$ cover $M_v$. Circumscribed cubes of edge-length $L$ provide the desired covering $\mathcal{K}_v$. To see that the multiplicity of this covering is bounded, consider a point $p \in M_v$. The ball centered at $p$ with radius $2L$ contains all the cubes that might cover it, and the number of balls of radius $\frac{1}{4}L$ that can pack into this ball is bounded, implying that the multiplicity of $\mathcal{K}_v$ is also bounded by some $m > 0$.

Now consider some such covering, with $L = 2$. By Lemma 7.1, there is an isoperimetric constant $\gamma$ such that

$$\text{area}(\partial(R_1 \cap K_i)) \geq \gamma(\text{vol}(R_1 \cap K_i))^{2/3},$$

and therefore, since $\max_k \text{vol}(R_1 \cap K_k) \geq \text{vol}(R_1 \cap K_i)$ for any $i$,

$$\text{area}(\partial(R_1 \cap K_i)) \geq \gamma \frac{\text{vol}(R_1 \cap K_i)}{(\max_k \text{vol}(R_1 \cap K_k))^{1/3}}. \tag{7.1}$$

Note that the total area of the surface is greater than $1/m$ times the sum of the areas in each cube, and the total volume enclosed is less than the sum of the volumes, so summing Equation 7.1 over all the cubes $K_i$ in the covering $\mathcal{K}_v$ yields

$$\text{area}(S_v) \geq \text{area}(\partial R_1) \geq m\gamma \frac{V_1}{(\max_k \text{vol}(R_1 \cap K_k))^{1/3}}$$

and

$$(\max_k(\text{vol}(R_1 \cap K_k)))^{1/3} \geq m\gamma \frac{V_1}{\text{area}(S_v)} \geq \delta > 0,$$

because $V_1 = 1$ and it is easy to show that $\text{area}(S_v)$ is bounded (since there is a bounded way of enclosing the volumes). Translate each $M_v$ so that the cube where the maximum occurs is centered

at the origin of $\mathbf{R}^N$, and rotate so that the tangent space of each $M_v$ at the origin is equal to a fixed $\mathbf{R}^3$ in $\mathbf{R}^N$. The limit of the $M_v$ will be equal to this $\mathbf{R}^3$. Since a cube with edge-length $L$ centered at the origin fits inside a ball of radius $2L$ centered at the origin, we have

$$\text{vol}(R_1 \cap B(0, 2L)) \geq \delta^3$$

for every $S_v$. By the compactness theorem for locally integral currents ([34, pp. 64,88], [48, Section 27.3, 31.2, 31.3]) we know that a subsequence of the $S_v$ has a limit, which we will call $D$, with the property that $\text{vol}(R_1) \geq \delta^3$. This completes the first step.

Since $D$ is contained in the limit of the $M_v$, namely, the copy of $\mathbf{R}^3$ chosen above, and each $S_v$ is minimizing for its volumes, a standard argument shows that the limit $D$ is the perimeter-minimizing way to enclose and separate the given volumes in $\mathbf{R}^3$ (cf. [34, 13.7]). In the limit, $V_2$ could be zero, in which case D is a round sphere. If both volumes are non-zero, D is the standard double bubble ([23] and [41], or see [34, Chapter 14]).

Our goal is now to prove that the double bubble of volumes $v$, $\lambda v$ lies inside a trivial ball, when $v$ is small enough. This can be accomplished using the monotonicity theorem for mass ratio [1, Section 5.1(1)], which implies that for a perimeter-minimizing bubble cluster, a small ball around any point on the surface contains some substantial amount of area. This will limit the number of disjoint balls we can place on the surface. The monotonicity theorem applies only to surfaces for which the mean curvature is bounded, *i.e.*, for which there is a $C$ such that for smooth variations

$$\frac{dA}{dV} \leq C.$$

Accordingly, the second major step in the proof is to obtain such bound on the curvature, for $v$ small. It suffices to show that all smooth variation vector fields have the property that changes in the volume of $S_v$ and in the area of $\partial S_v$ are controlled. Take a smooth variation vector field $F$ in $\mathbf{R}^n$ such that for $D$,

$$dV_1/dt = \int_{\partial R1} (F \cdot n)\, dA = c > 0$$

and

$$dV_2/dt = 0.$$

(Note that we need the first volume to be non-zero, or its variation could be zero.) For $v$ small enough the subsequence of $S_v$ headed towards the limit $D$ has the property that $dV_1/dt$ is approximately $c$ and $dV_2/dt$ is approximately 0.

By the argument in the first step, we can translate each $S_v$ similarly, so that a subsequence of the subsequence above converges to a minimizer $D'$ in $\mathbf{R}^3$ where the second volume is non-trivial. This time we take a smooth variation vector field $F'$, such that for $v$ small enough the subsequence of $S_v$ headed to $D'$ has the property that $dV_1/dt$ is approximately 0 and $dV_2/dt$ is approximately $c' > 0$. This proves that for this subsequence the change in volume is bounded below.

Now we need to show that the change in area is bounded above. This follows from the fact that every rectifiable set can be thought of as a varifold [34, Section 11.2]. By compactness for varifolds [1, Section 6], the $S_v$, situated so that the first volume does not disappear, converge as varifolds to some varifold, $J$. The first variation of the varifolds also converge, *i.e.*, $\delta S_v \to \delta J$, see [1]. The first variation of a varifold is a function representing the change in area. Therefore, far enough out in the sequence the change in area of the $S_v$ under $F$ is bounded close to the change in area of $J$ under $F$, which is finite. Similarly, the change in area of the $S_v$ under $F'$ is bounded.

We conclude that the mean curvature of $S_v$ is bounded for two independent directions in the two-dimensional space of volume variations, and hence for all variations. This completes the second step.

The third and final step is to show that all of the surface area is contained in some ball in $M_v$, of fixed radius for all $v$. Eventually, as $v$ shrinks and $M_v$ grows, this ball will have to be trivial in $M_v$. We will then use the result that the optimal double bubble in $\mathbf{R}^3$ is standard to show that our double bubble is standard as well.

As the $S_v$ are approaching the minimizer in $\mathbf{R}^3$, there must be a bound $A$ on the perimeters of

the $S_v$. By monotonicity of mass ratio [1, Section 5.1(1)], every unit ball centered at a point of $S_v$ contains perimeter $\delta > 0$. Therefore there are at most $A/\delta$ such disjoint balls.

We claim that the $S_v$ are eventually connected. There is an upper bound on the diameter of any component, $2A/\delta$. Since we are controlling curvature, our components cannot become too small. Since every unit ball contains at least $\delta$ area, we also have a lower bound on the area of each component, when unit balls are trivial. Unless eventually the $S_v$ are connected, you can arrange to get in the limit a disconnected minimizer in $\mathbf{R}^3$, a contradiction.

Hence $S_v$ is contained in a ball of radius $2A/\delta$ for all $v$. Since our original manifold has compact quotient by its isometry group, there is a radius such that balls in the original manifold of that radius or smaller are topologically trivial. Hence, as we expand the manifold, eventually balls of radius $2A/\delta$ can be lifted to $\mathbf{R}^3$, which means that they are Euclidean. Hence, $S_v$ is eventually contained in a Euclidean ball, and is therefore the standard double bubble ([41], [23], [34, Chapter 14]).

Finally, since $S_v \subset M_v$ is simply a scaled version of the original double bubble in $M$, we conclude that the original double bubble is standard as desired. QEF

*Remark 7.2.* Given $n$, $m$, similar arguments show that for any smooth $n$-dimensional Riemannian manifold with compact quotient by the isometry group, given $0 \leq \lambda \leq 1$, there are $C$, $\epsilon > 0$, such that for any $0 < v < \epsilon$, a minimizing cluster with $m$ prescribed volumes between $\lambda v$ and $v$ lies inside a ball of diameter at most $Cv^{1/n}$.

To further deduce that the cluster smoothly approximates a Euclidean minimizer would require knowing that convergence weakly and in measure, under bounded mean curvature, implies $C^1$ convergence, as is known for hypersurfaces without singularities ([1, Section 8], see [35, Section 1.2])

## 7.5 Special Tori

Changing the shape of the torus, by stretching it or by skewing some or all of its angles, would certainly change the phase diagram of Figure 7.2:

**Conjecture 7.8.** In the special case of a very long $T^3$ the Double Slab is optimal for most volumes.

Special tori may have special minimizers:

**Conjecture 7.9.** For the special case of a torus based on a relatively short $\frac{\pi}{3}$-rhombic right prism, the Hexagonal Honeycomb prism of Figure 7.4 is a perimeter-minimizing double bubble for which both regions and the exterior each have one third the volume, or when two volumes are equal and the third is close.

Indeed, for such volumes the Hexagonal Honeycomb ties the Double Slab, just as in the $\frac{\pi}{3}$-rhombic two-torus a Hexagonal Tiling ties the Double Band [9].

In the *triple bubble problem* for the Face Centered Cubic (FCC) and Body Centered Cubic (BCC) tori, if not for the standard cubic torus, we would expect to find minimizers that lift to $\mathbf{R}^3$ as periodic foams with cells of finite volume. For example, the Weaire-Phelan foam would be expected to appear as lifts of some solutions to the *seven* bubble problem in the BCC torus. The Weaire-Phelan foam, a counterexample to Kelvin's conjectured best way to divide space into unit volumes ([56], [27]; see [34, Chapter 15], [6]). Kelvin's foam might appear as a minimizer for the seven bubble problem in the FCC torus. It is interesting to note that the Weaire-Phelan foam has essentially the same shape as chlorine hydrate crystale.

In contrast with the triple bubble problem, it would be extremely unlikely if there was a solution to the double bubble problem in any torus that lifted to a division of $\mathbf{R}^3$ into finite volumes, since by regularity, singular curves meet in fours. This means that singular points look locally like the cone over a tetrahedral frame, and hence have four volumes coming together. Since a region will never be adjacent to another component of the same region (because the dividing wall could be removed

to decrease area and maintain volumes), a foam generated using a fundamental domain coming from a double double in the three-torus would have to exhibit the strange property that the singular curves never meet.

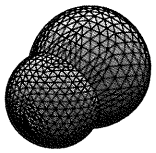
**Conjecture 7.10.** There are no least-area divisions of $\mathbf{T}^3$ into three volumes that lift to a foam in $\mathbf{R}^3$.

This conjecture suggests in addition that it is not likely that there will be are any other special minimizers.

**Conjecture 7.11.** The double bubbles of Figure 7.1 together with the Hexagonal Honeycomb of Figure 7.4 comprise the complete set of area minimizing double bubbles for all three tori.

As a final remark, in light of the fact that the adding more bubbles seems likely to produce many interesting candidates, we would like to mention one final conjecture.

**Conjecture 7.12.** For the triple bubble problem in a cubic $\mathbf{T}^3$ in the case where one of the volumes is small, the minimizers will look like the double bubbles of Figure 7.1 with a small ball attached. The phase diagram will look just like our Figure 7.2.



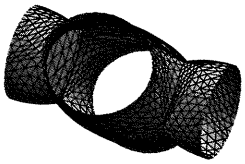
STANDARD DOUBLE BUBBLE



DÉLAUNEY CHAIN



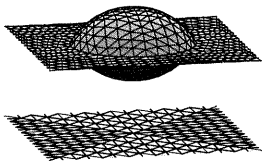
CYLINDER LENS



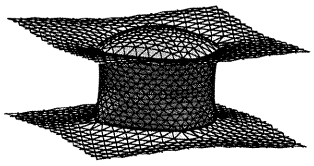
CYLINDER CROSS



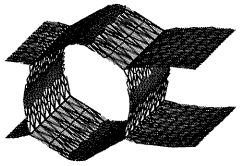
DOUBLE CYLINDER



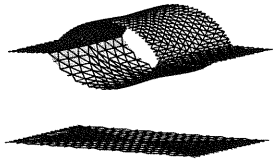
SLAB LENS



CENTER BUBBLE



CYLINDER STRING



SLAB CYLINDER



DOUBLE SLAB

Figure 7.1: Catalog of Conjectured Area-Minimizing Double Bubbles in the Flat Cubic 3-Torus.

SDB = Standard Double Bubble
DC = Delauney Chain
CL = Cylinder Lens
CC = Cylinder Cross
2C = Double Cylinder
SL = Slab Lens
CB = Center Bubble
CS = Cylinder String
SC = Slab Cylinder
2S = Double Slab

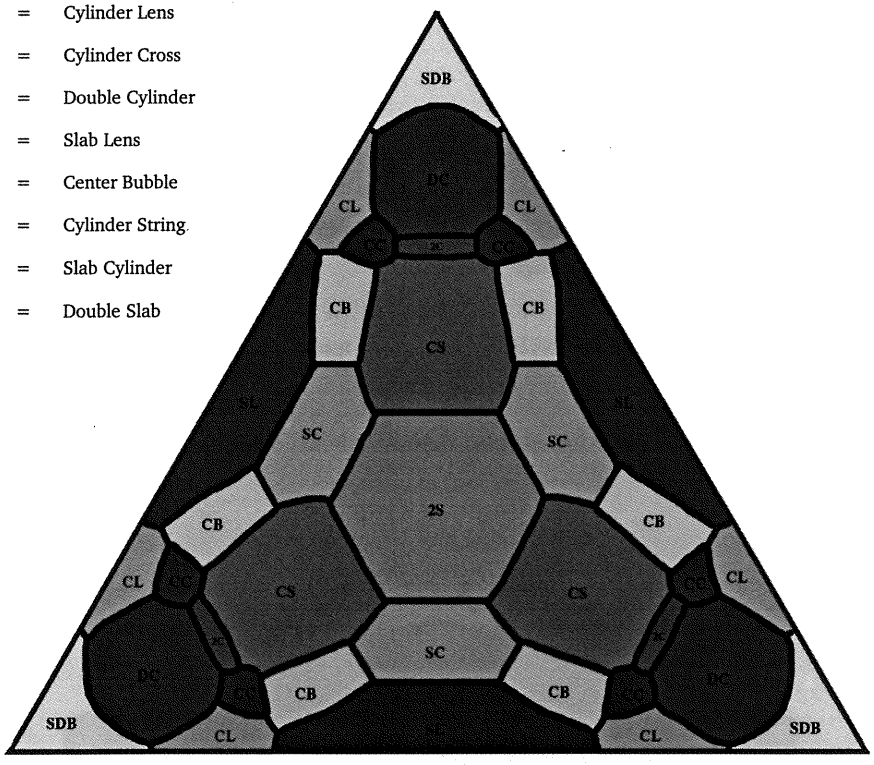
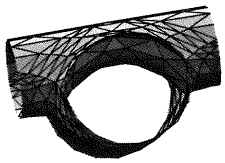
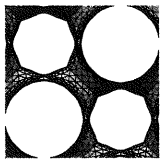


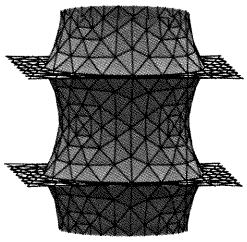

Figure 7.2: Phase portrait: volumes and corresponding double bubble. In the center both regions and the complement have one third of the total volume; along the edges one volume is small; in the corners two volumes are small.



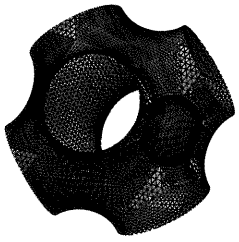
TRANSVERSE CYLINDERS



DOUBLE HYDRANT



CENTER CYLINDER



HYDRANT LENS

Figure 7.3: Inefficient double bubbles

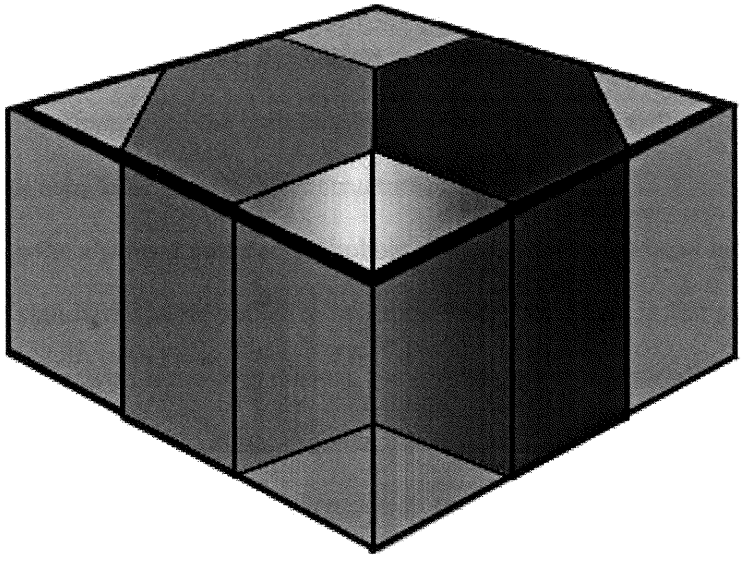


Figure 7.4: Hexagonal Honeycomb

# conjectural epilog

These are some of the thoughts I have had while working on this thesis.

Any resembalence of the things discussed here to the current views of my collaborators or of the mathematical community at large or to the actual facts of the matter has to be regarded as coincidental at best.

**Symmetry and algebraic structure** Comparing the results from the various chapters leads to the conjecture that area-minimizing double bubbles in constant curvature spaces always have some of the same symmetries as the space, and more generally maximize symmetry in some sense.

We now consider the possibility that by seeking to maximize symmetry, bubble clusters not only take on some of the symmetries of the ambient, but also exhibit certain symmetries (*i.e.* invariants) associated with their own 'intrinsic' algebraic structure.

If this is the case, the algebraic structure of bubble clusters is bound to be of interest. Recall from Chapter 4 that the standard double bubble is invariant under inversion in the separating cap. For standard bubble clusters with more components, we can again move between adjacent bubbles by inversions – although doing so sometimes requires two moves, since if $B_3$ curves into $B_2$, we might send a point $p \in B_1$ to a point $q \in B_3$ by inversion in the wall separating $B_1$ and $B_2$; all such '$q$'s are however mapped into $B_2$ by inversion in the wall separating $B_2$ and $B_3$. When there are more than two bubbles, points fixed by the entire group of these (generalized) adjacency transformations

(or, the *cluster group*) are always in the singular set. When there are more than four bubbles, there may be no points that are fixed by the entire cluster group, but all singular curves are fixed by some subgroup of order three, and all singular points are fixed by some subgroup of order four.

The intrinsic algebraic structure of foams may may be particularly interesting (or, possibly, but not likely, particularly easy to understand) since periodic foams have a natural group acting on them already.

Maximizing symmetry here should for the time being be understood in a weak sense, say, maximizing symmetry while taking into consideration certain other factors. The cluster group may be hard to define or even meaningless in clusters that have non-contractible components (although I suspect that that won't be much of an obstacle). When the boundary surfaces are constant-mean-curvature sufaces other than pieces of spheres, things certainly *sound* less attractive, but the important ideas still seem to carry through.

At the bottom of this line of inquiry lies the fundamental assumption that there are no factors besides symmetry that can come into play to reduce area.

As one specific example that might give evidence as to the truth or falsity of this assumption, here is a problem that I think is open: the Kelvin foam *seems* to have 'more symmetry' than the Weaire-Phelan foam that takes less area. However, it might be that the Weaire-Phelan foam actually has more symmetries when one looks at its cluster group. Decide this question.

**Is the $m$-bubble problem *inductive*?** The standard double bubble is clearly a relaxation of the solution to the single bubble problem in space minus a round ball. In addition, it seems that all of the other conjectured minimizers for the double bubble problem in $\mathbf{T}^k$ $(k = 1, 2, 3)$ are relaxations of the solution to the single bubble problem in the torus minus a single bubble solution. This seems to be plainly obvious, albeit ever-so-hard to prove, for everything but the hexagonal tiling and hexagonal honeycomb. And upon further reflection it does not seem at all unlikely that the hexagonal tiling (resp. hexagonal honeycomb) is a relaxation of the solution to the single bubble problem in a 'nut',

*i.e.* a hexagonal torus minus a round disk (resp. tubular neighborhood of a short geodesic).

These considerations lead to the following *meta-Conjecture*.

**Meta-Conjecture.**

1. The solution to the $m$-bubble problem in any space of constant curvature $C$ is bounded by regularity satisfying relaxations of some collection of (a) tubular neighborhoods of totally geodesic submanifolds (counting points as a degenerate case) and (b) tubular neighborhoods of totally geodesic submanifolds (not allowing points this time) regarded as manifolds with boundary that then have some noncontractible (with respect to their boundary) holes drilled out of them; moreover,

2. the boundary of the union of the closure of the holes in each tubular neighborhood $T_0$ in (1b) bounds a totally geodesic submanifold of $T_0$ (again not allowing points); and, ultimately,

3. the problem is inductive in the sense that the $m$-bubble problem is given by relaxing the solution to the 1-bubble problem for the current with free boundary given by designating some solution to the $(m-1)$-bubble problem in $C$ as boundary.

# appendices

## A Algebraic preliminaries

**Definition A.1.** A *vector space* is a quadruple $(X, F, +, *)$, where $X$ is a set, $F$ is a field, and $+$ and $*$ are operations such that

i. $\forall x \in X, f \in F : x * f \in X$,

ii. $\forall x, y \in X : x + y \in X$.

Furthermore, addition $(+)$ is commutative, there exists an additive identity (written 0) and there are additive inverses. Scalar multiplication $(*)$ is associative and distributes over addition both ways.

We write $V = (X, F, +, *)$, and when we say $v \in V$ we mean $v \in X$. The field $F$ is called the *base field*. In this thesis, the base field is usually the real or the complex numbers.

The notion of a *basis* and of how to do computations with a basis (using *matrices*) are assumed to be familiar to the reader, at least for finite dimensions.

**Definition A.2.** A *bilinear form* on a finite dimensional vector space $V$ is a map $b$ from $V \times V$ into the base field that is linear in each factor. If we chose a basis for the vector space, then the bilinear form is given by matrix multiplication as follows: $b : (x, y) \mapsto y^T B x$, for some matrix $B$.

**Remark A.1.** The two dimensional case is

$$((x_1, x_2), (y_1, y_2)) \mapsto \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2.$$

**Definition A.3.** An *inner product* on a vector space is a bilinear form that is also *positive definite* (which means that $b(x, x) \geq 0$ and is equal to zero only when $x = 0$) and *symmetric* ($b(x, y) = b(y, x)$).

**Remark A.2.** Given an abstractly defined linear map on a finite dimensional vector space, to determine the entries of the matrix representing this map it suffices to compute the action of the map on a basis. Similarly for bilinear forms and thus for inner products. The matrix representing a bilinear form $b$ with respect to the basis $\{x_i\}_{i=1}^n$ is $(b(x_i, x_j))$.

**Definition A.4.** A *vector space isomorphism* is a linear, bijective map between two vector spaces.

**Definition A.5.** The *dual* of a vector space $V$ is the set of linear maps from $V$ into the base field. We denote the dual of $V$ by the symbol $V^*$.

**Remark A.3.** It is easy to check that vector space duals are themselves also vector spaces. It is also easy to check that in the finite dimensional case a vector space is isomorphic to its dual. This is not generally true for infinite dimensional vector spaces.

**Definition A.6.** A *group* is a set $X$ together with a binary operation $*$ with the properties

i. $\forall x, y, z \in X : x * (y * z) = (x * y) * z$,

ii. $\exists e \in X : \forall x \in X : e * x = x * e = x$,

iii. $\forall x \in X : \exists x^{-1} \in X : x * x^{-1} = x^{-1} * x = e$,

We write $G = (X, *)$ and when we say $g \in G$ we mean $g \in X$. It is easy to deduce that a vector space is an *additive group* (that is, the operation $(+)$ in Definition A.1 plays the role of $(*)$ in Definition A.6). We now define another important example:

**Definition A.7.** The *symmetric group on $n$ elements* $\mathfrak{S}^n$ is equal to the set of permutations of $n$ elements together with the operation of composition.

**Remark A.4.** So, for example, $\mathfrak{S}^3$ can be written down as $\{(), (12), (23), (13), (123), (321)\}$ (do nothing, sent 1 to 2 and 2 to 1, send 2 to 3 and 3 to 2, etc.) and then it is easy to check that with the operation $\circ$, $\mathfrak{S}^3$ really is a group (for example, $(123)\circ(321) = ()$, so $(321) = (123)^{-1}$). The symmetric group is important in the theory of determinants (and many other places). Indeed, using the *sign* of permutations (even if the an even number of elements are permuted, odd if an odd number of elements are permuted) it is not too difficult to show (or at least beleive) that for an $n \times n$ matrix $A$,

$$det(A) = \sum_{\sigma \in \mathfrak{S}^n} \text{sign}(\sigma) A_{\sigma 1,1} \cdot \ldots \cdot A\sigma n, n);$$

see [20, Chapter 5] for the details.

**Definition A.8.** If a group $G$ acts on a set $X$ (in the sense that $\mathfrak{S}^n$ acts on the set $\{1, \ldots, n\}$) the *orbit* of a point $x \in X$ is defined to be the set of points in $X$ that are hit by applying elements of $G$ to $x$ (with repeated application of the same element allowed).

# B Topological preliminaries

**Definition B.1.** A *topological space* is a pair $(X, \tau)$ such that $\tau$ is comprised of subsets of $X$ and has the further additional properties

i. $\emptyset \in \tau$ and $X \in \tau$.

ii. finite intersections of elements of $\tau$ are again in $\tau$

iii. arbitrary unions of elements of $\tau$ are again in $\tau$.

A collection of subsets $\tau$ satisfying these conditions make $\tau$ into a *topology*. The elements of $\tau$ are then called the *open sets of $X$ in the $\tau$ topology*.

**Remark B.1.** If $T = (X, \tau)$ is a topological space, when we say $p \in T$ we mean $p \in X$.

**Definition B.2.** A map between topological spaces is *continuous* if the inverse image of every open set is an open set.

**Definition B.3.** A map between topological spaces is a *homeomorphism* if it is a continuous bijection such that in addition the image of every open set is an open set. (So, it is a bijection of points and it induces a bijection on open sets.)

Since they are a bijection of points and sets, we conclude that homeomorphisms respect topological structure. In this sense it would be appropriate to consider them to be 'topological isomorphisms'.

We will need the following two 'special' definitions.

**Definition B.4.** A topological space $T$ is *Hausdorff* if it has the property that any pair of distinct points $p, q \in T$ belong to disjoint open sets.

(Spaces without this property include easy examples like $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b, c\}\}$ and more complicated examples, like $(\mathbf{R} \cup (0, 1) \subset \mathbf{R}^2, \{U; U \cap \mathbf{R}$ such that $U$ is open in the metric topology on $\mathbf{R}$ and if $p \in U$ then $(U \cap \mathbf{R}) \cup 0$ is a neighborhood of 0 in $\mathbf{R}\})$. In general we will exclude non-Hausdorff spaces like this because they cause trouble.)

**Definition B.5.** A topological space $T = (X, \tau)$ is *paracompact* if there is a countable set $\mathcal{B}$ of open sets in $\tau$ with the property that every open set in $\tau$ is the union of elements of $\mathcal{B}$.

**Remark B.2.** In general, a set $\mathcal{B}$ that contains the set of open sets is called a basis for the topology. Every topology has a basis (any topology contains all of its open sets). We note that the basis *condition* is equivalent to the condition that any point in an open set is also in an element of $\mathcal{B}$. The *word* paracompact is synonymous with the phrase *second countable*.

This begs the question, what was *first countable*? This condition is not needed elsewhere in this thesis, but so that the reader leaves with a satisfied mind, here it is anyway: each point in $X$ must

have the property that there is a countable set $\mathcal{B}_p$ of open sets containing $p$ such that every open set containing $p$ also contains an element of $\mathcal{B}_p$. The set $\mathcal{B}_p$ is then called a *local basis*.

**Definition B.6.** A *metric space* is a pair $(X, d)$ such that $d$ is a function defined on $X \times X$ with the following additional properties:

i. $0 \leq d(x, y) < \infty$

ii. $d(x, y) = 0 \Leftrightarrow x = y$

iii. $d(x, y) = d(y, x)$

iv. $d(x, y) \leq d(x, z) + d(z, y)$

**Definition B.7.** The set of *metric balls* $\{\mathbf{D}_x(r) = \{y \in X; d(y, x) < r\}; x \in X, 0 \leq r \in \mathbf{R}\}$ form a topology for $X$. This is called the *metric topology*.

**Remark B.3.** We remark that $\mathbf{R}^n$ with the metric topology (the *standard topology on* $\mathbf{R}^n$) is both paracompact and Hausdorff. Metric balls with a rational radius centered at rational points form a countable basis for the topology, and by chosing $\rho$ small enough, any two points $p$ and $q$ can be enclosed in disjoint metric balls of radius $\rho$.

The dividing line between topology and geometry moves around a lot. It shows up right here this time.

# C Geometric Preliminaries

For our purposes, Riemannian geometry is the study of

I manifolds, with

II a differentiable structure,

III admitting the definition of vector fields,

IV which can themselves be differentiated;

V with also a notion of lengths and angles associated with vectors based at a point in the manifold,

VI leading to the notion of shortest paths in the manifold,

VII and to many other results having to do with global structure.

Item VII includes answers to questions like 'What are the least surface area objects in my manifold?'.

## Geometry of manifolds

Manifolds generalize the idea of parametrized surfaces. This is useful, because, for example, it is all fine and good to study the upper-hemisphere, considered as the graph of $\sqrt{r^2 - x_2 - y^2}$ over a disk of radius $r$, but if we can't talk coherently about the whole sphere, our mathematical lives will be considerably less interesting. Manifold theory lets us talk about the whole sphere and about lots of other things. One of the central ideas in the definition of a smooth manifold is that it stitches together 'local parameterizations' in a nice way.

**Definition C.1.** A *manifold* is a paracompact Hausdorff topological space such that about each point $p \in M$ there is some open set $U \ni p$ such that there exisits a homeomorphism $\varphi$ from $U$ to $\mathbf{R}^n$.

The pairs of open sets and homeomorphisms $(U, \varphi)$ from Definition C.1 are called *charts*. If we say that a point $p$ is in a chart $(U, \varphi)$, what we mean is that $p \in U$. All of the charts taken together are called an *atlas*. Our atlases are henceforth assumed to be *complete* in the sense that they are never contained in some other atlas. This is a viable assumption, because we can keep adding in charts until we complete the atlas. We denote the complete atlas for our manifold by $\mathcal{A}$.

We write $M$ for the triple $(X, \tau, \mathcal{A})$, and when we say $p \in M$ we mean $p \in X$. Because this convention is so prevalently and widely used, it is sometimes easy to lose sight of the important fact that the topology $\tau$ as well as the point-set $X$ are fixed; in particular, you should always be ready to ask 'with what topology?' if someone describes a manifold to you in terms of its point set only and their choice of topology was not obvious.

The atlas makes it possible to compute things; we can use a chart $(U, \varphi)$ to define a coordinate system on $U$ which we then use to study small pieces of the manifold analytically. If we are interested in what is going on around a point $p \in U \subset M$, we can assume $p$ is mapped by the associated homeomorphism $\varphi$ to $0 \in \mathbf{R}^n$, since wherever it is mapped we can compose with a translation to recenter $\mathbf{R}^n$ on the image of $p$. We then make the following extremely useful definition.

**Definition C.2.** If $x_1, \ldots, x_n$ are the functions which give the coordinates of the point $x$ in $\mathbf{R}^n$, we say that the set of functions $x_1 \circ \varphi, \ldots, x_n \circ \varphi$ are *local coordinates centered at p* or just *local coordinates at p*.

**Remark C.1.** For notational convenience, we will frequently make use of the alternate notation for local coordinates at $p$, namely $u_1 = x_1 \circ \varphi, \ldots u_n = x_n \circ \varphi$.

In order to compute anything very interesting, we need some more structure.

**Definition C.3.** A manifold's atlas is a *differentiable structure on the manifold* if the charts of Definition 1 satisfy the following property: if $(U_1, \varphi_1)$ is a chart about $p$ and $(U_2, \varphi_2)$ is a chart about $q$ then whenever $U_1 \cap U_2 \neq \emptyset$, the map $\varphi_1 \circ \varphi_2^{-1}$ from $\varphi_2(U_1 \cap U_2) \subset \mathbf{R}^n$ onto $\varphi_1(U_1 \cap U_2) \subset \mathbf{R}^n$ is differentiable.

We say a manifold has a $C^k$ *structure* if the transition functions $\varphi_j \circ \varphi_i^{-1}$ are $k$ times continuously differentiable. We say a manifold has a *smooth* structure if the transition functions are infinitely differentiable.

The existence of differentiable structures allows us to talk about differentiable and smooth maps between manifolds.

**Definition C.4.** If $M_1$ and $M_2$ are manifolds (say, of dimensions $m$ and $n$ respectively) then $f : M_1 \to M_2$ is said to be a *differentiable map* if for every chart $(U, \varphi)$ of $M_1$ and every chart $(V, \psi)$ of $M_2$ such that $f(U) \subset V$ the map $\psi \circ \varphi^{-1}$ from $\varphi(U) \subset \mathbf{R}^m$ into $\psi(V) \subset \mathbf{R}^n$ is $C^k$ as a map between Euclidean spaces (see section 2.1.1).

If we take $M_1 = J$ to be an open interval in $\mathbf{R}$ then a *differentiable curve* in $M_2$ is any differentiable map $\gamma$ of a closed interval in $M_1$ into $M_2$. For the rest of the appendix, curves will always be differentiable.

**Definition C.5.** Let $\gamma : [a, b] \to M$ be a smooth curve in $M$ that passes through the point $p$ at the time $t_0$. Let $\mathcal{F}$ be the set of differentiable functions from $M$ to $\mathbf{R}$. The *tangent vector to $\gamma$ at $p$* is a linear map $L_{\gamma,p}$ defined on $\mathcal{F}$ by the rule

$$L_{\gamma,p}(f) = \left.\frac{df(\gamma(t))}{dt}\right|_{t_0} \in \mathbf{R}.$$

**Remark C.2.** Tangent vectors give us a directional derivative on functions (we can use them to compute how fast a function is changing in a given direction). We can conveniently compute the tangent vector to a curve $\gamma$ at $p$ in terms of local coordinates at $p$. If $\gamma_i(t) = x_i \circ \varphi \circ \gamma(t)$ are the coordinates of the curve, and we know $f$ as a function of local coordinates at $p$, then the chain rule tells us that

$$\begin{aligned}\left.\frac{df(\gamma(t))}{dt}\right|_{t_0} &= \sum_{i=1}^{n} \left.\frac{df}{d(x_i \circ \varphi \circ \gamma)}\right|_{t_0} \left.\frac{d\gamma_i(t)}{dt}\right|_{t_0} \\ &= \sum_{i=1}^{n} \left.\frac{df}{d(x_i \circ \varphi)}\right|_{p=\gamma(t_0)} \left.\frac{d\gamma_i(t)}{dt}\right|_{t_0}\end{aligned}$$

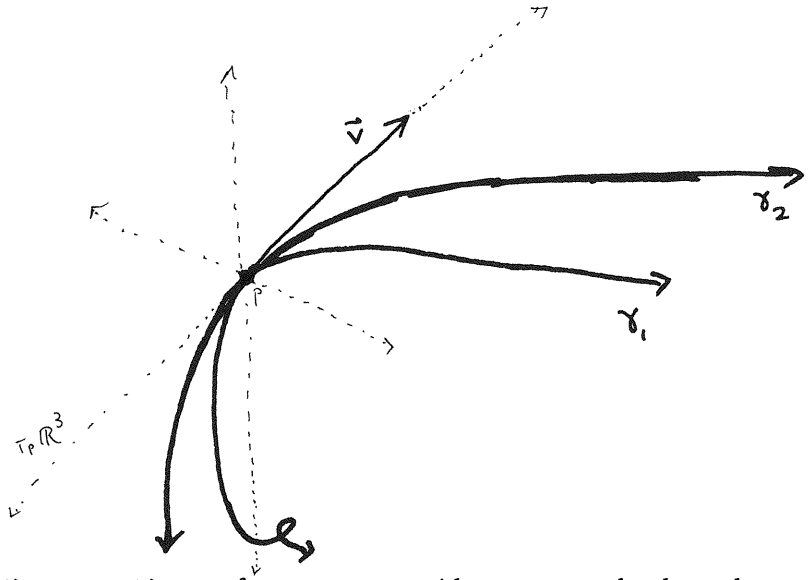



Figure 7.5: Picture of tangent space with two curves that have the same tangent vector.

$$= \sum_{i=1}^{n} \frac{df}{du_i}\bigg|_{p=\gamma(t_0)} \frac{d\gamma_i(t)}{dt}\bigg|_{t_0}.$$

**Definition C.6.** The *tangent space of $M$ at $p$*, denoted $T_pM$, is equal to the set of all tangent vectors of curves in $M$ through $p$.

**Remark C.3.** The computation of Remark C.2 shows that the tangent space at a point is an $n$-dimensional real vector space with basis $\{\frac{d}{du_i}\big|_p\}$. We frequently will identify this space with $\mathbf{R}^n$.

The functional definition of tangent vectors and tangent spaces given here is slightly less popular than the more geometric-seeming definition from subsection 2.1.1 in the text that identifies curves whose tangent vectors are the same to make the tangent space. The computation of Remark C.2 shows that these two definitions are equivalent, since if we regard tangent vectors as being given by a linear combination of '$\frac{d}{du_i}$'s, then to get a given tangent vector we must specify the right '$\frac{d\gamma_i(t)}{dt}$'s, that is, we must specify the derivatives found upon differentiating a curve in some equivalence class; whereas on the other hand, clearly if we specify on equivalence class of curves, we get a functional tangent vector (something that differentiates functions) using the given formula.

**Definition C.7.** The *tangent bundle $TM$* of a manifold is the union over $x$ in $M$ of all the the tangent spaces.

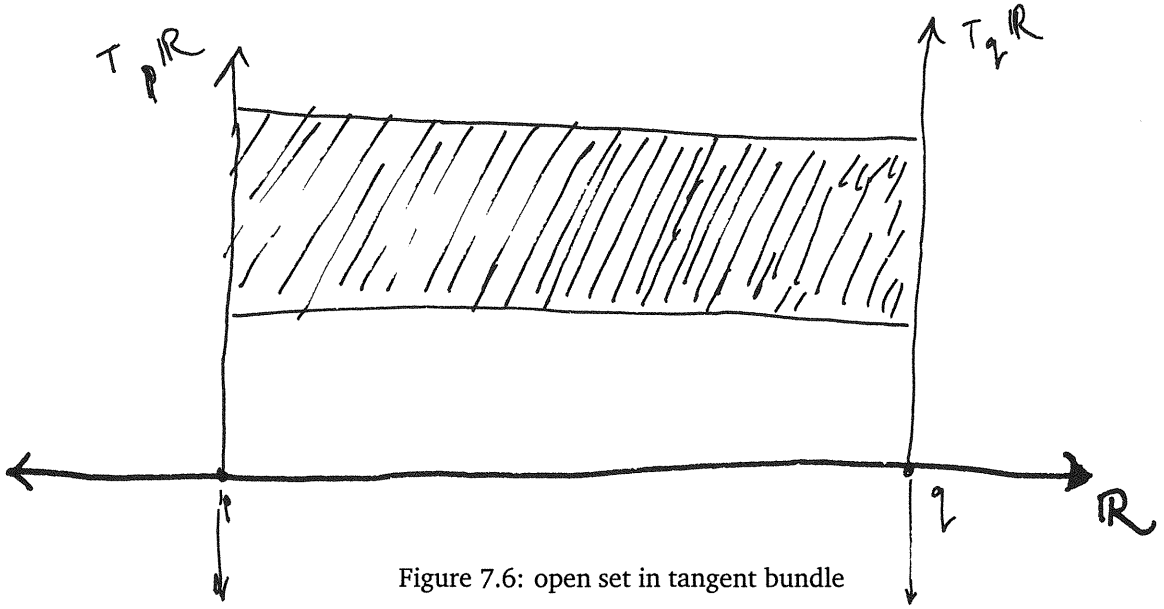



Figure 7.6: open set in tangent bundle

**Definition C.8.** We define the *standard projection operator* $\pi$ from the tangent bundle to the manifold which takes each vector to its base point.

**Remark C.4.** Typically we want to think of the tangent bundle as a manifold with a differentiable structure that can tell us something about the base manifold, not just as the set of all vectors tangent to the manifold 'bundled together' indiscriminately. We distiguish the points in $TM$ carefully by writing $TM = \{(p, v) : p \in M, v \in T_pM\}$; then a natural topology on $TM$ is induced by the topology of $M$, namely, an open set in $TM$ is given by $U \times V$ where $U$ is an open set of $M$ that comes from a chart and $V$ is an open set of $\mathbf{R}^n$. We are using the identification of the tangent spaces and $\mathbf{R}^n$. The picture is easy in one dimension; see Figure 7.6. The tangent bundle also has a natural smooth structure on it, since if $(U, \varphi)$ is a chart on $M$ then every element $u \in \{(p, v) : p \in U, v \in T_pM\} \subset TM$ can be written in the form $u = \sum_{i=1}^{n} a_i(v) \left.\frac{d}{du_i}\right|_{p=\pi(v)}$. The map

$$T_pM \ni u \mapsto (u_1(\pi(v)), \ldots, u_n(\pi(v)), a_1(v), \ldots, a_n(v)) \in \mathbf{R}^{2n}$$

is a homeomorphism from $\{(p, v) : p \in U, v \in T_pM\}$ to $\mathbf{R}^{2n}$.

One checks that the conditions of Definition C.3 are satisfied ([49, volume I, page 3-32]).

**Definition C.9.** A *vector field* on a manifold consists of a choice of one vector from each tangent space. Hence vector fields are functions from a base manifold to its tangent bundle. We write $\text{vect}(M) = \{\text{vector fields on } M\}$. Another way to talk about a vector field to call it a *section* of the tangent bundle, where the word section means a map $s$ from the manifold to the bundle such that $\pi \circ s$ is the identity. If $\mathcal{V}$ is a vector field on $M$, $\mathcal{V}|_x$ denotes the vector associated with $T_x M$.

**Remark C.5.** Since tangent vectors are differential operators, vector fields are too; they take differentiable functions defined on the manifold to functions on the manifold given by the directional derivatives specified by the tangent vector at each point. They also allow us to compute lots of other things, but as before, to compute anything really interesting, we want more structure.

**Definition C.10.** Refering to DefinitionC.3, we note that the smooth structure on $TM$ developed in Remark C.4 makes it possible to define

$$\text{vect}^k(M) = \{C^k \text{ vector fields on } M\}$$

and

$$\text{vect}^\infty(M) = \{C^\infty \text{ vector fields on } M\}.$$

We will usually just write $\text{vect}(M)$ and understand a degree of smoothness appropriate for the needs at hand.

**Remark C.6.** The idea of the nabla coming up is that somehow the derivative of a smooth section of $TM$ with respect to a smooth section of $TM$ is again a smooth section of $TM$. We don't necessarily expect that to happen; we have to work just the right way. (See Kobaishi and Nomizu [39] for a much more general way to go about doing this.)

**Definition C.11.** A *connection* is a map $\nabla : TM \times \text{vect}(M) \to TM$ which satisfies the following properties:

i. if $\xi \in TM, \mathcal{V} \in \text{vect}(M)$, then $\nabla(\xi, \mathcal{V})$ is in the same tangent space as $\xi$.

ii. if $a, b \in \mathbf{R}^n$, $x \in M$, $\xi, \eta \in T_x M$, and $\mathcal{V} \in \text{vect}(M)$, then

$$\nabla(a\xi + b\eta, \mathcal{V}) = a\nabla(\xi, \mathcal{V}) + b\nabla(\eta, \mathcal{V}) \tag{C.1}$$

iii. if $x \in M$, $\xi \in T_x M$, $\mathcal{V}, \mathcal{W} \in \text{vect}(M)$, and $f \in C^1(M)$, then

$$\nabla(\xi, \mathcal{V} + \mathcal{W}) = \nabla(\xi, \mathcal{V}) + \nabla(\xi, \mathcal{W}) \tag{C.2}$$

$$\nabla(\xi, f\mathcal{V}) = (\xi f)(\pi(\xi))\mathcal{V}|_{\pi(\xi)} + f(\pi(\xi))\nabla(\xi, \mathcal{V}) \tag{C.3}$$

iv. $\nabla$ is smooth in the sense that if $\mathcal{V}, \mathcal{W} \in \text{vect}^\infty(M)$ then $\cup_{x \in M} \nabla(\mathcal{V}|_x, \mathcal{W})$, which we call $\nabla(\mathcal{V}, \mathcal{W})$, is again in $\text{vect}^\infty(M)$.

**Remark C.7.** When it is clear from context where we are working, or when we want to be mobile, we use this more casual notation for the second part of item (iii):

$$\nabla(\xi, f\mathcal{V}) = (\xi f)\mathcal{V} + f\nabla(\xi, \mathcal{V}) \tag{C.4}$$

where it is understood that everything on the right-hand side that needs to be is to being evaluated at $\pi(\xi)$.

Definition C.11 is far from being the most general definition of a connection. In some ways it is much more like the definition of what is called a *covariant derivative associated with a connection*. We leave such matters aside; this definition will suffice for all our needs in this thesis. At any rate, it is clear from item (iii) above that whatever it is called, we now have a method for differentiating vector fields.

At this point we note that we must have

$$\nabla\left(\frac{d}{du_k}, \frac{d}{du_j}\right) = \sum_{i=1}^{n} \Gamma_{jk}^i \frac{d}{du_i} \tag{C.5}$$

for some set of '$\Gamma_{jk}^i$'s. (The order and placement of the various indices here is traditional.) If we 'know' a connection somehow, we know its action on all of $TM \times \text{vect}(M)$ and we can compute

these; alternatively, and more realistically, we can just specify them to begin with and get some connection as a result. In particular, this shows that connections exist.

In order to actually compute $\nabla(\xi, \mathcal{V})$ all we need to know is $\mathcal{V}$ restricted to some smooth curve $\gamma$ through $p = \pi(\xi)$ with tangent vector $\xi$ at $p$. In terms of the basis for the tangent space at $p$ that we chose in Remark C.1, $\xi = \sum_{i=1}^{n} \xi_i \frac{d}{du_i}$.

By the linearity in the first component (condition (ii) above) this means that

$$\nabla(\xi, \mathcal{V}) = \sum_{i=1}^{n} \xi_i \nabla(\frac{d}{du_i}, \mathcal{V}).$$

Since we know $\mathcal{V}$ along $\gamma$, for each $q \in \gamma$ we can express $\mathcal{V}$ at each point in terms of local coordinates at $q$ as $\mathcal{V}|_q = \sum_{i=1}^{n} \eta_i(q) \frac{d}{du_i}|_q$, or, more informally, $\mathcal{V} = \sum_{i=1}^{n} \eta_i \frac{d}{du_i}$. Combining this expression with the previous, we have

$$\nabla(\xi, \mathcal{V}) = \sum_{i=1}^{n} \xi_i \nabla \left( \frac{d}{du_i}, \sum_{j=1}^{n} \eta_j \frac{d}{du_j} \right).$$

We now enter a long chain of manipulations that will in the end give us a nice formula for $\nabla(\xi, \mathcal{V})$. First applying the first part of condition (iii) above

$$\sum_{i=1}^{n} \xi_i \nabla \left( \frac{d}{du_i}, \sum_{j=1}^{n} \eta_j \frac{d}{du_j} \right) = \sum_{i=1}^{n} \xi_i \sum_{j=1}^{n} \nabla \left( \frac{d}{du_i}, \eta_j \frac{d}{du_j} \right)$$

(and then the second part of condition (iii))

$$= \sum_{i=1}^{n} \xi_i \sum_{j=1}^{n} \left( \left( \frac{d\eta_i}{du_i} \right) \frac{d}{du_j} + \eta_j \nabla \left( \frac{d}{du_i}, \frac{d}{du_j} \right) \right)$$

(and the definition of the '$\Gamma_{jk}^{i}$'s)

$$= \sum_{i=1}^{n} \xi_i \left( \sum_{j=1}^{n} \left( \left( \frac{d\eta_j}{du_i} \right) \frac{d}{du_j} + \eta_j \sum_{k=1}^{n} \left( \Gamma_{ji}^{k} \frac{d}{du_k} \right) \right) \right)$$

(collecting like terms)

$$= \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \left( \left( \xi_i \frac{d\eta_j}{du_i} \right) \frac{d}{du_j} + \sum_{k=1}^{n} \left( (\xi_i \eta_j \Gamma_{ji}^{k}) \frac{d}{du_k} \right) \right) \right)$$

(rewriting the summation without quite so many parentheses)

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \left( \xi_i \frac{d\eta_j}{du_i} \right) \frac{d}{du_j} \right) + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \left( (\xi_i \eta_j \Gamma_{ji}^{k}) \frac{d}{du_k} \right)$$

(renaming one of the indices of summation does not effect the sum)

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\left(\xi_i \frac{d\eta_k}{du_i}\right)\frac{d}{du_k}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left((\xi_i\eta_j\Gamma_{ji}^k)\frac{d}{du_k}\right)$$

(and now rearranging the summation)

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n}\left(\left(\xi_i \frac{d\eta_k}{du_i}\right)\frac{d}{du_k}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\left((\xi_i\eta_j\Gamma_{ji}^k)\frac{d}{du_k}\right)\right)$$

(and finally, factoring out the elements of the basis)

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n}\left(\left(\xi_i \frac{d\eta_k}{du_i}\right)\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}(\xi_i\eta_j\Gamma_{ji}^k)\right)\frac{d}{du_k}$$

we get the promised formula for $\nabla(\xi, \mathcal{V})$. In terms of a curve $\gamma$ that goes through the point $p$ at time $t_0$ with the tangent vector $\xi$, this is

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\left(\frac{d(\eta_k \circ \gamma)}{dt}\bigg|_{t_0}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}(\xi_i\eta_j(p)\Gamma_{ji}^k(p))\right)\frac{d}{du_k}\bigg|_p . \tag{C.6}$$

So we have a word, 'connection', and we know that it is not a meaningless one, because all we need are some $n^3$ functions, the '$\Gamma_{ij}^k$'s to specify one. (The official, undescriptive and even misleading name for the '$\Gamma_{ij}^k$'s is the *Christoffel symbols* – I prefer to use the more descriptive but less popular name *connection coefficients*, but even this does not sufficiently stress the fact that these are functions that can change as we move around the manifold.) We also know a connection is not an *a priori* useless thing, because if we have a curve with the right tangent vector, we use it and the connection to compute something.

But the real reason we care about Definition C.11 will not be seen for a few pages, that is, until we work throught the most important example. Along the way we will introduce some further terminology that will take us closer to our goal, but understanding Definition C.11 will be our main objective for the moment. The main idea is that connections will enable us bring vectors from one tangent space to another tangent space to compare them. In this way a connection is sort of like a set of pnuematic tubes that run between every point on the manifold, through which we can send and receive tangent vectors. As one might guess things can happen to the tangent vectors while in

transit. In particular, one can send a tangent vector around a closed loop, and it will not necessarily come back the same as it was when it started. This is not a defect of the system, but one of its strengths.

For such a complicated structure to be useful, we first need a fruitful way to compare vectors in a fixed tangent space. There are a number of other things we could define without this added structure, but I suspect that they would all be better motivated if we make everything concrete first, rather than afterwards. In light of this change of mood, we will close the current subsection and begin a new one now.

## Riemannian manifolds

**Definition C.12.** A *Riemannian manifold* is a pair, a smooth manifold $M$, together with a map $g$ defined on the manifold that associates to each point an inner product, that is,

$$g(p) = \langle \cdot, \cdot \rangle_p : T_pM \times T_pM \to \mathbf{R};$$

$g$ must satisfy the additional property that for any open set $U \subset M$ and any fixed $\mathcal{V}, \mathcal{W} \in \text{vect}(U)$ the function

$$\langle \mathcal{V}, \mathcal{W} \rangle : U \ni p \mapsto g(p)(\mathcal{V}_p, \mathcal{W}_p) = \langle \mathcal{V}_p, \mathcal{W}_p \rangle_p \in \mathbf{R}$$

is differentiable on $U$. Any $g$ satisfing all of these conditions is called a *smooth Riemannian metric*.

**Remark C.8.** The idea of a Riemannian metric is (as the name suggests) the heart of Riemannian geometry. This is what lets us compare vectors at a point: in accordance with our experience from Euclidean space, which as we have remarked the tangent space *is*, the *length* of a vector $\xi$ based at the point $p$ is

$$|\xi|_p = \sqrt{\langle \xi, \xi \rangle_p}. \tag{C.7}$$

The *angle* between two vectors $\xi$ and $\eta$ based at the point $p$ is

$$\cos^{-1}\left(\frac{\langle \xi, \eta \rangle_p}{|\xi|_p |\eta|_p}\right). \tag{C.8}$$

The difference from what we might be used to is that the metric can change as we move from place to place. This is what makes comparing vectors from different tangent spaces such a subtle problem.

The Riemannian metric makes available to us for the first time some concrete geometric content. In addition to making local comparisons, we can begin to move around in the manifold and talk, for example, about the *length of a curve* $\gamma : [a, b] \to M$ in analogy with our experience in Euclidean space (see Chapter 2) as being

$$\int_a^b \left| \frac{d}{dt}\gamma(t) \right| \, dt. \tag{C.9}$$

If we define the *distance* between two points in the manifold by $d(p, q) = \inf_\gamma \text{length}(\gamma)$ then we get a distance metric on the manifold, and the manifold therefore can be viewed as a metric space. (Note that Riemannian metrics and distance metrics are very, very different things!)

As we made explicit in the section on Algebraic Preliminaries, each inner product on a finite dimensional vector space is determined by a matrix. We will write the matrix associated with the inner product at the point $p$ as $\left(g_{ij}(p)\right)$ where the '$g_{ij}(p)$'s are determined by the inner products of basis vectors as in remark A.2. We write $(g_{ij})$ when we want to be mobile.

The inverse matrix often comes up in computations. It is traditional to write $(g_{ij})^{-1} = (g^{ij})$. Since these matrices are inverses, the sum $\sum_{i=1}^{n} g^{ij} g_{jk} = \delta_{ik}$.

We are getting very close to an explicit statement of the 'correct' way to compare tangent vectors from different tangent spaces; in fact, there is just one more definition we need before we can state our big theorem.

**Definition C.13.** For $U \subset M$ an open subset, if $\mathcal{V}, \mathcal{W} \in \text{vect}(U)$ we define their *Lie Bracket* or *commutator* $[\mathcal{V}, \mathcal{W}]$ to be the unique vector field such that

$$\forall p \in U : [\mathcal{V}, \mathcal{W}]|_p = \mathcal{V}|_p \circ \mathcal{W}|_p - \mathcal{W}|_p \circ V|_p.$$

**Remark C.9.** Note that in this and in the previous definition, the reason we talk about an open set $U$ is that we will want to use local vector fields a lot of the time.

**Theorem C.1 (Levi-Civita).** If $M$ is a Riemannian manifold then there exists a unique connection $\nabla^{\mathrm{LC}}$ for which all $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathrm{vect}(M)$ satisfy

i. $\nabla^{\mathrm{LC}}(\mathcal{V}, \mathcal{W}) = \nabla^{\mathrm{LC}}(\mathcal{W}, \mathcal{V}) + [\mathcal{V}, \mathcal{W}]$

ii. $\mathcal{U}\langle \mathcal{V}, \mathcal{W}\rangle = \langle \nabla^{\mathrm{LC}}(\mathcal{U}, \mathcal{V}), \mathcal{W}\rangle + \langle \mathcal{V}, \nabla^{\mathrm{LC}}(\mathcal{U}, \mathcal{W})\rangle$.

**Remark C.10.** 'LC' stands of course for 'Levi-Civita'; $\nabla^{\mathrm{LC}}$ is called the *Levi-Civita connection for the Riemannian metric on* $M$. The two conditions say respectively that the Levi-Civita connection is *torsion-free* (in the sense that commutation is respected; this implies that the '$\Gamma^k_{ij}$'s depend on the metric in a particularly simple fashion, and are, as we will see, symmetric under exchange of $i$ and $j$) and that it is *compatible with the metric* in the sense that it behaves like a standard derivative of the metric (for example, if $v$ and $w$ are tangent vectors to a curve, $\frac{d\langle v,w\rangle}{dt} = \left\langle \frac{dv}{dt}, w\right\rangle + \left\langle v, \frac{dw}{dt}\right\rangle$).

*Proof of Theorem C.1.* Assuming such a connection $\nabla^{\mathrm{LC}}$ exists, we compute like mad (see Table C.1). Upon collecting like terms we have that

$$\begin{aligned}\langle \nabla^{\mathrm{LC}}(\mathcal{U}, \mathcal{V}), \mathcal{W}\rangle = \frac{1}{2}\big(&\mathcal{U}\langle \mathcal{V}, \mathcal{W}\rangle + \mathcal{V}\langle \mathcal{W}, \mathcal{U}\rangle - \mathcal{W}\langle \mathcal{U}, \mathcal{V}\rangle \\ &- \langle \mathcal{W}, [\mathcal{V}, \mathcal{U}]\rangle + \langle \mathcal{U}, [\mathcal{W}, \mathcal{V}]\rangle - \langle \mathcal{V}, [\mathcal{U}, \mathcal{W}]\rangle\big)\end{aligned} \tag{C.10}$$

which gives us the exact form of $\langle \nabla^{\mathrm{LC}}(\mathcal{U}, \mathcal{V}), \mathcal{W}\rangle$ and therefore fixes $\nabla^{\mathrm{LC}}(\mathcal{U}, \mathcal{V})$.

To get the form explicitly, we have to compute the '$\Gamma^k_{ij}$'s in terms of the metric; condition (i) of the theorem implies that

$$\nabla^{\mathrm{LC}}\left(\frac{d}{du_j}, \frac{d}{du_k}\right) = \nabla^{\mathrm{LC}}\left(\frac{d}{du_k}, \frac{d}{du_j}\right) + \left[\frac{d}{du_j}, \frac{d}{du_k}\right] \tag{C.11}$$

but $\left[\frac{d}{du_j}, \frac{d}{du_k}\right] = \frac{d}{du_j} \circ \frac{d}{du_k} - \frac{d}{du_k} \circ \frac{d}{du_j} = 0$ by equality of mixed partials*, so by definition (Equation C.5) we have $\Gamma^k_{ij} = \Gamma^k_{ji}$. Then Equation C.10 implies that

$$\left\langle \nabla^{\mathrm{LC}}\left(\frac{d}{du_i}, \frac{d}{du_j}\right), \frac{d}{du_k}\right\rangle = \frac{1}{2}\left(\frac{d}{du_i}\left\langle \frac{d}{du_j}, \frac{d}{du_k}\right\rangle + \frac{d}{du_j}\left\langle \frac{d}{du_k}, \frac{d}{du_i}\right\rangle\right.$$

*That is all this is, really.

$$
\begin{aligned}
& -\frac{d}{du_k}\left\langle \frac{d}{du_i}, \frac{d}{du_j} \right\rangle - \left\langle \frac{d}{du_k}, \left[ \frac{d}{du_j}, \frac{d}{du_i} \right] \right\rangle \\
& + \left\langle \frac{d}{du_i}, \left[ \frac{d}{du_k}, \frac{d}{du_j} \right] \right\rangle - \left\langle \frac{d}{du_j}, \left[ \frac{d}{du_i}, \frac{d}{du_k} \right] \right\rangle \Big) \\
= \; & \frac{1}{2}\left( \frac{d}{du_i}\left\langle \frac{d}{du_j}, \frac{d}{du_k} \right\rangle + \frac{d}{du_j}\left\langle \frac{d}{du_k}, \frac{d}{du_i} \right\rangle \right. \\
& \left. -\frac{d}{du_k}\left\langle \frac{d}{du_i}, \frac{d}{du_j} \right\rangle \right) \\
= \; & \frac{1}{2}\left( \frac{d}{du_i} g_{jk} + \frac{d}{du_j} g_{ki} - \frac{d}{du_k} g_{ij} \right) \qquad \text{(C.12)}
\end{aligned}
$$

But by the definition of the '$\Gamma^k_{ij}$'s and the properties of inner products, we also have

$$
\begin{aligned}
\left\langle \nabla^{\mathrm{LC}}\left( \frac{d}{du_i}, \frac{d}{du_j} \right), \frac{d}{du_k} \right\rangle &= \left\langle \sum_{i=1}^{n} \Gamma^i_{jk} \frac{d}{du_i}, \frac{d}{du_k} \right\rangle \\
&= \sum_{i=1}^{n} \left\langle \Gamma^i_{jk} \frac{d}{du_i}, \frac{d}{du_k} \right\rangle \\
&= \sum_{i=1}^{n} \Gamma^i_{jk} \left\langle \frac{d}{du_i}, \frac{d}{du_k} \right\rangle \\
&= \sum_{i=1}^{n} \Gamma^i_{jk} g_{ki} \qquad \text{(C.13)}
\end{aligned}
$$

Cleverly, we note that

$$
\sum_{k=1}^{n} g^{kl}\left( \sum_{i=1}^{n} \Gamma^i_{jk} g_{ki} \right) = \sum_{k=1}^{n}\sum_{i=1}^{n} g^{kl} g_{ki} \Gamma^i_{jk} = \sum_{k=1}^{n} \delta_{ik} \Gamma^i_{jk} = \Gamma^i_{jk}. \qquad \text{(C.14)}
$$

So combining everything we must have

$$
\Gamma^i_{jk} = \frac{1}{2}\sum_{j=1}^{n} g^{kj}\left( \frac{d}{du_i} g_{jk} + \frac{d}{du_j} g_{ki} - \frac{d}{du_k} g_{ij} \right) \qquad \text{(C.15)}
$$

On the other hand, these '$\Gamma^i_j k$'s specify a connection, and the reverse of the argument just given shows that this connection satisfies the conditions of the theorem. QEF

**Corollary C.1.** The connection coefficients for the Levi-Civita connection in $\mathbf{R}^n$ are all zero.

*Proof.* In Euclidean space the coefficients of the metric (the '$g_{ij}$'s) are all constant. Therefore in equation C.15 we are differentiating constants and summing. The result of this operation is always zero. QEF

$$
\begin{aligned}
\langle \nabla^{\mathrm{LC}}(u,v), w \rangle &= u\langle v, w\rangle - \langle v, \nabla^{\mathrm{LC}}(u,w)\rangle \\
&= u\langle v, w\rangle - \langle v, \nabla^{\mathrm{LC}}(w,u)\rangle - \langle v, [u,w]\rangle\rangle \\
&= u\langle v, w\rangle - w\langle v,u\rangle + \langle \nabla^{\mathrm{LC}}(w,v),u\rangle - \langle v,[u,w]\rangle\rangle \\
&= u\langle v, w\rangle - w\langle v,u\rangle + \langle \nabla^{\mathrm{LC}}(w,v),u\rangle + \langle [w,v],u)\rangle - \langle v,[u,w]\rangle\rangle \\
&= u\langle v, w\rangle - w\langle v,u\rangle + v\langle w,u\rangle - \langle w, \nabla^{\mathrm{LC}}(v,u)\rangle + \langle [w,v],u\rangle - \langle v,[u,w]\rangle\rangle \\
&= u\langle v, w\rangle - w\langle v,u\rangle + v\langle w,u\rangle - \langle w, \nabla^{\mathrm{LC}}(v,u)\rangle - \langle w,[v,u]\rangle + \langle [w,v],u\rangle - \langle v,[u,w]\rangle\rangle.
\end{aligned}
$$

Table C.1: Computing the form of the Levi-Civita connection

Having gone through the hard work of defining the 'right' connection, we can now begin to put our system to work. (We will then see why this connection is the right one.)

**Definition C.14.** Given a chart $(U, \varphi)$ and a smooth curve $\gamma : [a, b] \to U \subset M$ such that $U \supset \gamma$ we define the *derivative of a vector field* $\mathcal{V}$ *along* $\gamma$ in coordinates as follows: we write $\mathcal{V}(t) = \sum_{j=1}^{n} \xi_j(t) \frac{d\gamma}{du_j}|_{\gamma(t)}$ and then write

$$\nabla^{\mathrm{LC}}\left(\frac{d\gamma}{dt}, \mathcal{V}(t)\right) = \sum_{i=1}^{n}\left(\left(\left.\frac{dxi_i}{dt}\right|_{t_0} + \sum_{j=1}^{n}\sum_{k=1}^{n}(\Gamma^i_{jk} \circ \gamma)\xi_j(t_0)\left(\left.\frac{d\gamma_j}{dt}\right|_{t_0}\right)\right)\left.\frac{d\gamma}{du_i}\right|_{\gamma(t_0)}\right). \tag{C.16}$$

**Remark C.11.** The same formal definition works for other connections, in fact the definition is essentially the same, but since the connection coefficients are coming from the Levi-Civita connection, this derivative respects the metric and is torsion free.

**Definition C.15.** If $\gamma : (a, b) \to M$ is a path in $M$, we say that $\mathcal{V}$ is *parallel along* $\gamma$ if

$$\nabla^{\mathrm{LC}}\left(\frac{d\gamma}{dt}, \mathcal{V}(t)\right) = 0 \tag{C.17}$$

for all $t \in (a, b)$.

**Definition C.16.** We say a path $\gamma : (a, b) \to M$ is a *geodesic* if

$$\nabla^{\mathrm{LC}}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) = 0 \tag{C.18}$$

for all $t \in (a, b)$, *i.e.* if the tangent vector to $\gamma$ is parallel along $\gamma$ on the entire interval on which $\gamma$ is defined.

**Remark C.12.** To write the equation for a geodesic in local coordinates, we set $\gamma_j = x_j \circ \gamma$ as above. Then our definition for the derivative of a vector field along a curve (Definition C.14) implies that Equation C.18 becomes

$$\sum_{i=1}^{n}\left(\left(\left.\frac{d^2xi_i}{dt^2}\right|_{t_0} + \sum_{j=1}^{n}\sum_{k=1}^{n}(\Gamma^i_{jk} \circ \gamma)\left(\left.\frac{d\gamma_j}{dt}\right|_{t_0}\right)\left(\left.\frac{d\gamma_j}{dt}\right|_{t_0}\right)\right)\left.\frac{d\gamma}{du_i}\right|_{\gamma(t_0)}\right) = 0. \tag{C.19}$$

We can rewrite this second order equation as a first order system, and after that, rewrite the condition of being a geodesic in terms of the condition of being an integral curve for a certain vector field.

To each chart $(U, \varphi)$ on $M$ we associate a chart $(Q, \psi)$ on $TM$ by putting $Q = \pi^{-1}(U)$ and

$$\psi(\xi) = (\varphi \circ \pi(\xi), \xi(u_1), \ldots, \xi(u_n)) \in \mathbf{R}^{2n} \tag{C.20}$$

on $Q$. We write $q(\xi) = \varphi \circ \pi(\xi) \in \mathbf{R}^n$ and $\dot{q}(\xi) = (\xi(u_1), \ldots, \xi(u_n)) \in \mathbf{R}^n$. Using this notation we have

$$\xi = \sum_{j=1}^{n} \dot{q}_j(\xi) \left.\frac{d}{du_j}\right|_{\pi(\xi)}. \tag{C.21}$$

The differential equation above can be rewritten as

$$\frac{dq_j}{dt} = \dot{q}_j \tag{C.22}$$

$$\frac{d\dot{q}_j}{dt} = -\sum_{j=1}^{n}\sum_{k=1}^{n}(\Gamma^i_{jk} \circ \pi)\dot{q}_j\dot{q}_k. \tag{C.23}$$

The solution to this system are integal curves of the vector field $\mathcal{G}$ defined on $Q$ by

$$\mathcal{G} = \sum_{i=1}^{n}\left(\dot{q}_j\frac{d}{dq_j} - \sum_{j=1}^{n}\sum_{k=1}^{n}(\Gamma^i_{jk} \circ \pi)\dot{q}_j\dot{q}_k\frac{d}{d\dot{q}_i}\right). \tag{C.24}$$

**Definition C.17.** Let $\varphi(t, \xi)$ denote the geodesic flow of the vector field $\mathcal{G}$ of the previous remark (*i.e.* the map that tames points in $M$ to the points in the direction given by $\mathcal{G}$).

Let

$$\gamma_\xi : [0, 1] \ni t \mapsto \pi \circ \varphi(t, \xi) \in M.$$

Let $T^1M$ denote

$$\{\xi \in TM : 1 \in \text{maximal interval on which } \gamma_\xi \text{ is defined}\}.$$

We define

$$\exp : T^1M \ni \xi \mapsto \gamma_\xi(1) \in M.$$

**Remark C.13.** Frequently the manifolds we work with in this thesis will be *complete*, that is, their geodesics will run on forever (although they may, as in the case of spheres, come back and trace over themselves again, or, as in the case of tori, come back and intersect themselves infinitely often). *In the case of complete manifolds,* exp *is defined on all of* $TM$.

We note that one sometimes sees the related definition

$$\exp t\cdot : T^1M \ni \xi \mapsto \gamma_\xi(t) \in M, \tag{C.25}$$

for example in the proof of Theorem 2.1.

## Differential forms

Recall from the section on algebraic preliminaries that the dual of a finite dimensional vector space is the set of linear maps from that vector space to the base field, and that it is isomorphic to the original vector space. Since tangent spaces are finite dimensional real vector spaces, it is very easy to talk about their duals, which are called cotangent spaces[†]. Just to point out exactly where things stand, tangent vectors are a special kind of linear maps from functions on the manifold to the real numbers – and now we have cotangent vectors which are all of the linear maps taking that set of maps into the real numbers! The basis of the cotangent space that is dual to the standard basis of the tangent space is denoted $\{du^i\}$.

**Definition C.18.** The *cotangent bundle* of a manifold is the union over $x$ in $M$ of all the the cotangent spaces.

If we wanted to, we could write this in the careful way we wrote the tangent bundle above. All of the same ideas apply. In particular, we can define sections of the cotangent bundle. We are especially interested in the smooth sections, which we give a special name.

[†]This name has even less to do with the cotangent function from trigonometry than tangent spaces have to do with the tangent function – which is, admittedly, something, but not a lot.

**Definition C.19.** A smooth $m$-section of the cotangent bundle (that is, a choice of $m$ vectors in each cotangent space that varies smoothly with respect to the smooth structure on the cotangent bundle) is called a *differential $m$-form* (this is usually just shortened to $m$-form). We extend this in a slightly odd way by say that smooth functions on the manifold are *zero forms*[‡]. The collection of all $m$-forms ($m \in 0, 1, \ldots$) is the set of *differential forms* (or sometime, just *forms*). After lumping them all together, we sort them back out again by saying a differential form has *degree* $m$ if it is a $m$-form.

The fundamental binary operation on differential forms is the $\wedge$ product ('wedge product'). This is defined by the action of differential 1-forms on tangent vectors and its important because it defines the action of differential $m$-forms for higher $m$ on lists of tangent vectors, *i.e.* on sections of the tangent bundle. (The key point is that the $m$-form is associated with the wedge product of all of the 1-forms that make it up.)

**Remark C.14.** The standard short-hand for lists of $m$ vectors is *$m$-vectors*.

If $\omega$ is a $m$-form and $\upsilon$ is a $p$-form, their wedge product $\omega \wedge \upsilon$ takes an $(m+p)$-vector $(\xi_1, \ldots, \xi_{m+p})$ to the number

$$\frac{1}{m!p!} \sum_{\sigma \in \mathfrak{S}^{m+p}} \operatorname{sign}(\sigma) \omega(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(m)}) \upsilon(\xi_{\sigma(m+1)}, \ldots, \xi_{\sigma(m+p)}). \tag{C.26}$$

If $\omega$ and $\upsilon$ are *1-forms* this complicated expression reduces to

$$\sum_{\mathfrak{S}^2} \operatorname{sign}(\sigma) \omega(\xi_{\sigma(1)}) \upsilon(\xi_{\sigma(2)}) = \omega(\xi_1)\upsilon(\xi_2) - \omega(\xi_2)\upsilon(\xi_1); \tag{C.27}$$

In the case that one of the factors is a zero form, things get even easier since the wedge product reduces to standard multiplication.

In the case of a one form and a two form, the expression is almost as simple as it is for two one forms; and from here, various properties can be deduced[§] by induction.

[‡]Actually, I think that this might not be *so* odd; it might make sense for it to be standard practice to say that smooth functions are zero sections of any bundle

[§]Or, induced.

There are three fundamental unary operations on differential forms; their names are $\iota$, $d$, and $\delta$.

Actually, to be precise, $\iota$ is map defined from vector fields *into* the space of operators on differential forms. The definition is that $\iota(\mathcal{V})$ takes $m$-forms to $(m-1)$-forms that act by the following rule:

$$(\iota(\mathcal{V})\omega)(\xi_1, \ldots, \xi_{m-1}) = \omega(\mathcal{V}, \xi_1, \ldots, \xi_{m-1}).$$

The map $\iota$ is called the *substitution operator*. (At last, a nice, descriptive name! – the reason we use the *symbol* '$\iota$' is perhaps less clear at this point; the reason is that $\iota$ is associated with integration.)

The operator $d$ is called the *coboundary operator*; it is a map from $m$-forms to $(m+1)$-forms, and it is defined by these properties:

1. $d(\omega + \upsilon) = d\omega + d\upsilon$

2. $d(\omega \wedge \upsilon) = d\omega \wedge \upsilon + (-1)^{\text{degree}(\omega)} \wedge d\upsilon$

3. $dd\omega = 0$

4. for all zero forms $f$, $df = \sum_{i=1}^{n} \frac{df}{du_i} du_i$

The '$d$' appearing at the beginning of '$du_i$' is a special case of this '$d$'.

At this point we go on to define a couple of essential operations that have something to do with forms. *The following two definitions are almost impossible to keep straight, so pay attention*[¶].

**Definition C.20.** The *pushforward* of a map $\varphi : M \to N$ is a map $\varphi_* : TM \to TN$ defined so that so that if $\gamma$ is a curve in $M$ with tangent vector $\xi$ at time $t_0$, then $\varphi_*$ is the map that takes $\xi \in TM$ to

[¶] In the spirit of SOHCAHTOA, the acronym BOTCGNTTOFFEPF (Bottom On Tangents, Curve Gives New Tangent; Top On Forms, Form Eats Push Forward) may serve as a mneumonic tool to help the reader remember which one is which, and what they do.)

the vector in $TN$ that is tangent to the curve $\varphi(\gamma(t_0))$.

The map $\varphi_*$ is sometimes called the *differential of* $\varphi$ and written as $d\varphi$. One can explore the relationship of this '$d$' to the one above.

**Definition C.21.** The *pullback* of a differential form is defined whenever we have a smooth map $\varphi$ between two manifolds, say $\varphi : M \to N$. If $v$ is a form on $N$, then $\varphi^*(v)$ is a form on $M$ determined by

$$(\varphi^*(v))(\xi_1, \ldots, \xi_m) = v(\varphi_*\xi_1, \ldots, \varphi_*\xi_m).$$

Since have been talking about a *coboundary* operator, it might seem that we should have already at least mentioned the *boundary* operator. In fact, we have used this operator throughout the thesis, although we have not been terribly explicit about what we were doing. The boundary operator we are refering to just finds boundary of subsets in the manifold, in symbols, $\partial A = \bar{A} \setminus A^\circ$ for all $A \subset M$. It is a basic topological property that

$$\partial\partial A = \emptyset \tag{C.28}$$

for all subsets $A$. We are using the prefix 'co' approximately as we used it to talk about the cotangent bundle, *i.e.* we are talking about objects that are dual to one another in some sense. So far, we have defined the coboundary operator to act on the space of differential forms, which in turn act on the space of sections. We now use the definition of the action of differential forms on sections to define their action on domains. The way we let a differential $m$-form $\omega$ act on a $m$-dimensional domain $C$ is by integrating; we define $\int_\Omega \omega$ to be $\int_{\{p\in\Omega\}} \omega(\xi_1(c), \ldots, \xi_n(c))$, where $\{\xi_k(p)\}_{k=1}^n$ is the ordered basis for $T_p\Omega$.

The coboundary operator $d$ is dual to the boundary operator $\partial$ in the sense that

$$\int_{\partial\Omega} \omega = \int_\Omega d\omega. \tag{C.29}$$

This important equation has a name: it is called Stokes' Theorem. A nice proof of Stokes' Theorem is written down in [18]. We note that in the special case that $\Omega$ is the boundary of some set

$A$, C.28 implies that the left hand side of C.29 is zero. Hence, we deduce that the integral over a boundary of the coboundary of any form is zero.

Our last word on special operators applies to the operator $\delta$. It extremely interesting, and we can not get away without mentioning that it is extremely useful for talking about the shape and structure of manifolds, and that there is a dual operator that does the same thing for domains. But that is where we have to stop.

There are various inner products and norms defined on differential forms, and we close this section with a list of the most important ones.

**Definition C.22.** We say an $m$-form is *simple* if it can be written as a one-term sum.

We can define an inner product between simple forms $\omega = w_1 \wedge \ldots \wedge w_m$ and $v = v_1 \wedge \ldots \wedge v_m$ based at $p$ by

$$\langle\langle \omega, v \rangle\rangle = \sum_{\sigma \in \mathfrak{S}} \operatorname{sign}(\sigma) \prod_{k=1}^{m} \langle w_{\sigma(k)}, v_k \rangle_p. \tag{C.30}$$

(We will only use the double angle bracket notation in the appendix.)

The term on the right of Equation C.30 is clearly equal to

$$\sum_{\sigma \in \mathfrak{S}} \operatorname{sign}(\sigma^{-1}) \prod_{k=1}^{m} \langle w_k, v_{\sigma^{-1}(k)} \rangle_p,$$

and so our definition is symmetric. Linearity of the standard inner product on vectors extends the definition to all $m$-vectors. The norm induced by this inner product is called the *Riemannian norm*.

**Remark C.15.** Since we can easily take an $m$-vector and define a $m$-form (by taking the wedge product of all of the vectors in the list), we can use the above definition to define a product between $m$-forms and $m$-vectors, which we also denote by $\langle\langle \cdot, \cdot \rangle\rangle$.

**Definition C.23.** The *mass* of a differential $m$-form $\omega$ is

$$\operatorname{mass}(\omega) = \sup\{\langle\langle \omega, \Xi \rangle\rangle; \Xi \text{ is an } m-\text{vector, and } \operatorname{comass}(\Xi) \leq 1\}$$

where comass is about to be defined.

**Definition C.24.** The *comass* of a $m$-vector $\Xi$ is

$$\text{comass}(\Xi) = \sup\{\langle\langle \omega, \Xi \rangle\rangle; \omega \text{ is an } m-\text{form}, \xi \text{ is simple , and } |\omega| \leq 1\}$$

(here $|\cdot|$ denotes the Riemannian norm).

### Isometries

**Definition 7.1.** A map $\varphi$ from $M$ to $M$ is an *isometry* if for all $p \in M$, if $\xi, \eta \in T_pM$ then $g(\xi, \eta) = g(\varphi_*\xi, \varphi_*\eta)$.

We state without proof the fact that the set of isometries for a manifold form a group; this group is appropriately called the *isometry group*.

**Definition C.25.** The *quotient* of a manifold by its isometry group is the set obtained by identifying each point with the point(s) in its orbit(s) under the isometry group.

## D Analytic Preliminaries

**Charter** Review of the basic results of Analysis germane to our discussion.

### Measures

Note the similarities between the following definition and the definition of a topological space (see appendix on Topological Preliminaries).

**Definition D.1.** *Measureable space* This is a pair $(X, \mathcal{M})$ such that $\mathcal{M}$ is comprised of subsets of $X$ and has the further additional properties

i. $X \in \mathcal{M}$.

ii. The complement in $X$ of any $A \in \mathcal{M}$ is again in $\mathcal{M}$.

iii. countable unions of elements of $\mathcal{M}$ are again in $\mathcal{M}$.

If $\mathcal{M}$ satisfies all of these properties, it is called a *$\sigma$-algebra*. The elements of $\mathcal{M}$ are then the *measurable sets of $X$ with respect to $\mathcal{M}$* or just *measurable sets*.

**Definition D.2.** A *measure* is a function $\mu$ taking a $\sigma$-algebra $\mathcal{M}$ into the complex numbers $\mathbf{C}$ with the property that $\mu(\emptyset) = 0$, and that satisfies an additional criterion, namely that for all countable disjoint collections $\{A_i; A_i \in \mathcal{M}\}$,

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Functions with this property are said to satisfy *countable additivity*.

**Remark D.1.** Reasonably, we say that such a function $\mu$ is a *real* measure if its image is contained in the reals. Slightly less reasonably, but rather, according to convention, we say that a measure $\mu$ is *positive* if its image lies in the interval $[0, \infty]$.

The definition of a positive measure is unreasonable for two reasons: the first is easy to catch, namely it might be better to call such measures *non-negative* rather than positive, because zero is not positive. The second reason is that even though we allow $\infty$ into the range of a positive measure, $\infty \notin \mathbf{C}$, so by the above definition for measure it would never be hit. We make the special dispensation that $\mu$ may assign $\infty$ as the measure of a set as long as $\mu$ is positive.

We put this all together into one main definition as follows.

**Definition D.3.** A *measure space* is a measurable space $(X, \mathcal{M})$ together with a measure $\mu$ defined on $\mathcal{M}$.

**Remark D.2.** The definition of a measurable space is frequently written out as a triple, $(X, \mathcal{M}, \mu)$, rather than as a pair $((X, \mathcal{M}), \mu)$. Per usual, if $S = (X, \mathcal{M}, \mu)$ is a measure space, $p \in S$ means $p \in X$ (and in particular, a map defined on $S$ is a map defined on $X$.).

The structure of maps on measure spaces are understood in terms of the measurable sets of the underlying $\sigma$-algebra.

**Definition D.4.** A mapping from a measure space $(X, \mathcal{M}, \mu)$ into a topological space $(Y, \tau)$ is *measurable* if the inverse image of every open set $U \in \tau$ is an element of $\mathcal{M}$, *i.e.*, is a measurable set.

**Remark D.3.** When talking about a measure space, we say that a set is $\mu$-measurable if it is in the underlying $\sigma$-algebra. The terms 'almost everwhere', 'almost all', or, more specifically, '$\mu$-almost all', '$\mu$-almost everywhere', etc. when used to describe some property of a measure space mean that the property holds at all points except some set of $\mu$-measure zero.

**Example.** A *simple function* on a measure space $S$ is a map taking $S$ into $[0, \infty) \subset \mathbf{C}$ that has only finitely many points in its range.

Since the range of a simple function $\rho$ defined on $S$ is finite, $S$ splits into a finite number of subsets $A_1, \ldots, A_n$ such that $\rho(A_1) = a_1, \ldots, \rho(A_n) = a_n$. We can therefore write

$$\rho = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where the '$\chi$'s are *indicator functions*, defined in general by $\chi_A : S \to \{0, 1\}, p \mapsto 1$ if $p \in A$, $p \mapsto 0$ else.

A simple function $\rho$ is measurable if and only if all of the '$A_i$'s are measurable, since the inverse image of an open interval will be a union of some subset of the '$A_i$'s and so measurable if all of the '$A_i$'s are by item (iii) of the definition of a $\sigma$-algebra; on the other hand, if $A_1$, say, is not measurable, then the inverse image of some small open interval about $a_1$ is equal to $A_1$ and $\rho$ then not measurable.

(We can show that there is a sequence of simple functions that approximate any given measurable function if need be.)

## Abstract integration

Rudin [45] treats this very thoroughly; the reader is refered to him for most of the proofs.

**Definition D.5.** If $S = (X, \mathcal{M}, \mu)$ is a measure space, $E \in \mathcal{M}$ and $\rho = \sum_{i=1}^{n} a_i \chi_{A_i}$ is a measurable simple function on $S$, we define the *integral of $\rho$ over $E$ with respect to the measure $\mu$* by

$$\int_E \rho \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i \cap E).$$

In general, if $f : S \to [0, \infty]$ is measurable, we define the *(Lebesgue) integral of $f$ over $E$ with respect to the measure $\mu$* by

$$\int_E f \, d\mu = \sup_{\substack{\rho \text{ simple}, \\ 0 \le \rho \le f}} \int_E \rho \, d\mu.$$

In the case of measurable simple functions, it is easy to see that the two definitions give the same result

We get all the standard properties we expect of integrals (*i.e.*, integration respects majorization, is linear, etc.). These things are fairly easy to see for simple functions (we use countable additivity as needed) and then they carry over to the rest of the measurable positive functions because we can approximate such functions pointwise from below by a sequence of simple functions (so the supremum in the definition of the integral is ). A proof that we can approximate measurable functions by simple functions in this manner is not at all difficult, but for our purposes a sketch will suffice: the idea is to approximate $f$ from below wherever its values lie in $[0, n]$ by a simple function whose accuracy of approximation depends on $n$. Wherever $f$ takes values greater than $n$, it is approximated by a step of size $n$. As $n$ grows, we specify a finer and finer approximation on a growing finite interval, and so we converge wherever $f$ takes finite values. If $f$ takes the value infinity at some point or on some interval, it is approximated there by a sequence of steps of size $n$.

We also get a number of new consequences that show the superiority of the Lebesgue integral to the Riemann integral.

**Theorem D.1 (Monotone Convergence).** Let $\{f_n\}$ be a sequence of measurable functions on a measure space $S = (X, \mathcal{M}, \mu)$ and suppose that

i. $0 \leq f_1(x) \leq \ldots \leq \infty$ for every $x \in S$.

ii. $f_n(x) \to \infty$ as $n \to \infty$ for every $x \in S$.

Then $f$ is measurable and $\int_S f_n \, d\mu \to \int_S f \, d\mu$.

**Theorem D.2 (Fatou's Lemma).** If $\{f_n : S \to [0, \infty]\}$ is a sequence of measurable functions on a measure space $S = (X, \mathcal{M}, \mu)$, then for each positive integer $n$ we have

$$\int_S \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_S f_n \, d\mu$$

We now extend the definition of integration to cover integration of complex functions. This is the general definition of the Lebesgue integral.

**Definition D.6.** The *Lebesgue integral* over any measurable set is defined by breaking the function into its positive real, negative real, positive imaginary, and negative imaginary parts (which then all give positive functions), and integrating each seprarately, then multiplying by $1, -1, i$, or $-i$ as appropriate and summing.

Associated with this is the class of 'integrable functions'.

**Definition D.7.** The space $L^1(\mu)$ is the set of functions defined on all of $S = (X, \mathcal{M}, \mu)$ with the property that

$$\int_S |f| \, d\mu < \infty.$$

**Theorem D.3 (Dominated Convergence).** Suppose that $\{f_n : S \to \mathbf{C}\}$ is a measurable sequence of functions on a measure space $S = (X, \mathcal{M}, \mu)$ with the property that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in S$. We call this function $f(x)$. If there is a function $g \in L^1(\mu)$ such that for all positive integers $n$ and all $x \in S$ we have

$$|f_n(x)| \leq g(x)$$

then we conclude the following properties must hold for $f$:

1. $f \in L^1(\mu)$
2. $\lim_{n\to\infty} \int_S |f_n - f|\, d\mu = 0$
3. $\lim_{n\to\infty} \int_S f_n\, d\mu = \int_S f\, d\mu$.

Finally, we state one other related theorem:

**Theorem D.4 (Fubini's Theorem).** Let $(X, S, \mu)$ and $(Y, T, \lambda)$ be $\sigma$-finite measure spaces and let $f$ be an $(S \times T)$-measureable function on $X \times Y$.

- If $0 \leq f \leq \infty$ and if for all $x \in X$ and $y \in Y$ we have

$$\varphi(x) = \int_Y f_x\, d\lambda$$

and

$$\psi(x) = \int_X f^y\, d\mu$$

then $\varphi$ is $S$-measurable, $\psi$ is $T$-measurable, and

$$\int_X \varphi\, d\mu = \int_{X\times Y} f\, d(\mu \times \lambda) = \int_Y \varphi\, d\lambda$$

- If $f$ is complex and if

$$\varphi*(x) = \int_Y |f|_x\, d\lambda$$

and

$$\int_X \varphi*\, d\mu < \infty$$

then $f \in L^1(\mu \times \lambda)$

- If $f \in L^1(\mu \times \lambda)$, the $f_x \in L^1(\lambda)$ for almost all $x \in X$, $f^y \in L^1(\mu)$ for almost all $y \in Y$ and the functions $\varphi$ and $\psi$

## Lebesgue measure

**Definition D.8.** A *Borel measure* is a measure on open sets.

**Theorem D.5.** Up to a constant multiple, there is a unique translation invariant positive Borel measure on $\mathbf{R}^n$.

**Remark D.4.** This measure is called the *Lebesgue measure*. In general, an isometry invariant measure is called a *Harr measure*. Each of $\mathbf{S}^n$ and $\mathbf{H}^n$ has, again up to a constant multiple, a unique Harr measure.

**Remark D.5.** Note that the Harr measures are in some sort of abstract sense like the Levi-Civita connection. Both respect the basic geometrical structures of the spaces they are defined on.

## Miscellaneous

**Definition D.9.** The lim inf (lim sup) of a sequence of functions is...

**Definition D.10.** A function $f$ is *upper semi-continuous with respect to a metric* $\delta$, if

$$\delta(a_k, a) \to 0 \Rightarrow \limsup_{k\to\infty} f(a_k) \leq f(a).$$

**Definition D.11 (Alternative definition, see Rudin, p. 39).** A function $f$ on a manifold $M$ is *upper semi-continuous* if $\{x : f(x) < \alpha\}$ is open for every real $\alpha$.

**Definition D.12.** A function or form or whatever has *compact support* if the set on which it takes nonzero values is compact.

**Definition D.13.** A function from one metric space $(X)$ to another $(Y)$ is said to satisfy a Hölder condition of order $\alpha$ or to be *Hölder continuous* if there exist $M$ and $\alpha$ such that for all $x, y \in X$ we have $d_Y(x, y) < M(d_X(x, y))^\alpha$.

**Definition D.14.** A function is *Lipschitz* if the $\alpha$ in Definition D.13 is equal to one.

**Definition D.15.** A function is *Hölder contiuously differentiable* or $C^{1,\alpha}$ if is differentiable and the differentiated function satisfies a Hölder condition of order $\alpha$.

**Definition D.16.** A *semi-norm* is a real valued function $p(x)$ defined on a vector space such that

i. $0 \leq p(x) < \infty$,

ii. $p(x, y) \leq p(x) + p(y)$,

iii. $p(ax) = |a|p(x)$.

**Definition D.17.** A function is *real analytic* in a neighborhood of $p$ if there is a convergent power series with real coefficients with center $p$ such that at every point in the neighborhood the value of the function and the sum of the power series coincide. (Complex analytic functions are defined similarly.)

# E Variational Calculus

**Charter.** We should understand how to find extrema for functionals defined on a set of geometric objects.

We won't be able to fully deal with this charter, but we will do what we can. We begin with a definition that has little or nothing to do with variational calculus.

**Definition E.1.** The *second fundamenatal form* of an $n$-dimensional manifold $M$ is defined in terms of local coordinates at a point $p \in M$ by using the fact that by embedding $M$ in some higher-dimensional $\mathbf{R}^N$, we may view $M$ locally as a graph of some function $f_p : T_pM \to \mathbf{R}$, where at each

point $\mathbf{R}$ is viewed as a line in the direction of some normal vector to $M$ at $p$. Specifically, we put

$$\mathbf{Sec}_p(M) = \begin{pmatrix} \frac{\partial^2 f_p}{\partial^2 x_1}\Big|_p & \cdots & \frac{\partial^2 f_p}{\partial^2 x_1 \partial^2 x_m}\Big|_p \\ \frac{\partial^2 f_p}{\partial x_n \partial x_1}\Big|_p & \cdots & \frac{\partial^2 f_p}{\partial^2 x_n}\Big|_p \end{pmatrix} \tag{E.31}$$

where $x_n, \ldots, x_n$ are coordinates for $T_pM$. The matrix $\mathbf{Sec}_p(M)$ defines a bilinear form on $T_pM$, which we call by the same name.

**Remark E.1.** Since $\mathbf{Sec}_p(M)$ is symmetric, we can choose a basis for $T_pM$ so that it becomes diagonal (see for example [20, page 266], and think about the geometrical content of the Corollary there). The numbers on the diagonal are called the *principal curvatures of $M$ at $p$* and are traditionally labeled $\kappa_1, \ldots, \kappa_n$.

**Definition E.2.** The *mean curvature of $M$ at $p$* is defined to be

$$H = \text{trace}(\mathbf{Sec}_p(M)).$$

The name 'mean curvature' might make more sense of course if $H$ was defined to be $\text{trace}(\mathbf{Sec}_p(M))$; however, several nice equalities depend our definition.

**Definition E.3.** Given a smooth, compactly supported vectorfield $\mathcal{V}$ defined on $M$ *first variation of area for a manifold with respect to* $\mathcal{V}$ is defined to be

$$\delta^1(M) = \frac{d}{dt}\Big|_{t=0} \text{area}(\text{M} + \text{t}\mathcal{V}),$$

where $M + t\mathcal{V}$ means the geodesic flow at time $t$ of points in $M$ in the directions given by $\mathcal{V}$.

The first variation of area measures the infinitesimal change in area due to an infinitesimal change in the shape of the manifold.

**Theorem E.1.** The first variation of area is given by integrating against mean curvature as follows:

$$\delta^1(M) = -\int_M \mathcal{V} \cdot H\nu,$$

where $\nu$ is a unit normal to $M$

We need a simple lemma.

**Theorem E.1.** Changing the radius of a circle of radius $r = 1/\kappa$ by $dr$ changes the arclength of an infinitesimal sector from $ds$ to $(1 - \kappa dr)ds$.

*Proof.* Putting the new radius equal to $r' = 1/k - dr$ and the length of the new sector equal to $(ds)'$, we need to prove that $\frac{r'}{r} = \frac{(ds)'}{ds}$. But $ds = rd\theta$, so $(ds)' = r'd\theta$. QEF

*Proof of Theorem E.1.* Since integration is linear, we can decompose $\mathcal{V}$ into normal and tangential components, and integrate each separately. In the case of a purely tangential variation, the surface does not change area, so the theorem is correct. In the case of a purely normal variation, we may write $\mathcal{V} = V\nu$; then, by refering the the preceeding lemma to see what happens in each direction, we find that an infinitesimal area $du_1 \wedge \ldots \wedge$ is distorted to

$$(1 - V\kappa_1)du_1 \wedge \ldots \wedge (1 - V\kappa_n)du_n, \tag{E.32}$$

where the '$\kappa_i$'s are the principal curvatures. Equation E.32 is equal to

$$(1 - V\nu \cdot H\nu)du_1 \wedge \ldots \wedge \tag{E.33}$$

*i.e.*

$$(1 - \mathcal{V} \cdot H\nu)du_1 \wedge \ldots u_n \wedge . \tag{E.34}$$

The initial rate of change in area is measured by integrating this over the manifold and then subtracting off the integral of $du_1 \wedge \ldots u_n \wedge$ over the manifold. The theorem follows. QEF

With the idea of a variation now in mind, we might, if we had time, go on to prove lots of things. For example, geodesics can be studied from the variational viewpoint. However, we don't have time for that.

The main thing we are interested in showing is that $H = d(Area)/d(Volume)$; we use the result of Theorem E.1, and state this as a proposition.

**Theorem E.1.** The mean curvature of a surface $S$ at a point is given by the derivative of area with respect to volume $d(Area)/d(Volume)$.

*Proof.* Since we are concerned with the behavior at a point, we can assume that the change in volume comes from a variation vector field that is defined on an infinisimal piece of area, and so can be considered constant. From Theorem E.1, we then have $dA/dt = H \int_S f$. Since everything is inifintesimal, we have $dV/dt = \int_S f$. The chain rule $dA/dV = (dA/dt)(dt/dV) = (dA/dt)(dV/dt)$ then finishes the proof. QEF

This proposition means that for our problem, we can think of mean curvature as an exchange rate (how much area is obtained for a certain amount of volume, or vice versa). A little thought then shows that the each of the surfaces separating two given components of a minimizer for the least-area problem with volume constraints have *globally* constant mean curvature. Of course, as with the standard double bubble, different pieces of surface may have different curvatures, because as we showed at the end of Chapter 2, curvature is proportional to pressure difference.

The ideas behind the following two definitions are undercurrents of the entire thesis:

**Definition E.4.** A surface is *stable* for the least-area problem if small variations do not decrease area.

**Definition E.5.** A surface is in *equilibrium* for the least-area problem if for any variation that fixes volumes, the first variation of area is zero.

This has been a very quick and incomplete introduction to the subject of variational calculus, giving us only the bare minimum of what we need for this thesis. There are many places to read more; [?] is good for further applications to area-minimization and geometric measure theory. In particular, there (in Part B) we find a very general and elegant description of the category of variational problems – which again, there is not time to talk about here.

# indices

## Dramatis Personæ

| | |
|---|---|
| $\mathbf{R}^n$ | $n$-dimensional Euclidean space |
| $\mathbf{S}^n$ | $n$-dimensional Spherical space |
| $\mathbf{H}^n$ | $n$-dimensional Hyperbolic space |
| $\mathbf{D}_p(r)$ | a disk of radius $r$ centered at $p$ |
| $B$ | a double bubble |
| $I$ | an isoperimetric profile |
| $\mathit{\Pi}$ | a two-volume least-area function |
| $\mathbf{G}^n$ | Gauss space |
| $Y$ | the standard partition of $\mathbf{G}^n$ into three equal pieces |
| $\mathbf{T}^2$ | a two-torus |
| $\mathbf{T}^3$ | a three-torus |
| $\Theta$ | a standard double bubble |
| $\Xi$ | a double band |
| $\Phi$ | a band lens |
| O | an octagon square |

*the characters of the appendix*

| | |
|---|---|
| $X$ | a set of points |
| $\tau$ | a topology |
| $M$ | a manifold |
| $p$ | a point in $M$ |
| $T_pM$ | the tangent space to $M$ at $p$ |
| $f$ | a differentiable map |
| $\mathcal{M}$ | a $\sigma$-algebra |

*Also necessary throughout are various numbers, unary and binary operators, delimiters, etc.*

The play has been performed in its entirity only once, in the Great Oklahoma Theater, where it was directed by Joseph K. (see [26]). That production included a cast of thousands. An abriged version is to be presented in the Soo Bong Chae Memorial Auditorium, May 6, 2002.

# Index of notational conventions used in this thesis

## Euclidean space

Many authors (Dubrovin *et al.* [11] for example) use a notation such as $\mathbf{E}^n$ for what I am calling $\mathbf{R}^n$, and they write their definitions in terms of the structure of the isometry group. However, in one of the more interesting turns of mathematical fate not treated it this thesis due to time constraints, we can *recover* the isometry group structure from the geometrical/vector space definition.

## Subscripts

Whenever possible, I avoid them and write things out using parentheses. Exceptions include a finite enumerated set, a countable enumerated set, and an uncountable set indexed by a real parameter,

and vector spaces associated with points on a manifold ($T_pM$, $\mathcal{N}_pM$, etc.)

This results in a non-standard notation for connections and derivations.

## Volumes and Regions

We denote a volume by a lower case letter, and a given region containing the volume by the corresponding upper case letter. For example, if we are considering a double bubble containing volumes $v$ and $w$ in $\mathbf{S}^3$ we let $u = |\mathbf{S}^3| - v - w$ be the volume of '$U$', the complement of the double bubble. The space has then been divided into three regions, $U, V$ and $W$. Similarly with other compact spaces. For non-compact spaces, we typically refer to the complement of a bubble cluster as *the exterior*.

Note that the space itself does not have to follow the casing rule, e.g. as in this paragraph we will use $\mathbf{S}^3$ to denote the unit radius three-sphere, and $|\mathbf{S}^3|$ to denote its volume.

Finally, we note that if we start with $V$ (an old-style '$U$') and then go to $W$, it is only fitting to have the $m^{\text{th}}$ volume be $V^m$.

## Proofs

QEF stands for *quad erat faciendum* which translates as 'what was to be done'. (QED stands for *quad erat demonstratum*, 'what was to be demonstrated'.) Both used to be used, and were even found interspersed in the same document sometimes. One text that used both is Galileo's *Dialogs Concerning Two New Sciences* [15]. I use QEF to mark the end of proofs first as an homage to that book, and, secondly, because I feel that that if the thesis demonstrates anything, it does so as a whole, and certainly not as frequently as after every proof.

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## Technical details.

The final form of this thesis was composed using *xemacs*, LaTeX, *dvips*, and *xmodmap* on a Hewlett-Packard Pavillion 6330 running *Mandrake Linux 8* together with a keyboard from Kinesis Corporation. The line figures were produced using the *pstricks* package for LaTeX, except for the various conjectured minimizers appearing in Figure 7.1 which were drawn by *Surface Evolver*, and for a few which were drawn by hand.

Raytraced pictures of the tiled Euclidean plane and of the sphere appearing in Chapter 1 do so with the permission of their creators (in theory). The screenshot of *Asteroids* was taken with *xv* and edited slightly with the *GIMP*.

The phase diagrams in Chapter 6 were produced with *Mathematica*. The phase diagram in Chapter 7 was produced using *Surface Evolver*, *perl*, *GNUPlot*, and the *GIMP*. Computing the data for this phase plot in *Surface Evolver* took about 8 hours on 26 Dell PCs running in parallel.

**Typeface.** The text of this thesis is set in the "Charter" font available in standard LaTeX. Not only is this font nice and readible, but its name makes it fit nicely into one of the major themes of the thesis.